On the energy of sections of trivializable sphere bundles

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Abstract

Let $E \to M$ be a vector bundle with a metric connection over a Riemannian manifold M and consider on E the Sasaki metric. We find a condition for a section of the associated sphere bundle to be a critical point of the energy among all smooth unit sections. We apply the criterion to some particular cases where M is parallelizable, for instance $M = S^7$ or a compact simple Lie group G with a bi-invariant metric, and E is the trivial vector bundle with a connection induced by octonian multiplication or an irreducible real orthogonal representation of G, respectively. Generically, these bundles have no parallel unit sections.

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Introduction

Beginning with G. Wiegmink and C. M. Wood [5, 6], critical points of the energy of unit tangent fields have been extensively studied (see for instance in [1] the abundant bibliography on the subject). We are interested in a

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natural generalization, namely, critical points of the energy of sections of sphere bundles.

Let $\pi : E \to M$ be a vector bundle with a metric connection ∇ over an oriented Riemannian manifold, that is, each fiber has an inner product depending smoothly on the base point and

$$Z \langle V, W \rangle = \langle \nabla_Z V, W \rangle + \langle V, \nabla_Z W \rangle$$

for all vector fields Z on M and all smooth sections V, W of E.

On E one can define the canonical Sasaki metric associated with ∇ in such a way that the map

$$(d\pi, \mathcal{K})_\xi : T_\xi E \to T_q M \times E_q$$

is a linear isometry for each $\xi \in E$ (here $q = \pi(\xi)$ and $\mathcal{K}$ is the connection operator associated with ∇).

Let $\pi : E \to M$ be as before and denote by $E^1 = \{\xi \in E \mid \|\xi\| = 1\}$ the associated sphere bundle. Let N be a relatively compact open subset of M with smooth (possibly empty) boundary. Given a smooth section $V : M \to E^1$, the total bending of V on N is defined by

$$\mathcal{B}_N(V) = \int_N \|\nabla V\|^2,$$

where $(\nabla V)_p : T_p M \to E_p$, $\|\nabla V\|^2 = \operatorname{tr}(\nabla V)^*(\nabla V)$ and integration is taken with respect to the volume associated to the Riemannian metric of M.

Consider on E the Sasaki metric. As in the case of vector fields, there exist constants c_1 and c_2, depending only on the dimension and the volume of N, such that the energy $\mathcal{E}_N$ of the section V, thought of as map $V : N \to E$, is given by

$$\mathcal{E}_N(V) = c_1 + c_2 \, \mathcal{B}_N(V).$$

In the following we refer to the energy of the section instead of the bending, since that is a subject more commonly studied. In every example we will be concerned with the nonexistence of parallel unit sections, since they are trivial minima of the functional.

Definition. A smooth section $V : M \to E^1$ is said to be a harmonic section if for every relatively compact open subset N of M with smooth (possibly empty) boundary, V is a critical point of the functional $\mathcal{B}_N$ (or equivalently, of the energy $\mathcal{E}_N$) applied to smooth sections W of M satisfying $W|_{\partial N} = V|_{\partial N}$.

Notice that a harmonic section may be not a harmonic map from M to E^1 (see for example [2, 3], where $E = TM$).

The rough Laplacian Δ acts on smooth sections of E as follows:

$$(\Delta V)(p) = \sum_{i=1}^{n} (\nabla_{Z^i} \nabla_{Z^i} V)(p),$$

where $\{Z^i \mid i = 1, \ldots, n\}$ is any section of orthonormal frames on a neighborhood of p in M satisfying $(\nabla_{Z^i} Z^j)(p) = 0$ for all i, j.

Theorem 1 *Let $\pi : E \to M$ be a vector bundle with a metric connection over an oriented Riemannian manifold and consider on E the associated Sasaki metric. The section $V : M \to E^1$ is a harmonic section if and only if there is a smooth real function f on M such that*

$$\Delta V = fV.$$

Remark. This condition was proved for the particular case where E is the tangent bundle, by Wiegmink [5] and Wood [6] for compact manifolds and by Gil-Medrano [1] for general (not necessarily compact) manifolds (with a different presentation). Their proofs can be adapted to the present more general case.

