WEIGHTS FOR COMMUTATORS OF THE ONE-SIDED DISCRETE SQUARE FUNCTION, THE WEYL FRACTIONAL INTEGRAL AND OTHER ONE-SIDED OPERATORS.

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Abstract. The purpose of this paper is to prove strong type inequalities with one-sided weights for commutators (with symbol $b \in BMO$) of several one-sided operators, such as the one-sided discrete square function, the one-sided fractional operators, or one-sided maximal operators given by the convolution with a smooth function. We also prove that $b \in BMO$ is a necessary condition for the boundedness of commutators of these one-sided operators.

1. Introduction

There is a great amount of works that deal with the topic of commutators of different operators with BMO functions. If $T$ is the operator given by convolution with a kernel $K$, the commutator of $T$, with a locally integrable function $b$, called the symbol, is the operator

$$T_b f(x) = \int_{\mathbb{R}^n} (b(x) - b(y)) K(x - y) f(y) dy.$$ 

Sometimes the symbol appears inside the integral with absolute values (see Section 2 for precise definitions).

In 1976, Coifman, Rochberg and Weiss, [4], introduced the higher order commutators,

$$T_b^k f(x) = \int_{\mathbb{R}^n} (b(x) - b(y))^k K(x - y) f(y) dy, \quad k = 0, 1, 2, ...$$

Notice that $T_b^0 = T$. In [4], it is proved that if $K$ is a Calderón-Zygmund kernel then $T_b^k$ is bounded from $L^p(\mathbb{R}^n)$ to $L^p(\mathbb{R}^n)$, for $1 < p < \infty$ when $b \in BMO$. In fact, the condition $b \in BMO$ is also necessary in order to have $T_b$ bounded on $L^p(\mathbb{R}^n)$.

Later, many authors have studied strong and weak type inequalities for commutators with weights (see [3], [13], [14]). Furthermore, many of the

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results have been generalized to commutators of other operators, not only Calderón-Zygmund operators (see [16], [17], [18]).

In this paper we are interested in studying commutators of certain one-sided operators, such as the one-sided discrete square function that appears in [19], the fractional integrals of Weyl and Riemann-Liouville and a class of one-sided maximal operators given by the convolution with a smooth function. This work is highly inspired in the works of C. Segovia and J. L. Torrea [17] and [18]. The main tools for proving the weighted inequalities for these one-sided operators are extrapolation theorems proved by R. Macías and M.S. Riveros in [6]. We also prove that the condition $b \in BMO$ is necessary, i.e., even though we have one-sided operators and our weights are one-sided $A_p$ weights, the condition $b \in BMO$ can’t be weakened (is a two-sided condition).

Throughout this paper the letter $C$ will be a positive constant, not necessarily the same at each occurrence. If $1 \leq p \leq \infty$, then its conjugate exponent will be denoted by $p'$ and $A_p$ will be the classical Muckenhoupt’s class of weights (see [12]).

2. Definitions and statement of the results

**Definition 2.1.** For $f$ locally integrable, we define the one-sided discrete square function applied to $f$ by

$$S^+ f(x) = \left( \sum_{n \in \mathbb{Z}} |A_n f(x) - A_{n-1} f(x)|^2 \right)^{1/2},$$

where $A_n f(x) = \frac{1}{2^n} \int_x^{x+2^n} f(y) dy$.

It is not difficult to see that $S^+ f(x) = ||U^+ f(x)||_{l^2}$, where $U^+$ is the sequence valued operator

$$U^+ f(x) = \int_{\mathbb{R}} H(x-y) f(y) dy,$$

where

$$H(x) = \left\{ \frac{1}{2^n} \chi(-2^n,0)(x) - \frac{1}{2^{n-1}} \chi(-2^{n-1},0)(x) \right\}_{n \in \mathbb{Z}}.$$

(See [19].)

**Definition 2.2.** The one-sided Hardy-Littlewood maximal operators $M^+$ and $M^-$ are defined for locally integrable functions $f$ by

$$M^+ f(x) = \sup_{h > 0} \frac{1}{h} \int_x^{x+h} |f| \quad \text{and} \quad M^- f(x) = \sup_{h > 0} \frac{1}{h} \int_{x-h}^x |f|.$$

The good weights for these operators are the one-sided weights, $A^+_p$ and $A^-_p$:

$$\left( A^+_p \right) \sup_{a < b < c} \frac{1}{(c-a)^p} \int_a^b \omega \left( \int_b^c \omega'^{-p'} \right)^{p-1} < \infty, \quad 1 < p < \infty,$$
\( (A^+_1) \) 
\[ M^- \omega(x) \leq C \omega(x) \quad \text{a.e.} \]
and
\( (A^+_\infty) \) 
\[ A^+_\infty = \cup_{p \geq 1} A^+_p. \]

The classes \( A^+_p \) are defined in a similar way. It is interesting to note that \( A_p = A^+_p \cap A^-_p, A_p \not\subseteq A^+_p \) and \( A_p \not\subseteq A^-_p. \) (See [15], [7], [8], [9] for more definitions and results.)

It is proved in [19], that \( \omega \in A^+_p, \ 1 < p < \infty, \) if, and only if, \( S^+ \) is bounded from \( L^p(\omega) \) to \( L^p(\omega) \) and that \( \omega \in A^+_1 \), if, and only if, \( S^+ \) is of weak-type \((1,1)\) with respect to \( \omega \).

We shall also use for our purposes the following variant of the one-sided Hardy-Littlewood maximal operator:

**Definition 2.3.** Let \( \varphi \in C_c^\infty((-\infty,0], \varphi \geq 0 \) nondecreasing in \((-\infty,0]. \) For \( \varepsilon > 0 \) let \( \varphi_\varepsilon(x) = \varepsilon^{-1} \varphi(\varepsilon^{-1} x). \) The maximal operator associated to \( \varphi \) is defined by

\[ M_\varphi f(x) = \sup_{\varepsilon > 0} \varphi_\varepsilon * |f|(x). \]

It is not difficult to see that \( M_\varphi^+ f \) is pointwise equivalent to \( M^+ f. \) As a consequence, \( M_\varphi^+ \) is bounded from \( L^p(\omega) \) to \( L^p(\omega) \), for \( 1 < p < \infty \) and \( \omega \in A^+_p. \)

**Definition 2.4.** The one-sided maximal fractional operator \( M^+_\alpha, \ 0 < \alpha < 1, \) is defined, for locally integrable functions \( f, \) by

\[ M^+_\alpha f(x) = \sup_{h > 0} \frac{1}{h^{1-\alpha}} \int_x^{x+h} |f|. \]

It is proved in [2] that \( ||(M^+_\alpha f)\omega||_q \leq C ||f\omega||_p \) if and only if \( \omega \in A^+(p,q), \) for \( 1 < p < q, \ 1/p - 1/q = \alpha, \) where

\[ (A^+(p,q)) \quad \left( \frac{1}{h} \int_{x-h}^{x} \omega^q \right)^{1/q} \left( \frac{1}{h} \int_{x}^{x+h} \omega^{-p'} \right)^{1/p'} \leq C; \]

\[ (A^+(p,\infty)) \quad ||\omega \chi_{[x-h,x]}||_\infty \left( \frac{1}{h} \int_{x}^{x+h} \omega^{-p'} \right)^{1/p'} \leq C, \]

for all \( h > 0 \) and \( x \in \mathbb{R}. \)

