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On the geometry of the space of oriented lines of Euclidean space

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Abstract. We prove that the space of all oriented lines of the n-dimensional Euclidean space admits a pseudo-Riemannian metric which is invariant by the induced transitive action of a connected closed subgroup of the group of Euclidean motions, exactly when $n = 3$ or $n = 7$ (as usual, we consider Riemannian metrics as a particular case of pseudo-Riemannian ones). Up to equivalence, there are two such metrics for each dimension, and they are of split type and complete. Besides, we prove that the given metrics are Kähler or nearly Kähler if $n = 3$ or $n = 7$, respectively.

Key words. oriented lines – minitwistor – pseudo-Riemannian metric – quaternions – octonions – Kahler – nearly Kahler

1. Introduction

An oriented line in $\mathbb{R}^n$ is a pair $\ell(u,v) := (\{tu+v \mid t \in \mathbb{R}\}, u)$ for some $u, v \in \mathbb{R}^n$, $|u| = 1$, where u is the direction (orientation) of the oriented line. Let $\mathbb{T}^n$ denote the set of all oriented lines of $\mathbb{R}^n$ and

$$TS^{n-1} = \{(u,v) \in \mathbb{R}^n \times \mathbb{R}^n \mid |u| = 1, \langle u,v\rangle = 0\}$$

the tangent space of the $(n-1)$-dimensional sphere. Then $\ell : TS^{n-1} \to \mathbb{T}^n$ is a bijection whose inverse is given by

$$F : \mathbb{T}^n \to TS^{n-1}, \quad F(\ell(u,v)) = (u, v - \langle v,u\rangle u) \tag{1}$$

(here $v - \langle v,u\rangle u$ is the point on the line which is closest to the origin). This correspondence is called in [5] the minitwistor construction. By abuse of notation we sometimes identify $\mathbb{T}^n$ with TS^{n-1}.

The group $SO_n \ltimes \mathbb{R}^n$ of Euclidean motions of $\mathbb{R}^n$, with multiplication given by $(k,a)(k',a') = (kk', a+ka')$ acts transitively on $\mathbb{T}^n$ in the canonical way $(k,a)\cdot(\mathbb{R}u+v,u) = (\mathbb{R}ku+a+kv, ku)$. The action on TS^{n-1} induced by the identification F is

$$(k,a)\cdot(u,v) = (ku, a+kv - \langle a+kv, ku\rangle ku). \tag{2}$$

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2. Invariant metrics on $\mathbb{T}^n$

Two pseudo-Riemannian metrics g_1, g_2 on a smooth manifold M are said to be *equivalent* if there exists a diffeomorphism f and a constant $c \neq 0$ such that $f : (M, g_1) \to (M, cg_2)$ is an isometry. Given an inner product $\langle , \rangle$ we denote $\|x\| = \langle x, x \rangle$ and $|x| = \sqrt{|\langle x, x \rangle|}$. Let $\mathbb{A}$ denote either of the normed division algebras $\mathbb{H}$ or $\mathbb{O}$ (quaternions and octonions, respectively) and let $\times$ denote the cross product in $\operatorname{Im} \mathbb{A}$, the vector space of purely imaginary elements of $\mathbb{A}$. Let $K_{\mathbb{A}}$ be the group of automorphisms of $\times$, that is, $K_{\mathbb{H}} = SO_3$ and $K_{\mathbb{O}} = G_2$. For this subject we refer the reader to [4].

Theorem 1. *Suppose that $\mathbb{T}^n$ has a pseudo-Riemannian metric g which is invariant by the induced transitive action of a connected closed subgroup H of $SO_n \ltimes \mathbb{R}^n$. Then either $n = 3$ or $n = 7$.*

Moreover, there is a cross product $\times$ on $\mathbb{R}^n$, compatible with the Euclidean metric and the orientation, such that $(\mathbb{R}^n, \times)$ is isomorphic to $\operatorname{Im} \mathbb{A}$, which induces an identification of H with $K_{\mathbb{A}} \ltimes \mathbb{R}^n$ (where $\mathbb{A} = \mathbb{H}$ or $\mathbb{O}$ if $n = 3$ or 7, respectively) and g is equivalent to exactly one of the split pseudo-Riemannian metrics g_μ, $\mu \in \{0, 1\}$, whose associated norms are given by

$$\|(x, y)\|_\mu = \langle x, u \times y \rangle + \mu |x|^2 \tag{3}$$

for any $(x, y) \in T_{(u,v)} TS^{n-1} = T_{\ell(u,v)} \mathbb{T}^n$.