Applications

Let M be a parallelizable manifold with a fixed parallelization $\{X^1, \ldots, X^n\}$. Let $\mathcal{V}$ be a finite dimensional vector space with an inner product and $\mathfrak{o}(\mathcal{V})$ the set of all skew-symmetric endomorphisms of $\mathcal{V}$. Let $E = M \times \mathcal{V} \to M$ be the trivial vector bundle. For $v \in \mathcal{V}$, let $L_v : M \to E$ be the "constant" section $L_v(p) = (p, v)$.

Proposition 2 *Given a map $\theta : \{X^1, \ldots, X^n\} \to \mathfrak{o}(\mathcal{V})$, there exists a unique connection ∇ on $E \to M$ such that*

$$(\nabla_{X^i} L_v)(p) = L_{\theta(X^i)v}(p) \tag{1}$$

for all $p \in M$ and all $i = 1, \ldots, n$. Moreover, the connection is metric.

Proof. Let $\{v_1, \ldots, v_n\}$ be an orthonormal basis of $\mathcal{V}$. Let $X \in T_pM$ and $\sigma : M \to E$ be a smooth section. Then

$$X = \sum_{i=1}^{m} a_i X^i(p) \qquad \text{and} \qquad \sigma = \sum_{j=1}^{n} f_j L_{v_j}$$

for some numbers a_i and smooth functions $f_j : M \to \mathbf{R}$. A standard computation shows that

$$(\nabla_X \sigma)(p) = \sum_{i=1}^{m} \sum_{j=1}^{n} a_i \left(X_p^i (f_j) L_{v_j}(p) + f_j(p) \left(L_{\theta(X^i)v_j} \right)(p) \right)$$

defines a connection on E satisfying condition (1), which is metric since $\theta(X^i)$ is skew-symmetric for all i. □

Examples.

(1) The Levi-Civita connection of a Lie group G with a left invariant Riemannian metric may be obtained in this way: Let $\mathfrak{g}$ be the Lie algebra of G endowed with an arbitrary inner product. Let ∇ be the connection on $E = G \times \mathfrak{g} \to G$ induced by $\theta : \mathfrak{g} \to \mathfrak{o}(\mathfrak{g})$ given by

$$\theta(X) Y = \frac{1}{2} \left(\mathrm{ad}_X Y - (\mathrm{ad}_X)^* Y - (\mathrm{ad}_Y)^* X \right),$$

and any left invariant parallelization of G, where * means transpose with respect to the inner product at the identity. In this case the map

$$F : E \to TG, \qquad F(g, v) = d\ell_g(v) \tag{2}$$

(ℓ_g denotes left multiplication by g) is an affine vector bundle isomorphism, and moreover an isometry if E and TG carry the corresponding Sasaki metrics.

(2) A particular case of (1) is the following: If the metric on G is bi-invariant, or equivalently the inner product is $\mathrm{Ad}(G)$-invariant, we have

$$\theta(X) Y = \frac{1}{2} [X, Y].$$

(3) Let G be a compact connected Lie group and $(\mathcal{V}, \rho)$ a real orthogonal representation of G. Proposition 2 provides a connection ∇ on $E = G \times \mathcal{V} \to G$ induced by any left invariant parallelization and $\theta = \lambda\, d\rho$, for some $\lambda \in \mathbf{R}$.

Let $E = G \times \mathcal{V} \to G$ as in Example 3. For $v \in \mathcal{V}$, let R_v the section of E defined by

$$R_v(g) = \left(g, \rho\left(g^{-1}\right) v\right). \tag{3}$$

The sections L_v and R_v are called left and right invariant, respectively, since in the particular case where $\mathcal{V} = \mathfrak{g}$, $\rho = \mathrm{Ad}$ they correspond to left and right invariant vector fields, respectively, via the isomorphism (2).

Remark. Although the vector bundles $E \to G$ of Example 3 are topologically trivial (as for instance the tangent spaces of parallelizable manifolds are) in most cases they are not geometrically trivial, as shown in (b) of the following Theorem.