We also have a variant of the operator \( M^+_\alpha: \)

**Definition 2.5.** Let \( 0 < \alpha < 1 \) and let \( \varphi_\alpha \in C_c^\infty((-\infty,0]), \varphi_\alpha \geq 0, \) nondecreasing in \((-\infty,0] \) and such that \( |\varphi_\alpha(x-y) - \varphi_\alpha(x)| \leq C|y||x|^{-2+\alpha}, \) for all \( x, y \) such that \( |x| > 2|y|. \) For each \( \varepsilon > 0, \) set \( \varphi_{\alpha,\varepsilon}(x) = \varepsilon^{-1+\alpha} \varphi_\alpha(\varepsilon^{-1} x). \) We define the maximal operator associated to \( \varphi_\alpha \) by

\[ M_{\varphi_\alpha}^+ f(x) = \sup_{\varepsilon > 0} \varphi_{\alpha,\varepsilon} * |f|(x). \]

It is very easy to see that \( M_{\varphi_\alpha}^+ f(x) \leq CM_{\alpha}^+ f(x). \)
**Definition 2.6.** Let $b \in L^1_{loc}(\mathbb{R})$. We say that $b \in BMO$ if
\[
||b||_{BMO} = \sup_I \frac{1}{|I|} \int_I |b - b_I| < \infty,
\]
where $I$ denotes any bounded interval and $b_I = \frac{1}{|I|} \int_I b$.

**Definition 2.7.** Let $f$ be a locally integrable function. The one-sided sharp maximal function is defined by
\[
f^{\#,+}(x) = \sup_{h>0} \frac{1}{h} \int_x^{x+h} \left( f(y) - \frac{1}{h} \int_{x+h}^{x+2h} f \right)^+ dy.
\]

It is proved in [11] that
\[
f^{\#,+}(x) \leq \sup_{h>0} \inf_{a \in \mathcal{A}} \frac{1}{h} \int_x^{x+h} (f(y) - a)^+ dy + \frac{1}{h} \int_{x+h}^{x+2h} (a - f(y))^+ dy
\]
\[
\leq C||f||_{BMO}.
\]

Now we shall state our results.

**Theorem 2.8.** Let $b \in BMO$, $H$ as in (2.2) and $k = 0, 1, \ldots$. The $k$-th order commutator of the one-sided discrete square function is defined by
\[
S_{b,\cdot}^{+,k} f(x) = \left| \int_R (b(x) - b(y))^k H(x - y) f(y) dy \right|_{\ell^2}.
\]
Then for $1 < p < \infty$ and $\omega \in A_p^+$,
\[
\int_R |S_{b,\cdot}^{+,k} f|^p \omega \leq C \int_R |f|^p \omega,
\]
for all bounded functions $f$ with compact support.

**Theorem 2.9.** Let $0 < \alpha < 1$, $b \in BMO$, and $k = 0, 1, \ldots$. The $k$-th order commutator of the Weyl fractional integral is defined by
\[
I_{\alpha,b}^{+,k} f(x) = \int_x^\infty (b(x) - b(y))^k \frac{f(y)}{(y-x)^{1-\alpha}} dy.
\]
(The Weyl fractional integral is the corresponding one for $k = 0$) Then for all $\omega \in A^+(p,q)$, $1 < p < q < \infty$, $1/p - 1/q = \alpha$, we have
\[
\left( \int_R |I_{\alpha,b}^{+,k} f|^q \omega^q \right)^{1/q} \leq C \left( \int_R |f|^p \omega^p \right)^{1/p},
\]
for all bounded $f$ with compact support.

In the following theorems we prove that for commutators of one-sided operators given by convolution with a smooth function, $b \in BMO$ is also a necessary condition in order to have the commutator bounded on $L^p(\omega)$.

Let $\varphi \in C_c^\infty(-\infty,0]$, $\varphi \geq 0$ nondecreasing in $(-\infty,0]$. Then it is easy to see that there exists $C > 0$ such that $|\varphi(x-y) - \varphi(x)| \leq C|y||x|^{-2}$, for all $x, y$ such that $|x| > 2|y|$.

For $k = 0, 1, \ldots$, the $k$-th order commutator of $M_{\varphi}^+$ with symbol $b$ is defined by
\[
M_{\varphi,b}^{+,k} f(x) = \sup_{\varepsilon > 0} \int_x^\infty |b(x) - b(y)|^k \varphi(\varepsilon(x-y)) f(y) dy.
\]
**Theorem 2.10.** Let $1 < p < \infty$, $b \in \text{BMO}$ and $\omega \in A^+_p$. Then
\[ \int_{\mathbb{R}} |M_{\varphi,b}^{+,k} f|^p \omega \leq C \int_{\mathbb{R}} |f|^p \omega. \]

If we consider $\varphi$ as before and such that $\chi_{[-1,0]} \leq \varphi$, then we have
\[ M_{\varphi,b}^{+,k} f(x) \leq M_{\varphi,b} f(x), \]
where
\[ M_{\varphi,b} f(x) = \sup_{h>0} \frac{1}{h} \int_{x}^{x+h} |b(x) - b(y)| |f(y)| \, dy. \]

Therefore, if $b \in \text{BMO}$, Theorem 2.10 gives that $M_{\varphi,b}^{+,k}$ is bounded from $L^p(\omega)$ to $L^p(\omega)$, for $1 < p < \infty$ and $\omega \in A^+_p$. In fact, more can be said, the converse is also true.

**Theorem 2.11.** The following conditions are equivalent:

(i) $M_{\varphi,b}^{+,k}$ is bounded from $L^p(\omega)$ to $L^p(\omega)$, for all $p$ with $1 < p < \infty$ and $\omega \in A^+_p$.

(ii) $M_{\varphi,b}^{+,k}$ is bounded from $L^p(dx)$ to $L^p(dx)$, for some $p > 1$.

(iii) $b \in \text{BMO}$.

Analogous results hold for $M_{\varphi,\alpha}^+$. Let $0 < \alpha < 1$. Suppose $\varphi_\alpha \in C^\infty((-\infty,0])$, $\varphi_\alpha \geq 0$, nondecreasing in $(-\infty,0]$ and such that $|\varphi_\alpha(x-y) - \varphi_\alpha(x)| \leq C \|y\| |x|^{\alpha-2}$, for all $x,y$ such that $|x| > 2|y|$.