In particular, $H = SO_3 \ltimes \mathbb{R}^3$ or $H = G_2 \ltimes \mathbb{R}^7$, and the metric is of type $(2, 2)$ or $(6, 6)$, depending on whether $n = 3$ or 7.

Besides, the canonical projection of (TS^m, g_1) onto the standard round sphere S^m of radius one, is a pseudo-Riemannian submersion.

Notation. In the following we set $m = n - 1$ and consider the canonical orthonormal basis $\{e_0, e_1 \dots, e_m\}$ of $\mathbb{R}^n$.

Proposition 2. *Let H be a connected closed subgroup of $SO_n \ltimes \mathbb{R}^n$ such that the induced action of H on $\mathbb{T}^n$ is transitive, then $H = K \ltimes \mathbb{R}^n$, where K is a closed subgroup of SO_n such that the induced action on S^m is transitive.*

Proof. The canonical projection $\pi : SO_n \ltimes \mathbb{R}^n \to SO_n$ is a Lie group morphism, hence $K := \pi(H)$ is a connected subgroup of SO_n and $V := \{x \in \mathbb{R}^n \mid (1, x) \in H\} \simeq \operatorname{Ker}(\pi|_H)$ is a closed subgroup of $\mathbb{R}^n$. The group K acts transitively on S^m, since given $u \in S^m$ and $(k, a) \in H$ with $(k, a) \cdot \ell(e_0, 0) = \ell(u, 0)$, then $ke_0 = u$.

Next we see that V is invariant by the action of K on $\mathbb{R}^n$. Indeed, let $k \in K$ and $x \in V$, and take $a \in \mathbb{R}^n$ such that $(k^{-1}, a) \in H$. Then $(k^{-1}, a)^{-1} (1, x) (k^{-1}, a) = (1, kx) \in H$ and hence $kx \in V$.

Moreover, $V \neq \{(1, 0)\}$, since otherwise the group $H = K \subset SO_n$ would act transitively on TS^m. Since V is invariant by the action of K (and this group acts transitively on S^m), it contains a full sphere. But the only closed subgroups of $\mathbb{R}^n$ are congruent in $Gl(n, \mathbb{R})$ to $\mathbb{R}^s \times \mathbb{Z}^t$, which contain a full sphere only if $s = n$ and $t = 0$. Therefore $V = \mathbb{R}^n$. It follows that $H = K \ltimes \mathbb{R}^n$, since given $k \in K$, $x, y \in \mathbb{R}^n$ with $(k, y) \in H$, then $(1, x - y)(k, y) = (k, x) \in H$ (the other inclusion is clear). In particular K is closed in SO_n and hence compact. □

Let $\{1, i, j, k\}$ be the standard orthonormal basis of $\mathbb{H}$. Let $m = 2$ or $m = 6$ if $\mathbb{A} = \mathbb{H}$ or $\mathbb{O}$, respectively. Let $i^\perp$ denote the orthogonal complement of $\mathbb{R}i$ in $\operatorname{Im}\mathbb{A}$. Given any unit element $e \in \mathbb{O}$ orthogonal to $\mathbb{H} \subset \mathbb{O}$, we consider the orthonormal bases $\mathcal{B}_2 = \{j, k\}$ or $\mathcal{B}_6 = \{j, e, je, k, ie, i(je) = -ke\}$ of $i^\perp = T_i S^m$ and use them to identify this vector space with $\mathbb{R}^m$. Let us define

$$L_i : i^\perp \to i^\perp, \qquad L_i(z) = iz = i \times z, \tag{4}$$

whose matrix with respect to the basis $\mathcal{B}_m$ is $J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$. Identify as usual $(\mathbb{R}^m, L_i) = \mathbb{C}^{m/2}$ and consider the $\mathbb{C}$-bases $\mathcal{B}_1 = \{j\}$ and $\mathcal{B}_3 = \{j, e, je\}$ of $\mathbb{C}$ and $\mathbb{C}^3$.

