On the geometry of the space of oriented lines of the hyperbolic space

Marcos Salvai*

Abstract

Let H be the n-dimensional hyperbolic space of constant sectional curvature -1 and let G be the identity component of the isometry group of H. We find all the G-invariant pseudo-Riemannian metrics on the space $\mathcal{G}_n$ of oriented geodesics of H (modulo orientation preserving reparametrizations). We characterize the null, time- and space-like curves, providing a relationship between the geometries of $\mathcal{G}_n$ and H. Moreover, we show that $\mathcal{G}_3$ is Kähler and find an orthogonal almost complex structure on $\mathcal{G}_7$.

MSC 2000: 53A55, 53C22, 53C35, 53C50, 53D25.

Key words and phrases: hyperbolic space, space of geodesics, invariant metric, Kähler, octonions.

1. Introduction

Let M be a Hadamard manifold (a complete simply connected Riemannian manifold with nonpositive sectional curvature) of dimension $n+1$. An *oriented geodesic* c of M is a complete connected totally geodesic oriented submanifold of M of dimension one. We may think of c as the equivalence class of unit speed geodesics $\gamma : \mathbb{R} \to M$ with image c such that $\{\dot{\gamma}(t)\}$ is a positive basis of $T_{\gamma(t)}c$ for all t. Let $\mathcal{G} = \mathcal{G}(M)$ denote the space of all oriented geodesics of M. The space of geodesics of a manifold all of whose geodesics

*Partially supported by FONCyT, Antorchas, CIEM (CONICET) and SECyT (UNC).

are periodic with the same length is studied with detail in [1]. The geometry of the space of oriented lines of Euclidean space is studied in [3, 9, 10].

Let T^1M be the unit tangent bundle of M and ξ the spray of M, that is, the vector field on T^1M defined by $\xi(v) = d/dt|_0 \gamma_v'(t)$, where γ_v is the unique geodesic in M with initial velocity v. Clearly, $\mathcal{G}$ may be identified with the set of oriented leaves of the foliation of T^1M induced by ξ. By [7], if M is Hadamard, this foliation is regular in the sense of Palais [8]. Hence, $\mathcal{G}$ admits a unique differentiable structure of dimension $2n$ such that the natural projection $T^1M \to \mathcal{G}$ is a submersion.

Fix $o \in M$ and let $\mathrm{Exp} : T_oM \to M$ denote the geodesic exponential map. Let $S = \{v \in T_oM \mid \|v\| = 1\} \cong S^n$. We identify as usual $T_vS \cong v^\perp \subset T_oM$. Hence, $TS \cong \{(v, x) \mid v \in S \text{ and } \langle v, x\rangle = 0\}$. Let $F : TS \to \mathcal{G}$ be defined by

$$F(v, x) = [\gamma],$$

where γ is the unique geodesic in M with initial velocity $\tau_0^1 v$ (here τ denotes parallel transport along the geodesic $t \to \mathrm{Exp}(tx)$ of M). This is called the minitwistor construction in [5]. Keilhauer proved in [7] that F is a diffeomorphism.

2. The geometry of $\mathcal{G}$ for the hyperbolic space

Let $H = H^{n+1}$ be the hyperbolic space of constant sectional curvature -1 and dimension $n+1$. Consider on $\mathbb{R}^{n+2}$ the basis $\{e_0, e_1, \dots, e_{n+1}\}$ and the inner product whose associated norm is given by $\|x\| = \langle x, x\rangle = -x_0^2 + x_1^2 + \cdots + x_{n+1}^2$. Then $H = \{x \in \mathbb{R}^{n+2} \mid \|x\| = -1 \text{ and } x_0 > 0\}$ with the induced metric. Let G be the identity component of the isometry group of H, that is,

$$G = O_o(1, n+1) = \{g \in O(1, n+1) \mid (ge_0)_0 > 0 \text{ and } \det g > 0\}.$$

In the following we denote $\mathcal{G}_m = \mathcal{G}(H^m)$ (or simply $\mathcal{G}$ if no confusion is possible). The group G acts on $\mathcal{G}$ as follows: $g[\gamma] = [g \circ \gamma]$. This action is transitive, since H is two-point homogeneous, and smooth, since G acts smoothly on T^1H.

Let γ_o be the geodesic in H with $\gamma_o(0) = e_0$ and initial velocity $e_1 \in T_{e_0}H$. The isotropy subgroup of G at $c_o := [\gamma_o]$ is

$$G_o = \{\mathrm{diag}(T_t, A) \mid t \in \mathbb{R}, A \in SO_n\} \cong \mathbb{R} \times SO_n,$$

where $T_t = \begin{pmatrix} \cosh t & \sinh t \\ \sinh t & \cosh t \end{pmatrix}$. Therefore we may identify $\mathcal{G}$ with G/G_o in the usual way. Let $\mathfrak{g}$ be the Lie algebra of G and let

$$\mathfrak{g}_o = \{\mathrm{diag}\,(tR, A) \mid t \in \mathbb{R}, A \in so_n\}$$

be the Lie algebra of G_o (here $R = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$). Let B be the bilinear form on $\mathfrak{g}$ defined by $B\,(X, Y) = \frac{1}{2}\,\mathrm{tr}\,(XY)$, which is well-known to be a multiple of the Killing form of $\mathfrak{g}$, hence nondegenerate. Besides, the canonical projection $\pi : G \to H$, $\pi\,(g) = g\,(e_0)$, is a pseudo-Riemannian submersion.