Theorem 3 *Let G be a compact connected simple Lie group endowed with a bi-invariant Riemannian metric. Let $(\mathcal{V}, \rho)$ be an irreducible real orthogonal representation of G and let $E = G \times \mathcal{V}$ with the Sasaki metric induced by the connection associated to any left invariant parallelization of G and $\theta = \lambda d\rho$, for some $\lambda \in \mathbf{R}$. The following assertions are true:*

(a) *The left and right invariant unit sections are harmonic sections of $E^1 \to G$.*

(b) *If $\lambda = 0$ or $\lambda = 1$, then L_v or R_v, respectively, are parallel sections for all $v \in \mathcal{V}$. If $0 \neq \lambda \neq 1$, then the bundle $E \to G$ has no parallel unit sections.*

Remarks. (a) The result is still valid if G is semisimple and the metric of G is a negative multiple of the Killing form.

(b) If $(\mathcal{V}, \rho) = (\mathfrak{g}, \mathrm{Ad})$ and $\lambda = 1/2$, we have the well-known fact that the unit left invariant vector fields on G are harmonic sections of $T^1G \to G$, since they are Killing vector fields and G is Einstein [5] (see in [3, Section 4] the case where the bi-invariant metric is not Einstein).

We need the following Lemma to prove the Theorem.

Lemma 4 *Let ∇ be the connection on the bundle $E \to G$ in the hypothesis of* Theorem 3. *If Z is a left invariant vector field on G, then*

$$\left(\nabla_Z \nabla_Z R_v\right)(g) = \left(g, (\lambda - 1)^2 d\rho(Z)^2 \rho\left(g^{-1}\right) v\right) \tag{4}$$

for all $g \in G$, $v \in \mathcal{V}$.

Proof. Let V be a smooth section of $E \to G$ and suppose that $V(h) = (h, u(h))$. Denote $w(h) = (d/dt)_0 u(h \exp(tZ))$ and $\gamma(t) = g \exp(tZ)$ for $t \sim 0$. We may assume that $Z \neq 0$, otherwise the assertion is trivial. A smooth section W such that

$$W(\gamma(t)) = (\cos t) L_{u(g)}(\gamma(t)) + (\sin t) L_{w(g)}(\gamma(t))$$

satisfies $W(g) = V(g)$ and $(W \circ \gamma)'(0) = (V \circ \gamma)'(0)$. Hence, $(\nabla_Z V)(g) = (\nabla_Z W)(g)$, which by (1) equals

$$L_{\lambda\, d\rho(Z) u(g)}(g) + L_{w(g)}(g) = (g, \lambda\, d\rho(Z)\, u(g) + w(g)).$$

Applying this procedure to $V = R_v$, that is, $u(h) = \rho(h^{-1})\, v$ and $w(h) = -d\rho(Z)\, \rho(h^{-1})\, v$, one obtains

$$(\nabla_Z R_v)(g) = \left(g, (\lambda - 1)\, d\rho(Z)\, \rho\left(g^{-1}\right) v\right). \tag{5}$$

Finally, applying again the procedure to the section $V = \nabla_Z R_v$, one obtains (4). □

Proof of Theorem 3. (a) Let $\{Z_1, \ldots, Z_n\}$ be an orthonormal basis of $\mathfrak{g}$ and consider on G the associated left invariant parallelization. Given $v \in \mathcal{V}$, by (1) we compute

$$\begin{aligned}(\Delta L_v)(g) &= \sum_{i=1}^{n} (\nabla_{Z^i} \nabla_{Z^i} L_v)(g) = \sum_{i=1}^{n} L_{\lambda^2 d\rho(Z^i)^2 v}(g) \\ &= \left(g, \lambda^2 \sum_{i=1}^{n} d\rho\left(Z^i\right)^2 v\right) = \left(g, \lambda^2 \mathcal{C}_\rho(v)\right),\end{aligned}$$

where $\mathcal{C}_\rho$ is a multiple of the Casimir of the representation ρ (notice that the metric is a negative multiple of the Killing form). Now, the Casimir is a multiple of the identity, since ρ is irreducible (a direct application of Schur's Lemma). Hence, $\Delta L_v = \mu L_v$ for some μ and so L_v is a harmonic section of $E^1 \to G$ by Theorem 1. On the other hand, a straightforward computation shows that