The special unitary group SU_3 consists of all 3×3 complex matrices A with $A\bar{A}^t = 1$ and $\det A = 1$. Define

$$f : SU_3 \to SO_6, \quad f(x + iy) = \begin{pmatrix} x & -y \\ y & x \end{pmatrix} \quad \text{and} \quad G = f(SU_3).$$

Then f is a one to one morphism of Lie groups, and hence an isomorphism onto G, whose Lie algebra is

$$\mathfrak{g} = \left\{ z = \begin{pmatrix} x & -y \\ y & x \end{pmatrix} \mid x, y \in \mathbb{R}^{3\times 3},\ x + x^t = 0,\ y^t = y,\ \operatorname{tr} y = 0 \right\}. \tag{5}$$

Lemma 3. *A 6×6 real matrix w commutes with any $k \in G \subset SO_6$ if and only if $w = a1_6 + bJ$ for some $a, b \in \mathbb{R}$.*

Proof. Suppose that the matrix w with blocks $w_{s,t} \in \mathbb{R}^{3\times 3}$ $(1 \le s, t \le 2)$ commutes with every $k \in G \cong SU_3$. Then, by the well-known interplay between the actions of a Lie group and of its Lie algebra, it commutes with every $z \in \mathfrak{g} \subset so_6$, that is, with any z as in (5). Setting $y = 0$, one has that each $w_{s,t}$ commutes with every $x \in so_3$, or equivalently, that $w_{s,t}$ is a fixed point of the adjoint action of SO_3. Hence, $w_{s,t} = c_{s,t}1_3$ for any s, t. Setting now $x = 0$, one obtains that $c_{1,1} = c_{2,2}$ and $c_{1,2} = -c_{2,1}$, as desired. The converse is straightforward. □

Notation. We take $o := \ell(e_0, 0)$ as origin in $\mathbb{T}^n$. The isotropy subgroup at o of the action of H on $\mathbb{T}^n$ is $H_o := K_o \times \mathbb{R}e_0$, where $K_o = \{k \in K \mid ke_0 = e_0\}$, the isotropy subgroup at e_0 of the action of K on S^m.

Proof of Theorem 1. One can easily verify that

$$T_{(u,v)}TS^m = \{(x, y) \in \mathbb{R}^n \times \mathbb{R}^n \mid \langle x, u\rangle = 0, \langle x, v\rangle + \langle y, u\rangle = 0\}. \tag{6}$$

In particular, $T_oTS^m = e_0^\perp \times e_0^\perp = \mathbb{R}^m \times \mathbb{R}^m$. Next we compute the derivative at o of the action of H on TS^m. Given $(k,a) \in H$ and $(x,y) \in T_oTS^m$ let (u_t, v_t) be a curve in TS^m with $(u_0, v_0) = o$ and $(u_0', v_0') = (x,y)$. Using (2) we compute

$$\begin{aligned} (d(k,a))_o(x,y) &= \tfrac{d}{dt}\big|_0 (k,a)\cdot(u_t,v_t) \\ &= (kx, ky - \langle a, kx\rangle ke_0 - \langle a, ke_0\rangle kx). \end{aligned} \tag{7}$$

In particular, if $(k,a) \in H_o$, with $a = ce_0$,

$$(d(k,a))_o(x,y) = (kx, k(y - cx)). \tag{8}$$

Let χ be the endomorphism of $\mathbb{R}^m \times \mathbb{R}^m$ given by $\chi(x,y) = (y,-x)$ and let ω be the nondegenerate skew-symmetric bilinear form on $\mathbb{R}^m \times \mathbb{R}^m$ given by $\omega(\xi,\eta) = \langle \xi, \chi\eta\rangle$, which can easily be seen to be invariant by the action of H_o, using (8). (In fact, ω is the value at o of a constant multiple of the canonical symplectic form of TS^m, identified with T^*S^m via the metric on the sphere.) Hence, any H-invariant pseudo-Riemannian metric g on TS^m (provided it exists) is given at o by $g(\xi,\eta) = \omega(\xi, B\eta)$ for a nonsingular $B \in \operatorname{End}(\mathbb{R}^m \times \mathbb{R}^m)$ which commutes with the action (8) of the isotropy subgroup H_o. We write $B(x,y) = (\alpha x + \beta y, \gamma x + \delta y)$. Since g is symmetric and χ is skew-symmetric, we have that $\chi B + B^t \chi = 0$. This implies that β and γ are symmetric and $\delta = -\alpha^t$. The fact that B commutes with the action of H_o is equivalent to requiring that