Let $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{h}$ be the orthogonal decomposition with respect to B. Then

$$T_{c_o}\mathcal{G} = \mathfrak{h} := \{x_h + y_v \mid x, y \in \mathbb{R}^n\},$$

where for column vectors $x, y \in \mathbb{R}^n$,

$$x_h = \begin{pmatrix} 0_2 & (x,0)^t \\ (x,0) & 0_n \end{pmatrix} \quad \text{and} \quad y_v = \begin{pmatrix} 0_2 & (0,y)^t \\ (0,-y) & 0_n \end{pmatrix}$$

(here the exponent t denotes transpose and 0_m the $m \times m$ zero matrix). We chose this notation since x_h and y_v are horizontal and vertical, respectively, tangent vectors in $T_{(e_0,e_1)}\,(T^1H)$ with respect to the canonical projection $T^1H \to H$.

Theorem 1 *For each $n \geq 1$ there exists a G-invariant pseudo-Riemannian metric g_1 on $\mathcal{G}_{n+1}$ whose associated norm at c_o is given by*

$$\|x_h + y_v\|_1 = |x|^2 - |y|^2 .$$

For $n = 2$, if one identifies $\mathbb{R}^2 = \mathbb{C}$ as usual, there exists a G-invariant metric g_0 on $\mathcal{G}_3$ whose associated norm at c_o is given by

$$\|x_h + y_v\|_0 = \langle ix, y\rangle .$$

For $n \neq 2$, any G-invariant pseudo-Riemannian metric on $\mathcal{G}_{n+1}$ is homothetic to g_1. Any G-invariant pseudo-Riemannian metric on $\mathcal{G}_3$ is of the form $\lambda g_0 + \mu g_1$ for some $\lambda, \mu \in \mathbb{R}$ not simultaneously zero.

All the metrics are symmetric and have split signature (n, n). In particular, $\mathcal{G}$ does not admit any G-invariant Riemannian metric and the geodesics in $\mathcal{G}$ through c_o are exactly the curves $s \mapsto \exp_G(sX)\,c_o$, for $X \in \mathfrak{h}$.

Proof. One computes easily that $B(X,X)=\|X\|_1$ for all $X\in\mathfrak{h}$. Since B is G-invariant, g_1 defines a G-invariant metric on $\mathcal{G}$.

Let $Z=\mathrm{diag}\,(R,0_n)$, $\mathfrak{m}=\{\mathrm{diag}\,(0_2,A)\mid A\in so_n\}$ and $\mathfrak{g}_\lambda=\{U\in\mathfrak{g}\mid \mathrm{ad}_Z U=\lambda U\}$. One verifies that $\mathfrak{g}_0=\mathfrak{g}_o$ and $\mathfrak{g}_{\pm1}=\{x_h\pm x_v\mid x\in\mathbb{R}^n\}$. Moreover, one has the decompositions

$$\mathfrak{g}_0=\mathbb{R}Z\oplus\mathfrak{m}\quad\text{and}\quad\mathfrak{h}=\mathfrak{g}_1\oplus\mathfrak{g}_{-1},$$

which are preserved by the the action of $\mathfrak{m}$. Hence $\mathfrak{h}$ is $\mathfrak{g}_0$-invariant.

Since B is nondegenerate and G_o is connected, any other pseudo-Riemannian metric g on $\mathcal{G}$ has the form $g(U,V)=B(TU,V)$ for some $T:h\to h$ commuting with ad_Z and $\mathrm{ad}_{\mathfrak{m}}$. In particular, T preserves $\mathfrak{g}_{\pm1}$. We call $T_\pm$ the restrictions of T to the corresponding subspaces. Under the identification $\mathfrak{g}_{\pm1}\equiv\mathbb{R}^n$, $x_h\pm x_v\equiv x$, the action of $\mathfrak{m}\equiv so_n$ on $\mathbb{R}^n$ is the canonical one. If $T_\pm\in Gl(\mathfrak{g}_{\pm1})\equiv Gl(n,\mathbb{R})$ commutes with every $A\in so_n$, then either $T_\pm$ is a nonzero multiple of the identity or $n=2$ and $T_\pm=a_\pm I_2+b_\pm J$ where $J=\begin{pmatrix}0&-1\\1&0\end{pmatrix}$, for some not simultaneously zero constants $a_\pm$ and $b_\pm$. Next we consider the case $n=2$ and show that $a_+=a_-$ and $b_-=-b_+$. For $x\neq0$ we denote $x^\pm=x_h\pm x_v$ and compute

$$\begin{aligned}B\left(T\left(x^+\right),x^-\right)&=B\left(\left(a_+x+b_+ix\right)^+,x^-\right)\\&=a_+B\left(x^+,x^-\right)+b_+B\left(\left(ix\right)^+,x^-\right)\\&=2a_+\left|x\right|^2+0.\end{aligned}$$

Since T must be symmetric with respect to B, this expression coincides with $B(x^+,T(x^-))$, which by similar computations equals $2a_-\left|x\right|^2$. Hence $a_+=a_-$. Using again the symmetry of T in the case

$$B\left(T\left(x^-\right),(ix)^+\right)=B\left(x^-,T\left(ix\right)^+\right)$$

one obtains that $b_-=-b_+$. Finally, since $2\left(x_h+y_v\right)=\left(x+y\right)^++\left(x-y\right)^-$, one computes that the metric associated with T is homothetic to g_1 if $b_+=0$ and to g_o if $a_+=0$. The case $n\neq2$ is simpler since it does not involve $b_\pm$.

