# On the volume of unit vector fields on a compact semisimple Lie group

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**Abstract**

Let $G$ be a compact connected semisimple Lie group endowed with a bi-invariant Riemannian metric. We prove that maximal singular unit vector fields on $G$ are minimal, that is, they are critical points of the volume functional on unit vector fields on $G$. Besides, we give a lower bound for the number of nonequivalent minimal unit vector fields on $G$.

*Mathematics Subject Classification 2000:* 22E46, 53C30, 53C35, 53C42.

## Introduction

Let $M$ be a compact Riemannian manifold. The volume of a unit vector field on $M$ is by definition the volume of the submanifold that it determines in $T^1M$, the unit tangent bundle of $M$ endowed with the canonical (Sasaki) metric. A unit vector field on $M$ is said to be *minimal* if it is a critical point of the volume functional on unit vector fields on $M$. Gil-Medrano and Llinares-Fuster proved in [2] that a unit vector field is minimal if and only if it determines a minimal submanifold of $T^1M$ (in this approach, $M$ does not need to be compact). The characterization by Gluck and Ziller of Hopf vector fields as the unit vector fields on $S^3$ with minimum volume [4], motivated the study of minimal unit vector fields (see the abundant bibliography on this subject for instance in the references of [1]).

## Minimality of some left or right invariant unit vector fields

*Partially supported by FONCyT, CIEM (CONICET) and SECyT (UNC).

Let $G$ be a compact connected semisimple Lie group endowed with a bi-invariant Riemannian metric. In this note we deal with minimal unit vector fields on $G$. On other Lie groups with left invariant metrics, they have been studied in [3, 6, 9].

Let $\mathfrak{g} \cong T_eG$ be the Lie algebra of $G$ and let $S = T^1_eG$ denote the unit sphere in $\mathfrak{g}$. A unit vector $v \in \mathfrak{g}$ is said to be *maximal singular* if the $\mathrm{Ad}(G)$-orbit of $v$ has dimension strictly smaller than that of any other $\mathrm{Ad}(G)$-orbit through a unit vector in some neighborhood of $v$. A unit vector field on $G$ is said to be maximal singular if it is left or right invariant and its value at the identity is maximal singular.

**Remark 1** *Let $U$ and $V$ be unit vector fields on a compact Riemannian manifold $M$ such that $U \circ \psi = \Psi \circ V$, where $\psi$ and $\Psi$ are isometries of $M$ and $T^1M$, respectively. Clearly, $U$ and $V$ have the same volume and if one of them is minimal, so is the other one. In particular, this happens if $U, V$ are the following pairs of unit vector fields on a compact Lie group $G$ as above.*

(a) *If $U$ and $V$ are left invariant and $U_e = Ad(g)\, V_e$ for some $g \in G$: Take $\psi = L_g \circ R_{g^{-1}}$ and $\Psi = d\psi$, where $L_k$ and $R_k$ denote left and right multiplication by $k$, respectively.*

(b) *If $U$ is left invariant, $V$ is right invariant and $U_e = V_e$: Take $\psi =$ $\mathrm{Inv} : G \to G$, $\mathrm{Inv}(h) = h^{-1}$ (notice that $G$ is a symmetric space and* $\mathrm{Inv}$ *is the geodesic symmetry at $e$) and $\Psi = -d\,\mathrm{Inv}$ (the opposite of the identity on $T^1M$ is an isometry, by definition of the Sasaki metric).*

**Theorem 2** *Maximal singular unit vector fields on $G$ are minimal.*

**Proof.** By the previous remark, it suffices to prove the assertion for left invariant maximal singular unit vector fields on $G$.