$$d\rho(Z)\, \rho\left(g^{-1}\right) = \rho\left(g^{-1}\right) d\rho(\mathrm{Ad}\,(g)\, Z)$$

for all $g \in G$ and $Z \in \mathfrak{g}$. Hence, if we call $U^i = \mathrm{Ad}\,(g)\, Z^i$, we have by Lemma 4 that

$$\begin{aligned}(\Delta R_v)(g) &= \sum_{i=1}^{n} \left(g, (\lambda - 1)^2 \rho\left(g^{-1}\right) d\rho\left(U^i\right)^2 v\right) \\ &= \left(g, (\lambda - 1)^2 \rho\left(g^{-1}\right) \mathcal{C}_\rho(v)\right),\end{aligned}$$

since $\{U^i \mid i = 1, \ldots, n\}$ is an orthonormal basis of $\mathfrak{g}$ (the metric on G is bi-invariant). As before, $\mathcal{C}_\rho$ is a multiple $\bar{\mu}$ of the identity, hence $\Delta R_v = \bar{\mu}\,(\lambda - 1)^2 R_v$, which implies by Theorem 1 that R_v is a harmonic section of $E^1 \to G$.

(b) If $\lambda = 0$, clearly L_v is parallel by definition of the connection. If $\lambda = 1$, then R_v is parallel by (5). Suppose that a smooth unit section V with $V(e) = (e, v)$ is parallel. Then, for $X, Y \in \mathfrak{g}$ the curvature

$$\begin{aligned} R(X,Y)(e,v) &= \left(\nabla_X \nabla_Y L_v - \nabla_Y \nabla_X L_v - \nabla_{[X,Y]} L_v\right)(e) \\ &= (e, [\theta(X), \theta(Y)]\, v - \theta[X,Y]\, v) \\ &= \left(e, \lambda^2 [d\rho(X), d\rho(Y)]\, v - \lambda d\rho[X,Y]\, v\right) \\ &= (e, \lambda(\lambda - 1)\, d\rho[X,Y]\, v) \end{aligned}$$

vanishes. If G is semisimple, $[\mathfrak{g}, \mathfrak{g}] = \mathfrak{g}$. Hence, $0 \neq \lambda \neq 1$ implies that $d\rho(Z)\, v = 0$ for all $Z \in \mathfrak{g}$. This contradicts the fact that ρ is irreducible. □

Next we deal with an analogue of the particular case of Theorem 3 when $\mathcal{V} = \mathbf{H}$ is the algebra of quaternions, $G = S^3 = \{q \in \mathbf{H} \mid |q| = 1\}$ and $\rho(q)\, X = q.X$ (quaternion multiplication) for $X \in \operatorname{Im} \mathbf{H} = T_1 S^3$. (It is not a particular case of Theorem 3, since S^7 is not a Lie group.)

Let $\mathbf{O} \cong \mathbf{R}^8$ denote the octonians with the canonical inner product and let $S^7 = \{q \in \mathbf{O} \mid |q| = 1\}$ with the induced metric. The tangent space of S^7 at the identity may be identified with $\operatorname{Im} \mathbf{O}$, the purely imaginary octonians. Fix an orthonormal basis $\{x_1, \ldots, x_7\}$ of $\operatorname{Im} \mathbf{O}$ and consider the parallelization of S^7 consisting of the corresponding left invariant vector fields X^i's, that is, $X^i(q) = q.x^i \in q^\perp = T_q S^7$. By analogy with (3), given $v \in \mathbf{O}$, we define the section R_v of the trivial vector bundle $S^7 \times \mathbf{O} \to S^7$ by $R_v(q) = (q, \bar{q}v)$.

Theorem 5 *Let $E = S^7 \times \mathbf{O} \to S^7$ be the trivial vector bundle with the connection ∇ induced by*

$$\theta : \left\{X^1, \ldots, X^7\right\} \to \mathfrak{o}(\mathbf{O}), \quad \theta\left(X^i\right) v = \lambda\, x_i v,$$

with $\lambda \in \mathbf{R}$, and consider on E the Sasaki metric induced by ∇. The connection is independent of the choice of the orthonormal basis of $\operatorname{Im} \mathbf{O}$. If $v \in \mathbf{O}$ with $|v| = 1$, the following assertions are true for the sections L_v, R_v of the associated spherical bundle $E^1 \to S^7$.