Next we show that for any of the metrics above, $\mathcal{G}$ is a symmetric space. Let $G^\uparrow=\{g\in O(1,n+1)\mid (ge_0)_0>0\}$ be the isometry group of H and let $C=\mathrm{diag}\,(I_2,-I_n)\in G^\uparrow$, which induces an involutive diffeomorphism $\widetilde{C}$ of $\mathcal{G}$ by $\widetilde{C}[\gamma]=[C\circ\gamma]$ fixing exactly c_o. If $n=2$, $C\in G$, hence $\widetilde{C}$ is clearly an isometry for any G-invariant metric on $\mathcal{G}_3$. The same happens for

$n \neq 2$. Indeed, in this case, up to homotheties, we have seen that the unique metric on $\mathcal{G}_m$ with $m \neq 3$ comes from a multiple of the Killing form of $\mathfrak{g}$, which is invariant by the action of $G^{\uparrow}$. The statement regarding geodesics follows from the theory of symmetric spaces, since conjugation by C is an involutive automorphism of $\mathfrak{g}$ whose (-1)-eigenspace is $\mathfrak{g}_0$ and preserves the given metrics. □

Remarks. a) In contrast with the space of oriented lines of $\mathbb{R}^n$, which only for $n = 3, 7$ admits pseudo-Riemannian metrics invariant by the induced transitive action of a connected closed subgroup of the identity component of the isometry group (see [9]), $\mathcal{G}_n$ admits G-invariant metrics for all n.

b) The metric g_0 is the analogue of the metric defined in the Euclidean case in [11, 4]. We will see below that also in the hyperbolic case it admits a Kähler structure.

c) For any complete simply connected Riemannian manifold M of negative curvature, the space $\mathcal{G}(M)$ of its oriented geodesics has a canonical pseudo-Riemannian metric, which is in general only continuous, see [6]. If M is the hyperbolic space, then g_1 is the canonical metric on $\mathcal{G}$.

d) If H has dimension two, then $\mathcal{G}$ is isometric to the two-dimensional de Sitter sphere.

We recall some well-known facts about the imaginary border of the hyperbolic space and the action of G on it. For a geodesic γ in H, $\gamma(\infty)$ is defined to be the unique $z \in S^n$ such that $\lim_{t\to\infty} \gamma(t)/\gamma(t)_0 = e_0 + z \in \mathbb{R}^{n+2}$. One defines analogously $\gamma(-\infty)$. Sometimes we will identify $\mathbb{R}^{n+1}$ with $e_0^{\perp}$ and S^n with $\{e_0\} \times S^n$.

The group G acts on S^n by directly (that is, orientation preserving) conformal diffeomorphisms. More precisely, any $g \in G$ induces the directly conformal transformation $\widetilde{g}$ of S^n, well-defined by $\widetilde{g}(\gamma(\infty)) = (g \circ \gamma)(\infty)$, and any directly conformal transformation of S^n can be realized in this manner.

Proposition 2 *If S is a subgroup of G acting transitively on $\mathcal{G}$, then $S = G$.*

Proof. By the main result of [2], it suffices to show that S acts irreducibly on $\mathbb{R}^{n+2}$. Suppose that S leaves the nontrivial subspace V invariant. If V is degenerate, then V contains a null line, say $\mathbb{R}(e_0 + z)$, with $z \in \mathbb{R}^{n+1}$, $|z| = 1$. Hence S takes the oriented line $[\gamma]$ with $\gamma(\infty) = z$ to another line with the same point at ∞. If V is nondegenerate, either V or its complement (also S-invariant) intersects H. Let us call $H_1 \subsetneq H$ the intersection, which is a totally geodesic submanifold of H. Then S takes any oriented line contained in H_1 to a line contained in H_1. If H_1 is a point p, then S takes any line through p to a line through p. Therefore the action of S on $\mathcal{G}$ is not transitive. $\square$

Remark. The hyperbolic case contrasts with the Euclidean one: We found in [9] a pseudo-Riemannian metric on the space of oriented lines of $\mathbb{R}^7 = \operatorname{Im}\mathbb{O}$ which is invariant by the transitive action of $G_2 \ltimes \mathbb{R}^7$, where G_2 is the automorphism group of the octonions $\mathbb{O}$.

3. Null, space- and time-like curves

In order to give a geometric interpretation for a curve in $\mathcal{G}$ endowed with some of the G-invariant metrics to be null, space- or time-like, we introduce the following concept, which makes sense for any Hadamard manifold.

