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WEIGHTED INEQUALITIES FOR INTEGRAL OPERATORS WITH SOME HOMOGENEOUS KERNELS

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Abstract. In this paper we study integral operators of the form

$$Tf(x) = \int |x - a_1 y|^{-\alpha_1} \dots |x - a_m y|^{-\alpha_m} f(y)\,\mathrm{d}y,$$

$\alpha_1 + \dots + \alpha_m = n$. We obtain the $L^p(w)$ boundedness for them, and a weighted $(1,1)$ inequality for weights w in A_p satisfying that there exists $c \geqslant 1$ such that $w(a_i x) \leqslant c w(x)$ for a.e. $x \in \mathbb{R}^n$, $1 \leqslant i \leqslant m$. Moreover, we prove $\|Tf\|_{\mathrm{BMO}} \leqslant c\|f\|_\infty$ for a wide family of functions $f \in L^\infty(\mathbb{R}^n)$.

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1. Introduction

In [7] the authors study the boundedness on $L^2(\mathbb{R})$ of the operator

$$Tf(x) = \int |x - y|^{-\alpha} |x + y|^{\alpha - 1} f(y)\,\mathrm{d}y,$$

$0 < \alpha < 1$.

In [3] the authors study integral operators of the form

$$Tf(x) = \int_{\mathbb{R}^n} |x - y|^{-\alpha} |x + y|^{-n+\alpha} f(y)\,\mathrm{d}y,$$

$0 < \alpha < n$. They obtain the $L^p(\mathbb{R}^n, \mathrm{d}x)$ boundedness and the weak type $(1,1)$ of them.

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In this paper we consider integral operators defined for f belonging to the Schwartz class $S(\mathbb{R}^n)$ by

$$Tf(x) = \int_{\mathbb{R}^n} |x - a_1 y|^{-\alpha_1} \dots |x - a_m y|^{-\alpha_m} f(y)\,\mathrm{d}y, \tag{1.1}$$

$\alpha_1 + \ldots + \alpha_m = n$, $\alpha_i > 0$ and $a_i \in \mathbb{R} - \{0\}$ for $i = 1, \ldots, m$.

We take the Hardy-Littlewood maximal function as

$$Mf(x) = \sup_Q \frac{1}{|Q|} \int_Q |f(x)|\,\mathrm{d}x$$

where the supremum is taken along all cubes Q such that x belongs to Q. We recall that a weight w is a measurable, non negative and locally integrable function. It is well known that, for $p > 1$, M is bounded on $L^p(w)$ if and only if there exists $c > 0$ such that

$$\sup_Q \left(\frac{1}{|Q|} \int_Q w \right) \left(\frac{1}{|Q|} \int_Q w^{-1/(p-1)} \right)^{p-1} \leqslant c. \tag{1.2}$$

The class of functions that satisfy (1.2) is denoted by A_p. For $p = 1$, the class A_1 is defined by

$$Mw(x) \leqslant cw(x)$$

for a.e. $x \in \mathbb{R}^n$ and for some positive constant c. The weak type (1,1) of the maximal function is equivalent to $w \in A_1$. These classes A_p have been defined by Muckenhoupt (see [6]) in the one dimensional case and for higher dimensions by Coifmann and Fefferman (see [1]).

In this paper we obtain the boundedness of T on $L^p(\mathbb{R}^n, w)$ and a weighted (1,1) inequality for a wide class of weights w in A_p. We prove the following result:

Theorem 1. *Let T be defined by (1.1). Suppose there exists $c \geqslant 1$ such that $w(a_i x) \leqslant cw(x)$ for $1 \leqslant i \leqslant m$ and for almost every $x \in \mathbb{R}^n$.*

a) *If $w \in A_p$, $1 < p < \infty$, then T is bounded on $L^p(\mathbb{R}^n, w)$.*

b) *If $w \in A_1$ then there exists $k > 0$ such that, for $\lambda > 0$ and $f \in S(\mathbb{R}^n)$,*

$$w(\{x\colon\ |Tf(x)| > \lambda\}) \leqslant \frac{k}{\lambda} \int |f(x)| w(x)\,\mathrm{d}x.$$

We also analyze the boundedness of the operator T from L^∞ into BMO, the classical space consisting of functions with bounded mean oscillation, defined by

John and Nirenberg in [5]. Precisely, we say that $f \in L^1_{\text{loc}}$ belongs to BMO if there exist $c > 0$ such that

$$\frac{1}{|Q|}\int\left|f(x) - \frac{1}{|Q|}\int f\right| \mathrm{d}x \leqslant c$$

for all cubes $Q \subset \mathbb{R}^n$. The smallest bound c for which the above inequality holds is called $\|f\|_*$. From the techniques used, the following result follows inmediately:

Theorem 2. *Let T be defined by* (1.1). *Then there exists $c > 0$ such that*

$$\|Tf\|_* \leqslant c\|f\|_\infty$$

for all $f \in S(\mathbb{R}^n)$.