The product $G \times G$ acts on $T^1G$ by isometries as follows: $(g, h)\, u = dL_g dR_{h^{-1}} u$. Let $v$ be a unit vector in $\mathfrak{g}$. The orbit $(G \times G)\, v$ is a submanifold of $T^1G$ whose dimension equals $\dim G + \dim \mathrm{Ad}(G)\, v$. Let us consider the action $\rho$ of $G$ on $(G \times G)\, v$ defined by $\rho\,(g, u) = dL_g u$. Notice that any orbit of $\rho$ is the submanifold of $T^1M$ determined by the left invariant vector field taking the value $\mathrm{Ad}(g)\, v$ at the identity, for some $g \in G$. Hence, all orbits of $\rho$ in $(G \times G)\, v$ have the same dimension $\dim G$ and also the same volume, by Remark 1 (a). Therefore, by the main result of Hsiang and Lawson in [8], they are all minimal in $(G \times G)\, v$, and so the mean curvature vector field $H_u$ of the orbit $\rho\,(G)\, u$ is perpendicular to $(G \times G)\, v$ for all $u \in \mathrm{Ad}(g)\, v$. Let $u \in \mathrm{Ad}(G)\, v$. Now, $T_u (G \times G)\, v \subset T_u T^1 G$ includes the horizontal space, since parallel transport is realized by the action of the group ($G$ is a symmetric space). Therefore, $H_u(u)$ is vertical and we have an $\mathrm{Ad}(G)$-invariant vector field $V(u) = H_u(u)$ on $\mathrm{Ad}(G)\, v$, tangent to $S$. If $v$ is maximal singular, $V$ vanishes identically, since otherwise, for all $s \sim 0$, $\{\mathrm{Exp}_u sV(u) \mid u \in \mathrm{Ad}(G)\, v\}$ would be an orbit of $\mathrm{Ad}(G)$ close to $\mathrm{Ad}(G)\, v$ with the same dimension, but different from it if $s \neq 0$ ($\mathrm{Exp}_u : T_u S \to S$ denotes the geodesic exponential function of $S$ at $u$). By $G$-invariance, $H_v \equiv 0$ and hence the left invariant vector field taking the value $v$ at the identity is minimal. □

Remark 5 provides another proof of the Theorem.

The following criterion for a unit vector field on a unimodular Lie group to be minimal, obtained by Tsukada and Vanhecke, will be useful below.

**Criterion.** [9, Proposition 2.2] *Let $G$ be a unimodular Lie group endowed with a left invariant metric. Let $f : S \to \mathbb{R}$ be defined by*

$$f(v) = \mathrm{vol}(V) / \mathrm{vol}(G),$$

*where $V$ is a left invariant vector field with $V(e) = v$. Then a left invariant unit vector field $V$ is minimal if and only if $df_{V(e)} = 0$.*

For the basic facts from the theory of compact semisimple Lie groups we refer the reader to [7]. Let $\mathfrak{t}$ be a maximal abelian subalgebra of $\mathfrak{g}$ and $\Delta$ the corresponding root system. Let $C$ be a Weyl chamber and $\Phi = \{\alpha_1, \ldots, \alpha_n\}$ the associated basis of $\Delta$. For each $0 \leq k < n$ let $S_k$ be the unit sphere in the $(k+1)$-dimensional wall $W = \{w \in \mathfrak{t} \mid \alpha_i(w) = 0 \text{ for all } i > k+1\}$ in $\mathfrak{t}$.

**Proposition 3** *If $v \in S_k$ is a critical point of the restriction of $f$ to $S_k$, then a left or right invariant vector field $V$ on $G$ with $V_e = v$ is minimal.*

**Proof.** First we show that $d\, f|_{T_v(\mathfrak{t} \cap S)} = 0$. Let $H_i$ the vector dual to $\alpha_i$, $\hat{H}_i = H_i / \|H_i\|$ and let $\rho_i$ be the reflection of $\mathfrak{t}$ fixing $\mathrm{Ker}(\alpha_i)$. Let us recall that the action of the Weyl group

$$\{g \in G \mid \mathrm{Ad}(g)\, \mathfrak{t} = \mathfrak{t}\} \,/\, \{g \in G \mid \mathrm{Ad}(g)\, u = u \text{ for all } u \in \mathfrak{t}\}$$

on $\mathfrak{t}$ is generated by the reflections with respect to the kernels of the roots in $\Phi$, hence any such a reflection $\rho$ may be written as $\rho = \mathrm{Ad}(g)|_{\mathfrak{t}}$ for some