Definition. Let H be a Hadamard manifold. Given a smooth curve c in $\mathcal{G}$ defined on the interval I, a function $\varphi : \mathbb{R} \times I \to H$ is said to be a *standard presentation of* c if $s \mapsto \alpha_t(s) := \varphi(s,t)$ is a unit speed geodesic of H satisfying $c(t) = [\alpha_t]$ and $\left\langle \dot\beta(t), \dot\alpha_t(0) \right\rangle = 0$ for all $t \in I$, where $\beta(t) = \varphi(0,t)$.

Proposition 3 *Given a smooth curve $c : I \to \mathcal{G}$ and p a point in the image of some (any) geodesic in the equivalence class $c(t_o)$, there exists a standard presentation φ of c such that $\varphi(0,t_o) = p$.*

Proof. Consider the submersion $\Pi : T^1H \to \mathcal{G}$, $\Pi(v) = [\gamma_v]$. Let $v(t)$ be a lift of $c(t)$ to T^1H with $v(t_o) \in T^1_pH$, and let $\psi : \mathbb{R} \times I \to H$ be defined by $\psi(s,t) = \gamma_{v(t)}(s)$. We look for a function $f : I \to \mathbb{R}$ such that

$$\varphi(s,t) = \psi(s + f(t), t)$$

satisfies the required properties. Clearly $\alpha_t(s) = \varphi(s,t)$ has unit speed and

$$c(t) = \Pi(v(t)) = \Pi\left(\gamma'_{v(t)}(f(t))\right) = [\alpha_t].$$

One can verify easily that taking as f the solution of the differential equation

$$f'(t) = -\frac{\langle \psi_t(f(t),t), \psi_s(f(t),t)\rangle}{\|\psi_s(f(t),t)\|^2}$$

(subindexes denote partial derivatives) with $f(t_o) = 0$, then $\varphi(0, t_o) = p$ and $\left\langle \dot{\beta}(t), \dot{\alpha}_t(0) \right\rangle = 0$ for all $t \in I$, where β is as in the definition of the standard presentation. $\square$

The following Proposition characterizes the null, time- and space-like curves of $\mathcal{G}$, providing a relationship between the geometries of $\mathcal{G}$ and H.

Proposition 4 *For the metric g_1, a smooth curve c in $\mathcal{G}_n$ is null (respectively, space-, time-like) if and only if, for any standard presentation, the rate of variation of the directions, that is, $\left\| \frac{D}{dt}\dot{\alpha}_t(0) \right\|$, coincides with (respectively, is smaller, larger than) the rate of displacement $\left\| \dot{\beta}(t) \right\|$ for all t (here $\frac{D}{dt}$ denotes covariant derivative along β).*

For the metric g_0 on $\mathcal{G}_3$, a smooth curve c in $\mathcal{G}_3$ is null (respectively, space-, time-like) if and only if, for any standard presentation,

$$\left\{ \dot{\beta}(t), \frac{D}{dt}\dot{\alpha}_t(0), \dot{\alpha}_t(0) \right\}$$

is linearly dependent (respectively, positively, negatively oriented) for all t.

Proof. Let $[\gamma]$ be an oriented geodesic of a Hadamard manifold and let $\mathcal{J}_\gamma$ be the space of Jacobi fields along γ orthogonal to $\dot{\gamma}$. First we show that $L_\gamma : \mathcal{J}_\gamma \to T_{[\gamma]}\mathcal{G}$ given by

$$L_\gamma(J) = (d/dt)_0 [\gamma_t],$$

where γ_t is a variation of γ by unit speed geodesics associated with the Jacobi field J, is a well-defined vector space isomorphism. Indeed, let $\Pi : T^1M \to \mathcal{G}$ be as above the canonical projection, which is a smooth submersion, by definition of the differentiable structure on $\mathcal{G}$. We compute

$$(d/dt)_0 [\gamma_t] = (d/dt)_0 \Pi(\dot{\gamma}_t(0)) = d\Pi_{\dot{\gamma}(0)}((d/dt)_0 \dot{\gamma}_t(0)).$$

Now, let $\mathrm{p} : T^1H \to H$ be the canonical projection and $\mathcal{K} : T_{\dot{\gamma}(0)}(T^1H) \to \dot{\gamma}(0)^\perp \subset T_{\gamma(0)}H$ the connection operator. It is well-known that $(d\mathrm{p}, \mathcal{K}) : T_{\dot{\gamma}(0)}(T^1H) \to T_{\gamma(0)}H \oplus \dot{\gamma}(0)^\perp$ is a bijection and

$$(d/dt)_0 \dot{\gamma}_t(0) = (d\mathrm{p}, \mathcal{K})^{-1}(J(0), J'(0))$$

(see for instance [1]). Therefore, L_γ is well-defined.

