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TWO WEIGHT NORM INEQUALITIES FOR COMMUTATORS OF ONE-SIDED SINGULAR INTEGRALS AND THE ONE-SIDED DISCRETE SQUARE FUNCTION

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Abstract

The purpose of this paper is to prove strong type inequalities with pairs of related weights for commutators of one-sided singular integrals (given by a Calderón-Zygmund kernel with support in $(-\infty, 0)$) and the one-sided discrete square function. The estimate given by C. Segovia and J. L. Torrea is improved for these one-sided operators giving a wider class of weights for which the inequality holds.

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1. Introduction

Many operators in Real Analysis have one-sided versions for which the class of weights is wider than the one of Muckenhoupt. It is well known that in Ergodic Theory there are many situations that require one-sided operators. In this paper we study one-sided singular integrals and the one-sided discrete square function. A one-sided singular integral is a Calderón-Zygmund singular integral whose kernel K has support in $(-\infty, 0)$ or $(0, \infty)$.

In [1], Aimar, Forzani and Martín-Reyes have studied these operators. They proved that the maximal operators which control them are the one-sided Hardy-Littlewood maximal operators M^+ and M^-, and that the good weights for these operators are the one-sided weights introduced by Sawyer [12].

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For one-sided singular integrals it is possible to improve many weighted inequalities in two ways, by putting on the right hand side a smaller operator or by allowing a wider class of weights for which the inequalities hold (see, for example, [1, 4, 10]).

In this paper we study inequalities with pairs of related weights for commutators of one-sided singular integrals and the one-sided discrete square function (studied by de la Torre and Torrea in [15]). Our starting point is the work of Segovia and Torrea, [13].

Throughout this paper the letter C will denote a positive constant, not necessarily the same at each occurrence and M will denote the Hardy-Littlewood maximal function, $Mf(x) = \sup_{h>0} 1/(2h)\int_{x-h}^{x+h}|f|$. If $1 \le p \le \infty$, then its conjugate exponent will be denoted by p' and A_p will be the classical Muckenhoupt's class of weights (see [9] for finite p and [3] for the definition of A_∞). Finally, given an interval $I = (x, x+h)$ $(h > 0)$, then $I^+ = (x+h, x+2h)$, $I^- = (x-h, x)$, $I^{++} = (x+2h, x+3h), \dots$.

2. Definitions and statement of the results

DEFINITION 2.1. We shall say that a function K in $L^1_{\mathrm{loc}}(\mathbb{R}\setminus\{0\})$ is a *Calderón-Zygmund kernel* if the following properties are satisfied:

(a) There exists a finite constant B_1 such that $\left|\int_{\varepsilon<|x|<N} K(x)\,dx\right| \le B_1$, for all ε and all N with $0 < \varepsilon < N$, and furthermore, there exists the limit $\lim_{\varepsilon\to 0^+}\int_{\varepsilon<|x|<1} K(x)\,dx$.

(b) There exists a finite constant B_2 such that $|K(x)| \le B_2/|x|$, for all $x \ne 0$.

(c) There exists a finite constant B_3 such that $|K(x-y) - K(x)| \le B_3|y||x|^{-2}$, for all x and y with $|x| > 2|y|$.

Given a Calderón-Zygmund kernel K, the singular integral associated to K is defined by

$$Tf(x) = \int_{\mathbb{R}} K(x-y)\,f(y)\,dy,$$

in the principal value sense. A one-sided singular integral T^+ (respectively T^-) is a singular integral associated to a Calderón-Zygmund kernel K with support in $(-\infty, 0)$ (respectively $(0,\infty)$); therefore, in that case,

$$T^+ f(x) = \lim_{\varepsilon\to 0^+}\int_{x+\varepsilon}^{\infty} K(x-y)\,f(y)\,dy.$$

An example of such kernels is $K(x) = \sin(\log|x|)/(x\log|x|)\chi_{(-\infty,0)}(x)$ (see [1]).

DEFINITION 2.2. For f locally integrable, we define the one-sided discrete square function applied to f by

$$S^+ f(x) = \left(\sum_{n\in\mathbb{Z}} |A_n f(x) - A_{n-1} f(x)|^2\right)^{1/2},$$

where $A_n f(x) = (1/2^n) \int_x^{x+2^n} f(y)\,dy$.

It is not difficult to see that $S^+ f(x) = \|U^+ f(x)\|_{\ell^2}$, where U^+ is the sequence valued operator

$$U^+ f(x) = \int_{\mathbb{R}} H(x-y) f(y)\,dy, \tag{2.1}$$

where

$$H(x) = \left\{ \frac{1}{2^n} \chi_{(-2^n,0)}(x) - \frac{1}{2^{n-1}} \chi_{(-2^{n-1},0)}(x) \right\}_{n\in\mathbb{Z}} \tag{2.2}$$

(see [15]).

DEFINITION 2.3. Let T^+ be a one-sided singular integral with kernel K and let S^+ be the one-sided discrete square function. For an appropriate b, we define the commutator of T^+ and S^+ by

$$T_b^+ f(x) = \int_x^\infty (b(x) - b(y))\, K(x-y)\, f(y)\,dy = b(x) T^+ f(x) - T^+(bf)(x)$$

and

$$S_b^+ f(x) = \left\| \int_{\mathbb{R}} (b(x) - b(y)) H(x-y) f(y)\,dy \right\|_{\ell^2},$$

where H is as in (2.2).

DEFINITION 2.4. The one-sided Hardy-Littlewood maximal operators M^+ and M^- are defined, for locally integrable functions f, by

$$M^+ f(x) = \sup_{h>0} \frac{1}{h} \int_x^{x+h} |f| \quad \text{and} \quad M^- f(x) = \sup_{h>0} \frac{1}{h} \int_{x-h}^{x} |f|.$$

The good weights for these operators are the one-sided weights, A_p^+ and A_p^-

$$\sup_{a<b<c} \frac{1}{(c-a)^p} \int_a^b \omega \left(\int_b^c \omega^{1-p'} \right)^{p-1} < \infty, \quad 1 < p < \infty; \tag{A_p^+}$$

$$M^- \omega(x) \le C \omega(x) \quad \text{a.e.} \tag{A_1^+}$$

There exist positive numbers C and δ such that for all numbers $a < b < c$ and all measurable sets $E \subset (b, c)$,

$$\frac{|E|}{c-a} \le C \left(\frac{\omega(E)}{\int_a^b \omega} \right)^\delta. \tag{A_∞^+}$$

It is known (see [8]) that $A_\infty^+ = \bigcup_{p\ge 1} A_p^+$. The classes A_p^- are defined in a similar way. (See [7, 8, 12] for more definitions and results.)