$g \in G$. Now, the orthogonal complement of $T_v S_k$ in $T_v (S \cap \mathfrak{t})$ is the subspace spanned by $\{H_i \mid i > k+1\}$ (empty if $k = n$). For $k+1 < i \leq n$ we compute

$$\begin{aligned} df_v\left(-\hat{H}_i\right) &= \left.\frac{d}{dt}\right|_0 f\left((\cos t)\, v - (\sin t)\, \hat{H}_i\right) \\ &= \left.\frac{d}{dt}\right|_0 f\left(\rho_i\left((\cos t)\, v + (\sin t)\, \hat{H}_i\right)\right) \\ &= \left.\frac{d}{dt}\right|_0 f\left(\mathrm{Ad}\ (g_i)\left((\cos t)\, v + (\sin t)\, \hat{H}_i\right)\right) \\ &= \left.\frac{d}{dt}\right|_0 f\left((\cos t)\, v + (\sin t)\, \hat{H}_i\right) = df_v\left(\hat{H}_i\right). \end{aligned}$$

Hence, $d\, f|_{T_v(\mathfrak{t}\cap S)} = 0$, since $\left(df|_{S_k}\right)_v = 0$ by hypothesis. Now we prove that $df_v = 0$. The following decomposition of $\mathfrak{g}$ is well-known.

$$\begin{aligned} \mathfrak{g} &= \mathrm{Ker}\ (\mathrm{ad}_{\,v}) \oplus \mathrm{Image}\ (\mathrm{ad}_{\,v}) \\ &= \bigcup \mathrm{Ad}\ (g)\,(\mathfrak{t}) \oplus \left\{ \left.\frac{d}{dt}\right|_0 \mathrm{Ad}\ (\exp tX)\, v \mid X \in \mathfrak{g} \right\}, \end{aligned} \tag{1}$$

where the union is over $G_v = \{g \in G \mid \mathrm{Ad}\ (g)\, v = v\}$. Let $u$ be a unit vector in $T_v (\mathfrak{t} \cap S)$ and let $g \in G_v$. We already know that $df_v (u) = 0$. Therefore

$$\begin{aligned} df_v (\mathrm{Ad}\ (g)\, u) &= \left.\frac{d}{dt}\right|_0 f\,(\mathrm{Ad}\ (g)\,((\cos t)\, v + (\sin t)\, u)) \\ &= \left.\frac{d}{dt}\right|_0 f\,((\cos t)\, v + (\sin t)\, u) = df_v (u) = 0. \end{aligned}$$

Similarly, if $X \in \mathfrak{g}$, we have $df_v\left(\left.\frac{d}{dt}\right|_0 \mathrm{Ad}\ (\exp tX)\, v\right) = 0$. By (1), the preceding clearly implies that $df_v$ vanishes identically. Thus, $V$ is minimal by the criterion of Tsukada and Vanhecke. $\square$

Given $\alpha \in \Phi$, let $v_\alpha$ be the unique unit vector in $\mathfrak{t}$ satisfying $\alpha(v_\alpha) > 0$ and $\beta(v_\alpha) = 0$ for all $\beta \in \Phi - \{\alpha\}$. The vector $v_\alpha$ is a vertex of the spherical simplex $\overline{C} \cap S$ (the unit vectors in the closure of the Weyl chamber $C$) and is maximal singular. Moreover, any maximal singular unit vector in $\mathfrak{g}$ is in the Ad$(G)$-orbit of exactly one of these vertices. Let $V_\alpha$ denote the left invariant vector field on $G$ taking the value $v_\alpha$ at the identity. As a corollary of Proposition 3, we have the following result.