Next we show that for any $J \in \mathcal{J}_\gamma$ one has

$$\begin{aligned} \left\| L_\gamma (J) \right\|_1 &= \left\| J(0) \right\|^2 - \left\| J'(0) \right\|^2 \\ \left\| L_\gamma (J) \right\|_0 &= \left\langle \dot\gamma (0) \times J(0), J'(0) \right\rangle . \end{aligned} \tag{1}$$

We may suppose without loss of generality that $c = c_o$ and $\gamma = \gamma_o$. Let $c'(0) = x_h + y_v$ with $x, y \in \mathbb{R}^n$. Then the Jacobi field along γ_o satisfying $L_{\gamma_o}(J) = c'(0)$ is the one determined by

$$J(0) = d\pi_I(x_h) \text{ and } J'(0) = d\pi_I(y_h),$$

where $\pi : G \to H$ is as before the canonical projection. In fact, clearly, $\gamma_t(s) = \exp(tx_h)\exp(ty_v)\gamma_o(s)$ is a variation of γ_o by unit speed geodesics. Let us see that the associated Jacobi field is J. Indeed,

$$J(0) = \left. \frac{d}{dt} \right|_0 \gamma_t(0) = \left. \frac{d}{dt} \right|_0 \exp(tx_h)\, e_0 = d\pi_I(x_h),$$

since $\gamma_o(0) = e_0$, which is fixed by $\exp(ty_v)$. If $\frac{D}{dt}$ denotes covariant derivative along $t \mapsto \gamma_t(0)$ and Z is as in the beginning of the proof of Theorem , then

$$\begin{aligned} J'(0) &= \left. \frac{D}{dt} \right|_0 \dot\gamma_t(0) = \left. \frac{D}{dt} \right|_0 d\left(\exp(tx_h)\exp(ty_v)\right)_{\pi(I)} e_1 \\ &= \left. \frac{D}{dt} \right|_0 d\exp(tx_h)\, d\pi_I \mathrm{Ad}(\exp ty_v)\, Z \\ &= d\pi_I \left. \frac{d}{dt} \right|_0 e^{t \operatorname{ad} y_v} Z = d\pi_I [y_v, Z] = d\pi_I(y_h), \end{aligned}$$

since $d\exp(tx_h)$ realizes the parallel transport and $d\pi_I(Z) = e_1$. Therefore (1) is true by Theorem . Finally, suppose that φ is a standard presentation of c and let α_t, β be as above. Let J_t denote the Jacobi field along α_t associated with the variation φ. Clearly, $\dot c(t) = L_{\alpha_t}(J_t)$, $J_t(0) = \frac{d}{dt}\varphi(0,t) = \dot\beta(t)$ and

$$J_t'(0) = \left. \frac{D}{ds} \right|_0 \frac{d}{dt}\varphi(s,t) = \frac{D}{dt} \left. \frac{d}{ds} \right|_0 \varphi(s,t) = \frac{D}{dt} \dot\alpha_t(0).$$

Consequently, the proposition follows from (1). □

4. A geometric invariant of $\mathcal{G}$

We have mentioned in the introduction that $\mathcal{G}(H^n)$ is diffeomorphic to $\mathbb{T}^n$, the space of all oriented lines of $\mathbb{R}^n$. For $n = 3$ and $n = 7$, we found in [9] pseudo-Riemannian metrics on $\mathbb{T}^n$ invariant by the induced transitive action of a connected closed subgroup of $SO_n \ltimes \mathbb{R}^n$ (only for those dimensions such metrics exist).

Proposition 5 *For $n = 3, 7$, no metric on $\mathcal{G}_n$ invariant by the identity component of the isometry group of H^n is isometric to $\mathbb{T}^n$ endowed with any of the metrics above.*

Proof. We compute now a pseudo-Riemannian invariant of $\mathcal{G}_n$ involving its periodic geodesics. For any $c \in \mathcal{G}$, let A denote the subset of $T_c\mathcal{G}$ consisting of the velocities of periodic geodesics of $\mathcal{G}$ though c. We show next that the frontier of A in $T_c\mathcal{G}$ is the union of two subspaces of half the dimension of $\mathcal{G}$ intersecting only at zero. By homogeneity we may suppose that $c = c_o$. Since by the proposition below $A = \{\lambda x_h + x_v \mid x \in \mathbb{R}^n,\ |\lambda| < 1\}$, the frontier of A is $\mathfrak{g}_1 \cup \mathfrak{g}_{-1}$. On the other hand, we have computed in [10] that the analogue invariant for $\mathbb{T}^n$ $(n = 3, 7)$ is a subspace of half the dimension of $\mathbb{T}^n$. Hence the proposition follows. $\square$

Remarks. a) Of course we could have considered more standard invariants, like the curvature or the isometry group, but we chose this one since the geodesics can be described so easily.

b) Clearly the difference in the invariants is related to the fact that the two horospheres through a point associated with opposite directions coincide in the Euclidean case but are different in the hyperbolic case.

Proposition 6 *A geodesic in $\mathcal{G}$ with initial velocity $x_h + y_v$ is periodic if and only if $x = \lambda y$ for some $\lambda \in \mathbb{R}$ with $|\lambda| < 1$.*

Proof. We may suppose that $x_h + y_v \neq 0$. We compute that $\operatorname{Ad}\left(e^{tZ}\right)(x_h + y_v) = x^t_v + y^t_v$, where

$$x^t = (\cosh t)\, x + (\sinh t)\, y \quad \text{and} \quad y^t = (\sinh t)\, x + (\cosh t)\, y.$$

Now, there exists s such that $\langle x^s, y^s \rangle = 0$ (take $\tanh(2s) = -\frac{2\langle x, y\rangle}{|x|^2 + |y|^2}$). Hence $[x^s_h, y^s_v] = 0$ and consequently

$$\pi \exp\left(t\left(x^s_h + y^s_v\right)\right) = \pi \exp\left(t x^s_h\right) \exp\left(t y^s_v\right) = \pi \exp\left(t x^s_h\right),$$

which is a geodesic in H, in particular it is periodic only if it is constant, or equivalently, only for $x^s = 0$.

