ON THE VOLUME AND ENERGY OF SECTIONS OF A CIRCLE BUNDLE OVER A COMPACT LIE GROUP

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Let G be a compact simply connected semisimple Lie group endowed with a bi-invariant Riemannian metric and let $E \to G$ be a vector bundle with two-dimensional fibers and a G-invariant metric connection (generically, it has no parallel unit sections). We prove that if E carries the Sasaki metric, then the constant unit sections are exactly those of minimum volume and minimum energy among all smooth sections of the associated circle bundle.

Gluck and Ziller proved in the much cited paper [5] that Hopf vector fields on S^3 are exactly those having minimum volume among all unit vector fields. This article motivated the study of the volume, and later of the energy [11,12] of unit tangent fields on various Riemannian manifolds, mainly the critical points of the functionals (see the abundant bibliography on the subject for instance in the references to [4]). Recently, Brito [1] proved the analogue of the result by Gluck and Ziller for the energy instead of the volume. We are interested in a natural generalization, namely, volume and energy of sections of sphere bundles. An important source of examples is the following:

Let G be a Lie group with Lie algebra $\mathfrak{g}$. Let $\mathcal{V}$ be a finite dimensional vector space with an inner product and $\mathfrak{o}(\mathcal{V})$ the set of all skew-symmetric endomorphisms of $\mathcal{V}$. Let $E = G \times \mathcal{V} \to G$ be the trivial vector bundle. For $v \in \mathcal{V}$, let $L_v : G \to E$ be the "constant" section $L_v(g) = (g, v)$.

Given a linear map $\theta : \mathfrak{g} \to \mathfrak{o}(\mathcal{V})$, there exists a unique connection ∇ on $E \to G$ such that

$$(\nabla_Z L_v)(g) = L_{\theta(Z)v}(g)$$

for all $g \in G$ and all left invariant vector fields Z on G. Moreover, the connection is metric. If G has a left invariant Riemannian metric, one can define on E the canonical Sasaki metric induced by ∇, in such a way that the map

$$(d\pi, \mathcal{K})_\xi : T_\xi E \to T_g G \times E_g$$

is a linear isometry for each $\xi \in E$ (here $g = \pi(\xi)$ and $\mathcal{K}$ is the connection operator associated with ∇). The vector bundle $E \to G$ with the connection above and the Sasaki metric is called the *Riemannian vector bundle over G induced by θ*.

For example, the situation in the paper of Gluck and Ziller is a particular case of the preceding: Think of S^3 as the Lie group of unit quaternions with the canonical bi-invariant (round) metric. If E is the Riemannian vector bundle over S^3 induced by

$$\theta = \tfrac{1}{2}\,\mathrm{ad} \;= \tfrac{1}{2}\, d\,\mathrm{Ad} \tag{1}$$

and ℓ_g denotes left invariant multiplication by g, then

$$F : E \to TG, \quad F\,(g, v) = d\ell_g\,(v)$$

is an affine vector bundle isomorphism, and moreover an isometry if E and TG carry the corresponding Sasaki metrics. Via this isomorphism, Hopf vector fields are congruent to constant unit sections. Notice that the involved vector space is three-dimensional, so the result of Gluck and Ziller is not a consequence of the result in this paper, which deals with circle bundles.

As far as we know, the following results are the unique ones concerning minima of the volume and energy of sections of sphere bundles, apart from the trivial case where parallel unit sections exist.

a) A detailed study on the minimum of the volume and energy of unit vector fields on tori can be found in [8] and [11], respectively.

b) Hopf vector fields on S^3 are exactly those unit vector fields on S^3 with minimum volume [5] and minimum energy [1].

c) In [9] we prove an analogue of the main result of [5], also in the setting above, for the Riemannian vector bundle over S^3 induced by $\theta = \frac{1}{2}\, d\rho$ (cf (1)), where $\mathcal{V}$ is the algebra of quaternions and ρ is the representation of S^3 on $\mathcal{V}$ given by $\rho\,(g)\,h = hg^{-1}$ (quaternionic multiplication). We also use calibrations.

d) For $n \geq 1$, no unit tangent vector field on S^{2n+1} has minimum energy [2,3].

e) The two distinguished left invariant unit vector fields on a Berger three-sphere are exactly those of minimum energy [6].