(a) *If $\lambda = 0$, then L_v and R_v are harmonic sections. If $\lambda \neq 0$, then L_v is a harmonic section and R_v is a harmonic section if and only if $v = \pm 1$.*

(b) *If* $0 \neq \lambda \neq 1$, *then the bundle* $E^1 \to S^7$ *has no parallel sections. The section* L_v *is parallel if and only if* $\lambda = 0$, *and* R_v *is parallel if and only if* $\lambda = 1$ *and* $v = \pm 1$.

Before proving the theorem we recall from Chapter 6 of [4] some facts about the octonians $\mathbf{O}$ (also called Cayley numbers), which are a non-associative normed algebra with identity, isomorphic to $\mathbf{R}^8$ as an inner product vector space. The algebra $\mathbf{O}$ is $\mathbf{H} \times \mathbf{H}$, with the multiplication given by

$$(a, b)(c, d) = \left(ac - \bar{d}b, da + b\bar{c}\right). \tag{6}$$

Setting $1 = (1, 0)$ and $e = (0, 1)$, one writes $(a, b) = a + be$. If $u = a + x$ with $a \in \mathbf{R}.1$ and $\langle x, 1\rangle = 0$, the conjugate of u is $\bar{u} = a - x$ and $\langle u, v\rangle = \operatorname{Re}(u\bar{v})$ holds for all $u, v \in \mathbf{O}$. If $x \in \operatorname{Im}\mathbf{O} = 1^{\perp}$ with $|x| = 1$, then

$$x^2 = -x\bar{x} = -|x|^2 = -1. \tag{7}$$

Moreover, if $\langle u, v\rangle = 0$, then

$$u(\bar{v}w) = -v(\bar{u}w) \tag{8}$$

for all w. From Lemma 6.11 of [4] and its proof we have that the associator

$$[u, v, w] = (uv)w - u(vw)$$

is an alternating 3-linear form which vanishes either if one of the arguments is real or if two consecutive arguments are conjugate. In particular, if $x \in \operatorname{Im}\mathbf{O}$ with $|x| = 1$, we have by (7) that for all v,

$$x(xv) = \left(x^2\right)v - [x, x, v] = -v + [x, \bar{x}, v] = -v. \tag{9}$$

Lemma 6 *Let* $z = x_\ell$ *be an element of the basis of* $\operatorname{Im}\mathbf{O}$ *considered above and denote* $Z = X_\ell$. *Then for unit octonians* v *and* q *one has*

$$(\nabla_Z R_v)(q) = (q, \lambda z(\bar{q}v) - (z\bar{q})v) \tag{10}$$

and

$$(\nabla_Z \nabla_Z R_v)(q) = -\left(1 + \lambda^2\right)R_v(q) - 2\lambda(q, z((z\bar{q})v)). \tag{11}$$

Proof. The assertions follow proceeding as in the proof of Lemma 4, setting $\rho(q)X = qX$ and $d\rho(z)X = zX$, taking into account that $\mathbf{O}$ is not associative and using (9). $\square$

Proof of Theorem 5. (a) First we show that $\theta(X^i)$ is skew symmetric for all $i = 1, \ldots, 7$. Indeed, given $v \in \mathbf{O}$, since $x_i \in \operatorname{Im}\mathbf{O}$, then

$$\langle \lambda x_i v, v \rangle = \lambda \operatorname{Re}\left((x_i v)\bar{v}\right) = \lambda \operatorname{Re}\left([x_i, v, \bar{v}] - x_i |v|^2\right) = 0,$$

by one of the properties of the associator mentioned above. On the other hand, by definition of the connection and (9), we compute

$$\begin{aligned} (\Delta L_v)(q) &= \sum_{i=1}^{7} \left(\nabla_{X^i}\nabla_{X^i} L_v\right)(q) = \sum_{i=1}^{7} L_{\lambda^2 x^i (x^i v)}(q) = \\ &= \left(q, -\sum_{i=1}^{7} \lambda^2 v\right) = \left(q, -7\lambda^2 v\right) = -7\lambda^2 L_v(q). \end{aligned}$$