**Theorem 4** *If $\alpha \neq \beta \in \Phi$ and the maximal singular unit vector fields $V_\alpha$ and $V_\beta$ have the same volume, then there exists $v$ in the shortest geodesic arc in $S$ joining $v_\alpha$ with $v_\beta$, $v \neq v_\alpha, v_\beta$, such that the left invariant vector field $V$ with $V(e) = v$ is minimal.*

**Proof.** Let $u(s)$, $0 \leq s \leq 1$ be a parametrization of the shortest geodesic arc in $S$ joining $v_\alpha$ with $v_\beta$, which is an edge of the spherical simplex $\overline{C} \cap S$. Suppose that the basis of the root system is $\{\alpha_1, \ldots, \alpha_n\}$, with $\alpha = \alpha_1$ and $\beta = \alpha_2$. Then $u(s)$ is a regular curve in $S_1$ as in Proposition 3, which in this case is a circle. Since $f(v_\alpha) = f(v_\beta)$ by hypothesis, there exists $s_o$ with $(f \circ u)'(s_o) = 0$. Hence, taking $v = u(s_o)$, we have that $\left(d\, f|_{S_1}\right)_v = 0$. Therefore, the statement follows from Proposition 3. $\square$

**Remark 5** *With similar arguments we can give another proof of Theorem 2: Suppose that $\alpha = \alpha_1 \in \Phi$. By definition of $v_\alpha$, we have that $S_0$ has only two elements $\pm v_\alpha$. Hence $\left(d\, f|_{S_0}\right)_{v_\alpha} = 0$ trivially. Thus, $V_\alpha$ is a minimal unit vector field by Proposition 3.*

## Counting nonequivalent minimal unit vector fields

In order to have a clear statement of Theorem 7 below, we say that two unit vector fields $U, V$ on a Riemannian manifold $M$ are *equivalent* if there exists $\phi$ in the identity component of the isometry group of $M$ such that $U \circ \phi = d\phi \circ V$.

**Remark 6** *Two left invariant unit vector fields $U$ and $V$ on $G$ are equivalent if and only if $U_e = \operatorname{Ad}(g) V_e$ for some $g \in G$, since the identity component of the isometry group of $G$ consists of $\{L_g \circ R_h \mid g, h \in G\}$. In particular, by Remark 1 (a), equivalent unit vector fields have the same volume and if one of them is minimal, so is the other one. Moreover, no right invariant unit vector field is equivalent to a left invariant one, since the center of $G$ is finite.*

If $\Phi = \{\alpha_1, \ldots, \alpha_n\}$ is as above a basis of the root system, then $\mathcal{V} = \{V_{\alpha_1}, \ldots, V_{\alpha_n}\}$ consists of nonequivalent left invariant unit vector fields. Suppose that $\{V_1, \ldots, V_{k_1}\}$, $\{V_{k_1+1}, \ldots, V_{k_2}\}$, etc., are $\ell$ disjoint subsets of $\mathcal{V}$ consisting of vector fields with the same volume. Next, we give a lower bound for the number of nonequivalent minimal unit vector fields on $G$ in terms of the numbers $k_j$.

**Theorem 7** *Let $n$ be the rank of $G$ and let $k_1, \ldots, k_\ell$ be the cardinalities of disjoint sets of nonequivalent maximal singular left invariant unit vector fields having the same volume. Then the number of nonequivalent minimal unit vector fields on $G$ is not smaller than $2n + \sum_{i=1}^{\ell} k_i (k_i - 1)$.*

**Proof.** By Theorem 4, the number of nonequivalent minimal left invariant unit vector fields on $G$ is not smaller than $n + \sum_{i=1}^{\ell} \binom{k_i}{2}$. By Remark 6, this lower bound doubles if one considers the corresponding right invariant vector fields. $\square$

## An expression for the volume of a left invariant unit vector field

We show later that in some cases, a lower bound as in Theorem 7 can be obtained without actually computing the volume of maximal singular unit vector fields, but just using the symmetries of the associated root system. Nevertheless, if it has no symmetries (or if it has, but one wants to try to improve the lower bound), Proposition 8 below gives a simple formula for the volume of maximal singular unit vector fields in terms of the root system (see the examples).

