On the energy of sections of trivializable sphere bundles

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Abstract

Let $E \to M$ be a vector bundle with a metric connection over a Riemannian manifold $M$ and consider on $E$ the Sasaki metric. We find a condition for a section of the associated sphere bundle to be a critical point of the energy among all smooth unit sections. We apply the criterion to some particular cases where $M$ is parallelizable, for instance $M = S^7$ or a compact simple Lie group $G$ with a bi-invariant metric, and $E$ is the trivial vector bundle with a connection induced by octonian multiplication or an irreducible real orthogonal representation of $G$, respectively. Generically, these bundles have no parallel unit sections.


Introduction

Beginning with G. Wiegmink and C. M. Wood [5, 6], critical points of the energy of unit tangent fields have been extensively studied (see for instance in [1] the abundant bibliography on the subject). We are interested in a

*Partially supported by FONCYT, CIEM (CONICET) and SECYT (UNC)
natural generalization, namely, critical points of the energy of sections of sphere bundles.

Let \( \pi : E \to M \) be a vector bundle with a metric connection \( \nabla \) over an oriented Riemannian manifold, that is, each fiber has an inner product depending smoothly on the base point and

\[
Z \langle V, W \rangle = \langle \nabla_Z V, W \rangle + \langle V, \nabla_Z W \rangle
\]

for all vector fields \( Z \) on \( M \) and all smooth sections \( V, W \) of \( E \).

On \( E \) one can define the canonical Sasaki metric associated with \( \nabla \) in such a way that the map

\[
(d\pi, \mathcal{K})_\xi : T_\xi E \to T_q M \times E_q
\]

is a linear isometry for each \( \xi \in E \) (here \( q = \pi (\xi) \) and \( \mathcal{K} \) is the connection operator associated with \( \nabla \)).

Let \( \pi : E \to M \) be as before and denote by \( E^1 = \{ \xi \in E \mid \|\xi\| = 1 \} \) the associated sphere bundle. Let \( N \) be a relatively compact open subset of \( M \) with smooth (possibly empty) boundary. Given a smooth section \( V : M \to E^1 \), the total bending of \( V \) on \( N \) is defined by

\[
B_N (V) = \int_N \|\nabla V\|^2 ,
\]

where \( (\nabla V)_p : T_p M \to E_p, \|\nabla V\|^2 = \text{tr} (\nabla V)^* (\nabla V) \) and integration is taken with respect to the volume associated to the Riemannian metric of \( M \).

Consider on \( E \) the Sasaki metric. As in the case of vector fields, there exist constants \( c_1 \) and \( c_2 \), depending only on the dimension and the volume of \( N \), such that the energy \( \mathcal{E}_N \) of the section \( V \), thought of as map \( V : N \to E \), is given by

\[
\mathcal{E}_N (V) = c_1 + c_2 B_N (V) .
\]

In the following we refer to the energy of the section instead of the bending, since that is a subject more commonly studied. In every example we will be concerned with the nonexistence of parallel unit sections, since they are trivial minima of the functional.

**Definition.** A smooth section \( V : M \to E^1 \) is said to be a harmonic section if for every relatively compact open subset \( N \) of \( M \) with smooth (possibly empty) boundary, \( V \) is a critical point of the functional \( B_N \) (or equivalently, of the energy \( \mathcal{E}_N \)) applied to smooth sections \( W \) of \( M \) satisfying \( W|_{\partial N} = V|_{\partial N} \).
Notice that a harmonic section may be not a harmonic map from $M$ to $E^1$ (see for example [2, 3], where $E = TM$).

The rough Laplacian $\Delta$ acts on smooth sections of $E$ as follows:

$$(\Delta V)(p) = \sum_{i=1}^{n} (\nabla Z_i \nabla Z_i V)(p),$$

where $\{Z^i \mid i = 1, \ldots, n\}$ is any section of orthonormal frames on a neighborhood of $p$ in $M$ satisfying $(\nabla Z_i Z_i)(p) = 0$ for all $i, j$.