$$k(\alpha x + \beta y) = \alpha kx + \beta k(y - cx) \tag{9}$$

$$k\big(\gamma x - \alpha^t y - c(\alpha x + \beta y)\big) = \gamma kx - \alpha^t k(y - cx) \tag{10}$$

for all $x, y \in \mathbb{R}^m$, $c \in \mathbb{R}$ and $k \in K_o$. Setting $x = 0$, $k = \mathrm{id}$ and $c = 1$ in (10) we obtain that $\beta = 0$. This implies, looking at (9), that α commutes with every $k \in K_o$. Setting $y = 0$ and $c = 0$ in (10), we have that γ commutes with every $k \in K_o$. Now, setting $y = 0$, $k = \mathrm{id}$ and $c = 1$ in (10), one has that α is skew-symmetric.

Therefore m must be even, since otherwise α is degenerate and hence B is singular. Now, by [7] (see also [1]) the only compact connected groups acting effectively and transitively on m-dimensional spheres, with m even, are SO_n, or congruent to G_2, if $m = 6$. The isotropy subgroup at e_0 of the action of SO_n on S^m is SO_m. Since the adjoint action of SO_m on its Lie algebra has no nonzero fixed points for $m \geq 3$, the nonsingular matrix α can commute with every element of SO_m only if $m = 2$. Thus, only $m = 2$ and $m = 6$ are admitted.

Since we look for invariant metrics on $\mathbb{T}^n$ up to isometries, we may identify $\mathbb{R}^3$ and $\mathbb{R}^7$ with $\operatorname{Im}\mathbb{A}$, where $\mathbb{A} = \mathbb{H}$ or $\mathbb{O}$ endowed with its standard cross product, respectively, and set also $e_0 = i$. Thus $H = K_{\mathbb{A}} \ltimes \mathbb{R}^m$ by Proposition 2. For $m = 2, 7$, let S^m be as above the unit sphere in $\mathbb{A}$, then $T_iS^m = \mathbb{R}^m = i^\perp$. The isotropy subgroup at $e_0 = i$ of SO_3 on S^2 is $SO_2 = U_1$, and of G_2 on S^6, by [4], the group $G \cong SU_3$ defined before (5).

Now let $m = 6$. If $\alpha \in \operatorname{End}\left(\mathbb{R}^6\right)$ is nonsingular, skew-symmetric and commutes with every $k \in G \subset SO_6$, then, by Lemma 3, $\alpha = \frac{\lambda}{2} L_i$ for some $\lambda \neq 0$. If γ is a symmetric $m \times m$ matrix commuting with every $k \in G$, then, again by Lemma 3, $\gamma = \mu 1$ for some $\mu \in \mathbb{R}$. For $m = 2$ similar statements hold, by elementary reasons.

Consequently, since L_i is skew-symmetric, we have that up to an isometry, $B(x, y) = \left(\frac{\lambda}{2} L_i x, \mu x + \frac{\lambda}{2} L_i y\right)$ for some $\lambda, \mu \in \mathbb{R}$, $\lambda \neq 0$. For $(x, y) \in T_o TS^m$, the norm associated to g is

$$\begin{aligned}\|(x,y)\|_{\mu,\lambda} &= g\left((x,y),(x,y)\right) = \langle (x,y), \chi B(x,y)\rangle \\ &= \left\langle (x,y), \left(\mu x + \tfrac{\lambda}{2} i \times y, -\tfrac{\lambda}{2} i \times x\right)\right\rangle \\ &= \lambda \langle x, i \times y\rangle + \mu |x|^2 .\end{aligned}$$

