# GEODESICS OF THE SPACE OF ORIENTED LINES OF EUCLIDEAN SPACE *

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**Abstract**

For $n = 3$ or $n = 7$ let $\mathbb{T}^n$ be the space of oriented lines in $\mathbb{R}^n$. In a previous article we characterized up to equivalence the metrics on $\mathbb{T}^n$ which are invariant by the induced transitive action of a connected closed subgroup of the group of Euclidean motions (they exist only in such dimensions and are pseudo-Riemannian of split type) and described explicitly their geodesics. In this short note we present the geometric meaning of the latter being null, time- or space-like.

On the other hand, it is well-known that $\mathbb{T}^n$ is diffeomorphic to $\mathcal{G}(H^n)$, the space of all oriented geodesics of the $n$-dimensional hyperbolic space. For $n = 3$ and $n = 7$, we compute now a pseudo-Riemannian invariant of $\mathbb{T}^n$ (involving its periodic geodesics) that will be useful to show that $\mathbb{T}^n$ and $\mathcal{G}(H^n)$ are not isometrically equivalent, provided that the latter is endowed with any of the metrics which are invariant by the canonical action of the identity component of the isometry group of $H$.

## THE SPACE OF ORIENTED LINES OF $\mathbb{R}^n$.

We begin by recalling the definitions and some notation and results from [4]. An oriented line in $\mathbb{R}^n$ is a pair $\ell(u,v) := (\{tu + v \mid t \in \mathbb{R}\}, u)$ for some $u, v \in \mathbb{R}$, $|u| = 1$, where $u$ is the direction (orientation) of the oriented line. Let $\mathbb{T}^n$ denote the set of all oriented lines of $\mathbb{R}^n$ and

$$TS^{n-1} = \{(u,v) \in \mathbb{R}^n \times \mathbb{R}^n \mid |u| = 1, \langle u, v\rangle = 0\}$$

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the tangent space of the $(n-1)$-dimensional sphere. Then $\ell : TS^{n-1} \to \mathbb{T}^n$ is a bijection whose inverse is given by

$$F : \mathbb{T}^n \to TS^{n-1}, \quad F(\ell(u,v)) = (u, v - \langle v, u\rangle u) \tag{1}$$

(here $v - \langle v, u\rangle u$ is the point on the line which is closest to the origin). This correspondence is called in [2] the minitwistor construction. By abuse of notation we sometimes identify $\mathbb{T}^n$ with $TS^{n-1}$.

The group $SO_n \ltimes \mathbb{R}^n$ of Euclidean motions of $\mathbb{R}^n$, with multiplication given by $(k,a)(k',a') = (kk', a + ka')$, acts transitively on $\mathbb{T}^n$ in the canonical way $(k,a) \cdot (\mathbb{R}u + v, u) = (\mathbb{R}ku + a + kv, ku)$.

Two pseudo-Riemannian metrics $g_1, g_2$ on a smooth manifold $M$ are said to be *equivalent* if there exists a diffeomorphism $f$ and a constant $c \neq 0$ such that $f : (M, g_1) \to (M, cg_2)$ is an isometry. Given an inner product $\langle , \rangle$ we denote $\|x\| = \langle x, x\rangle$ and $|x| = \sqrt{|\langle x, x\rangle|}$. Let $\mathbb{A}$ denote either of the normed division algebras $\mathbb{H}$ or $\mathbb{O}$ (quaternions and octonions, respectively) and let $\times$ denote the cross product in $\operatorname{Im}\mathbb{A}$, the vector space of purely imaginary elements of $\mathbb{A}$. Let $K_{\mathbb{A}}$ be the group of automorphisms of $\times$, that is, $K_{\mathbb{H}} = SO_3$ and $K_{\mathbb{O}} = G_2$.

## INVARIANT METRICS ON $\mathbb{T}^n$ FOR $n = 3$ AND $n = 7$.

For $n = 3$ or $n = 7$ we identify $\mathbb{R}^n$ with $\operatorname{Im}\mathbb{H}$ or $\operatorname{Im}\mathbb{O}$, respectively. For $\mu \in \mathbb{R}$ we defined in [4] the split pseudo-Riemannian metric $g_\mu$ on $\mathbb{T}^n$ as the one whose associated norm is given by

$$\|(x,y)\|_\mu = \langle x, u \times y\rangle + \mu |x|^2 \tag{2}$$

for any $(x,y) \in T_{(u,v)}TS^{n-1} = T_{\ell(u,v)}\mathbb{T}^n$. The metric $g_\mu$ is of type $(2,2)$ or $(6,6)$ and is invariant by the induced action of $H = SO_3 \ltimes \mathbb{R}^3$ or $H = G_2 \ltimes \mathbb{R}^7$ on $\mathbb{T}^n$, depending on whether $n = 3$ or $n = 7$.