For $k = 0,1,\ldots$, the $k$-th order commutator of $M_{\varphi,\alpha}^+$ with symbol $b$ is defined by
\[ M_{\varphi,\alpha,b}^{+,k} f(x) = \sup_{\varepsilon>0} \int_{x}^{\infty} |b(x) - b(y)|^k \varphi_{\alpha,\varepsilon}(x-y) |f(y)| \, dy. \]

Then we have the following:

**Theorem 2.12.** Let $b \in \text{BMO}$ and let $(p,q)$ be such that $1 < p < q < \infty$ and $\frac{1}{p} - \frac{1}{q} = \alpha$. Then, for all $\omega \in A^+(p,q)$, we have that
\[ \left( \int_{\mathbb{R}} |M_{\varphi,\alpha,b}^{+,k} f|^q \omega^q \right)^{1/q} \leq C \left( \int_{\mathbb{R}} |f|^p \omega^p \right)^{1/p}. \]

Since $M_{\varphi,\alpha}^+ f(x) \leq CM_{\varphi,\alpha}^+ f(x)$, the case $k = 0$ is a consequence of a result of Andersen and Sawyer [2]. And if we choose $\varphi_\alpha$ such that $\chi_{[-1,0]} \leq \varphi_\alpha$, then $M_{\varphi,\alpha,b}^{+,k} f(x) \leq M_{\varphi,\alpha}^{+,k} f(x)$, where
\[ M_{\varphi,\alpha,b}^{+,k} f(x) = \sup_{h>0} \frac{1}{h^{1-\alpha}} \int_{x}^{x+h} |b(x) - b(y)|^k |f(y)| \, dy. \]

For this operator we can also prove that $b \in \text{BMO}$ is a necessary condition.

**Theorem 2.13.** The following conditions are equivalent:

(i) $M_{\varphi,b}^{+,k}$ is bounded from $L^p(\omega^p)$ to $L^q(\omega^q)$ for all pairs $(p,q)$ such that $\frac{1}{p} - \frac{1}{q} = \alpha$, $1 < p < q < \infty$ and $\omega \in A^+(p,q)$.

(ii) $M_{\varphi,b}^{+,k}$ is bounded from $L^p(dx)$ to $L^q(dx)$ for some pair $(p,q)$ such that $\frac{1}{p} - \frac{1}{q} = \alpha$ and $1 < p < q < \infty$.

(iii) $b \in \text{BMO}$.
3. Proof of the Results

The main tools for proving our results are two extrapolation theorems that appeared in [6], with slight modifications.

**Theorem 3.1.** Let $T$ be a sublinear operator defined in $C^\infty_c(\mathbb{R})$. Assume that for all $\omega$ such that $\omega^{-1} \in A_{-1}$, there exists $C = C(\omega)$ such that
\[
|||\omega Tf|||_\infty \leq C|||f\omega|||_\infty.
\]
Then, for all $\omega \in A_1^+$, $1 < p < \infty$, there exists $C = C(\omega)$ such that
\[
\left(\int |Tf|^p \omega\right)^{1/p} \leq C \left(\int |f|^p \omega\right)^{1/p},
\]
provided that the left hand side is finite.

**Theorem 3.2.** Let $1 < p_0 < \infty$ and $T$ be a sublinear operator defined in $C^\infty_c(\mathbb{R})$. Assume that for all $\omega \in A_{p_0}^+$ there exists $C = C(\omega)$ such that
\[
|||\omega Tf|||_\infty \leq C|||f\omega|||_{p_0}.
\]
Then, for all pairs $(p, q)$ such that $1 < p < p_0$, $\frac{1}{p} - \frac{1}{q} = \frac{1}{p_0}$ and all $\omega \in A_{p, q}^+$, there exists $C = C(\omega)$ such that
\[
|||\omega Tf|||_q \leq C|||f\omega|||_p,
\]
provided that the left hand side is finite.

We will also need the following result of Martín–Reyes and de la Torre (theorem 4 in [11]):

**Theorem 3.3.** Let $1 < p < \infty$. If $\omega \in A_1^+$ and $M^+ f \in L^p(\omega)$, then there exists $C = C(\omega)$ such that
\[
\int_{\mathbb{R}} (M^+ f)^p \omega \leq C \int_{\mathbb{R}} (f^{#,+})^p \omega.
\]

An other result that will be used often is the following (see [15]):

**Theorem 3.4.** Let $\omega \in A_1^{-1}$. Then there exists $s > 1$ such that $\omega^r \in A_1^{-1}$, for all $r$ such that $1 < r \leq s$.

**Proof of Theorem 2.8.** Let $\omega \in A_1^+$. For $b \in L^\infty \subset BMO$ and $f$ bounded of compact support, we have that $S_{b}^{+, k} f \in L^p(\omega)$. Then, by Theorem 3.3,
\[
\int_{\mathbb{R}} |S_{b}^{+, k} f|^p \omega \leq C \int_{\mathbb{R}} |M^+(S_{b}^{+, k} f)|^p \omega \leq C \int_{\mathbb{R}} |(S_{b}^{+, k} f)^{#,+}|^p \omega.
\]

To prove the theorem for any $b \in BMO$ we proceed in the same way as in [5]. We will control $(S_{b}^{+, k} f)^{#,+}$ by some one-sided maximal operators. Using Theorem 3.1, we shall prove that they are bounded from $L^p(\omega)$ to $L^p(\omega)$. 
Let \( \lambda \) be an arbitrary constant. Then \( b(x) - b(y) = (b(x) - \lambda) - (b(y) - \lambda) \) and

\[
S_{b}^{+,k} f(x) = \left\| \int_{\mathbb{R}} (b(x) - b(y))^k H(x - y)f(y)dy \right\|_{\ell^2} \\
= \left\| \sum_{j=0}^{k} C_{j,k} (b(x) - \lambda)^j \int_{\mathbb{R}} (b(y) - \lambda)^{k-j} H(x - y)f(y)dy \right\|_{\ell^2} \\
\leq \left\| \int_{\mathbb{R}} (b(y) - \lambda)^k H(x - y)f(y)dy \right\|_{\ell^2} \\
+ \left\| \sum_{j=1}^{k} C_{j,k} (b(x) - \lambda)^j \int_{\mathbb{R}} (b(y) - \lambda)^{k-j} H(x - y)f(y)dy \right\|_{\ell^2} \\
= S^+((b - \lambda)^k f)(x) \\
+ \sum_{j=1}^{k} C_{j,k} (b(x) - \lambda)^j \int_{\mathbb{R}} (b(y) - \lambda)^{k-j} H(x - y)f(y)dy \\
\leq S^+((b - \lambda)^k f)(x) + \sum_{m=0}^{k-1} C_{k,m} |b(x) - \lambda|^{k-m} S_{b}^{+,m} f(x),
\]

where \( C_{j,k} \) (respectively \( C_{j,k,s} \)) are absolute constants depending only on \( j \) and \( k \) (respectively \( j, k \) and \( s \)). Let \( x \in \mathbb{R}, h > 0 \). Let \( i \in \mathbb{Z} \) be such that \( 2^i \leq h < 2^{i+1} \) and set \( J = [x, x + 2^{i+3}] \). Then, write \( f = f_1 + f_2 \), where \( f_1 = f \chi_J \) and set \( \lambda = b_j \). Then

\[
\frac{1}{h} \int_{x}^{x+2h} |S_{b}^{+,k} f(y) - S^+((b - b_j)^k f_2)(x)|dy \\
\leq \frac{1}{h} \int_{x}^{x+2h} |S^+((b - b_j)^k f_1)(y)|dy \\
+ \frac{1}{h} \int_{x}^{x+2h} |S^+((b - b_j)^k f_2)(y) - S^+((b - b_j)^k f_2)(x)|dy \\
+ \sum_{m=0}^{k-1} C_{k,m} \frac{1}{h} \int_{x}^{x+2h} |b(y) - b_j|^{k-m} |S_{b}^{+,m} f(y)|dy \\
= I(x) + II(x) + III(x).
\]