It is proved in [1] and [15] respectively that if $\omega \in A_p^+$, $1 < p < \infty$, then T^+ and S^+ are bounded from $L^p(\omega)$ to $L^p(\omega)$ and that, if $\omega \in A_1^+$, then T^+ and S^+ are of weak-type $(1, 1)$ with respect to ω.

DEFINITION 2.5. Let $b \in L^1(\mathbb{R})$ and $\nu \in A_\infty$. We say that $b \in BMO_\nu$ if

$$\|b\|_{BMO_\nu} = \sup_I \frac{1}{\nu(I)} \int_I |b - b_I| < \infty,$$

where I denote any bounded interval and $b_I = (1/|I|)\int_I b$. (Observe that if $\nu = 1$ then we get the classical BMO space.)

Now we are ready to establish our main results.

THEOREM 2.1. *Let* $1 < p < \infty$, $\alpha \in A_p$, $\beta \in A_p^+$, $\nu = (\alpha/\beta)^{1/p} \in A_\infty$ *and* $b \in BMO_\nu$. *Let* K *be a Calderón-Zygmund kernel with support in* $(-\infty, 0)$ *and let* T^+ *be the one-sided singular integral associated to* K. *Then, there exists* $C > 0$ *such that* $\int_{\mathbb{R}} |T_b^+ f|^p \beta \le C \int_{\mathbb{R}} |f|^p \alpha$, *for all bounded* f *with compact support.*

THEOREM 2.2. *Let* $1 < p < \infty$, $\alpha \in A_p$, $\beta \in A_p^+$, $\nu = (\alpha/\beta)^{1/p} \in A_\infty$ *and* $b \in BMO_\nu$. *Then, there exists* $C > 0$ *such that* $\int_{\mathbb{R}} |S_b^+ f|^p \beta \le C \int_{\mathbb{R}} |f|^p \alpha$, *for all bounded* f *with compact support.*

REMARK. The result of Theorem 2.1 for two-sided Calderón-Zygmund singular integrals is due to Segovia and Torrea [13]. They proved the boundedness of Calderón-Zygmund singular integrals from $L^p(\alpha)$ to $L^p(\beta)$ for both $\alpha, \beta \in A_p$. (Their result is highly more general, it is applied to many other operators. For the Hilbert transform, see Bloom [2].) The improvement in Theorem 2.1 for one-sided singular integrals is that it takes into consideration a wider class of weights. Taking $\beta \in A_p^+$, one improves not only in the left hand side of the inequality, but also in the right hand side, by noticing the fact that $\alpha = \nu^p \beta$ gives

$$\int_{\mathbb{R}} |T_b^+ f|^p \beta \le C \int_{\mathbb{R}} |f|^p \alpha = C \int_{\mathbb{R}} |f\nu|^p \beta.$$

An example that our class of weights is wider is the following: Set $\alpha(x) = 1$ for $x \le 1$ and $\alpha(x) = x^s$ for $x > 1$, where $-1 < s < p - 1$; set $\beta(x) = 1$ for $x \le 1$ and $\beta(x) = x^{p-1}$ for $x > 1$. Then $\beta \in A_p^+$ since it is nondecreasing, but $\beta \notin A_p$. On the other hand, $\alpha \in A_p$ and $\nu = (\alpha/\beta)^{1/p} \in A_2 \subset A_\infty$. We suspect that Theorems 2.1 and 2.2 hold for $\alpha \in A_p^+$, for this is what is needed in their proofs (see, for instance, the last step in the proof of Theorem 2.1). However, one of the key points to prove those theorems is Lemma 3.3, and there, what is needed is, precisely, that $\alpha \in A_p^-$. That is why we require $\alpha \in A_p$.

3. Preliminaries

We introduce some further definitions and results that we need to prove the main results.

DEFINITION 3.1. Let $g \geq 0$ be a locally integrable function (that is, g is a weight). We define *the maximal operator* M_g^+ by

$$M_g^+ f(x) = \sup_{h>0} \frac{1}{\int_x^{x+h} g} \int_x^{x+h} |f| g.$$

It is proved in [7] that for a weight u and $s > 1$, M_g^+ is bounded from $L^s(u)$ to $L^s(u)$ if and only if $u \in A_s^+(g)$:

$$(A_s^+(g)) \qquad \sup_{a<b<c} \left(\frac{1}{\int_a^c g} \int_a^b u \right)^{1/s} \left(\frac{1}{\int_a^c g} \int_b^c u^{1-s'} g^{s'} \right)^{1/s'} < \infty.$$

DEFINITION 3.2. Let f be a locally integrable function. The *one-sided sharp maximal function* is defined by

$$f_{\#,+}(x) = \sup_{h>0} \frac{1}{h} \int_x^{x+h} \left(f(y) - \frac{1}{h} \int_{x+h}^{x+2h} f \right)^+ dy.$$

It is proved in [6] that

$$(3.1) \qquad \begin{aligned} f_{\#,+}(x) &\leq \sup_{h>0} \inf_{a\in\mathbb{R}} \frac{1}{h} \int_x^{x+h} (f(y) - a)^+ \, dy + \frac{1}{h} \int_{x+h}^{x+2h} (a - f(y))^+ \, dy \\ &\leq C \|f\|_{BMO}. \end{aligned}$$

Another result that will be used often is the following ([6, Theorem 4]): if $\omega \in A_p^+$ and $M^+ f \in L^p(\omega)$, then $\int_{\mathbb{R}} (M^+ f)^p \omega \leq C \int_{\mathbb{R}} (f_{\#,+})^p \omega$.

DEFINITION 3.3. Let $1 < r < \infty$. We say that a weight ω belongs to the class RH_r^+ if there exists C such that for any $a < b$

$$\int_a^b \omega^r \leq C (M(\omega \chi_{(a,b)})(b))^{r-1} \int_a^b \omega.$$

The definition of RH_r^- is the expected one. (See [5] and [11] for more definitions and results.)

It is proved in [8] that $\omega \in A_\infty^+$ if and only if there exists $r > 1$ such that $\omega \in RH_r^+$. Something more can be said: if $\omega \in A_p^+$ then $\omega^{1-p'} \in A_{p'}^-$; as a consequence, there exists $\delta > 0$ such that $\omega^{1-p'} \in RH^-_{1+\delta}$. If we take $s = 1 + p/(p'(1+\delta)) = (p+\delta)/(1+\delta)$, then $\omega \in A^+_q$, for all $s \le q \le p$ (see the proof of Proposition 3 in [5]).