By Theorem 1, L_v is a harmonic section of $E^1 \to S^7$ for any λ and using (11) and (9), R_v is a harmonic section if $\lambda = 0$ or $v = \pm 1$. Now we consider the case $\lambda \neq 0$. If R_v is a harmonic section, by Theorem 1 and (11) there exists a smooth function f on S^7 such that

$$\sum_{\ell=1}^{7} x_\ell \left((x_\ell \bar{q}) v\right) = f(q)\,\bar{q}v \tag{12}$$

for all $q \in S^7$. By Proposition 6.40 in [4], based on a theorem of Artin, we may suppose without loss of generality that $v = a + bi$, with $a^2 + b^2 = 1$. We must show that $b = 0$. Take $\bar{q} = c + dj$ with $c^2 + d^2 = 1$ and suppose that $\{x_\ell \mid \ell = 1, \ldots, 7\}$ is the canonical basis $\{i, j, k, e, ie, je, ke\}$. Now a straightforward computation using (6) and (9) yields that $\sum_{\ell=1}^{7} x_\ell \left((x_\ell j) i\right) = -k$. Setting $\xi = ac + cbi + adj$, equality (12) becomes

$$-7\xi - dbk = f(c - dj)(\xi - dbk).$$

Suppose that $b \neq 0$. If $b = \pm 1$ (so $a = 0$), taking $c = d \neq 0$, one has $1 = f(c - dj) = -7$. If $b \neq \pm 1$ (so $a \neq 0$), taking $c = 0, d = 1$, one gets also a contradiction. Thus, $b = 0$ as desired.

(b) By definition of the connection, L_v is parallel if and only if $\lambda = 0$. Suppose that $0 \neq \lambda \neq 1$. As in the proof of Theorem 3 (b), we show that for any $v \in \mathbf{O}$, $v \neq 0$, there exist an orthonormal set $\{x, y\} \subset T_1 S^7 = \operatorname{Im} \mathbf{O}$ such that the curvature $R(x, y) v \neq 0$. Let X, Y be the left invariant vector fields on S^7 corresponding to x and y, respectively. By Proposition 6.40 of [4], based on a theorem of Artin, the span H of $\{1, x, y, xy\}$ is a normed subalgebra isomorphic to the quaternions. Hence, one can think of X, Y as left invariant vector fields on the Lie group $S^3 = H \cap S^7$. Therefore $[X, Y](1) = xy - yx$. Using (8) we compute

$$\begin{aligned} R(x, y) v &= \left(\nabla_X \nabla_Y L_v - \nabla_Y \nabla_X L_v - \nabla_{[X,Y]} L_v\right)(1) \\ &= \lambda^2 x(yv) - \lambda^2 y(xv) - \lambda(xy - yx) v \\ &= 2\lambda(\lambda x(yv) - (xy) v) \\ &= 2\lambda((\lambda - 1)(xy) v - \lambda [x, y, v]). \end{aligned}$$

If $v = \pm 1$, for any orthonormal set $\{x, y\} \subset \operatorname{Im} \mathbf{O}$ one has clearly

$$R(x, y) v = \pm 2\lambda(\lambda - 1) xy \neq 0.$$

If $v \neq \pm 1$, then $u := \operatorname{Im} v \neq 0$ and taking an orthonormal set $\{x, y\}$ in $\operatorname{Im} \mathbf{O}$, with $y = \bar{u} / |u|$, by the properties of the associator given after (8), one has $R(x, y) v = 2\lambda(\lambda - 1)(xy) v \neq 0$. Finally, by (10), R_v is not parallel if $\lambda = 0$, and if $\lambda = 1$, then $(\nabla_Z R_v)(q) = (q, -[z, \bar{q}, v])$ for all $q \in S^7$, $\operatorname{Re} z = 0$. Similar arguments yield that in this case R_v is parallel if and only if $v = \pm 1$. This concludes the proof of (b). □

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