If $(x, y) \in T_{(u,v)} S^m$, let us denote $g_{\mu,\lambda}(u, v, x, y) = \mu |x|^2 + \lambda \langle x, u \times y\rangle$. This is the norm associated to a pseudo-Riemannian metric on TS^m. We have that $g_{\mu,\lambda} = \| \; \|_{\mu,\lambda}$, since they clearly coincide at o and $g_{\mu,\lambda}$ is invariant by the action of H. Indeed, given $(x, y) \in T_o TS^m$ and $(k, a) \in H$, one can show using (7) that

$$g_{\mu,\lambda}\left((d(k,a))_o (x,y)\right) = g_{\mu,\lambda}\left((e_0, 0, x, y)\right),$$

since k is orthogonal and preserves the product $\times$, and $\langle e_0 \times x, x\rangle = 0$, $x \times x = 0$ for all x. Next we verify that for any fixed λ and μ, the map

$$\phi : (TS^m, g_{\mu,\lambda}) \to (TS^m, g_{\mu,1}), \qquad \phi(u, v) = (u, \lambda v), \tag{11}$$

is an isometry. Indeed,

$$\left\|d\phi_{(u,v)}(x,y)\right\|_{\mu,1} = \|(u, \lambda v, x, \lambda y)\|_{\mu,1} = \langle x, u \times \lambda y\rangle + \mu |x|^2 ,$$

which equals $= \|(u, v, x, y)\|_{\mu,\lambda}$. Hence, all the norms $\| \; \|_{\mu,\lambda}$ on $\mathbb{T}^n$ are isometric to $g_\mu := \| \; \|_{\mu,1}$. But g_μ is isometric to $g_{\mu'}$ only if $\mu = \mu'$ by Proposition 6 below. Up to homothety we may suppose that $\mu = 0$ or $\mu = 1$. Therefore any H-invariant metric on $\mathbb{T}^n$ $(n = 3, 7)$ is equivalent to exactly one of the metrics g_0, g_1. Let us note that by polarization of (3), if $\xi = (x, y)$, $\eta = (x', y') \in T_{(u,v)} TS^m$, then

$$4 \langle \xi, \eta\rangle_\mu = \langle x, u \times y'\rangle + \langle x', u \times y\rangle + \mu \langle x, x'\rangle . \tag{12}$$

Finally, given $(u, v) \in TS^m$, one has that $\operatorname{Ker} d\pi_{(u,v)} = \left\{(0, y) \mid y \in u^\perp\right\}$. By (6) and (12), its orthogonal complement in $T_{(u,v)} TS^m$ with respect to the metric g_μ is $\mathcal{H}_{(u,v)} := \{(x, -\langle x, v\rangle u) \mid \langle x, u\rangle = 0\}$. The last assertion follows from the fact that $\|(x, -\langle x, v\rangle u)\|_1 = |x|^2 = \left|d\pi_{(u,v)}(x, -\langle x, v\rangle u)\right|^2$. $\square$

Remarks. a) Although the map ϕ defined in (11) is an isometry from one H-homogeneous space to another, it is not H-equivariant if $\lambda \neq 1$, since by (8), given $c \neq 0$ and $x \neq 0$, we have

$$d\left(\phi \circ (1, ce_0)\right)_o (x, 0) = (x, \lambda c x) \neq (x, cx) = d\left((1, ce_0) \circ \phi\right)_o (x, 0) .$$

b) The metric g_0 on $\mathbb{T}^3$ is equivalent to those defined in [10] and [2] (in the first article lines without orientation are considered).

c) The space of oriented geodesics of a non-Euclidean space form M of any dimension admits pseudo-Riemannian metrics invariant by the canonical action of the group of orientation preserving isometries of M. Indeed, the space of oriented geodesics of the sphere S^n is the Grassmannian of oriented planes in $\mathbb{R}^{n+1}$, which admits SO_{n+1}-invariant metrics; if M is the hyperbolic space, this issue has been studied in [9].

d) Let $n = 3$ or $n = 7$. For the sake of simplicity of the computations we work with TS^m as a submanifold of $\mathbb{R}^{2n}$, as presented in (6). But, in fact, the expression for the metric g_μ given in Theorem 1 does not change if one identifies as usual $T_{(u,v)}TS^m \cong u^\perp \oplus u^\perp$: Let $\pi : TS^m \to S^m$ be the canonical projection and let $\mathcal{K}_{(u,v)} : T_{(u,v)}TS^m \to T_u S^m = u^\perp$ be the connection operator, that is,

$$\mathcal{K}_{(u,v)}(x,y) = \tfrac{D}{dt}\big|_0 (u_t, v_t) = y - \langle y, u\rangle u,$$

where (u_t, v_t) is a curve in TS^m with initial velocity $(x,y) \in T_{(u,v)}TS^m$ and $\frac{D}{dt}$ denotes covariant derivative along the curve u_t. We observe that its kernel coincides with the subspace $\mathcal{H}_{(u,v)}$ defined at the end of the proof of Theorem 1. The map

$$\psi_{(u,v)} := \left(d\pi_{(u,v)}, \mathcal{K}_{(u,v)}\right) : T_{(u,v)}TS^m \to u^\perp \times u^\perp \tag{13}$$

is a linear isomorphism leaving invariant the expression (3).