In order to prove the main theorems we still need four preliminary results that we are going to establish and prove now.

LEMMA 3.1. *Let $1 < p < \infty$ and let $\beta \in A^+_p$. Then, there exists $\delta > 0$ such that for all r with $p' \le r \le p'(1+\delta)$, $\beta^{-r/p} \in A^-_r$.*

PROOF. Since $\beta \in A^+_p$, there exists $\delta > 0$ such that $\beta^{-p'/p} = \beta^{1-p'} \in RH^-_{1+\delta}$. Let r be such that $p' \le r \le p'(1+\delta)$. By Hölder's inequality we have $\beta^{-p'/p} \in RH^-_{r/p'}$.

Let us prove that $\beta^{-r/p} \in A^-_r$. Consider $a < b < c < d$ such that $d - c = c - b = b - a$. Then, Hölder's inequality and the facts that $\beta^{-p'/p} \in RH^-_{r/p'}$ (see [11, Lemma 2.5]) and $\beta \in A^+_p$ give

$$\begin{aligned}
&\left(\frac{1}{b-a}\int_c^d \beta^{-r/p}\right)^{1/r}\left(\frac{1}{b-a}\int_a^b \beta^{-r(1-r')/p}\right)^{1/r'}\\
&\quad\le C\left(\frac{1}{b-a}\int_c^d (\beta^{-p'/p})^{r/p'}\right)^{1/r}\left(\frac{1}{b-a}\int_a^b \beta\right)^{1/p}\\
&\quad\le C\left(\frac{1}{b-a}\int_b^c \beta^{-p'/p}\right)^{1/p'}\left(\frac{1}{b-a}\int_a^b \beta\right)^{1/p} \le C.
\end{aligned}$$

By [11, Lemma 2.6], this finishes the proof of Lemma 3.1. □

LEMMA 3.2. *Let $1 < p < \infty$ and let $\beta \in A^+_p$. Then there exists $\delta > 0$ such that for all r with $p' < r < p'(1+\delta)$, $\beta \in A^+_{p/r'}(\beta^{r'/p})$.*

PROOF. As before, there exists $\delta > 0$ such that $\beta^{-p'/p} \in RH^-_{1+\delta}$ and, as we have noticed above, $\beta \in A^+_q$, for all q in the range $s = 1 + p/(p'(1+\delta)) \le q \le p$. Therefore, for r such that $p' < r < p'(1+\delta)$, we have $\beta \in A^+_{1+p/r}$.

We have to prove that $\beta \in A^+_{p/r'}(\beta^{r'/p})$, that is,

$$\int_a^b \beta\left(\int_a^c \beta^{r'/p}\right)^{-p/r'}(c-b)^{(p-r')/r'} \le C,$$

for all $a < b < c$.

So, let $a < b < c$. Then, since $-p/r' < 0$, we have

$$\left(\int_a^c \beta^{r'/p}\right)^{-p/r'} \le \left(\int_b^c \beta^{r'/p}\right)^{-p/r'}. \tag{3.2}$$

On the other hand, using Hölder's inequality with exponents (r, r'), we get

$$1 = \left(\frac{1}{c-b}\int_b^c \beta^{-1/p}\beta^{1/p}\right)^p \le \left(\frac{1}{c-b}\int_b^c \beta^{-r/p}\right)^{p/r} \left(\frac{1}{c-b}\int_b^c \beta^{r'/p}\right)^{p/r'}.$$

This implies that

$$\left(\int_b^c \beta^{r'/p}\right)^{-p/r'} \le (c-b)^{-p}\left(\int_b^c \beta^{-r/p}\right)^{p/r}. \tag{3.3}$$

Putting together inequalities (3.2) and (3.3) and using the fact that $\beta \in A^+_{1+p/r}$, we obtain

$$\begin{aligned}\int_a^b \beta \left(\int_a^c \beta^{r'/p}\right)^{-p/r'} (c-b)^{(p-r')/r'} &\le \int_a^b \beta \left(\int_b^c \beta^{-r/p}\right)^{p/r} (c-b)^{(p-r')/r'-p} \\ &\le C(c-a)^{1+p/r}(c-b)^{(p-r')/r'-p}.\end{aligned}$$

If $c - b \ge b - a$ we have

$$C(c-a)^{1+p/r}(c-b)^{(p-r')/r'-p} \le C(c-b)^{1+p/r+(p-r')/r'-p} = C,$$

and we would have finished the proof.

In the case that $c - b < b - a$ we partition the interval $[a, c]$ by points $x_0 = a < x_1 < \cdots < x_n < b \le x_{n+1} < x_{n+2} = c$, such that $x_{i+1} - x_i = c - b$, $i = 0, 1, \ldots, n$. Therefore, for $i < n$, we have

$$\int_{x_i}^{x_{i+1}} \beta(c-b)^{(p-r')/r'} \le \left(\int_{x_i}^{x_{i+2}} \beta^{r'/p}\right)^{p/r'}$$

and, since $b - x_n < c - b$,

$$\int_{x_n}^{b} \beta(c-b)^{(p-r')/r'} \le \left(\int_{x_n}^{c} \beta^{r'/p}\right)^{p/r'}.$$

Thus,

$$\begin{aligned}\int_a^b \beta(c-b)^{(p-r')/r'} &\le \sum_{i=0}^{n}\int_{x_i}^{x_{i+1}} \beta(c-b)^{(p-r')/r'} \\ &\le \sum_{i=0}^{n}\left(\int_{x_i}^{x_{i+2}} \beta^{r'/p}\right)^{p/r'} \le \left(2\int_a^c \beta^{r'/p}\right)^{p/r'},\end{aligned}$$

which finishes the proof of Lemma 3.2. □

LEMMA 3.3. *Let* $1 < p < \infty$, $\alpha \in A_p$, $\beta \in A_p^+$, $\nu = (\alpha/\beta)^{1/p} \in A_\infty$ *and* $b \in BMO_\nu$. *Then, there exists* $\varepsilon > 0$ *so that, for all* r *with* $p' \le r \le p' + \varepsilon$,

$$\int_I |b - b_I|^r \alpha^{-r/p} \le C \int_{I^+} \beta^{-r/p}.$$

PROOF. By [13, Lemma 2], there exists $\varepsilon > 0$ such that for all r in the range $p' \le r \le p' + \varepsilon$, $\alpha^{-r/p} \in A_r$. Let us fix such r and take $s' > 1$ such that $\alpha^{-r/p} \in RH_{s'}$. It then follows that

$$\begin{aligned}
\frac{1}{|I|}\int_I |b - b_I|^r \alpha^{-r/p} &\le \left(\frac{1}{|I|}\int_I |b - b_I|^{rs}\right)^{1/s} \left(\frac{1}{|I|}\int_I \alpha^{-rs'/p}\right)^{1/s'} \\
&\le C \left(\frac{1}{|I|}\int_I |b - b_I|^{rs}\right)^{1/s} \frac{1}{|I|}\int_I \alpha^{-r/p} \\
&\le C \left(\frac{\nu(I)}{|I|}\right)^r \frac{1}{|I|}\int_I \alpha^{-r/p}.
\end{aligned} \tag{3.4}$$