We proved in the same article that only for those dimensions there exist a pseudo-Riemannian metric which is invariant by the induced transitive action of a connected closed subgroup of $SO_n \ltimes \mathbb{R}^n$ (as usual, we consider Riemannian metrics as a particular case of pseudo-Riemannian ones). The metrics $g_\mu$ are not isometric to each other. Moreover, for $\mu \neq 0$, $g_\mu$ is equivalent to $g_1$ and not equivalent to $g_0$.

We recall some further notation from [4].

**Notation.** In the following we set $m = n-1$ and consider the canonical orthonormal basis $\{e_0, e_1 \dots, e_m\}$ of $\mathbb{R}^n$. We take $o := \ell(e_0, 0)$ as origin in $\mathbb{T}^n$.

The isotropy subgroup at $o$ of the action of $H$ on $\mathbb{T}^n$ is $H_o := K_o \times \mathbb{R}e_0$, where $K_o = \{k \in K \mid ke_0 = e_0\}$, the isotropy subgroup at $e_0$ of the action of $K$ on $S^m$, that is, $K_o = SO_2$ or $K_o = SU_3$ for $m = 2$ or $m = 6$, respectively The infinitesimal isotropy action of $H_o$ is given by

$$(d(k, ce_0))_o (x,y) = (kx, k(y - cx)) \tag{3}$$

for any $(x,y) \in \mathbb{R}^m \times \mathbb{R}^m = T_o\mathbb{T}^n$.

Let $\mathfrak{h}$, $\mathfrak{h}_o$, $\mathfrak{k}$, $\mathfrak{k}_o$ be the Lie algebras of $H$, $H_o$, $K$ and $K_o$, respectively. We have the following direct sum decompositions: $\mathbb{R}^n = \mathbb{R}e_0 + \mathbb{R}^m$, $\mathfrak{h}_o = \mathfrak{k}_o + \mathbb{R}e_0$ and also, since $K$ acts transitively on $S^m$, $\mathfrak{k} = \mathfrak{k}_o + \mathfrak{m}$, where $\mathfrak{m} = \{\widetilde{x} \mid x \in \mathbb{R}^m\}$, with $\widetilde{x} = \begin{pmatrix} 0 & -x^t \\ x & 0_m \end{pmatrix} \in \mathfrak{k}$. Hence $\mathfrak{h}$ decomposes as $\mathfrak{h} = \mathfrak{h}_o \oplus \mathfrak{p}$, with $\mathfrak{p} = \mathfrak{m} \oplus \mathbb{R}^m$ (by abuse of notation we denote the subgroup $\{1\} \times \mathbb{R}^n$ of $H$ by $\mathbb{R}^n$, and use the same notation for its subgroups).

## NULL, TIME- AND SPACE-LIKE GEODESICS OF $\mathbb{T}^n$.

We obtained in [4] the complete description of the geodesics of $(\mathbb{T}^n, g_\mu)$ for $n = 3$ and $n = 7$:

**Proposition 1** *For $n = 3$ or $n = 7$, the geodesics in $(\mathbb{T}^n, g_\mu)$ through $o$ are exactly the curves $s \mapsto \exp_H(sX) \cdot o$, for $X \in \mathfrak{p}$. In particular they are defined on the whole real line and do not depend on $\mu$.*

In this short note we present the geometric meaning of a geodesic being null, time- or space-like. We begin by stating a relationship with the ruled (parametrized) surface associated to it. The following proposition, which holds for all $n \in \mathbb{N}$, is elementary and well-known, we include it and its proof for the sake of completeness.