The proof of Theorem 1 will be complete as soon as we show that g_μ is isometric to $g_{\mu'}$ only if $\mu = \mu'$. This will be achieved using the lengths of the periodic geodesics in $(\mathbb{T}^n, g_\mu)$ for $n = 3$ and $n = 7$. For that reason we study next geodesics in these spaces.

As an immediate corollary of Theorem 1 in [8] we have the following criterion for a vector field along a curve in a pseudo-Riemannian manifold to be parallel.

Lemma 4. *If $p : M \to B$ is a pseudo-Riemannian submersion, α is a horizontal curve in M and X is a horizontal vector field along α whose covariant derivative has vanishing horizontal component, then the vector field $dp(X)$ along $p \circ \alpha$ is parallel.*

Let $\mathfrak{h}$, $\mathfrak{h}_o$, $\mathfrak{k}$, $\mathfrak{k}_o$ be the Lie algebras of H, H_o, K and K_o, respectively. We have the following direct sum decompositions: $\mathbb{R}^n = \mathbb{R}e_0 + \mathbb{R}^m$, $\mathfrak{h}_o = \mathfrak{k}_o + \mathbb{R}e_0$ and also, since K acts transitively on S^m, $\mathfrak{k} = \mathfrak{k}_o + \mathfrak{m}$, where $\mathfrak{m} = \{\widetilde{x} \mid x \in \mathbb{R}^m\}$, with $\widetilde{x} = \begin{pmatrix} 0 & -x^t \\ x & 0_m \end{pmatrix} \in \mathfrak{k}$. Hence $\mathfrak{h}$ decomposes as $\mathfrak{h} = \mathfrak{h}_o \oplus \mathfrak{p}$, with $\mathfrak{p} = \mathfrak{m} \oplus \mathbb{R}^m$ (by abuse of notation we denote the subgroup $\{1\} \times \mathbb{R}^n$ of H by $\mathbb{R}^n$, and use the same notation for its subgroups).

Let $p : H \to TS^m$ be the projection $p(k,a) = (k,a) \cdot (e_0, 0)$. The kernel of $dp_{(1,0)} : \mathfrak{h} \to T_oTS^m \cong \mathbb{R}^m \times \mathbb{R}^m$ is clearly $\mathfrak{h}_o$. If we call ρ the restriction of $dp_{(1,0)}$ to $\mathfrak{p}$, then $\rho(\widetilde{x}, y) = (x, y)$.

For any μ consider on H the left invariant pseudo-Riemannian metric Γ_μ defined at the identity as follows: The subspaces $\mathfrak{h}_o$ and $\mathfrak{p}$ are orthogonal with respect to Γ_μ, set any nondegenerate inner product on $\mathfrak{h}_o$, and on $\mathfrak{p}$ the inner product such that ρ is a linear isometry, where T_oTS^m has the metric g_μ. Then p is a pseudo-Riemannian submersion.

The canonical isomorphism of the group H with the matrix group

$$\bar{H} = \left\{ \begin{pmatrix} 1 & 0 \\ a & k \end{pmatrix} \mid k \in K, a \in \mathbb{R}^n \right\}$$

induces an isomorphism of Lie algebras. The element $(\widetilde{x}, y) \in \mathfrak{p}$ is identified with the matrix $\begin{pmatrix} 0 & 0 \\ \widehat{y} & \widetilde{x} \end{pmatrix}$, where $\widehat{y} = \begin{pmatrix} 0 \\ y \end{pmatrix}$. Using this identification one obtains that

$$[\mathfrak{p}, \mathfrak{p}]_{\mathfrak{p}} = \{0\}, \tag{14}$$

where the subindex $\mathfrak{p}$ stands for the $\mathfrak{p}$-component of a vector in $\mathfrak{h}$.