The last inequality is a consequence of John-Nirenberg's inequality (see the proof of Proposition 6, [14, Chapter III]).

Now, we use Hölder's inequality and the facts that $\nu \in A_\infty \subset A_\infty^+$ and $\alpha^{-r/p} \in A_r \subset A_r^+$, to obtain

$$\begin{aligned}
\left(\frac{\nu(I)}{|I|}\right)^r \frac{1}{|I|}\int_I \alpha^{-r/p} &\le C \left(\frac{1}{|I|}\int_{I^+} \nu\right)^r \frac{1}{|I|}\int_I \alpha^{-r/p} \\
&\le C \left(\frac{1}{|I|}\int_{I^+} \alpha^{r'/p}\right)^{r/r'} \frac{1}{|I|}\int_{I^+} \beta^{-r/p} \frac{1}{|I|}\int_I \alpha^{-r/p} \\
&\le C \frac{1}{|I|}\int_{I^+} \beta^{-r/p}.
\end{aligned} \tag{3.5}$$

Putting together inequalities (3.4) and (3.5) we obtain the desired result. □

LEMMA 3.4. *Suppose that we are under the same hypotheses of Lemma* 3.3. *Let* $x \in \mathbb{R}$, $h > 0$, $l \in \mathbb{N}$ *and let* $I = (x, x + 2^l h)$. *For* $k \in \mathbb{N}$, *let* $I_k = (x + 2^k h, x + 2^{k+1} h)$. *Then, there exists* $C > 0$ *independent of* x, h, l *and* k *such that*

$$|b_I - b_{I_k}| \le Ck \max_{l-1 \le j \le k-2} \frac{1}{|I_j|}\int_{I_j} \nu.$$

PROOF. Let $k \in \mathbb{N}$, $k > l$. We shall estimate $|b_I - b_{I_k}|$. Clearly,

$$|b_I - b_{I_k}| \le |b_I - b_{I_l}| + \sum_{j=l}^{k-1} |b_{I_j} - b_{I_{j+1}}|. \tag{3.6}$$

For the first summand on the right hand side, we observe that if $b \in BMO_\nu$, then there exists C such that

$$\frac{1}{\nu(J)} \int_J |b - b_{J^+}| \le C, \tag{3.7}$$

for any interval J. In fact, this sort of estimate, for all J, characterizes that $b \in BMO_\nu$, as well as this other one

$$\frac{1}{\nu(J)} \int_{J \cup J^+} |b - b_{J^+}| \le C. \tag{3.8}$$

Consequently, since $I_l = I^+$, we get

$$|b_I - b_{I_l}| \le C \frac{1}{|I|} \nu(I) \le Ck \max_{l-1 \le j \le k-2} \frac{1}{|I_j|} \int_{I_j} \nu.$$

For the rest of the sum, we note that $I_{j+1}^- \supset I_j$, then the above remark and the fact that $\nu \in A_\infty$ give

$$\begin{aligned}\sum_{j=l}^{k-1} |b_{I_j} - b_{I_{j+1}}| &\le C \sum_{j=l}^{k-1} \frac{1}{|I_j|} \nu(I_j) = C \sum_{j=l}^{k-1} \frac{\nu(I_j)}{\nu(I_{j-1})} \frac{\nu(I_{j-1})}{|I_{j-1}|} \frac{|I_{j-1}|}{|I_j|} \\ &\le C \sum_{j=l}^{k-1} \frac{1}{|I_{j-1}|} \int_{I_{j-1}} \nu \le Ck \max_{l-1 \le j \le k-2} \frac{1}{|I_j|} \int_{I_j} \nu.\end{aligned}$$

□

4. Proof of the results

PROOF OF THEOREM 2.1. The following pointwise estimate is the key to prove Theorem 2.1. We claim that there exist $\delta_1 > 0$, $\delta_2 > 0$ and $q > 1$ such that for all r in the range

$$\left(\frac{p}{q}\right)' < r < \min \left\{ \frac{1}{q} p'(1 + \delta_1), \left(\frac{p}{q}\right)' (1 + \delta_2) \right\},$$

the following inequality holds

$$\begin{aligned}(T_b^+ f)_{\#,+}(x) \le C \Bigg\{ &\left(M^+_{\beta^{qr'/p}}(|f\nu|^{qr'})(x) \right)^{1/qr'} \\ &+ \left(M^+_{\beta^{r'/p}}(|\nu T^+ f|^{r'})(x) \right)^{1/r'} + M^+(\nu M^+ f)(x) \Bigg\},\end{aligned} \tag{4.1}$$

for all bounded f with compact support.

Let us prove this claim. We have

$$(T_b^+ f)_{\#,+}(x) \leq C \sup_{h>0} \inf_{c\in\mathbb{R}} \frac{1}{h}\int_x^{x+2h} |T_b^+ f(y) - c|\, dy. \tag{4.2}$$

Let $x \in \mathbb{R}$ and $h > 0$ be fixed. Set $I = (x, x+h)$, $J = (x, x+8h)$, $f_1 = f\chi_J$, $f_2 = f - f_1$ and $C_J = \int_{\mathbb{R}} K(x+2h-z)(b_J - b(z))\, f_2(z)\, dz$. Observe that

$$\begin{aligned} T_b^+ f(y) &= \int_{\mathbb{R}} (b(y) - b_J + b_J - b(t))K(y-t)\, f(t)\, dt \\ &= (b(y) - b_J)T^+ f(y) - \int_{\mathbb{R}} (b(t) - b_J)K(y-t)\, f(t)\, dt \\ &= (b(y) - b_J)T^+ f(y) - T^+((b-b_J)\, f_1)(y) - T^+((b-b_J)\, f_2)(y). \end{aligned}$$