Let $\Delta^+$ be the set of positive roots with respect to $\Phi$. Given a root $\phi \in \Delta^+$, there exist nonnegative integers $m_\beta(\phi)$, $\beta \in \Phi$, such that $\phi = \sum_{\beta \in \Phi} m_\beta(\phi)\, \beta$.

**Proposition 8** *The volume of a left or right invariant unit vector field $V$ on $G$ with $V_e = v$ is given by*

$$\operatorname{vol}(G) \prod_{\phi \in \Delta^+} \left(1 + \tfrac{1}{4} \phi(v)^2\right).$$

*In particular, the volume of a maximal singular unit vector field $V_\alpha$ is given by*

$$\operatorname{vol}(G) \prod_{\phi \in \Delta^+} \left(1 + \tfrac{1}{4} \alpha(v_\alpha)^2 m_\alpha(\phi)^2\right).$$

**Proof.** It is well-known that the volume of a unit vector field $V$ on a compact Riemannian manifold $M$ is given by

$$\int_M \sqrt{\det} \left(I + (\nabla V)^* (\nabla V)\right),$$

where integration is taken with respect to the Riemannian volume form on $M$. Let $V$ be a left invariant unit vector field on $G$ with $V_e = v$. The Levi-Civita connection of the bi-invariant metric on $G$ satisfies that $\nabla_W V = \frac{1}{2}[W, V]$ for any left invariant vector field $W$. Since $\operatorname{ad} v$ is skew-symmetric with respect to any bi-invariant metric, we have then

$$\begin{aligned} \operatorname{vol}(V) &= \operatorname{vol}(G)\sqrt{\det}\left(I + (\nabla V)^* (\nabla V)\right) \\ &= \operatorname{vol}(G)\sqrt{\det}\left(I + \tfrac{1}{4}(\operatorname{ad} v)^* (\operatorname{ad} v)\right) \\ &= \operatorname{vol}(G)\sqrt{\det}\left(I - \tfrac{1}{4}(\operatorname{ad} v)^2\right). \end{aligned}$$

Let $\mathfrak{m}$ be the orthogonal complement of $\mathfrak{t}$ in $\mathfrak{g}$ with respect to the Killing form. There exists an orthonormal basis $\{x_\phi | \phi \in \Delta^+\} \cup \{y_\phi | \phi \in \Delta^+\}$ of $\mathfrak{m}$ such that

$$\operatorname{ad}_v x_\phi = \phi(v)\, y_\phi \quad \text{and} \quad \operatorname{ad}_v y_\phi = -\phi(v)\, x_\phi$$

for all $v \in \mathfrak{t}$ and $\phi \in \Delta^+$. The matrix of $-\left.(\operatorname{ad} v)^2\right|_{\mathfrak{m}}$ with respect to that basis consists then of blocks $\phi(v)^2 I$, $\phi \in \Delta^+$, where $I$ is the $2 \times 2$ identity matrix. Moreover, clearly, $\operatorname{ad}_v = 0$ on $\mathfrak{t}$. Therefore, the stated formula for $\operatorname{vol}(V)$ is valid. The expression for $\operatorname{vol}(V_\alpha)$ follows, since by definition of $v_\alpha$ we have that $\phi(v_\alpha) = m_\alpha(\phi)\,\alpha(v_\alpha)$. □

## Examples

In the following, compact semisimple Lie groups are supposed to be endowed with a bi-invariant Riemannian metric. A reference for the symmetries of the Dynking diagrams referred to below is for instance [7, Ch. X, Theorem 3.29]. For $\alpha_k$ belonging to a basis of a root system we denote $v_k = v_{\alpha_k}$ and $V_k = V_{\alpha_k}$.