Proposition 5. *For $n = 3$ or $n = 7$, the geodesics in $(\mathbb{T}^n, g_\mu)$ through o are exactly the curves $s \mapsto \exp_H(sX) \cdot o$, for $X \in \mathfrak{p}$. In particular they are defined on the whole real line and do not depend on μ.*

Proof. Since the metric Γ_μ on H is left-invariant, calling ∇ its Levi Civita connection, one has

$$2\Gamma_\mu(\nabla_X Y, Z) = \Gamma_\mu([X,Y], Z) + \Gamma_\mu([Y,Z], X) - \Gamma_\mu([Z,X], Y) \tag{15}$$

for any left invariant vector fields X, Y, Z. Consider the pseudo-Riemannian submersion $p : H \to \mathbb{T}^n$ as above. Let $X \in \mathfrak{p}$ and let $\alpha(s) = \exp_H(sX)$, which is clearly a horizontal curve in H. By Lemma 4 and homogeneity, to see that $p \circ \alpha$ is a geodesic in $\mathbb{T}^n$, it suffices to show that $\Gamma_\mu(\nabla_X X, Z) = 0$ for any $Z \in \mathfrak{p}$, but this is clear from (15) and (14). □

Proposition 6. *Let $n = 3$ or $n = 7$. If $\mu > 0$, every periodic geodesic in $(\mathbb{T}^n, g_\mu)$ has length $2\pi\sqrt{\mu}$. In particular, $(\mathbb{T}^n, g_\mu)$ is not isometric to $(\mathbb{T}^n, g_{\mu'})$ if $\mu \neq \mu'$.*

Proof. Let γ be a nonconstant geodesic in $\mathbb{T}^n \cong TS^m$. Since the action of H on $\mathbb{T}^n$ is transitive, we may suppose that $\gamma(0) = o$. Hence $\gamma'(0) = (x, y) \in \mathbb{R}^m \times \mathbb{R}^m$. We may suppose additionally that $\langle x, y \rangle = 0$. Indeed, this is clear if $x = 0$; if not, setting $k = 1$ and $c = \langle x, y \rangle / |x|$ in (8) on sees that γ is conjugate by an element in H_o to a geodesic satisfying this condition.

Now, if $(\widetilde{x}, y) \in \mathfrak{p}$ and $\langle x, y \rangle = 0$, we have by definition of the multiplication in $SO_n \ltimes \mathbb{R}^n$ that $\exp_H t(\widetilde{x}, y) = (\exp_K(t\widetilde{x}), ty)$. By Proposition 5 we have

$$\begin{aligned}\gamma(t) &= \exp_H t(0,y)\cdot o = (1,ty)\cdot \ell(e_0,0) = \ell(e_0,ty)\\ \gamma(t) &= \exp_H t(\widetilde{x},y)\cdot o = (\exp_K(t\widetilde{x}),ty)\cdot \ell(e_0,0)\\ &= \ell((\cos t)e_0 + (\sin t)x, ty)\end{aligned}$$

if $x=0$ or $|x|=1$, respectively (if $x\neq 0$ we may suppose that $|x|=1$ by considering a reparametrization of γ). Therefore a geodesic in $\mathbb{T}^n$ is periodic if and only if it is congruent in H to a constant speed reparametrization of the geodesic $\sigma(t)=\ell((\cos t)e_0+(\sin t)x,0)$ for some unit $x\perp e_0$. The lenght of σ is $2\pi\sqrt{\mu}$ since its period is 2π and $\|\sigma'(0)\|_\mu = \|(x,0)\|_\mu = \langle x, e_0\times 0\rangle + \mu|x|^2 = \mu$. □

3. Additional geometric structures on $\mathbb{T}^n$

An almost Hermitian structure on a pseudo-Riemannian manifold (M,g) is a smooth tensor field J of type $(1,1)$ on M such that J_p is an orthogonal transformation of (T_pM,g_p) and satisfies $J_p^2=-1$ for all $p\in M$. If ∇ is the Levi Civita connection of (M,g), then (M,g,J) is said to be Kähler if $\nabla J=0$ and nearly Kähler if $(\nabla_X J)X=0$ for any vector field X on M.