Thus,

$$\begin{aligned} \frac{1}{h}\int_x^{x+2h} |T_b^+ f(y) - C_J|\, dy &\leq \frac{1}{h}\int_x^{x+2h} |b(y) - b_J|\, |T^+ f(y)|\, dy \\ &\quad + \frac{1}{h}\int_x^{x+2h} |T^+((b-b_J)\, f_1)(y)|\, dy \\ &\quad + \frac{1}{h}\int_x^{x+2h} |T^+((b-b_J)\, f_2)(y) - C_J|\, dy \\ &= \mathrm{I} + \mathrm{II} + \mathrm{III}\,. \end{aligned} \tag{4.3}$$

By Lemmas 3.1–3.3, there exists $\delta_1 > 0$ such that for all r in the range $p' < r < p'(1+\delta_1)$, it holds that $\beta^{-r/p} \in A_r^-$, $\beta \in A^+_{p/r'}(\beta^{r'/p})$ and

$$\int_I |b - b_I|^r \alpha^{-r/p} \leq C \int_{I^+} \beta^{-r/p}.$$

Let $q > 1$, close enough to 1, such that $\beta^q \in A_p^+$, $\beta \in A^+_{p/q}$ and $(p/q)' < p'(1+\delta_1)/q$. Therefore, since $\beta \in A^+_{p/q}$, there exists $\delta_2 > 0$ such that, for all r in the range $(p/q)' < (p/q)'(1+\delta_2)$, it holds that $\beta \in A^+_{p/qr'}(\beta^{qr'/p})$. Let r be such that

$$\left(\frac{p}{q}\right)' < r < \min\left\{\frac{1}{q}p'(1+\delta_1), \left(\frac{p}{q}\right)'(1+\delta_2)\right\}.$$

Then, by Hölder's inequality and the above remarks,

$$\begin{aligned} \mathrm{I} &\leq \left(\frac{1}{h}\int_x^{x+8h} |b - b_J|^r \alpha^{-r/p}\right)^{1/r} \left(\frac{1}{h}\int_x^{x+8h} |T^+ f|^{r'} \alpha^{r'/p}\right)^{1/r'} \\ &\leq C\left(\frac{1}{h}\int_{x+8h}^{x+16h} \beta^{-r/p}\right)^{1/r} \left(\frac{1}{h}\int_x^{x+8h} |\nu T^+ f|^{r'} \beta^{r'/p}\right)^{1/r'} \end{aligned} \tag{4.4}$$

$$\leq C\left(\int_x^{x+8h}\beta^{r'/p}\right)^{-1/r'}\left(\int_x^{x+8h}|\nu T^+ f|^{r'}\beta^{r'/p}\right)^{1/r'}$$
$$\leq C\left(M^+_{\beta^{r'/p}}(|\nu T^+ f|^{r'})(x)\right)^{1/r'}.$$

To control II, we observe that, since

$$\frac{p'}{q} < p' < \left(\frac{p}{q}\right)' < r < \min\left\{\frac{1}{q}p'(1+\delta_1), \left(\frac{p}{q}\right)'(1+\delta_2)\right\},$$

we have $p' < rq < p'(1+\delta_1)$. Then, Hölder's inequality, the fact that T^+ is bounded from $L^q(dx)$ to $L^q(dx)$, Lemma 3.3 and the fact that $\beta^{-qr/p} \in A_r^-$ give

$$\text{II} \leq \left(\frac{1}{h}\int_x^{x+2h}|T^+((b-b_J)f_1)(y)|^q\,dy\right)^{1/q} \tag{4.5}$$
$$\leq C\left(\frac{1}{h}\int_x^{x+8h}|b-b_J|^q|f|^q\right)^{1/q}$$
$$\leq C\left(\frac{1}{h}\int_x^{x+8h}|b-b_J|^{qr}\alpha^{-qr/p}\right)^{1/qr}\left(\frac{1}{h}\int_x^{x+8h}|f|^{qr'}\alpha^{qr'/p}\right)^{1/qr'}$$
$$\leq C\left(\frac{1}{h}\int_{x+8h}^{x+16h}\beta^{-qr/p}\right)^{1/qr}\left(\frac{1}{h}\int_x^{x+8h}|f\nu|^{qr'}\beta^{qr'/p}\right)^{1/qr'}$$
$$\leq C\left(\frac{1}{h}\int_x^{x+8h}\beta^{qr'/p}\right)^{-1/qr'}\left(\frac{1}{h}\int_x^{x+8h}|f\nu|^{qr'}\beta^{qr'/p}\right)^{1/qr'}$$
$$\leq C\left(M^+_{\beta^{qr'/p}}(|f\nu|^{qr'})(x)\right)^{1/qr'}.$$

Next, we use condition (c) of the kernel to obtain

$$\text{III} = \frac{1}{h}\int_x^{x+2h}\left|\int_{\mathbb{R}}(b(z)-b_J)(K(y-z)-K(x+2h-z))f_2(z)\,dz\right|dy \tag{4.6}$$
$$\leq C\frac{1}{h}\int_x^{x+2h}\int_{x+8h}^{\infty}\frac{x+2h-y}{(z-(x+2h))^2}|b(z)-b_J||f(z)|\,dz\,dy$$
$$\leq C\frac{1}{h}\int_x^{x+2h}h\sum_{k=3}^{\infty}\int_{x+2^kh}^{x+2^{k+1}h}\frac{|b(z)-b_J|}{(z-(x+2h))^2}|f(z)|\,dz\,dy$$
$$\leq Ch\sum_{k=3}^{\infty}\frac{2^{k+1}}{(2^k-2)^2h^2}\frac{1}{2^{k+1}}\int_{I_k}|b(z)-b_J||f(z)|\,dz$$
$$\leq C\sum_{k=3}^{\infty}\frac{2^{k+1}}{(2^k-2)^2}\frac{1}{2^{k+1}h}\int_{I_k}|b(z)-b_{I_k}||f(z)|\,dz$$

$$+ C \sum_{k=3}^{\infty} \frac{2^{k+1}}{(2^k-2)^2} \frac{1}{2^{k+1}h} |b_{I_k} - b_J| \int_{I_k} |f(z)|\, dz = \mathrm{IV} + \mathrm{V}\,.$$