Since $Z \in \mathfrak{g}_0$ and the metric is G-invariant, the geodesics with initial velocities $x_h^t + y_v^t$ are simultaneously periodical or not periodical for all t. Now, one verifies that $x^s = 0$ if and only if $x = \lambda y$ for some $\lambda \in \mathbb{R}$ with $|\lambda| < 1$ and the proposition follows. □

5. Additional geometric structures on $\mathcal{G}$

An almost Hermitian structure on a pseudo-Riemannian manifold (M, g) is a smooth tensor field J of type $(1,1)$ on M such that J_p is an orthogonal transformation of (T_pM, g_p) and satisfies $J_p^2 = -\operatorname{id}$ for all $p \in M$. If ∇ is the Levi Civita connection of (M, g), then (M, g, J) is said to be Kähler if $\nabla J = 0$.

A Kähler structure on $\mathcal{G}(H^3)$

Let $\mathcal{G} = \mathcal{G}_3$ and let j_o be the endomorphism of $\mathfrak{h} \equiv T_{c_o}\mathcal{G} \equiv \mathbb{C} \times \mathbb{C}$ given by $j_o(z, w) = (iz, iw)$. One checks that j_o commutes with the action of G_o, is orthogonal for g_0 and g_1 and $j_o^2 = -\operatorname{id}$. Therefore j_o defines an orthogonal almost complex structure on $\mathcal{G}_3$ for any G-invariant metric on it.

Proposition 7 *The space $(\mathcal{G}_3, J)$ is Kähler for any pseudo-Riemannian G-invariant metric on $\mathcal{G}_3$.*

Proof. We show that for every geodesic γ in $\mathcal{G}_3$ and any parallel vector field Y along γ, the vector field JY along γ is parallel. By homogeneity we may suppose that $\gamma(0) = c_o$. Suppose that $\gamma(t) = \exp(tX)\, c_o$ for some $X \in \mathfrak{h}$. By a well-known property of symmetric spaces, $Y = d\exp(tX)_{c_o} Y_{c_o}$. Since J is G-invariant, $JY = d\exp(tX)_{c_o} JY_{c_o}$ and thus JY is parallel along γ, as desired. □

An orthogonal almost complex structure on $\mathcal{G}_7$

We present another model of $\mathcal{G}_{n+1}$ endowed with the metric g_1 and use it to define an orthogonal almost complex structure on $\mathcal{G}_7$.

In the following we use the notations given before Proposition of concepts related to the imaginary border of H. We recall that $g \in G$ is called a transvection of H if it preserves a geodesic γ of H and dg realizes the parallel transport along γ, that is, $g(\gamma(t)) = \gamma(t+s)$ for all t and some s and $dg_{\gamma(t)}$ realizes the parallel transport between $\gamma(t)$ and $\gamma(t+s)$ along γ. For any unit $v \in T_{e_0}H = e_0^\perp = \mathbb{R}^{n+1}$ the transvections through $e_0 \in H$ preserving the geodesic with initial velocity v form a one parameter subgroup ϕ_t such that the corresponding one parameter group $\widetilde{\phi}_t$ of conformal transformations of S^n (which we also call transvections, by abuse of notation) is the flow of the vector field on S^n defined at $q \in S^n$ as the orthogonal projection of the constant vector field v on $\mathbb{R}^{n+1}$ onto $T_qS^n = q^\perp$. In particular $\widetilde{\phi}_t$ fixes $\pm v \in S^n$. For $\tau = \widetilde{\phi}_t$ we will need specifically the following standard facts:

$*$) If $u \in S^n$ is orthogonal to v, then $v \in T_uS^n$ and if $\tau(u) = (\cos\theta)u + (\sin\theta)v$, then $(d\tau)_u v$ is a vector in $T_{\tau(u)}S^n$ spanned by u and v of length $\cos\theta$.

$**$) There exists a positive constant c such that $(d\tau)_{\pm v}$ is a multiple $c^{\pm 1}$ of the identity map on $T_{\pm v}S^n = v^\perp$.

Let $\Delta_n = \{(p,p) \mid p \in S^n\}$ denote the diagonal in $S^n \times S^n$. The map

$$\psi : \mathcal{G}_{n+1} \to (S^n \times S^n) \setminus \Delta_n, \qquad \psi([\gamma]) = (\gamma(-\infty), \gamma(\infty)) \tag{2}$$

is a well-defined diffeomorphism. We denote by $\widehat{g}$ the induced action of $g \in G$ on $(S^n \times S^n) \setminus \Delta_n$, that is $\widehat{g}(p,q) = (\widetilde{g}(p), \widetilde{g}(q))$. Given distinct points $p, q \in S^n$, let $T_{p,q}$ denote the reflection on $\mathbb{R}^{n+1}$ with respect to the hyperplane orthogonal to $p - q$.