In this note we obtain one more result in this direction:

Theorem. *Let G be a compact simply connected semisimple Lie group endowed with a bi-invariant Riemannian metric and let $E \to G$ be the Riemannian vector bundle over G induced by a linear map $\theta : \mathfrak{g} \to \mathfrak{o}\,(\mathcal{V})$, where $\mathcal{V}$ is a two-dimensional vector space with an inner product. Then the constant unit sections are exactly those of minimum volume and minimum energy among all smooth unit sections.*

Corollary. *Under the hypotheses of the theorem, if $\theta \neq 0$ there are no parallel unit sections.*

Remark. In fact, the proof shows that a stronger statement is true: The result is still valid if the metric is only *left* invariant and some (or any) generator Z of $(\operatorname{Ker}\theta)^{\perp}$ satisfies that $\exp(tZ)$ acts on G on the *right* by isometries (or equivalently, Z is a Killing vector field of G).

Critical points of the energy of unit sections of Riemannian vector bundles induced by a map θ as above (and more general) have been studied in [10].

Given a smooth unit section $V : G \to E$, the volume of V is defined to be the volume of the submanifold $V(G)$ of E, with the induced metric. A section is in particular a map from a Riemannian manifold to another, hence one has the notion of the energy $\mathcal{E}(V)$ of the section V. As in the case of vector fields, there exist constants c_1 and c_2, depending only on the dimension and the volume of G, such that

$$\mathcal{E}(V) = c_1 + c_2\,\mathcal{B}(V)\,, \tag{2}$$

where $\mathcal{B}(V) = \int_G \|\nabla V\|^2\,\omega$ is the total bending of V on G (here $(\nabla V)_g : T_gM \to E_g$, $\|\nabla V\|^2 = \operatorname{tr}(\nabla V)^*(\nabla V)$, the star denotes transpose and integration is taken with respect to the volume form ω associated to the Riemannian metric of G).

Lemma. *Let G be a compact Riemannian manifold and $f : G \to \mathbf{R}$ a smooth function satisfying $\int_G f\,\omega = 0$. Then*

$$\int_G \sqrt{1+(f+c)^2}\,\omega \geq \operatorname{vol}(G)\,\sqrt{1+c^2} \tag{3}$$

and
$$\int_G (f+c)^2\,\omega \geq c^2\operatorname{vol}(G) \tag{4}$$

for any constant $c \in \mathbf{R}$. Moreover, equality holds if and only if f vanishes identically.

Proof. We may suppose without loss of generality that G has unit volume. We verify (3): The function $\phi(x) = \sqrt{1+x^2}$ is convex, hence, by Jensen's inequality [7] we have

$$\int_G \sqrt{1+(f+c)^2}\,\omega \geq \phi\left(\int_G (f+c)\,\omega\right) = \sqrt{1+c^2}.$$

The other inequality follows in a similar manner, using instead of ϕ the convex function $\psi(x) = x^2$. In both cases, since ϕ and ψ are strictly convex, equality holds if and only if $f + c$ (or equivalently f) is constant, and this clearly happens if and only if $f \equiv 0$. $\square$

We prove the Theorem by carefully computing and taking appropriate lower bounds, as in [1].

Proof of the Theorem. We may suppose that $\theta \neq 0$, otherwise for a unit section it is equivalent being parallel, constant and realizing the minimum of both functionals. Let $\{u, v\}$ be an orthonormal basis of $\mathcal{V}$ and $\lambda \in \mathfrak{g}^*$ satisfying $\theta(Z)u = \lambda(Z)v$ and $\theta(Z)v = -\lambda(Z)u$ for all $Z \in \mathfrak{g}$. Let V be a smooth unit section of $E \to G$. Since G is simply connected, there exists a smooth function $\alpha : G \to \mathbf{R}$ such that

$$\begin{aligned} V(g) &= (g, \cos(\alpha(g))\, u + \sin(\alpha(g))\, v) \\ &= \cos\alpha(g)\, L_u(g) + \sin\alpha(g)\, L_v(g)\,. \end{aligned} \tag{5}$$

By the definition of ∇, we have for any left invariant vector field Z on G that

$$\nabla_Z L_u = L_{\theta(Z)u} = \lambda(Z)\, L_v \quad \text{and} \quad \nabla_Z L_v = L_{\theta(Z)v} = -\lambda(Z)\, L_u. \tag{6}$$

From (5) and (6) we have

$$\nabla_Z V = (Z(\alpha) + \lambda(Z))\,(-(\sin\alpha)\, L_u + (\cos\alpha)\, L_v)\,. \tag{7}$$