To estimate IV, we introduce a modified version of Lemma 3.3. If $I = (x, x+h)$, we denote by I^2 the interval $(x + h/2, x+h)$. It is very easy to prove that Lemma 3.3 holds changing b_I by b_{I^2}. Consequently, arguing as in the estimate of II, and using this version of Lemma 3.3, we get

$$\begin{aligned} \mathrm{IV} &\le C \sum_{k=3}^{\infty} \frac{2^{k+1}}{(2^k-2)^2} \frac{1}{2^{k+1}h} \int_x^{x+2^{k+1}h} |b(z) - b_{I_k}|\, |f(z)|\, dz \\ &\le C \left(M^+_{\beta^{qr'/p}}(|f\nu|^{qr'})(x) \right)^{1/qr'}. \end{aligned} \tag{4.7}$$

Now, let $I_{j(k)}$ be such that

$$\max_{2 \le j \le k-2} \frac{1}{|I_j|} \int_{|I_j|} \nu = \frac{1}{|I_{j(k)}|} \int_{I_{j(k)}} \nu.$$

Then, for all $s \in I_{j(k)}$, we have

$$\begin{aligned} \frac{1}{2^{k+1}h} \int_{I_k} |f(z)|\, dz &\le \frac{1}{2^{k+1}h} \int_s^{x+2^{k+1}h} |f(z)|\, dz \\ &\le C \frac{1}{x + 2^{k+1}h - s} \int_s^{x+2^{k+1}h} |f(z)|\, dz \le C M^+ f(s). \end{aligned} \tag{4.8}$$

This conclusion and Lemma 3.4 give us

$$\begin{aligned} \mathrm{V} &\le C \sum_{k=3}^{\infty} \frac{2^{k+1}}{(2^k-2)^2} k \frac{1}{|I_{j(k)}|} \int_{I_{j(k)}} \nu \frac{1}{2^{k+1}h} \int_{I_k} |f(z)|\, dz \\ &\le C \sum_{k=3}^{\infty} \frac{k 2^{k+1}}{(2^k-2)^2} \frac{1}{|I_{j(k)}|} \int_{I_{j(k)}} \nu(s) M^+ f(s)\, ds \\ &\le C M^+(\nu M^+ f)(x) \sum_{k=3}^{\infty} \frac{k 2^{k+1}}{(2^k-2)^2} = C M^+(\nu M^+ f)(x). \end{aligned} \tag{4.9}$$

In the last inequality we have used

$$\frac{1}{|I_{j(k)}|} \int_{I_{j(k)}} \nu M^+ f \le \frac{2}{2^{j(k)+1}} \int_x^{x+2^{j(k)+1}} \nu M^+ f.$$

Collecting all these inequalities, we complete the proof of (4.1).

Next, we are going to prove that we can apply [6, Theorem 4], that is, we have to prove that $M^+(T_b^+ f) \in L^p(\beta)$. Since $\beta \in A_p^+$, it suffices to show that $T_b^+ f \in L^p(\beta)$.

If b is bounded, then $b \in L^\infty \subset BMO$ and, by a result in [4],

$$\int_{\mathbb{R}} |T_b^+ f|^p \beta \le C \int_{\mathbb{R}} |f|^p \beta < \infty.$$

In the general case, let $b_m = b$ if $-m \le b \le m$, $b_m = m$ if $b \ge m$ and $b_m = -m$ if $b \le -m$. Then, it is not difficult to see that $b_m \in BMO_\nu$ and $\|b_m\|_{BMO_\nu} \le C\|b\|_{BMO_\nu}$, with C independent of m. Then, for each b_m, we have

$$\int_{\mathbb{R}} |T_{b_m}^+ f|^p \beta \le C \int_{\mathbb{R}} |f|^p \beta,$$

with C independent of m. Using now the dominated convergence theorem, we get that $\{b_m f\}$ converges to bf in $L^1(dx)$, as m tends to infinity and, since T^+ is of weak type $(1, 1)$ with respect to the Lebesgue measure, $\{T^+(b_m f)\}$ converges to $T^+(bf)$ in measure (dx). Therefore, there exists a subsequence that converges almost everywhere. We shall continue denoting this subsequence by $\{T^+(b_m f)\}$. On the other hand, $\{b_m T^+ f\}$ converges to $bT^+ f$ almost everywhere. Then, by Fatou's Lemma,

$$\begin{aligned}\int_{\mathbb{R}} |T_b^+ f|^p \beta &= \int_{\mathbb{R}} \lim_{m\to\infty} |T_{b_m}^+ f|^p \beta \\ &\le \liminf_{m\to\infty} \int_{\mathbb{R}} |T_{b_m}^+ f|^p \beta \le C \int_{\mathbb{R}} |f|^p \beta < \infty.\end{aligned}$$

As a consequence, [6, Theorem 4] gives that

$$\int_{\mathbb{R}} |T_b^+ f|^p \beta \le \int_{\mathbb{R}} (M^+(T_b^+ f))^p \beta \le C \int_{\mathbb{R}} ((T_b^+ f)_{\#,+})^p \beta.$$

Now, using (4.1), we get

$$\begin{aligned}\int_{\mathbb{R}} |T_b^+ f|^p \beta &\le C \int_{\mathbb{R}} \left(M^+_{\beta^{qr'/p}}(|f\nu|^{qr'}) \right)^{p/qr'} \beta \\ &\quad + C \int_{\mathbb{R}} \left(M^+_{\beta^{r'/p}}(|\nu T^+ f|^{r'}) \right)^{p/r'} \beta + C \int_{\mathbb{R}} (M^+(\nu M^+ f))^p \beta \\ &= \mathrm{I} + \mathrm{II} + \mathrm{III}\,.\end{aligned}$$

Since $\beta \in A^+_{p/qr'}(\beta^{qr'/p})$, it follows that

$$\mathrm{I} \le C \int_{\mathbb{R}} (|f\nu|^{qr'})^{p/qr'} \beta = C \int_{\mathbb{R}} |f|^p \alpha$$

and

$$\text{II} \le C \int_{\mathbb{R}} (|\nu T^+ f|^{r'})^{p/r'} \beta = C \int_{\mathbb{R}} |T^+ f|^p \alpha \le C \int_{\mathbb{R}} |f|^p \alpha,$$

since $\alpha \in A_p \subset A_p^+$.