Let \( U^+ \) be as in (2.1). Then

\[
II(x) \leq \frac{1}{h} \int_{x}^{x+2^{i+3}} ||U^+((b - b_j)^k f_2)(y) - U^+((b - b_j)^k f_2)(x)||_{\ell^2} dy,
\]

and

\[
||U^+((b - b_j)^k f_2)(y) - U^+((b - b_j)^k f_2)(x)||_{\ell^2} \\
\leq \int_{x+2^{i+3}}^{\infty} |b(t) - b_j|^k |f(t)||H(y - t) - H(x - t)||_{\ell^2} dt.
\]
Consider the following sublinear operators defined in $C^\infty_c$:

$$M_1^+ f(x) = \sup_{i \in \mathbb{Z}} \frac{1}{2^i} \int_{x}^{x+2^{i+2}} |S^+((b - b_J)^k f \chi_J)(y)|dy;$$

$$M_2^+ f(x) = \sup_{i \in \mathbb{Z}} \frac{1}{2^i} \int_{x}^{x+2^{i+3}} \int_{x+2^{i+3}}^{\infty} |b(t) - b_{J_J}^i|^k |f(t)||H(y-t) - H(x-t)||_{L^2} dt dy;$$

and

$$M_{3,m}^+ f(x) = \sup_{h>0} \frac{1}{h} \int_{x}^{x+2h} |b(y) - b_{[x,x+8h]}(y)|^{k-m} |f(y)|dy, \quad 0 \leq m \leq k-1,$$

where, for each $i$, $J$ denotes the interval $[x,x+2^{i+3}]$.

Inequalities (3.3), (3.4), (3.5) and the above definitions give that

$$(S_b^{+,k} f)_{#,+}(x) \leq C \left( M_1^+ f(x) + M_2^+ f(x) + \sum_{m=0}^{k-1} M_{3,m}^+ (S_b^{+,m} f)(x) \right).$$

We shall prove, using Theorem 3.1, that these operators are bounded from $L^p(\omega)$ to $L^p(\omega)$, $\omega \in A^+_p$, $1 < p < \infty$.

Boundedness of $M_1^+$: Let $\omega$ be a weight such that $\omega^{-1} \in A^-_1$. Let $1 < q < \infty$ and $1 < s < \infty$ be such that $\omega^{-qs} \in A^-_1$ (Theorem 3.4). By Hölder’s and John-Nirenberg’s inequalities and the fact that $S^+$ is bounded from $L^q(dx)$ to $L^q(dx)$ we get

$$(3.6) \quad \frac{1}{2^i} \int_{x}^{x+2^{i+2}} |S^+((b - b_J)^k f \chi_J)(y)|dy \leq C \left( \frac{1}{2^i} \int_{x}^{x+2^{i+2}} |S^+((b - b_J)^k f \chi_J)(y)|^q dy \right)^{1/q}$$

$$\leq C \left( \frac{1}{2^i} \int_{x}^{x+2^{i+3}} |(b(y) - b_J)^k f(y)|^q dy \right)^{1/q}$$

$$\leq C ||f\omega||_{L^q} \left( \frac{1}{2^i} \int_{x}^{x+2^{i+3}} |(b(y) - b_J)^{k-1}(y)|^q dy \right)^{1/q}$$

$$\leq C ||f\omega||_{L^q} \left( \frac{1}{2^i} \int_{x}^{x+2^{i+3}} |b - b_J|^{kqs'} \right)^{1/qs'} \left( \frac{1}{2^i} \int_{x}^{x+2^{i+3}} \omega^{-qs} \right)^{1/sq}$$

$$\leq C ||f\omega||_{L^q} ||b||_{BMO} \omega^{-1}(x).$$

Then, for all $\omega$ such that $\omega^{-1} \in A^-_1$,

$$||\omega M_1^+ f||_{L^p} \leq C ||b||_{BMO} ||f\omega||_{L^p},$$

and by Theorem 3.1, we obtain that for all $\omega \in A^+_p$, $1 < p < \infty$,

$$||M_1^+ f||_{L^p,\omega} \leq C ||b||_{BMO} ||f||_{L^p,\omega,p}.$$
Boundedness of $M_2^+$: Let $\omega$ be such that $\omega^{-1} \in A_1^−$. Then, if $I_j = [x, x + 2^{j+1}]$, we have that

\begin{equation}
(3.7)
\int_{x+2^{j+3}}^{\infty} |b(t) - b_{I_j}|^k |f(t)||H(t) - H(x-t)||_{L^2} dt \\
\leq C \sum_{j = i+3}^\infty \int_{x+2^{j}}^{x+2^{j+1}} |b(t) - b_{I_j}|^k |f(t)||H(t) - H(x-t)||_{L^2} dt \\
+ C \sum_{j = i+3}^\infty |b_{I_j} - b_{I_j}|^k \int_{x+2^{j}}^{x+2^{j+1}} |f(t)||H(t) - H(x-t)||_{L^2} dt \\
= IV(x) + V(x).
\end{equation}

We choose $(s, s')$ and $(t, t')$ such that $\omega^{-s} \in A_1^–$ and $\omega^{-st'} \in A_1^–$. Then, by Hölder’s inequality with exponents $(s, s')$ and $(t, t')$,

\begin{equation}
IV(x) \leq C \sum_{j = i+3}^\infty \left( \int_{I_j} |b - b_{I_j}|^{k\omega^{-s} \omega^s |f|^s} \right)^{1/s} \\
\times \left( \int_{x+2^{j}}^{x+2^{j+1}} |H(t) - H(x-t)||_{L^2} dt \right)^{1/s'}
\end{equation}

\begin{equation}
(3.8)
\leq C ||f\omega||_\infty \sum_{j = i+3}^\infty \left( \int_{I_j} |b - b_{I_j}|^{kst} \right)^{1/st} \left( \int_{I_j} \omega^{-st'} \right)^{1/st'} \\
\times \left( \int_{x+2^{j}}^{x+2^{j+1}} |H(t) - H(x-t)||_{L^2} dt \right)^{1/s'}.
\end{equation}

It is proved in theorem 1.6 of [19] that for all $y \in [x, x + 2^{i+3}]$ the kernel $H$ satisfies

\begin{equation}
(3.9)
\left( \int_{x+2^{j}}^{x+2^{j+1}} |H(t) - H(x-t)||_{L^2} dt \right)^{1/s'} \leq C \frac{2^{j/s'}}{2^{j}}.
\end{equation}