**Theorem 1** Let $\pi : E \rightarrow M$ be a vector bundle with a metric connection over an oriented Riemannian manifold and consider on $E$ the associated Sasaki metric. The section $V : M \rightarrow E^1$ is a harmonic section if and only if there is a smooth real function $f$ on $M$ such that

$$\Delta V = fV.$$

**Remark.** This condition was proved for the particular case where $E$ is the tangent bundle, by Wiegmink [5] and Wood [6] for compact manifolds and by Gil-Medrano [1] for general (not necessarily compact) manifolds (with a different presentation). Their proofs can be adapted to the present more general case.

**Applications**

Let $M$ be a parallelizable manifold with a fixed parallelization $\{X^1, \ldots, X^n\}$. Let $\mathcal{V}$ be a finite dimensional vector space with an inner product and $\mathfrak{o}(\mathcal{V})$ the set of all skew-symmetric endomorphisms of $\mathcal{V}$. Let $E = M \times \mathcal{V} \rightarrow M$ be the trivial vector bundle. For $v \in \mathcal{V}$, let $L_v : M \rightarrow E$ be the “constant” section $L_v(p) = (p, v)$.

**Proposition 2** Given a map $\theta : \{X^1, \ldots, X^n\} \rightarrow \mathfrak{o}(\mathcal{V})$, there exists a unique connection $\nabla$ on $E \rightarrow M$ such that

$$(\nabla_{X^i} L_v)(p) = L_{\theta(X^i)v}(p)$$

for all $p \in M$ and all $i = 1, \ldots, n$. Moreover, the connection is metric.
Proof. Let \( \{v_1, \ldots, v_n\} \) be an orthonormal basis of \( V \). Let \( X \in T_pM \) and \( \sigma : M \to E \) be a smooth section. Then

\[
X = \sum_{i=1}^{m} a_i X^i(p) \quad \text{and} \quad \sigma = \sum_{j=1}^{n} f_j L_{v_j}
\]

for some numbers \( a_i \) and smooth functions \( f_j : M \to \mathbb{R} \). A standard computation shows that

\[
(\nabla_X \sigma)(p) = \sum_{i=1}^{m} \sum_{j=1}^{n} a_i \left( X^i_p(f_j) L_{v_j}(p) + f_j(p) \left( L_{\theta(X^i)v_j}(p) \right) \right)
\]

defines a connection on \( E \) satisfying condition (1), which is metric since \( \theta(X^i) \) is skew-symmetric for all \( i \). \( \square \)

Examples.

(1) The Levi-Civita connection of a Lie group \( G \) with a left invariant Riemannian metric may be obtained in this way: Let \( \mathfrak{g} \) be the Lie algebra of \( G \) endowed with an arbitrary inner product. Let \( \nabla \) be the connection on \( E = G \times \mathfrak{g} \to G \) induced by \( \theta : \mathfrak{g} \to \mathfrak{so}(\mathfrak{g}) \) given by

\[
\theta(X)Y = \frac{1}{2} (\text{ad}_X Y - (\text{ad}_X)^* Y - (\text{ad}_Y)^* X),
\]

and any left invariant parallelization of \( G \), where \( \ast \) means transpose with respect to the inner product at the identity. In this case the map

\[
F : E \to TG, \quad F(g, v) = d\ell_g(v)
\]

(\( \ell_g \) denotes left multiplication by \( g \)) is an affine vector bundle isomorphism, and moreover an isometry if \( E \) and \( TG \) carry the corresponding Sasaki metrics.