**Example 1.** *The number of nonequivalent minimal unit vector fields on $SU(n+1)$ is not smaller than $2n + \sum_{i=1}^{[n/2]} 2 = 2n + 2[n/2]$, where $[a]$ denotes the integral part or $a$.*

It follows from Theorem 7, since $\operatorname{vol}(V_k) = \operatorname{vol}(V_{n-k+1})$ by the $Z_2$ symmetry of the Dynking diagram of $SU(n+1)$.

**Example 2.** *For $SO(5)$ and $G_2$ (the only groups of rank 2 which remain to analyze), two nonequivalent maximal singular unit vectors have different volumes. Thus, in these cases, Theorem 7 yields the lower bound 4 and so it does not improve Theorem 2.*

The statement for $SO(5)$ follows from Example 4 below for $n = 2$. On the other hand, a basis of the root system of $\mathfrak{g}_2$ (contained in $\mathbb{R}^2$ with the canonical metric) consists of $\alpha_1 = r(2, 0)$ and $\alpha_2 = r\left(-3, \sqrt{3}\right)$ for some $r > 0$. The positive roots are $\alpha_1$, $\alpha_2$, $\alpha_1 + \alpha_2$, $2\alpha_1 + \alpha_2$, $3\alpha_1 + \alpha_2$ and $3\alpha_1 + 2\alpha_2$. One easily computes $v_1 = \left(1, \sqrt{3}\right)/2$ and $v_2 = (0, 1)$. Hence $\alpha_1(v_1) = r$ and $\alpha_2(v_2) = \sqrt{3}r$. Therefore, by Proposition 8,

$$\begin{aligned}
\operatorname{vol}(V_1) &= \operatorname{vol}(G)\left(1 + r^2\right)\left(1 + \frac{r^2}{4}\right)^2\left(1 + \frac{9r^2}{4}\right)^2, \\
\operatorname{vol}(V_2) &= \operatorname{vol}(G)\left(1 + 3r^2\right)\left(1 + \frac{3r^2}{4}\right)^4.
\end{aligned}$$

With the aide of the computer one obtains easily that $\operatorname{vol}(V_2) > \operatorname{vol}(V_1)$ for any $r > 0$.

**Example 3.** *The group $SO(8)$ has at least 14 nonequivalent minimal unit vector fields. Besides, for this group, no lower bound as in Theorem 7 is sharp.*

Let $\{\alpha_1, \alpha_2, \alpha_3, \alpha_4\}$ be a basis of the root system of $SO(8)$, where $\alpha_4$ is the distinguished root, that is, there is an action of the group of permutations of three elements by root system isomorphisms (corresponding to the symmetries of the Dynking diagram), fixing $\alpha_4$ and interchanging the rest of the roots. Hence the left invariant vector fields $V_1, V_2, V_3$ have the same volume. Thus, the first assertion follows from Theorem 7. Next we check the validity of the second one. In fact, for $SO(8)$ there exists at least one minimal unit vector field not detected by Theorem 7 and its proof: Let $\overline{C}$ be the closure of the Weyl chamber associated to the given basis and let $S_2$ be as in Proposition 3. Then $S_2 \cap \overline{C}$ is an equilateral spherical triangle with vertices $v_1, v_2, v_3$. Now, by the symmetry of the root system, the center $v$ of the spherical triangle is a critical point of $f$ restricted to $S_2$. By Proposition 3, a left or right invariant vector field on $SO(8)$ taking the value $v$ at the identity is minimal.