Then, using that $b \in BMO$, the fact that $\omega^{-st'} \in A_1^–$ and (3.9), we get

\begin{equation}
IV(x) \leq C ||f\omega||_\infty \sum_{j = i+3}^\infty ||b||_{BMO}^k (2^{j})^{1/st} \omega^{-1}(x)(2^{j+1})^{1/st'} \frac{2^{j/s'}}{2^{j}}
\end{equation}

\begin{equation}
(3.10)
\leq C \omega^{-1}(x)||f\omega||_\infty ||b||_{BMO}^k \sum_{j = i+3}^\infty \left( \frac{2^{j}}{2^{j}} \right)^{1/s'}
\end{equation}

\begin{equation}
\leq C \omega^{-1}(x)||f\omega||_\infty ||b||_{BMO}^k.
\end{equation}

Using again Hölder’s inequality, (3.9), the fact that $w^{-s} \in A_1^–$ and lemma
1 in [5], we get

\[ V(x) = C \sum_{j=i+3}^{\infty} |b_{I_j} - b_j|^k \int_{x+2^j}^{x+2^{j+1}} |f(t)|||H(y-t) - H(x-t)||_{L^2} dt \]

\[ \leq C \sum_{j=i+3}^{\infty} ||b||_{BMO}^k (2(j-i))^k ||f\omega||_\infty \left( \int_{I_j} \omega^{-s} \right)^{1/s} \]

\[ \times \left( \int_{x+2^j}^{x+2^{j+1}} ||H(y-t) - H(x-t)||_{L^2}^{s'} dt \right)^{1/s'} \]

\[ \leq C ||b||_{BMO}^k ||f\omega||_\infty \omega^{-1}(x) \sum_{j=i+3}^{\infty} (2(j-i))^k (2^{j+1})^{1/s} \frac{2i/s'}{2^j} \]

\[ \leq C \omega^{-1}(x) ||b||_{BMO}^k ||f\omega||_\infty \sum_{j=i+3}^{\infty} \frac{(j-i)^{k2i/s'}}{2^{j/s'}} \]

\[ \leq C \omega^{-1}(x) ||b||_{BMO}^k ||f\omega||_\infty. \]

Then, by (3.7), (3.10) and (3.11), we get that, for all \( \omega \) such that \( \omega^{-1} \in A_1^- \),

\[ ||\omega M^+_2 f||_\infty \leq C ||f\omega||_\infty. \]

Then, by Theorem 3.1, for all \( 1 < p < \infty \) and \( \omega \in A_p^+ \),

\[ ||M^+_2 f||_{p,\omega} \leq C ||f||_{p,\omega}. \]

Boundedness of \( M^+_{3,m} \): Let \( \omega \) be such that \( \omega^{-1} \in A_1^- \) and let \( q > 1 \) be such that \( \omega^{-q} \in A_1^- \). Then, using again Hölder’s and John-Nirenberg’s inequalities we obtain

\[ \frac{1}{h} \int_{x}^{x+2h} |b(y) - b_{[x,x+8h]}|^{k-m} |f(y)| dy \]

\[ \leq \left( \frac{1}{h} \int_{x}^{x+2h} |b(y) - b_{[x,x+8h]}|^{(k-m)q'} dy \right)^{1/q'} \left( \frac{1}{h} \int_{x}^{x+2h} |f(y)|^q dy \right)^{1/q} \]

\[ \leq C ||b||_{BMO}^{k-m} \left( \frac{1}{h} \int_{x}^{x+2h} |f(y)|^q dy \right)^{1/q} \]

\[ \leq C ||b||_{BMO}^{k-m} ||f\omega||_\infty \left( \frac{1}{h} \int_{x}^{x+2h} \omega^{-q} dy \right)^{1/q} \]

\[ \leq C ||b||_{BMO}^{k-m} ||f\omega||_\infty \omega^{-1}(x). \]

Then, for all \( \omega \) such that \( \omega^{-1} \in A_1^- \),

\[ ||\omega M^+_{3,m} f||_\infty \leq C ||b||_{BMO}^{k-m} ||f\omega||_\infty. \]
Therefore, by Theorem 3.1, we have that, for all \( \omega \in A^+_p \), \( 1 < p < \infty \),

\[
||M^+_{3,m}f||_{\omega,p} \leq C ||b||^{k-m}_{BMO} ||f||_{\omega,p}.
\]

Using now the induction principle (the case \( k = 0 \) was proved in [19]), we obtain that, for all \( \omega \in A^+_p \), \( 1 < p < \infty \),

\[
||M^+_{3,m}(S^+_bf)||_{\omega,p} \leq C ||b||^{k-m}_{BMO} ||S^+_bf||_{\omega,p} \leq C ||b||^{k}_{BMO} ||f||_{\omega,b}.
\]

**Proof of Theorem 2.9.** This proof follows the same pattern as the preceding one. Let \( b \in BMO \) bounded, and \( \lambda \in \mathbb{R} \), then, as in (3.2), we can write

\[
I^{k+}_m f(x) = I^+_\alpha((b - \lambda)^k f)(x) + \sum_{m=0}^{k-1} C_{k,m}(b(x) - \lambda)^{k-m} I^{+,m}_\alpha f(x).
\]

Let \( x \in \mathbb{R} \), \( h > 0 \) and \( J = [x, x + 4h] \). Write \( f = f_1 + f_2 \), where \( f_1 = f \chi_J \) and set \( \lambda = b_J \). Then,

\[
(3.12)
\]

\[
\frac{1}{h} \int_x^{x+2h} |I^{+,k}_\alpha f(y) - I^{+,k}_\alpha((b - b_J)^k f_2)(x + 2h)| dy
\]

\[
\leq \frac{1}{h} \int_x^{x+2h} |I^{+,k}_\alpha((b - b_J)^k f_1)(y)| dy
\]

\[
+ \frac{1}{h} \int_x^{x+2h} |I^{+,k}_\alpha((b - b_J)^k f_2)(y) - I^{+,k}_\alpha((b - b_J)^k f_2)(x + 2h)| dy
\]

\[
+ \sum_{m=0}^{k-1} C_{k,m} \frac{1}{h} \int_x^{x+2h} |b(y) - b_J|^{k-m} |I^{+,m}_\alpha f(y)| dy
\]

\[
= I(x) + II(x) + III(x).
\]

It is clear that

\[
III(x) \leq \sum_{m=0}^{k-1} C_{k,m} M^+_{3,m}(I^{+,m}_\alpha f)(x),
\]

where \( M^+_{3,m} \) is as in the proof of Theorem 2.8. Then, we already know that \( M^+_{3,m} \) is bounded from \( L^p(\omega) \) to \( L^p(\omega) \), provided \( \omega \in A^+_p \) and \( 1 < p < \infty \). So, if \( \omega \in A^+(p,q) \), \( 1/p - 1/q = \alpha \), then \( \omega^q \in A^+_{q'} \), and by induction (see \( k=0 \) in [10]), we obtain

\[
||M^+_{3,m}(I^{+,m}_\alpha f)||_{\omega^q,p} \leq C ||b||^{k-m}_{BMO} ||I^{+,m}_\alpha f||_{\omega^q,q} \leq C ||b||^k_{BMO} ||f||_{\omega^p,p},
\]

for all \( f \in C^\infty_c(\mathbb{R}) \).