Finally, using that $\beta \in A_p^+$ and using again that $\alpha \in A_p \subset A_p^+$, we get

$$\text{III} \le C \int_{\mathbb{R}} (\nu M^+ f)^p \beta = C \int_{\mathbb{R}} (M^+ f)^p \alpha \le C \int_{\mathbb{R}} |f|^p \alpha. \qquad \square$$

PROOF OF THEOREM 2.2. This proof follows the same pattern as the preceding one. As above, the essential step is the pointwise boundedness of the one-sided sharp of the operator. In this case we claim the following: Let $\delta_1 > 0$, $\delta_2 > 0$ and $q > 1$ be as in the proof of Theorem 2.1. Assume also that q is close enough to 1 to ensure that $\alpha \in A_{p/q}$ and that δ_1 is such that the conclusion of [13, Lemma 2] holds for $\varepsilon = p'\delta_1$. Then, for all r in the range

$$\left(\frac{p}{q}\right)' < r < \min\left\{\frac{1}{q} p'(1+\delta_1), \left(\frac{p}{q}\right)'(1+\delta_2)\right\},$$

the following inequality holds

$$(4.10) \qquad (S_b^+ f)_{\#,+}(x) \le C \left\{ \left(M^+_{\beta^{qr'/p}}(|f\nu|^{qr'})(x) \right)^{1/qr'} + \left(M^+_{\beta^{r'/p}}(|\nu S^+ f|^{r'})(x) \right)^{1/r'} + M^+(\nu (M^+|f|^q)^{1/q})(x) \right\},$$

for all bounded f with compact support.

Let us prove the claim. Let $x \in \mathbb{R}$ and $h > 0$. Let $i \in \mathbb{Z}$ be such that $2^i \le h < 2^{i+1}$. Set $J = (x, x+2^{i+3})$, $f_1 = f\chi_J$, $f_2 = f - f_1$ and $C_J = S^+(b - b_J) f_2(x)$. As above, we have

$$\begin{aligned}(4.11) \qquad \frac{1}{h}\int_x^{x+2h} |S_b^+ f(y) - C_J|\,dy &\le \frac{1}{h}\int_x^{x+2^{i+3}} |b(y) - b_J|\,|S^+ f(y)|\,dy \\ &\quad + \frac{1}{h}\int_x^{x+2^{i+3}} |S^+((b-b_J) f_1)(y)|\,dy \\ &\quad + \frac{1}{h}\int_x^{x+2^{i+3}} |S^+((b-b_J) f_2)(y) - C_J|\,dy \\ &= \text{I} + \text{II} + \text{III}.\end{aligned}$$

Clearly, I and II are estimated as in the proof of Theorem 2.1.

Let U^+ be as in (2.1). Then

$$\text{III} = \frac{1}{h}\int_x^{x+2^{i+3}} \Big| \|U^+((b-b_J)\, f_2)(y)\|_{\ell^2} - \|U^+((b-b_J)\, f_2)(x)\|_{\ell^2}\Big|\, dy \tag{4.12}$$
$$\leq \frac{1}{h}\int_x^{x+2^{i+3}} \|U^+((b-b_J)\, f_2)(y) - U^+((b-b_J)\, f_2)(x)\|_{\ell^2}\, dy.$$

If H is as in (2.2), then

$$\|U^+((b-b_J)\, f_2)(y) - U^+((b-b_J)\, f_2)(x)\|_{\ell^2} \tag{4.13}$$
$$\leq \int_{x+2^{i+3}}^{\infty} |b(t)-b_J|\,|f(t)|\|H(y-t)-H(x-t)\|_{\ell^2}\, dt$$
$$\leq \sum_{k=i+3}^{\infty}\int_{x+2^k}^{x+2^{k+1}} |b(t)-b_{I_k}|\,|f(t)|\|H(y-t)-H(x-t)\|_{\ell^2}\, dt$$
$$+ \sum_{k=i+3}^{\infty}|b_{I_k}-b_J|\int_{x+2^k}^{x+2^{k+1}} |f(t)|\|H(y-t)-H(x-t)\|_{\ell^2}\, dt$$
$$= \text{IV}+\text{V}.$$

By Hölder's inequality with exponents (q, q') and (r, r'),

$$\text{IV} \leq \sum_{k=i+3}^{\infty}\left(\int_{I_k}|b-b_{I_k}|^q\alpha^{-q/p}\alpha^{q/p}|f|^q\right)^{1/q} \tag{4.14}$$
$$\times\left(\int_{I_k}\|H(y-t)-H(x-t)\|_{\ell^2}^{q'}\, dt\right)^{1/q'}$$
$$\leq \sum_{k=i+3}^{\infty}\left(\int_{I_k}|b-b_{I_k}|^{qr}\alpha^{-qr/p}\right)^{1/qr}\left(\int_{I_k}|f\nu|^{qr'}\beta^{qr'/p}\right)^{1/qr'}$$
$$\times\left(\int_{I_k}\|H(y-t)-H(x-t)\|_{\ell^2}^{q'}dt\right)^{1/q'}.$$

Then, by Lemma 3.3 and the fact that $\beta^{-qr/p}\in A_r^-$,

$$\left(\int_{x+2^k}^{x+2^{k+1}}|b-b_{I_k}|^{qr}\alpha^{-qr/p}\right)^{1/qr} \leq C\left(\int_{x+2^{k+1}}^{x+2^{k+2}}\beta^{-qr/p}\right)^{1/qr} \tag{4.15}$$
$$\leq C(2^k)^{1/q}\left(\int_{x+2^k}^{x+2^{k+1}}\beta^{qr'/p}\right)^{-1/qr'}.$$

Putting together inequalities (4.14) and (4.15), we obtain

$$\text{IV} \leq C\sum_{k=i+3}^{\infty}(2^k)^{1/q}\left(M^+_{\beta^{qr'/p}}(|f\nu|^{qr'})(x)\right)^{1/qr'} \tag{4.16}$$

$$\times \left(\int_{I_k} \|H(y-t) - H(x-t)\|_{\ell^2}^{q'} \, dt \right)^{1/q'}.$$

It is proved in [15, Theorem 1.6] that the kernel H satisfies

$$\left(\int_{I_k} \|H(y-t) - H(x-t)\|_{\ell^2}^{q'} dt \right)^{1/q'} \leq C \frac{2^{i/q'}}{2^k}. \tag{4.17}$$

Inequalities (4.16), (4.17) and the fact that $\sum_{k=i+3}^{\infty} (2^k)^{1/q} 2^{i/q'}/2^k = C$, give

$$\mathrm{IV} \leq C \left(M^+_{\beta^{qr'/p}}(|f\nu|^{qr'})(x) \right)^{1/qr'}.$$

Now we observe that Lemma 3.4 yields

$$|b_{I_k} - b_J| \leq C(k-i) \max_{3 \leq j \leq k-2} \frac{1}{|I_j|} \int_{I_j} \nu = C(k-i) \frac{1}{|I_{j(k)}|} \int_{I_{j(k)}} \nu, \tag{4.18}$$

since $I_k = (x + 2^i 2^{k-i}, x + 2^i 2^{k-i+1})$.