Proposition 8 *If $\mathcal{G}_{n+1}$ is endowed with the metric g_1 and one considers on $(S^n \times S^n) \setminus \Delta_n$ the pseudo-Riemannian metric whose associated norm is*

$$\|(x,y)\|_{(p,q)} = 4\langle T_{p,q}x, y\rangle / |q-p|^2 \tag{3}$$

for $x \in p^\perp$, $y \in q^\perp$, then the diffeomorphism ψ of (2) *is an isometry.*

Proof. Clearly ψ is G-equivariant. Since the metric g_1 on $\mathcal{G}_{n+1}$ is G-invariant, it is sufficient to show that the metric (3) on $(S^n \times S^n) \setminus \Delta_n$ is G-invariant as well and that $d\psi_{[\gamma_o]}$ is a linear isometry.

Given distinct points $p_\pm \in S^n$, we show first that for any $g \in G$ with $\widetilde{g}(e_{\pm 1}) = p_\pm$, $d\widehat{g}_{(-e_1, e_1)}$ is a linear isometry. A straightforward computation shows that the given metric on $(S^n \times S^n) \setminus \Delta_n$ is invariant by the action of SO_{n+1}, since for all k in this group, $T_{k(p),k(q)} \circ k = k \circ T_{p,q}$ for all $p, q \in S^n$, $p \neq q$. Hence we may suppose without loss of generality that $p_\pm =$

$\pm(\cos\theta)\, e_1 + (\sin\theta)\, e_2$ for some $\theta \in [0, \pi/2)$. Now, any directly conformal transformation $\widetilde{g}$ as above may be written as a composition $\tau^2 \circ \tau^1 \circ R$, where R is a rotation fixing e_1 and τ^1 and τ^2 are transvections fixing $(-e_1, e_1)$ and $(-e_2, e_2)$, respectively.

The assertion $(**)$ above, with $v = e_1$ and $\tau = \tau^1$, implies that $d\widehat{\tau^1}_{(-e_1,e_1)}$ is a linear isometry. Now we use the assertion $(*)$ with $v = e_2$ and $u = e_1$ to see that $d\widehat{\tau^2}_{(-e_1,e_1)} : e_1^\perp \times e_1^\perp \to p_-^\perp \times p_+^\perp$ is a linear isometry. Let $\lambda_\pm v + x_\pm \in T_{\pm u} S^n = u^\perp$, with $\lambda_\pm$ real numbers and $\langle x_\pm, v\rangle = 0$. One computes

$$\begin{aligned} \left\|(\lambda_- v + x_-, \lambda_+ v + x_+)\right\|_{(-u,u)} &= 4\left(\lambda_-\lambda_+ + \langle x_-, x_+\rangle\right) / \left|2u\right|^2 \\ &= \left(\lambda_-\lambda_+ + \langle x_-, x_+\rangle\right). \end{aligned} \tag{4}$$

On the other hand, call $d\tau^2_{\pm u}(v) = v_\pm$ and $d\tau^2_{\pm u}(x_\pm) = y_\pm$. Hence $|v_\pm| = \cos\theta$. Since $d\tau^2_{\pm u}$ is conformal, $y_\pm$ is orthogonal to $v_\pm$ and has length $|x_\pm|\cos\theta$. Also, $y_\pm$ is orthogonal to u, hence it is left fixed by T_{p_-,p_+}. Therefore one computes

$$\left\|(\lambda_- v_- + y_-, \lambda_+ v_+ + y_+)\right\|_{(-u,u)} = \frac{4\cos^2\theta}{\left|p_- - p_+\right|^2}\left(\lambda_-\lambda_+ + \langle x_-, x_+\rangle\right),$$

which coincides with (4) since $|p_- - p_+| = 2\cos\theta$. This completes the proof that $d\widehat{g}_{(-e_1,e_1)}$ is a linear isometry. It remains only to show that $d\psi_{[\gamma_o]}$ is a linear isometry.

We have that $\gamma_o(t) = (\cosh t, \sinh t, 0) \in \mathbb{R}^{n+2}$. Let J be the Jacobi field along γ_o orthogonal to γ_o and satisfying $J(0) = x$ and $J'(0) = y$, both in $T_{e_0}H$ orthogonal to $e_1 = \gamma'(0)$. We show next that

$$d\psi_{[\gamma_o]} L_{\gamma_o}(J) = (x - y, x + y),$$

where L_{γ_o} was defined in (**??**). By invariance of ψ by rotations it is sufficient to see that

$$d\psi_{[\gamma_o]} L_{\gamma_o}(J_\pm) = (\pm e_2, e_2), \tag{5}$$

where $J_-(0) = 0$, $J_-'(0) = e_2$, $J_+(0) = e_2$ and $J_+'(0) = 0$. Let now

$$A_s = \begin{pmatrix} \cos s & -\sin s \\ \sin s & \cos s \end{pmatrix} \text{ and } B_s = \begin{pmatrix} \cosh s & 0 & \sinh s \\ 0 & 1 & 0 \\ \sinh s & 0 & \cosh s \end{pmatrix}.$$