To control \( I(x) \) let us define

\[
M^+_1 f(x) = \sup_{h>0} \frac{1}{h} \int_x^{x+2h} |I^+_\alpha((b - b_J)^k f \chi_J)(y)| dy,
\]

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where, for each $h > 0$, $J = [x, x + 4h]$. It is not difficult to see that $M^+_4$ is a sublinear operator in $C_c^\infty$ and it is clear that $I(x) \leq M^+_4 f(x)$. Let us prove that $M^+_4$ is bounded from $L^p(\omega^\mu)$ to $L^q(\omega^\nu)$ using Theorem 3.2.

Let $\omega \in A^+\left(\frac{1}{\alpha}, \infty\right)$, then $\omega^{1/h} \in A_1^\ast$. Therefore, there exist $t_1 > 1$ and $t_2 > 1$ such that $\omega^{-1/h} \in A_1^\ast$ and $\omega^{-1/t_2} \in A_1^-$. Let $s > 1$ and $r > 1$ be such that $s = t_1/(1 - \alpha)$ and $1/r - 1/s = \alpha$. Then, using Hölder’s inequality and the fact that $I^+_\alpha$ is bounded from $L^r(\mathbb{R})$ to $L^s(\mathbb{R})$, we get

\begin{equation}
\frac{1}{h} \int_x^{x+2h} |I^+_{\alpha}(b - b_J)^k f \chi_J(y)| \, dy \leq \left( \frac{1}{h} \int_x^{x+2h} |I^+_{\alpha}(b - b_J)^k f \chi_J(y)|^s \, dy \right)^{1/s} \leq C h^\alpha \left( \frac{1}{h} \int_x^{x+4h} |(b(y) - b_J)^k f(y)|^{r \omega^r} \omega^{-r} \, dy \right)^{1/r} \leq C h^\alpha \left( \frac{1}{h} \int_x^{x+4h} |(b - b_J)^k^{r \omega^{-r}} \omega^{-r} \right)^{1/r} \left( \frac{1}{h} \int_x^{x+4h} |f|^\frac{1}{s} \omega^\frac{1}{s} \right)^{\alpha}.
\end{equation}

(3.13)

Therefore, using Hölder’s and John-Nirenberg’s inequalities and the fact that $\omega^{-st_2} \in A_1^\ast$, the chain of inequalities in (3.13) can be continued as follows:

\begin{equation}
\begin{aligned}
&\leq C ||f\omega||^\frac{1}{\alpha} \left( \frac{1}{h} \int_x^{x+4h} |b - b_J|^{k \omega^{-s}} \right)^{\frac{1}{\alpha}} \\
&\leq C ||f\omega||^\frac{1}{\alpha} \left( \frac{1}{h} \int_x^{x+4h} |b - b_J|^{k s t_2} \right)^{\frac{1}{\alpha t_2}} \left( \frac{1}{h} \int_x^{x+4h} \omega^{-s t_2} \right)^{\frac{1}{\alpha t_2}} \\
&\leq C ||f\omega||^\frac{1}{\alpha} ||b||^k_{BMO} \omega^{-1}(x).
\end{aligned}
\end{equation}

(3.14)

As a consequence,

$$||\omega M^+_4 f||_{\infty} \leq C ||b||^k_{BMO} ||f\omega||^\frac{1}{\alpha}.$$ 

Then, by Theorem 3.2, for all $\omega \in A^+(p, q), \frac{1}{p} - \frac{1}{q} = \alpha$,

$$||M^+_4 f||_{\omega^{p}, q} \leq C ||b||^k_{BMO} ||f||_{\omega^{p}, p}.$$ 

Finally, we shall estimate $II(x)$. We have that

$$II(x) = \frac{1}{h} \int_x^{x+2h} \left| \int_{x+4h}^{\infty} \sigma(t,y) dt \right| dy,$$

where

$$\sigma(t,y) = (b(t) - b_J)^k f(t) \left( \frac{1}{(y-t)^{1-\alpha}} - \frac{1}{(x+2h-t)^{1-\alpha}} \right).$$
Consider the following sublinear operator in $C^\infty_c(\mathbb{R})$:

$$M^+_h f(x) = \sup_{h > 0} \frac{1}{h} \int_x^{x+2h} \left| \int_x^\infty \sigma(t,y)dy \right| dt.$$

For each $j \in \mathbb{N}$, set $I_j = [x, x + 2^{j+1}h]$. Then,

$$\frac{1}{h} \int_x^{x+2h} \left| \int_x^\infty \sigma(t,y)dy \right| dt \leq \frac{1}{h} \int_x^{x+2h} \sum_{j=2}^{\infty} \int_{x+2^j h}^{x+2^{j+1}h} |\sigma(t,y)|dtdy \leq C \sum_{j=2}^{\infty} \int_{x+2^j h}^{x+2^{j+1}h} |b(t) - b_{I_j}|^k |f(t)| \frac{2h}{(h(2^j - 2) - 2)^{\alpha/2}} dt$$

(3.15)

$$\leq C \sum_{j=2}^{\infty} \frac{h^\alpha}{(2^{j-1})^{1-\alpha}} \left( \frac{2}{2^{j-1}} \int_{x+2^j h}^{x+2^{j+1}h} |b(t) - b_{I_j}| |f(t)| dt \right)$$

$$\leq C \sum_{j=2}^{\infty} \frac{h^\alpha}{(2^{j-1})^{1-\alpha}} \left( \int_{x+2^j h}^{x+2^{j+1}h} |b(t) - b_{I_j}|^k |f(t)| dt \right) + C \sum_{j=2}^{\infty} \frac{h^\alpha}{(2^{j-1})^{1-\alpha}} \left( \int_{x+2^j h}^{x+2^{j+1}h} |b_{I_j} - b_{I_j}|^k |f(t)| dt \right)$$

$$\leq C \sum_{j=2}^{\infty} \frac{h^\alpha}{(2^{j-1})^{1-\alpha}} (IV(x) + V(x)).$$

Let $\omega \in A^+(\frac{1}{\alpha}, \infty)$. Then $\omega^{\frac{1}{\alpha}} \in A_1^-$. Choose $r > 1$ such that $\omega^{\frac{1}{r-\alpha}} \in A_1^-$. Then, by Hölder’s and John-Nirenberg’s inequalities,

$$IV(x) \leq \left( \frac{2}{2^{j-1}} \int_{x+2^j h}^{x+2^{j+1}h} |f|^{\frac{1}{\alpha}} \omega^{\frac{1}{\alpha}} \right)^{\alpha} \times \left( \frac{2}{2^{j-1}} \int_{x+2^j h}^{x+2^{j+1}h} |b(t) - b_{I_j}|^{\frac{k}{1-\alpha}} \omega^{\frac{1}{1-\alpha}} \right)^{1-\alpha}$$