**Example 4.** *Let $G = SO(2n + 1)$ be endowed with the metric $-\lambda^2 B$, where $B$ is the Killing form, and let $\{\alpha_1, \dots, \alpha_n\}$ be a basis of the root system with Dynking diagram*

$$\begin{array}{ccccccc}
\overset{2}{\circ} & - & \overset{2}{\circ} & - \cdots - & \overset{2}{\circ} & = & \overset{1}{\circ} \\
\alpha_1 & & \alpha_2 & & \alpha_{n-1} & & \alpha_n
\end{array}.$$

*Then, for $1 \leq k \leq n$, the volume of $V_k$ is given by*

$$\mathrm{vol}\,(G)\left(1+\frac{\rho^2}{4k}\right)^{k+2k(n-k)}\left(1+\frac{\rho^2}{k}\right)^{k(k-1)/2}, \tag{2}$$

*where $2\,(2n-1)\,\rho^2 = \lambda^2$. If $\lambda$ and the rank $n$ are given concretely, using this expression, one can compare the values of $\mathrm{vol}(V_k)$ with the aide of a computer. If $n = 3$, one sees that for all but two values of $\lambda^2$, three nonequivalent maximal singular unit vector fields have different volumes, while for the remaining values of $\lambda^2$, two of them have the same volume.*

Let $\{e_1, \ldots, e_n\}$ be an orthogonal basis of $\mathbb{R}^n$ with $\|e_i\| = r$ for all $i$. The roots

$$\alpha_i = e_i - e_{i+1} \quad (1 \leq i < n) \quad \text{and} \quad \alpha_n = e_n$$

form a basis of the root system of $so\,(2n+1)$, corresponding to the given Dynking diagram, whose positive roots are (see [7, p 462])

$$\begin{array}{ll} e_i = \alpha_i + \cdots + \alpha_n & \text{for } 1 \leq i \leq n, \\ e_i - e_j = \alpha_i + \cdots + \alpha_{j-1} & \text{for } 1 \leq i < j \leq n, \\ e_i + e_j = \alpha_i + \cdots + \alpha_{j-1} + 2\alpha_j + \cdots + 2\alpha_n & \text{for } 1 \leq i < j \leq n. \end{array}$$

Denoting $m_{\alpha_k}$ by $m_k$, one has then for $1 \leq k \leq n$ that

$$\begin{array}{rcl} m_k\,(e_i - e_j) & = & \left\{ \begin{array}{ll} 1 & \text{if } i \leq k < j \\ 0 & \text{if not} \end{array} \right. , \quad m_k\,(e_i + e_j) = \left\{ \begin{array}{ll} 1 & \text{if } i \leq k < j \\ 2 & \text{if } k \geq j \\ 0 & \text{if not} \end{array} \right. \\ \text{and} \;\; m_k\,(e_i) & = & \left\{ \begin{array}{ll} 1 & \text{if } k \geq i \\ 0 & \text{if not} \end{array} \right. . \end{array}$$

Solving the equation for $v_k$ one obtains $v_k = x_k\,(e_1 + \cdots + e_k)$, with $k x_k^2 r^2 = 1$, for $1 \leq k \leq n$. Hence, $\alpha_k\,(v_k)^2 = x_k^2 r^4 = r^2/k$. Now, since for $1 \leq k \leq n$ the cardinalities of the sets

$$\{(i,j) \mid 1 \leq i \leq k < j \leq n\} \quad \text{and} \quad \{(i,j) \mid 1 \leq i < j \leq k\}$$

are $k\,(n-k)$ and $k\,(k-1)\,/2$, respectively, we have by Proposition 8 that the volume of $V_k$ equals

$$\mathrm{vol}\,(G)\left(1+\frac{r^2}{4k}\right)^{k+2k(n-k)}\left(1+\frac{r^2}{k}\right)^{k(k-1)/2}.$$

Finally, since the opposite of the Killing form corresponds to $2\,(2n-1)\,r^2 = 1$, formula (2) is true (set $\rho^2 = \lambda^2 r^2$).

**Acknowledgments.** I would like to thank the referee for making me aware of a bad mistake in the first version, and Carlos Olmos for his helpful comments.

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