The field J_- is associated to the variation of γ_o corresponding to the one parameter group of isometries $s \mapsto A_s^- = \operatorname{diag}(1, A_s, I_{n-1})$. One computes $A_s^-(\gamma_o(t)) = (\cosh t)\, e_0 + \sinh t\, ((\cos s)\, e_1 + (\sin s)\, e_2) \in H$. Hence

$$\begin{aligned}\left(A_s^- \circ \gamma_o\right)(\pm\infty) &= \lim_{t\to\pm\infty} (\tanh t)\, ((\cos s)\, e_1 + (\sin s)\, e_2) \\ &= \pm (\cos s)\, e_1 \pm (\sin s)\, e_2,\end{aligned}$$

whose derivative at $s = 0$ is $\pm e_2$. Therefore $(d/ds|_0)\, \psi\, [A_s^- \circ \gamma_o] = (-e_2, e_2)$. Using $B_s^+ = \operatorname{diag}(B_s, I_{n-1})$ instead of A_s^- one verifies the remaining identity of (5). Finally, since T_{-e_1,e_1} clearly fixes x, y, the norm (3) of $(x - y, x + y)$ at $(-e_1, e_1)$ is $4\, \langle x - y, x + y\rangle / |2e_1|^2 = |x|^2 - |y|^2$, which coincides with the norm of $L_{\gamma_o}(J)$ by (1). This shows that $d\psi_{[\gamma_o]}$ is a linear isometry. $\square$

Let $\mathbb{O}$ denote the normed division algebra of the octonions and let $\mathbb{R}^7 = \operatorname{Im} \mathbb{O}$ endowed with its canonical cross product $\times$. Let j be the almost complex structure of S^6 defined by $j_p(x) = p \times x$ if $x \in T_p S^6 = p^\perp$. For $q \in S^6$, $q \neq p$, let $j_{p,q}$ be the linear operator on $T_q S^6 = q^\perp$ defined by $j_{p,q} = T_{p,q} \circ j_p \circ T_{p,q}$.

Proposition 9 *For all $x \in p^\perp, y \in q^\perp$,*

$$J_{(p,q)}(x, y) = (j_p(x), j_{p,q}(y))$$

defines an orthogonal almost complex structure on $(S^6 \times S^6) \backslash \Delta_n$ with the metric above.

Proof. First we check that J is an almost complex structure. Indeed,

$$\langle j_{p,q}(y), q\rangle = \langle j_p T_{p,q}(y), T_{p,q}(q)\rangle = \langle p \times T_{p,q}(y), p\rangle = 0$$

and $J^2 = -\operatorname{id}$ holds as well, since $j_p^2 = -\operatorname{id}$ and $T_{p,q}^2 = \operatorname{id}$. Finally, J is orthogonal since both j_p and $T_{p,q}$ are so. $\square$

Remarks. a) By Proposition there exists no proper subgroup of G acting transitively on $\mathcal{G}$ leaving J invariant, as it is the case of the analogous almost complex structure defined in [9] on the space of oriented lines of $\mathbb{R}^7$.

b) The structure J is not integrable, since $(S^6 \backslash \{p\}) \times \{p\}$ is an almost complex submanifold for any p, whose induced almost complex structure is $q \mapsto j_q$, which is not integrable.

Acknowledgment. I would like to thank Eduardo Hulett for his help concerning the symmetry of $\mathcal{G}$ and Antonio Di Scala for the statement and the idea of the proof of Proposition .

References

[1] A. Besse, Manifolds all of whose geodesics are closed, Ergebnisse der Mathematik und ihre Grenzgebiete no. 93 (Springer, 1978).

[2] A. Di Scala and C. Olmos, The geometry of homogeneous submanifolds of hyperbolic space, Math. Z. **237** (2001), 199-209.

[3] B. Guilfoyle and W. Klingenberg, On the space of oriented affine lines in $\mathbb{R}^3$, Archiv Math. **82** (2004), 81–84.

[4] B. Guilfoyle and W. Klingenberg, An indefinite Kähler metric on the space of oriented lines, J. London Math. Soc. **72** (2005), 497–509.

[5] N. J. Hitchin, Monopoles and geodesics, Comm. Math. Phys. **83** (1982), 579–602.

[6] M. Kanai, Geodesic flows of negatively curved manifolds with smooth stable and unstable foliations, Ergodic Theory Dyn. Syst. **8** (1988), 215–239.

[7] G. Keilhauer, A note on the space of geodesics, Rev. Unión Mat. Argent. **36** (1990), 164–173.

[8] R. Palais, A global formulation of the Lie theory of transformation groups, Memoirs A.M.S. no. 22 (AMS, 1957).

[9] M. Salvai, On the geometry of the space of oriented lines in Euclidean space, Manuscr. Math. **118** (2005), 181–189.

[10] M. Salvai, Geodesics in the space of oriented lines in Euclidean space, Proceedings of Egeo 2005, Rev. Unión Mat. Argent. (2) **47** (2006), 109–114.

[11] M. Shepherd, Line congruences as surfaces in the space of lines. Diff. Geom. Appl. **10** (1999), 1–26.

Marcos Salvai, FAMAF - CIEM, Ciudad Universitaria, 5000 Córdoba, Argentina.
E-mail: salvai@mate.uncor.edu