(3.16)

$$\leq C(2^j h)^{-\alpha} ||f\omega||^{\frac{1}{\alpha}} \left( \frac{1}{2^j h} \int_{x+2^j h}^{x+2^{j+1}h} |b(t) - b_{I_j}|^{\frac{k}{1-\alpha}} \right)^{1-\alpha} \times \left( \frac{1}{2^j h} \int_{x+2^j h}^{x+2^{j+1}h} \omega^{\frac{1}{1-\alpha}} \right)^{\frac{1-\alpha}{r}}$$

$$\leq C ||b||_{BMO}^{k}(2^j h)^{-\alpha} ||f\omega||^{\frac{1}{\alpha}} \omega^{1}(x).$$
Using again lemma 1 in [5] and Hölder’s inequality,

\[ V(x) \leq \frac{1}{2^j h} |b_{I_j} - b_J|^k \int_x^{x+2^{j+1}h} |f(t)| dt \]
\[ \leq C(2j)^k ||b||_{BMO}^k \left( \frac{1}{2^j h} \int_x^{x+2^{j+1}h} |f| \theta^\alpha \right)^\alpha \]
\[ \times \left( \frac{1}{2^j h} \int_x^{x+2^{j+1}h} \theta^{\frac{1}{1-\alpha}} \right)^{1-\alpha} \]
\[ \leq C(2j)^k (2^j h)^{-\alpha} ||b||_{BMO}^k ||f\omega||_\frac{1}{\alpha} \omega^{-1}(x). \]

Putting together inequalities (3.15), (3.16) and (3.17), we get that

\[ (3.17) \]

Taking supremums first on \( h > 0 \) and then on \( x \in \mathbb{R} \), we get

\[ ||\omega M_\sigma^+ f||_\infty \leq C ||b||_{BMO}^k ||f\omega||_\frac{1}{\alpha}. \]

So, by Theorem 3.2, for all \( \omega \in A^+(p,q), \frac{1}{p} - \frac{1}{q} = \alpha, \)

\[ ||M_\sigma^+ f||_{\omega^q,p} \leq C ||b||_{BMO}^k ||f||_{\omega^p,p}. \]

\[ \square \]

**Proof of Theorem 2.10.**

This proof also follows the same pattern as the preceding ones. Let \( b \in BMO \) bounded and let \( \lambda \in \mathbb{R} \). Then, as in (3.2), we have

\[ M_{\varphi,b}^{+,k} f(x) \leq M_\varphi^+ ((b - \lambda)^k f)(x) + \sum_{m=0}^{k-1} C_{k,m} |b(x) - \lambda|^{k-m} M_{\varphi,b}^{+,m} f(x). \]

Let us fix \( x \in \mathbb{R} \) and \( h > 0 \) and let \( J = [x,x+8h] \). Write \( f = f_1 + f_2 \),

where \( f_1 = f \chi_J \), and also write \( \lambda = b_J \). Then, as in (3.3) and (3.12), it follows that

\[ \frac{1}{h} \int_x^{x+2h} |M_{\varphi,b}^{+,k} f(y) - M_\varphi^+ ((b - b_J)^k f_2)(x+2h) | dy \]
\[ \leq \frac{1}{h} \int_x^{x+2h} |M_\varphi^+ ((b - b_J)^k f_1)(y) | dy \]
\[ + \frac{1}{h} \int_x^{x+2h} |M_\varphi^+ ((b - b_J)^k f_2)(y) - M_\varphi^+ ((b - b_J)^k f_2)(x+2h) | dy \]
\[ + \sum_{m=0}^{k-1} C_{k,m} \frac{1}{h} \int_x^{x+2h} |b(y) - b_J|^{k-m} |M_{\varphi,b}^{+,m} f(y) | dy \]
\[ = I(x) + II(x) + III(x). \]
It is clear that $III(x) \leq \sum_{m=0}^{k-1} C_{k,m} M_{3,m}^+(M_{\varphi,b}^+ f)(x)$, being $M_{3,m}^+$ the same as in the proofs of Theorems 2.8 and 2.9, and

$$ (3.19) \quad \text{II}(x) \leq C \frac{1}{h} \int_x^{x+2h} \int_{x+8h}^{\infty} \frac{x + 2h - y}{(z - (x + 2h))^2} |b(z) - b_J|^k |f(z)| dz dy,$$

by the conditions imposed on the kernel $\varphi$.

Consider the following sublinear operators defined on $C_{c}^\infty$:

$$ M_0^+ f(x) = \sup_{h > 0} \frac{1}{h} \int_x^{x+2h} |M_\varphi^+ ((b - b_J)^k f \chi_J)(y)| dy $$

and

$$ M_7^+ f(x) = \sup_{h > 0} \frac{1}{h} \int_x^{x+2h} \int_{x+8h}^{\infty} \frac{x + 2h - y}{(z - (x + 2h))^2} |b(z) - b_J|^k |f(z)| dz dy,$$

where, for each $h > 0$, $J$ is the interval $[x, x + 8h]$.

The above inequalities and definitions give that

$$ (M_{\varphi,b}^+ f)^{(k)}(x) \leq C \left( M_6^+ f(x) + M_7^+ f(x) + \sum_{m=0}^{k-1} M_{3,m}^+(M_{\varphi,b}^{+m} f)(x) \right). $$

Let $\omega \in A_p^+$, $1 < p < \infty$. Then, acting as in the boundedness of $M_1^+$, in the proof of Theorem 2.8, we get that $M_6^+$ is bounded from $L^p(\omega)$ to $L^p(\omega)$. On the other hand, we already know that $M_{3,m}^+$ is bounded from $L^p(\omega)$ to $L^p(\omega)$ and, to proceed with the induction, we observe that the case $k = 0$ is a consequence of the fact that $M_7^+ f$ is pointwise equivalent to $M^+ f$. Therefore, we only have to prove that $M_7^+$ is bounded from $L^p(\omega)$ to $L^p(\omega)$, for $\omega \in A_p^+$, $1 < p < \infty$.