Let $E=\left\{(u,v,x,y)\in\mathbb{R}^{4n} \mid (u,v)\in TS^m, \langle x,u\rangle=\langle y,u\rangle=0\right\}$. Then $\psi: TTS^m\to E$ defined in (13) is a vector bundle isomorphism over the identity on TS^m. For $m=2$ or $m=6$, let j_o be the canonical almost complex structure on the round sphere S^m, that is $j_o(x)=u\times x$ if $x\in u^\perp = T_uS^m$, which is known to be Kähler (in particular integrable) if $m=2$ and nearly Kähler but not integrable if $m=6$. Let J be the induced almost complex structure on TS^m, that is, $J=\psi^{-1}\circ(j_o,j_o)\circ\psi$. Since $\psi^{-1}_{(u,v)}(x,y) = (x, y-\langle x,v\rangle u)$, one computes $J_{(u,v)}(x,y) = (u\times x, u\times y - \langle u\times x, v\rangle u)$. One can check using (7) that J is invariant by the action of H.

Next we see that J is not integrable if $m=6$. Let N_o and N denote the Nijenhuis tensors of j_o and J, respectively. Since it is well-known that in this case j_o is not integrable, there exist vector fields X_1,X_2 on S^6 such that $N_o(X_1,X_2)\neq 0$. Let $\zeta: S^6\to TS^6$ be the zero section and $\mathcal{S}$ its image. For $i=1,2$, let Y_i be a vector field on TS^6 extending the tangent vector field $d\zeta\circ X_i\circ\pi$ on $\mathcal{S}$. We have that $N(Y_1,Y_2)\neq 0$, since Y_i is ζ-related to X_i and is horizontal on $\mathcal{S}$. Therefore, J is not integrable.

Proposition 7. *For $\mu=0,1$, $(\mathbb{T}^n,J,g_\mu)$ is Kähler if $n=3$, and nearly Kähler but not Kähler if $n=7$.*

Remark. The Kähler structure $\left(\mathbb{T}^3,g_0,J\right)$ is equivalent to that defined in [3]. The Kähler structure $\left(\mathbb{T}^3,g_1,J\right)$ is probably new in this setting and has the additional property stated at the end of Theorem 1.

Proof. Let $n=3$. We know that J is integrable. The metric g_0 is Kähler by [3]. To show that g_1 is Kähler, we verify that the associated Hermitian form is closed (see [6]). Let Ω^μ the Hermitian form of g_μ. For $\xi = (x, y)$, $\eta = (x', y')$ in T_oTS^2 we have by (12) that

$$4\Omega^1_o(\xi, \eta) = 4\langle \xi, J\eta\rangle_1 = \langle x, L_i^2 y'\rangle + \langle L_i x', L_i y\rangle + \langle x, L_i x'\rangle = \\ = -\langle x, y'\rangle + \langle x', y\rangle + \langle x, L_i x'\rangle = 4\Omega^0_o(\xi, \eta) - \Omega_o(\xi, \eta),$$

where Ω is the pull-back of the standard volume form θ of S^2. Indeed, at $i \in S^2$,

$$(\pi^*\theta)(\xi, \eta) = \theta(x, x') = \langle L_i x, x'\rangle,$$

and since J and all the bilinear forms involved are H-invariant, we have that $4\Omega^1 = 4\Omega^0 - \Omega$. Hence Ω^1 is closed, since Ω^0 is so (we already know that g_0 is Kähler) and Ω is clearly closed. Therefore (g_1, J) is Kähler.

Now we consider the case $n = 7$. We have already verified that J is not integrable and hence g_μ is not Kähler. To see that it is nearly Kähler we show that for every geodesic γ in $\mathbb{T}^7$ the vector field $J\gamma'$ along γ is parallel. By homogeneity we may suppose that $\gamma(0) = o$. We know from Proposition 5 and its previous paragraphs that $p : (H, \Gamma_\mu) \to (\mathbb{T}^7, g_\mu)$ is a pseudo-Riemannian submersion and that any geodesic γ has the form $\gamma(s) = p(\alpha(s))$ with $\alpha(s) = \exp(sX)$ for some $X \in \mathfrak{p}$. Since J is H-invariant, by Lemma 4, it suffices to show that the horizontal component of $\nabla_X(J\alpha')$ vanishes. This is true, since if Y denotes the left invariant vector field on H such that $Y \circ \alpha = J\alpha'$ and Z is any horizontal left invariant vector field, then $\Gamma_\mu(\nabla_X Y, Z) = 0$ by (15) and (14). □

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