(2) A particular case of (1) is the following: If the metric on \( G \) is bi-invariant, or equivalently the inner product is \( \text{Ad}(G) \)-invariant, we have

\[
\theta(X)Y = \frac{1}{2} [X, Y].
\]

(3) Let \( G \) be a compact connected Lie group and \( (\mathcal{V}, \rho) \) a real orthogonal representation of \( G \). Proposition 2 provides a connection \( \nabla \) on \( E = G \times \mathcal{V} \to G \) induced by any left invariant parallelization and \( \theta = \lambda d\rho \), for some \( \lambda \in \mathbb{R} \).
Let $E = G \times \mathcal{V} \to G$ as in Example 3. For $v \in \mathcal{V}$, let $R_v$ the section of $E$ defined by

$$R_v(g) = (g, \rho(g^{-1})v).$$  \hspace{1cm} (3)

The sections $L_v$ and $R_v$ are called left and right invariant, respectively, since in the particular case where $\mathcal{V} = \mathfrak{g}$, $\rho = \text{Ad}$ they correspond to left and right invariant vector fields, respectively, via the isomorphism (2).

**Remark.** Although the vector bundles $E \to G$ of Example 3 are topologically trivial (as for instance the tangent spaces of parallelizable manifolds are) in most cases they are not geometrically trivial, as shown in (b) of the following Theorem.

**Theorem 3** Let $G$ be a compact connected simple Lie group endowed with a bi-invariant Riemannian metric. Let $(\mathcal{V}, \rho)$ be an irreducible real orthogonal representation of $G$ and let $E = G \times \mathcal{V}$ with the Sasaki metric induced by the connection associated to any left invariant parallelization of $G$ and $\theta = \lambda d\rho$, for some $\lambda \in \mathbb{R}$. The following assertions are true:

(a) The left and right invariant unit sections are harmonic sections of $E^1 \to G$.

(b) If $\lambda = 0$ or $\lambda = 1$, then $L_v$ or $R_v$, respectively, are parallel sections for all $v \in \mathcal{V}$. If $0 \neq \lambda 
eq 1$, then the bundle $E \to G$ has no parallel unit sections.

**Remarks.** (a) The result is still valid if $G$ is semisimple and the metric of $G$ is a negative multiple of the Killing form.

(b) If $(\mathcal{V}, \rho) = (\mathfrak{g}, \text{Ad})$ and $\lambda = 1/2$, we have the well-known fact that the unit left invariant vector fields on $G$ are harmonic sections of $T^1G \to G$, since they are Killing vector fields and $G$ is Einstein [5] (see in [3, Section 4] the case where the bi-invariant metric is not Einstein).

We need the following Lemma to prove the Theorem.

**Lemma 4** Let $\nabla$ be the connection on the bundle $E \to G$ in the hypothesis of Theorem 3. If $Z$ is a left invariant vector field on $G$, then

$$(\nabla_Z \nabla_Z R_v)(g) = (g, (\lambda - 1)^2 d\rho(Z)^2 \rho(g^{-1})v)$$  \hspace{1cm} (4)

for all $g \in G$, $v \in \mathcal{V}$.
Proof. Let $V$ be a smooth section of $E \to G$ and suppose that $V(h) = (h, u(h))$. Denote $w(h) = (d/dt)_0 u(h \exp(tZ))$ and $\gamma(t) = g \exp(tZ)$ for $t \sim 0$. We may assume that $Z \neq 0$, otherwise the assertion is trivial. A smooth section $W$ such that

$$W(\gamma(t)) = (\cos t) L_{u(g)}(\gamma(t)) + (\sin t) L_{w(g)}(\gamma(t))$$

satisfies $W(g) = V(g)$ and $(W \circ \gamma)'(0) = (V \circ \gamma)'(0)$. Hence, $(\nabla_Z V)(g) = (\nabla_Z W)(g)$, which by (1) equals

$$L_{\lambda d\rho(Z) u(g)} + L_{w(g)}(g) = (g, \lambda d\rho(Z) u(g) + w(g)).$$

Applying this procedure to $V = R_v$, that is, $u(h) = \rho(h^{-1}) v$ and $w(h) = -d\rho(Z) \rho(h^{-1}) v$, one obtains

$$(\nabla_Z R_v)(g) = (g, (\lambda - 1) d\rho(Z) \rho(g^{-1}) v) \quad (5)$$

Finally, applying again the procedure to the section $V = \nabla_Z R_v$, one obtains (4).