**Proposition 2** *If $\sigma(s) = \ell(u_s, v_s)$ is a curve in $\mathbb{T}^n$ with $u'_s \neq 0$ for all $s$, then there exists a unique curve*

$$\alpha_\sigma(s) = v_s - \tau(s)\, u_s$$

*in the parametrized (possible singular) ruled surface $\phi_\sigma(s,t) = v_s + t u_s$ in $\mathbb{R}^n$, satisfying $\langle u', \alpha'_\sigma \rangle = 0$. This curve is called the* striction line *of $\phi_\sigma$.*
*Moreover, if $\sigma(0) = o$ and $(F \circ \sigma)'(0) = (x, y)$, then*

$$\alpha_\sigma(0) = 0 \iff \langle x, y \rangle = 0 \iff |\mathcal{J}| \text{ takes its minimum at } t = 0,$$

*where $\mathcal{J}$ is the Jacobi field along the parametrization $t \mapsto te_0$ of $\sigma(0)$ associated to the variation by geodesics determined by $\sigma$.*

**Proof.** Take $\tau = \langle u', v' \rangle / |u'|^2$ and use that $|u| = 1$ implies $\langle u, u' \rangle = 0$. Uniqueness is clear. The first equivalence of the second assertion is a consequence of $(F \circ \sigma)'(0) = (u'_0, v'_0 - \langle v'_0, e_0 \rangle e_0)$, which follows from (1) since $u_0 = e_0 \perp u'_0$ and $v_0 = 0$. Finally, the Jacobi field along the given parametrization of $\sigma(0)$ is $\mathcal{J}(t) = \frac{d}{ds}\big|_0 v_s + t u_s$ and satisfies $\left(|\mathcal{J}|^2\right)'(t) = 2\langle u'_0, v'_0 \rangle + 2t\,|u'_0|^2$. □

Let now again $n = 3$ or $n = 7$ and suppose as before that $\mathbb{R}^n = \operatorname{Im} \mathbb{A}$, with $\mathbb{A} = \mathbb{H}$ or $\mathbb{A} = \mathbb{O}$. If $\sigma$ is a curve in $\mathbb{T}^n$ as in the Proposition above, the $\times$-*pitch* of $\sigma$ is the function $\rho = \langle u \times u', v' \rangle / |u'|^2$, which is well-defined, since the expression does not change if one substitutes $v$ with $v + \tau u$, where $\tau$ is any smooth function.
For example, if $\sigma$ describes a helicoid passing through the origin, that is, $\phi_\sigma(s,t) = sv + tu_s$, where $u$ describes, with unit angular speed, a unit circle in a plane orthogonal to $v$, then its striction line is $\alpha_\sigma(s) = sv$. (By abuse of notation we admit

degenerate helicoids, in the case $v = 0$.) For $n = 3$ its $\times$-pitch is the constant $\rho$ such that $2\pi\rho$ is the (signed) length travelled along the striction line whilst $u$ gives one complete positive turn around it. For $n = 7$, one has to consider instead the (signed) length travelled along the projection of the striction line onto the $\times$-normal to the oriented plane determined by the oriented circle $u$ (here, the $\times$-normal to the oriented plane determined by an orthonormal set $\{x, y\}$ is $x \times y$).
According to the definition, if two curves in $\mathbb{T}^n$ are $H$-congruent, then they have the same $\times$-pitch, but if $n = 7$, they might have different pitches if they are just congruent by an element of $SO_7 \ltimes \mathbb{R}^7$.

Next we make explicit the identification of $\mathbb{R}^n$ with $\operatorname{Im}\mathbb{H}$ and $\operatorname{Im}\mathbb{O}$, if $n = 3$ or $n = 7$, respectively. Let $\{1, i, j, k\}$ be the standard orthonormal basis of $\mathbb{H}$. Let $i^{\perp}$ denote the orthogonal complement of $\mathbb{R}i$ in $\operatorname{Im}\mathbb{A}$. Given any unit element $e \in \mathbb{O}$ orthogonal to $\mathbb{H} \subset \mathbb{O}$, we consider the orthonormal bases $\mathcal{B}_2 = \{j, k\}$ or $\mathcal{B}_6 = \{j, e, je, k, ie, ke\}$ of $i^{\perp} = T_i S^m$ and use them to identify this vector space with $\mathbb{R}^m$. Let $L_i : i^{\perp} \to i^{\perp}$ be defined by $L_i(z) = iz = i \times z$. We identify as usual $(\mathbb{R}^m, L_i) = \mathbb{C}^{m/2}$.
In the following Lemma we consider on $\mathbb{C}^3$ the canonical real inner product of the underlying six-dimensional Euclidean space.