Let us use again Theorem 3.1. Let $\omega$ be such that $\omega^{-1} \in A_1^-$. For each $j \in \mathbb{N}$, let $I_j = [x, x + 2^j h]$. Then

$$ \frac{1}{h} \int_x^{x+2h} \int_{x+8h}^{\infty} \frac{x + 2h - y}{(z - (x + 2h))^2} |b(z) - b_J|^k |f(z)| dz dy $$

$$ \leq C \frac{1}{h} \int_x^{x+2h} h \sum_{j=3}^{\infty} \int_{x+2^j h}^{x+2^{j+1} h} \frac{|b(z) - b_J|^k}{(z - (x + 2h))^2} |f(z)| dz dy $$

$$ \leq C \int_{I_j} |f\omega| \sup_{j=3}^{\infty} \frac{2^{j+1}}{(2^j - 2)^2 h^2} \int_{I_j} |b(z) - b_J|^k \omega^{-1}(z) dz $$

$$ \leq C \int_{I_j} |f\omega| \sup_{j=3}^{\infty} \frac{2^{j+1}}{(2^j - 2)^2} \int_{I_j} |b_J|^k \omega^{-1}(z) dz $$

$$ + \frac{1}{2^{j+1} h} \int_{I_j} |b_J - b_J|^k \omega^{-1}(z) dz $$

(3.20) $$ \leq C \int_{I_j} |f\omega| \sum_{j=3}^{\infty} \frac{2^{j+1}}{(2^j - 2)^2} (IV(x) + V(x)). $$
Let \( q > 1 \) be such that \( \omega^{-q} \in A_1^- \) (Theorem 3.4). Then, Hölder’s and John-Nirenberg’s inequalities give

\[
IV(x) \leq \left( \frac{1}{2^{j+1}h} \int_x^{x+2^{j+1}h} |b - b_{I_j}|^{kq'} \left( \frac{1}{2^{j+1}h} \int_x^{x+2^{j+1}h} \omega^{-q} \right)^{1/q} \right)^{1/q'} \leq C ||b||_{BMO}^{k} \omega^{-1}(x).
\]

In [5], it is proved that \( |b_{I_j} - b_{I_j}|^k \leq C|2^j|^k||b||_{BMO}^k \) for all \( 3 \leq j \). Then, using this fact and using that \( \omega^{-1} \in A_1^- \), we obtain

\[
(3.22) \quad V(x) = \frac{1}{2^{j+1}h} \int_{I_j} |b_{I_j} - b_{I_j}|^k \omega^{-1} \leq C|2^j|^k||b||_{BMO}^k \omega^{-1}(x).
\]

Then by (3.20), (3.21) and (3.22) we get that, for all \( \omega \) such that \( \omega^{-1} \in A_1^- \),

\[
||\omega M_\tau^+ f||_\infty \leq C||b||_{BMO}^k||f||_\infty,
\]

and, by Theorem 3.1, for all \( \omega \in A_1^+, 1 < p < \infty \),

\[
||M_\tau^+ f||_{\omega,p} \leq C||b||_{BMO}^k||f||_{\omega,p}.
\]

Now, we only have to take into account (3.19) and Theorem 3.3 to conclude the proof of Theorem 2.10.

**Remark 3.5.** Following the same steps as in the proof of Theorem 2.10 we get Corollary 1 of [5], i.e., the boundedness of the commutator of one-sided singular integrals (introduced in [1]), from \( L^p(\omega) \) to \( L^p(\omega), \omega \in A_1^+, 1 < p < \infty \).

**Proof of Theorem 2.11.**

(iii) ⇒ (i) It is a consequence of Theorem 2.10.

(i) ⇒ (ii) Let \( p > 1 \) and set \( \omega \equiv 1 \).

(ii) ⇒ (iii) Set \( I = (a, b), I^+ = (b, c) \), and \( |I| = |I^+| \). Then

\[
\frac{1}{|I|} \int_I |b(y) - b_{I^+}| dy \leq \left( \frac{1}{|I|} \int_I |b(y) - b_{I^+}|^k dy \right)^{1/k} = \left( \frac{1}{|I|} \int_I \left( \frac{1}{|I^+|} \int_{I^+} (b(y) - b(x)) dx \right)^k \right)^{1/k} \leq \left( \frac{1}{|I|} \int_I \left( \frac{1}{|I^+|} \int_{I^+} |b(y) - b(x)|^k dx \right) dy \right)^{1/k}.
\]

Observe that, for \( y \in I \),

\[
\frac{1}{|I^+|} \int_{I^+} |b(x) - b(y)|^k dx = \frac{1}{|I^+|} \int_{I^+} |b(x) - b(y)|^k \chi_{I^+}(x) dx \leq CM_b^{+,k} \chi_{I^+}(y).
\]
Then, by Hölder’s inequality and (ii),

$$\frac{1}{|I|} \int_I |b(y) - b_{I^+}|dy \leq C \left( \frac{1}{|I|} \int_I M_{b,x}^{+,k} \chi_{I^+} \, dy \right)^{1/k} \leq C \left( \frac{1}{|I|} \int_I |M_{b,x}^{+,k} \chi_{I^+}| \, dy \right)^{1/pk} \leq C \left( \frac{|I^+|}{|I|} \right)^{1/pk} = C.$$

So $b \in BMO$. \hfill \square

Proof of Theorem 2.12. Observe that by the conditions given on $\varphi_\alpha$, $M_{\varphi_\alpha,b}^{+,k}$ can be treated in the same way as $I_{\alpha,b}^{+,k}$, the commutator of the one-sided fractional operator. Observe that the case $k = 0$ is a consequence of the fact that $M_{\varphi_\alpha} f(x) \leq CM_{\alpha} f(x)$ and the result in [2]. \hfill \square

Proof of Theorem 2.13.

(iii) ⇒ (i) It is a consequence of Theorem 2.12.

(i) ⇒ (ii) Given an appropriate pair $(p, q)$, set $\omega \equiv 1$.

(ii) ⇒ (iii) Set $I = (a, b)$, $I^+ = (b, c)$, and $|I| = |I^+|$. Then

$$\frac{1}{|I|} \int_I |b - b_{I^+}|dy \leq \left( \frac{1}{|I|} \int_I \left( \frac{1}{|I^+|} \int_{I^+} |b(x) - b(y)|^k \, dx \right)dy \right)^{1/k} \leq \left| \frac{|I|^{-\alpha}}{|I|} \int_I \left( \frac{1}{|I^+|^{1-\alpha}} \int_{I^+} |b(x) - b(y)|^k \, dx \right)dy \right|^{1/k}.$$

Observe that, for $y \in I$,

$$\frac{1}{|I^+|^{1-\alpha}} \int_{I^+} |b(x) - b(y)|^k \, dx = \frac{1}{|I^+|^{1-\alpha}} \int_y^c |b(x) - b(y)|^k \chi_{I^+} \, dx \leq CM_{\alpha,b}^{+,k} \chi_{I^+}(y).$$

Then, by Hölder’s inequality and (ii),

$$\frac{1}{|I|} \int_I |b - b_{I^+}|dy \leq C \left( \frac{|I|^{-\alpha}}{|I|} \int_I M_{\alpha,b}^{+,k} \chi_{I^+} \, dy \right)^{1/p} \leq C \left( \frac{1}{|I|} \int_I \left( \frac{1}{|I^+|} \int_{I^+} |M_{\alpha,b}^{+,k} \chi_{I^+}| \, dy \right)^{1/p} \right)^{1/p} \leq C \left( \frac{1}{|I|} \int_I \left( \int_{\mathbb{R}} |\chi_{I^+}| \, dy \right)^{1/p} \right)^{1/p} \leq C \left( |I|^{-\alpha - \frac{1}{q} + \frac{1}{p}} \right)^{1/k} = C.$$

So $b \in BMO$. \hfill \square

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References


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