Proof of Theorem 3. (a) Let $\{Z_1, \ldots, Z_n\}$ be an orthonormal basis of $\mathfrak{g}$ and consider on $G$ the associated left invariant parallelization. Given $v \in V$, by (1) we compute

$$(\Delta L_v)(g) = \sum_{i=1}^n (\nabla_{Z_i} \nabla_{Z_i} L_v)(g) = \sum_{i=1}^n L_{\lambda^2 d\rho(Z_i^i) v}(g)$$

$$= \left(g, \lambda^2 \sum_{i=1}^n d\rho(Z_i^i)^2 v\right) = (g, \lambda^2 C_\rho(v)), \quad (6)$$

where $C_\rho$ is a multiple of the Casimir of the representation $\rho$ (notice that the metric is a negative multiple of the Killing form). Now, the Casimir is a multiple of the identity, since $\rho$ is irreducible (a direct application of Schur’s Lemma). Hence, $\Delta L_v = \mu L_v$ for some $\mu$ and so $L_v$ is a harmonic section of $E^1 \to G$ by Theorem 1. On the other hand, a straightforward computation shows that

$$d\rho(Z) \rho(g^{-1}) = \rho(g^{-1}) d\rho(\text{Ad}(g) Z)$$

for all $g \in G$ and $Z \in \mathfrak{g}$. Hence, if we call $U^i = \text{Ad}(g) Z^i$, we have by Lemma 4 that

$$(\Delta R_v)(g) = \sum_{i=1}^n \left(g, (\lambda - 1)^2 \rho(g^{-1}) d\rho(U^i)^2 v\right)$$

$$= \left(g, (\lambda - 1)^2 \rho(g^{-1}) C_\rho(v)\right), \quad (6)$$
Theorem 5

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vanishes. If G is semisimple, [g, g] = g. Hence, 0 ≠ λ ≠ 1 implies that
dρ(Z)v = 0 for all Z ∈ g. This contradicts the fact that ρ is irreducible. □

Next we deal with an analogue of the particular case of Theorem 3 when

V = H is the algebra of quaternions, G = S³ = \{q ∈ H | |q| = 1\} and

ρ(q)X = q.X (quaternion multiplication) for X ∈ ImH = T₁S³. (It is not a
particular case of Theorem 3, since S³ is not a Lie group.)

Let O ≅ R⁸ denote the octonians with the canonical inner product and let

S⁷ = \{q ∈ O | |q| = 1\} with the induced metric. The tangent space of S⁷ at
the identity may be identified with ImO, the purely imaginary octonians. Fix
an orthonormal basis \{x₁, . . . , x₇\} of ImO and consider the parallelization
of S⁷ consisting of the corresponding left invariant vector fields Xᵢ’s, that is,

Xᵢ(q) = q.xᵢ ∈ q⊥ = TᵦS⁷. By analogy with (3), given v ∈ O, we define the
section Rᵥ of the trivial vector bundle S⁷ × O → S⁷ by Rᵥ(q) = (q, qv).

Theorem 5 Let E = S⁷ × O → S⁷ be the trivial vector bundle with the
connection ∇ induced by

\[ \theta : \{X^1, . . . , X^7\} \to o(O), \quad \theta(X^i)v = λx_iv, \]

with λ ∈ R, and consider on E the Sasaki metric induced by ∇. The connection
is independent of the choice of the orthonormal basis of ImO. If v ∈ O with
|v| = 1, the following assertions are true for the sections Lᵥ, Rᵥ of the
associated spherical bundle E⁷ → S⁷.