**Lemma 3** *Let $x, y \in \mathbb{C}^3$, with $x \neq 0$ and $\langle x, y\rangle = 0$. Then there exists $g \in SU_3$ and $a, b, c \in \mathbb{R}$, $b, c > 0$, such that $g(x) = cj$ and $g(y) = ak + be$.*

**Proof.** Let $c = |x|$ and write $y = a'x + \frac{a}{c}ix + y'$, with $y' \perp \mathbb{C}x$. Clearly $a' = 0$ since $\langle x, y\rangle = 0$. Since $SU_3$ acts transitively on $S^5$, there exists $g_1 \in SU_3$ such that $g_1(x) = cj$. Hence $g_1(ix) = ck$ and $g_1(y') \in (\mathbb{C}j)^{\perp} \cong \mathbb{C}^2$ (with the induced orientation). Since $SU_2$ acts transitively on $S^3$, there exists $g_2 \in SU_3$ fixing $j$ (and hence also $k$) such that $g_2(g_1(y')) = be$ for some $b \geq 0$. Thus, $g = g_2 \circ g_1$ satisfies the requirements. □

**Proposition 4** *Let $n = 3$ or $n = 7$. Any nonconstant geodesic in $\mathbb{T}^n$ is congruent by the action of $H$ (up to orientation preserving reparametrization) to exactly one of the following geodesics*

$$\sigma_0(s) = \ell(i, sk), \qquad \sigma(s) = \ell((\cos s)\, i + (\sin s)\, j, s(ak + be))$$

*for some $a, b \in \mathbb{R}$, $b \geq 0$. Moreover, $\sigma_0$ is a null geodesic for any $\mu$ and its corresponding ruled surface is a plane. The number $a$ is the $\times$-pitch of the ruled surface determined by $\sigma$ (a helicoid) and $\|\sigma'\|_{\mu} = \mu - a$. That is, $\sigma$ is a space-, time-like or null geodesic if and only if the $\times$-pitch of the corresponding ruled surface if smaller, bigger or equal to $\mu$, respectively.*

**Proof.** First we show that $\sigma_0$ and $\sigma$ are geodesics. We call $y = ak + be$, consider $(0, k)$ and $(\widetilde{\jmath}, y)$ as elements of $\mathfrak{p} \subset \mathfrak{h}$ and observe that

$$\sigma_0(s) = \ell(i, sk) = (1, sk) \cdot \ell(i, 0) = \exp_H(s(0, k)) \cdot o.$$

We also have that $\exp_K(s\widetilde{\jmath})\, i = i \cos s + j \sin s$. Moreover, by definition of the multiplication on $H$, $\exp_H s(\widetilde{\jmath}, y) = (\exp_K(s\widetilde{\jmath}), sy)$, since $\langle y, j\rangle = 0$. Hence,

$$\sigma(s) = \ell(i \cos s + j \sin s, sy) = \exp_H s(\widetilde{\jmath}, y) \cdot o.$$

Therefore, $\sigma_0$ and $\sigma$ are geodesics by Proposition 1.
Given a nonconstant geodesic $\gamma$ in $\mathbb{T}^n$, since the action of $H$ on $\mathbb{T}^n$ is transitive, we may suppose that $\gamma(0) = o$. Hence $(F \circ \gamma)'(0) = (x, y) \in \mathbb{R}^m \times \mathbb{R}^m$. If $x = 0$, there exists $g \in K_o$ (which acts transitively on $S^{m-1}$) with $g(y) = ck$. Hence, $(g \circ \gamma)(s) = \sigma_0(cs)$ for all $s$. If $x \neq 0$, by looking at the action (3) of $H_o$ on $T_o\mathbb{T}^n$, one may suppose additionally that $\langle x, y \rangle = 0$ (see the geometric meaning of this condition in Proposition 2). If $n = 7$, by Lemma 3, there exists $g \in K_o = G \cong SU_3$ such that $g \circ \gamma$ and $\sigma$ differ in an orientation preserving reparametrization. The case $n = 3$, where $K_o = SO_2 \cong U_1$, is clear. The curve $\sigma_0$ is not $H$-congruent to a reparametrization of $\sigma$, since by (3) the $H_o$-orbit of $\sigma_0'(0)$ consists of the elements $(0, y)$ with $y$ in a sphere. On the other hand, one has

$$\begin{aligned} \|\sigma'(0)\|_\mu &= \|(j, ak + be)\|_\mu = \langle j, i \times (ak + be) \rangle + \mu |j|^2 \\ &= \langle j, -aj + bie \rangle + \mu = \mu - a, \end{aligned}$$

and the $\times$-pitch of $\sigma$ is $\rho(s) = \langle (i \cos s + j \sin s) \times (j \cos s - i \sin s), ak + be \rangle = a$. Hence, the last assertion is true. □

**Remark.** For $n = 3$ and $\mu = 0$, the geometric interpretation given above of a geodesic in $(\mathbb{T}^n, g_\mu)$ being null, time- or space-like is of course a rephrasing of that given in [1] involving angular momentum.