On the other hand, for all $z \in I_{j(k)}$,

$$\begin{aligned} \left(\int_{I_k} |f|^q \right)^{1/q} &\leq C (2^k)^{1/q} \left(\frac{1}{x + 2^{k+1} - z} \int_z^{x+2^{k+1}} |f|^q \right)^{1/q} \\ &\leq C (2^k)^{1/q} \left(M^+(|f|^q)(z) \right)^{1/q}. \end{aligned} \tag{4.19}$$

Taking into account inequalities (4.18), (4.19) and using again Hölder's inequality and (4.17), we get

$$\begin{aligned} \mathrm{V} &\leq C \sum_{k=i+3}^{\infty} (k-i) \frac{1}{|I_{j(k)}|} \int_{I_{j(k)}} \nu \left(\int_{I_k} |f|^q \right)^{1/q} \\ &\quad \times \left(\int_{I_k} \|H(y-t) - H(x-t)\|_{\ell^2}^{q'} \, dt \right)^{1/q'} \\ &\leq C \sum_{k=i+3}^{\infty} (k-i) \frac{(2^k)^{1/q} 2^{i/q'}}{2^k} \frac{1}{|I_{j(k)}|} \int_{I_{j(k)}} \nu(z) \left(M^+(|f|^q)(z) \right)^{1/q} dz \\ &\leq C \sum_{k=i+3}^{\infty} (k-i) \frac{(2^k)^{1/q} 2^{i/q'}}{2^k} M^+(\nu (M^+(|f|^q))^{1/q})(x) \\ &= C M^+(\nu (M^+(|f|^q))^{1/q})(x). \end{aligned}$$

Our next task will be to prove that [6, Theorem 4] can be applied in this setting. Assuming it for the moment, we obtain

$$\int_{\mathbb{R}} |S_b^+ f|^p \beta \leq C \int_{\mathbb{R}} ((S_b^+ f)_{\#,+})^p \beta,$$

which ensures the desired result having into consideration inequality (4.10) and the choice of r, t and s.

Let us prove now that [6, Theorem 4] can be applied. If $b \in L^\infty$, the result in [15] gives

$$\int_{\mathbb{R}} |S_b^+ f|^p \beta \le C \int_{\mathbb{R}} |bS^+ f|^p \beta + C \int_{\mathbb{R}} |S^+(bf)|^p \beta$$
$$\le C\|b\|_\infty^p \int_{\mathbb{R}} |f|^p \beta + C \int_{\mathbb{R}} |bf|^p \beta \le C\|b\|_\infty^p \int_{\mathbb{R}} |f|^p \beta < \infty.$$

Thus, the above argument works and we obtain

$$\int_{\mathbb{R}} |S_b^+ f|^p \beta \le C \int_{\mathbb{R}} |f|^p \alpha.$$

In the general case, take b_m as in the proof of Theorem 2.1, and obtain

$$\int_{\mathbb{R}} |S_{b_m}^+ f|^p \beta \le C \int_{\mathbb{R}} |f|^p \alpha$$

with a constant C not depending on m, since $\|b_m\|_{BMO_\nu} \le C\|b\|_{BMO_\nu}$. Now, an argument similar to the one used in the proof of Theorem 2.1 shows that, a subsequence of $\{S_{b_m}^+ f\}$ converges to $S_b^+ f$ a.e., so by Fatou's Lemma again,

$$\int_{\mathbb{R}} |S_b^+ f|^p \beta \le \liminf \int_{\mathbb{R}} |S_{b_m}^+ f|^p \beta \le C \int_{\mathbb{R}} |f|^p \alpha,$$

which finishes the proof of Theorem 2.2. □

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References

[1] H. Aimar, L. Forzani and F. J. Martín-Reyes, 'On weighted inequalities for one-sided singular integrals', *Proc. Amer. Math. Soc.* **125** (1997), 2057–2064.
[2] S. Bloom, 'A commutator theorem and weighted *BMO*', *Trans. Amer. Math. Soc.* **292** (1985), 103–122.
[3] R. R. Coifman and C. Fefferman, 'Weighted norm inequalities for maximal functions and singular integrals', *Studia Math.* **51** (1974), 241–250.
[4] M. Lorente and M. S. Riveros, 'Weighted inequalities for commutators of one-sided singular integrals', *Comment. Math. Univ. Carolin.* **43** (2002), 83–101.

[5] F. J. Martín-Reyes, 'New proofs of weighted inequalities for the one-sided Hardy-Littlewood maximal functions', *Proc. Amer. Math. Soc.* **117** (1993), 691–698.

[6] F. J. Martín-Reyes and A. de la Torre, 'One sided *BMO* spaces', *J. London Math. Soc.* **49** (1994), 529–542.

[7] F. J. Martín-Reyes, P. Ortega and A. de la Torre, 'Weighted inequalities for one sided maximal functions', *Trans. Amer. Math. Soc.* **319** (1990), 517–534.

[8] F. J. Martín-Reyesa, L. Pick and A. de la Torre, 'A_∞^+ condition', *Canad. J. Math.* **45** (1993), 1231–1244.

[9] B. Muckenhoupt, 'Weighted norm inequalities for the Hardy maximal function', *Trans. Amer. Math. Soc.* **165** (1972), 207–226.

[10] M. S. Riveros and A. de la Torre, 'Norm inequalities relating one-sided singular integrals and the one-sided maximal function', *J. Austral. Math. Soc. (Series A)* **69** (2000), 403–414.

[11] ———, 'On the best ranges for A_p^+ and RH_r^+', *Czechoslovak Math. J.* **51** (2001), 285–301.

[12] E. T. Sawyer, 'Weighted inequalities for the one-sided Hardy-Littlewood maximal functions', *Trans. Amer. Math. Soc.* **297** (1986), 53–61.

[13] C. Segovia and J. L. Torrea, 'Vector-valued commutators and applications', *Indiana Univ. Math. J.* **38** (1989), 959–971.

[14] J. Strömberg and A. Torchinsky, *Weighted Hardy spaces*, Lecture Notes in Math. (Springer, 1989).

[15] A. de la Torre and J. L. Torrea, 'One-sided discrete square function', *Studia Math.* **156** (2003), 243–260.

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