(a) If λ = 0, then Lᵥ and Rᵥ are harmonic sections. If λ ≠ 0, then Lᵥ is a
harmonic section and Rᵥ is a harmonic section if and only if v = ±1.
(b) If $0 \neq \lambda \neq 1$, then the bundle $E^1 \to S^7$ has no parallel sections. The section $L_v$ is parallel if and only if $\lambda = 0$, and $R_v$ is parallel if and only if $\lambda = 1$ and $v = \pm 1$.

Before proving the theorem we recall from Chapter 6 of [4] some facts about the octonians $\mathbb{O}$ (also called Cayley numbers), which are a non-associative normed algebra with identity, isomorphic to $\mathbb{R}^8$ as an inner product vector space. The algebra $\mathbb{O}$ is $\mathbb{H} \times \mathbb{H}$, with the multiplication given by

\[(a, b) (c, d) = (ac - \overline{db}, da + b\overline{c}).\]  

Setting $1 = (1, 0)$ and $e = (0, 1)$, one writes $(a, b) = a + be$. If $u = a + x$ with $a \in \mathbb{R}$ and $\langle x, 1 \rangle = 0$, the conjugate of $u$ is $\overline{u} = a - x$ and $\langle u, v \rangle = \text{Re} (uv)$ holds for all $u, v \in \mathbb{O}$. If $x \in \text{Im} \mathbb{O} = 1^\perp$ with $|x| = 1$, then

\[x^2 = -x\overline{x} = -|x|^2 = -1.\]  

Moreover, if $\langle u, v \rangle = 0$, then

\[u (\overline{w}v) = -v (\overline{uw})\]  

for all $w$. From Lemma 6.11 of [4] and its proof we have that the associator

\[[u, v, w] = (uv) w - u (vw)\]

is an alternating 3-linear form which vanishes either if one of the arguments is real or if two consecutive arguments are conjugate. In particular, if $x \in \text{Im} \mathbb{O}$ with $|x| = 1$, we have by (7) that for all $v$,

\[x (xv) = (x^2) v - [x, x, v] = -v + [x, \overline{x}, v] = -v.\]  

**Lemma 6** Let $z = x_\ell$ be an element of the basis of $\text{Im} \mathbb{O}$ considered above and denote $Z = X_\ell$. Then for unit octonians $v$ and $q$ one has

\[(\nabla_Z R_v) (q) = (q, \lambda z (\overline{qv}) - (zq) v)\]  

and

\[(\nabla_Z \nabla_Z R_v) (q) = - (1 + \lambda^2) R_v (q) - 2\lambda (q, z ((zq) v)).\]
Proof. The assertions follow proceeding as in the proof of Lemma 4, setting \( \rho(q) X = qX \) and \( d\rho(z) X = zX \), taking into account that \( O \) is not associative and using (9).