On the other hand, let Z_1 be a unit generator of $(\operatorname{Ker}\theta)^\perp = (\operatorname{Ker}\lambda)^\perp$. We compute

$$Z_1(\alpha)\,\omega = d\alpha(Z_1)\,\omega = d\alpha \wedge i_{Z_1}\omega = d(\alpha\, i_{Z_1}\omega) - \alpha.d\,(i_{Z_1}\omega)$$

(i denotes interior multiplication). By Stokes' Theorem,

$$\int_G Z_1(\alpha)\,\omega = -\int_G \alpha.d\,(i_{Z_1}\omega) = -\int_G \alpha.(\operatorname{div} Z_1)\,\omega = 0, \tag{8}$$

since $\operatorname{div} Z_1 = 0$ (Z_1 is a Killing vector field on G because $\exp(tZ_1)$ acts on G on the right by isometries).

At each $g \in G$ we have the operator $(\nabla V)_g : T_gG \to E_g = \{g\} \times \mathcal{V} \cong \mathcal{V}$. In the following we do not write explicitly the basepoint g. Let $\{Z_1, \ldots, Z_n\}$ be an orthonormal left invariant parallelization, where Z_1 is as above, and let A be the matrix of ∇V with respect to the bases $\{Z_1, \ldots, Z_n\}$ and $\{u, v\}$. By (7) and the choice of Z_1, we have

$$A = \begin{pmatrix} -(\sin\alpha)\, X \\ (\cos\alpha)\, X \end{pmatrix},$$

with $X = (Z_1(\alpha) + \lambda(Z_1), Z_2(\alpha), \ldots, Z_n(\alpha))$. Hence $A^*A = X^*X$ and so, if x_i is the i-th component of X, we have

$$(A^*A)_{i,j} = x_i x_j. \tag{9}$$

Energy: By (2), instead of the energy of V, we may consider its total bending, which is given by

$$\mathcal{B}(V) = \int_G \|\nabla V\|^2 \omega = \int_G \mathrm{tr}\,(A^*A)\,\omega = \int_G \sum_{i=1}^n x_i^2\,\omega.$$

Therefore, by (8) and (4) with $f = Z_1(\alpha)$ and $c = \lambda(Z_1)$, we have

$$\mathcal{B}(V) \geq \int_G x_1^2\,\omega = \int_G (Z_1(\alpha) + \lambda(Z_1))^2\,\omega \geq \mathrm{vol}\,(G)\,\lambda(Z_1)^2,$$

which equals the total bending of any constant unit section.

Volume: As for tangent bundles, we have

$$\mathrm{vol}\,(V) = \int_G \sqrt{\det\left(\mathrm{Id}\; + (\nabla V)^*(\nabla V)\right)}\,\omega,$$

where Id is the identity on TG. By (9), if e_i is the i-th element of the canonical basis of $\mathbf{R}^n$, we have

$$\begin{aligned}
\det\,(\mathrm{I} + A^*A) &= \det\,(e_1 + x_1X, \ldots, e_n + x_nX)\\
&= \det\,(e_1, \ldots, e_n) + \sum_{i=1}^n \det\,(e_1, \ldots, e_{i-1}, x_iX, e_{i+1}, \ldots, e_n)\\
&= 1 + x_1^2 + \cdots + x_n^2,
\end{aligned}$$

since det is n-linear and vanishes whenever two entries are proportional.

Therefore, by (8) and (3), with $f = Z_1(\alpha)$ and $c = \lambda(Z_1)$, we have

$$\begin{aligned}
\mathrm{vol}\,(V) &= \int_G \sqrt{1 + x_1^2 + \cdots + x_n^2}\\
&\geq \int_G \sqrt{1 + x_1^2}\\
&\geq \mathrm{vol}\,(G)\,\sqrt{1 + \lambda(Z_1)^2},
\end{aligned}$$

which equals the volume of any constant unit section.

By the Lemma, both for the total bending and the volume, equality holds if and only if $Z_i(\alpha) = 0$ for all $i = 1, \ldots, n$, that is, if and only if α (or equivalently the unit section V) is constant. □

Proof of the Corollary. Looking at the expressions for the volume and the energy of unit sections, it is clear that these functionals attain the minimum at parallel unit sections, provided they exist. By the Theorem they can exist only if $\theta = 0$, since otherwise the constant unit sections are not parallel. $\square$

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