## A GEOMETRIC INVARIANT OF $\mathbb{T}^n$

It is well-known that $\mathbb{T}^n$ is diffeomorphic to $\mathcal{G}(H)$, the space of all oriented geodesics of $H$, for any Hadamard manifold of dimension $n$ (see [3]). For $n = 3$ and $n = 7$, we compute now a pseudo-Riemannian invariant of $\mathbb{T}^n$ (involving its periodic geodesics) that will be useful in [5] to show that if $H$ is the $n$-dimensional hyperbolic space, then $\mathbb{T}^n$ and $\mathcal{G}(H)$ are not isometrically equivalent, provided that the latter is endowed with any of the metrics which are invariant by the canonical action of the identity component of the isometry group of $H$.
We remark that in [4] we obtained the geodesics of $\mathbb{T}^n$ without needing to compute explicitly the Levi-Civita connection. That is why we give this pseudo-Riemannian invariant instead of a more standard one, like the curvature, since the computation of the latter would have been probably rather cumbersome.
For $n = 3$ or $n = 7$ and $\ell \in \mathbb{T}^n$ let $A$ denote the subset of $T_\ell\mathbb{T}^n$ consisting of the initial velocities of periodic geodesics of $\mathbb{T}^n$ though $\ell$.

**Proposition 5** *The frontier of $A$ in $T_\ell\mathbb{T}^n$ is a subspace of dimension $m$.*

**Proof.** Since $\mathbb{T}^n$ is homogeneous we may suppose that $\ell = o$. Clearly the geodesic $\sigma$ in Proposition 4 is periodic if and only if $a = b = 0$, while $\sigma_0$ is not periodic. By that proposition, $A$ is the orbit of the isotropy action (3) of the multiples of the initial velocity of $\sigma(s) = \ell((\cos s) i + (\sin s) j, 0)$. Under the identification $T_o\mathbb{T}^n \cong \mathbb{R}^m \times \mathbb{R}^m$ one has $\sigma'(0) = (j, 0)$. Therefore $A = \{(x, cx) \in \mathbb{R}^m \times \mathbb{R}^m \mid c \in \mathbb{R}\}$, since $K_o$ acts transitively on the unit sphere in $\mathbb{R}^m$.
We show that the frontier of $A$ equals $\{0\} \times \mathbb{R}^m$. Since clearly $(0, y) \notin A$ if $y \neq 0$ and $(0, y) = \lim_{n\to\infty}(y/n, ny/n)$ for all $y \in \mathbb{R}^m$, we have that $\{0\} \times \mathbb{R}^m$ is contained in the

frontier of $A$. Next we verify the other inclusion. Suppose that $\lim_{n\to\infty}(x_n, c_n x_n) = (x, y)$. If $x = 0$ we are done. If $x \neq 0$, we have $c_n |x_n|^2 = \langle c_n x_n, x_n \rangle$. Hence $\lim_{n\to\infty} c_n = \langle y, x \rangle / |x|^2 := c$. Therefore $(x, y) = (x, cx)$, which belongs to the interior of $A$. This completes the proof of the proposition. □

# References


[1] B. Guilfoyle & W. Klingenberg, *An indefinite Kähler metric on the space of oriented lines*, J. London Math. Soc. **72** (2005), 497–509.

[2] N. J. Hitchin, *Monopoles and geodesics*, Comm. Math. Phys. **83** (1982), 579–602.

[3] G. Keilhauer, *A note on the space of geodesics*. Rev. Unión Mat. Argent. **36** (1990), 164–173.

[4] M. Salvai, *On the geometry of the space of oriented geodesics of Euclidean space*, Manuscr. Math. **118** (2005), 181–189.

[5] M. Salvai, *On the geometry of the space of oriented geodesics of the hyperbolic space*, preprint.