Proof of Theorem 5. (a) First we show that \( \theta(X^i) \) is skew symmetric for all \( i = 1, \ldots, 7 \). Indeed, given \( v \in O \), since \( x_i \in \text{Im} O \), then
\[
\langle \lambda x_i v, v \rangle = \lambda \text{Re} \left( (x_i v) \bar{v} \right) = \lambda \text{Re} \left( [x_i, v, \bar{v}] - x_i |v|^2 \right) = 0,
\]
by one of the properties of the associator mentioned above. On the other hand, by definition of the connection and (9), we compute
\[
(\Delta L_v) (q) = \sum_{i=1}^{7} (\nabla_{X^i} \nabla_{X^i} \nabla_{X^i}) (q) = \sum_{i=1}^{7} L_{x_i^2 (x_i v)} (q) = (q, -7\lambda^2 v) = -7\lambda^2 L_v(q).
\]
By Theorem 1, \( L_v \) is a harmonic section of \( E^1 \to S^7 \) for any \( \lambda \) and using (11) and (9), \( R_v \) is a harmonic section if \( \lambda = 0 \) or \( v = \pm 1 \). Now we consider the case \( \lambda \neq 0 \). If \( R_v \) is a harmonic section, by Theorem 1 and (11) there exists a smooth function \( f \) on \( S^7 \) such that
\[
\sum_{\ell=1}^{7} x_{\ell} ((x_{\ell} q) v) = f (q) \bar{q} v
\]
for all \( q \in S^7 \). By Proposition 6.40 in [4], based on a theorem of Artin, we may suppose without loss of generality that \( v = a + bi \), with \( a^2 + b^2 = 1 \). We must show that \( b = 0 \). Take \( \bar{q} = c + dj \) with \( c^2 + d^2 = 1 \) and suppose that \( \{x_{\ell} \mid \ell = 1, \ldots, 7\} \) is the canonical basis \( \{i, j, k, e, ie, je, ke\} \). Now a straightforward computation using (6) and (9) yields that \( \sum_{\ell=1}^{7} x_{\ell} ((x_{\ell} j) i) = -k \). Setting \( \xi = ac + bci + adj \), equality (12) becomes
\[
-7\xi - dbk = f (c - dj) (\xi - dbk).
\]
Suppose that \( b \neq 0 \). If \( b = \pm 1 \) (so \( a = 0 \)), taking \( c = d \neq 0 \), one has \( 1 = f (c - dj) = -7 \). If \( b \neq \pm 1 \) (so \( a \neq 0 \)), taking \( c = 0, d = 1 \), one gets also a contradiction. Thus, \( b = 0 \) as desired.

(b) By definition of the connection, \( L_v \) is parallel if and only if \( \lambda = 0 \). Suppose that \( 0 \neq \lambda \neq 1 \). As in the proof of Theorem 3 (b), we show that for
any \( v \in O, v \neq 0 \), there exist an orthonormal set \( \{x, y\} \subset T_x S^7 = \text{Im } O \) such that the curvature \( R(x, y) v \neq 0 \). Let \( X, Y \) be the left invariant vector fields on \( S^7 \) corresponding to \( x \) and \( y \), respectively. By Proposition 6.40 of [4], based on a theorem of Artin, the span \( H \) of \( \{1, x, y, xy\} \) is a normed subalgebra isomorphic to the quaternions. Hence, one can think of \( X, Y \) as left invariant vector fields on the Lie group \( S^3 = H \cap S^7 \). Therefore \( [X, Y] (1) = xy - yx \).

Using (8) we compute

\[
R(x, y) v = (\nabla_X \nabla_Y L_v - \nabla_Y \nabla_X L_v - \nabla_{[X,Y]} L_v) (1) \\
= \lambda^2 x (yv) - \lambda^2 y (xv) - \lambda (xy - yx) v \\
= 2 \lambda (\lambda x (yv) - (xy) v) \\
= 2 \lambda ((\lambda - 1) (xy) v - \lambda [x, y, v]).
\]

If \( v = \pm 1 \), for any orthonormal set \( \{x, y\} \subset \text{Im } O \) one has clearly

\[
R(x, y) v = \pm 2 \lambda (\lambda - 1) xy \neq 0.
\]

If \( v \neq \pm 1 \), then \( u := \text{Im } v \neq 0 \) and taking an orthonormal set \( \{x, y\} \) in \( \text{Im } O \), with \( y = \bar{u}/|u| \), by the properties of the associator given after (8), one has \( R(x, y) v = 2 \lambda (\lambda - 1) (xy) v \neq 0 \). Finally, by (10), \( R_v \) is not parallel if \( \lambda = 0 \), and if \( \lambda = 1 \), then \( (\nabla_z R_v)(q) = (q, -[z, \bar{q}, v]) \) for all \( q \in S^7, \text{Re } z = 0 \). Similar arguments yield that in this case \( R_v \) is parallel if and only if \( v = \pm 1 \).

This concludes the proof of (b). \( \square \)

References