If f is a positive constant then $Tf(x) = \infty$ for all $x \in \mathbb{R}^n$, so we cannot expect a general boundedness from L^∞ into BMO. With techniques similar to those developed in [8], we obtain

Theorem 3. *Let T be defined by* (1.1).

a) *If $f \in L^\infty$ and $T|f|(x_0) < \infty$ for some $x_0 \in \mathbb{R}^n$ then $Tf(x)$ is well defined for all $x \neq 0$ and $Tf \in L^1_{\text{loc}}(\mathbb{R}^n)$.*
b) *There exists $c > 0$ such that*

$$\|Tf\|_* \leqslant c\|f\|_\infty$$

for all f as in a).

By c we denote a positive constant, not the same at each occurrence.

Proof of the main results

We follow the argument developed in [2, p. 144] where the case of the Calderón-Zygmund operators is treated. As there we define, for $f \in L^1_{\text{loc}}(\mathbb{R}^n)$, the sharp maximal function by

$$M^\#(f)(x) = \sup_{Q\colon x\in Q} \frac{1}{|Q|}\int_Q |f - f_Q|(y)\,\mathrm{d}y$$

with $f_Q = |Q|^{-1}\int_Q f$.

We denote $D = \max\limits_{1\leqslant i\leqslant m} |a_i^{-1}|$ and $d = \min\limits_{1\leqslant i\leqslant m} |a_i^{-1}|$. We need the following result:

Lemma 1.3. *If T is defined by* (1.1) *and $s > 1$ then there exists $c > 0$ such that for all $f \in S(\mathbb{R}^n)$,*

$$M^{\#}(Tf)(x) \leqslant c[(Mf^s(a_1^{-1}x))^{1/s} + \ldots + (Mf^s(a_m^{-1}x))^{1/s}].$$

Proof. We first observe that T is a bounded operator on $L^p(\mathbb{R}^n, \mathrm{d}x)$, $1 < p < \infty$ (see [4]), so for $f \in S(\mathbb{R}^n)$, $Tf \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$ and $M^{\#}(Tf)(x)$ is well defined for all $x \in \mathbb{R}^n$. We take $x \in \mathbb{R}^n$ such that $T|f|(x) < \infty$ and Q a cube that contains x. We set $l(Q)$ as the length of the side of Q, denote by $\overline{Q}$ the cube with the same center as Q, such that $l(\overline{Q}) \geqslant 2D/d \cdot l(Q)$ and, for $1 \leqslant i \leqslant m$, we also set $\overline{Q}_i = a_i^{-1}\overline{Q}$. We decompose $f = f_1 + f_2$, $f_1 = f\chi_{\bigcup_{1 \leqslant k \leqslant m} \overline{Q}_k}$ and take $a = Tf_2(x)$. Then

$$\frac{1}{|Q|}\int_Q |Tf(y) - a| \, \mathrm{d}y \leqslant \frac{1}{|Q|}\int_Q |Tf_1(y)| \, \mathrm{d}y + \frac{1}{|Q|}\int_Q |Tf_2(y) - Tf_2(x)| \, \mathrm{d}y.$$

If $s > 1$ then T is bounded on $L^s(\mathbb{R}^n, \mathrm{d}x)$ (see [4]), so

$$\begin{aligned}
\frac{1}{|Q|}\int_Q |Tf_1(y)| \, \mathrm{d}y &\leqslant \Big(\frac{1}{|Q|}\int_Q |Tf_1(y)|^s \, \mathrm{d}y\Big)^{1/s} \\
&\leqslant c\bigg(\Big(\frac{1}{|Q|}\int_{\overline{Q}_1} |f(y)|^s \, \mathrm{d}y\Big)^{1/s} + \ldots + \Big(\frac{1}{|Q|}\int_{\overline{Q}_m} |f(y)|^s \, \mathrm{d}y\Big)^{1/s}\bigg) \\
&\leqslant c'\big[(Mf^s(a_1^{-1}x))^{1/s} + \ldots + (Mf^s(a_m^{-1}x))^{1/s}\big].
\end{aligned}$$

On the other hand,

$$\frac{1}{|Q|}\int_Q |Tf_2(y) - Tf_2(x)| \, \mathrm{d}y \leqslant \frac{1}{|Q|}\int_Q \bigg|\int_{\left(\bigcup_{1 \leqslant k \leqslant m} \overline{Q}_k\right)^c} (K(y,z) - K(x,z)) f(z) \, \mathrm{d}z\bigg| \, \mathrm{d}y$$

where we denote by $K(x,y)$ the kernel $|x - a_1 y|^{-\alpha_1} \ldots |x - a_m y|^{-\alpha_m}$.

We now estimate $|K(y,z) - K(x,z)|$.

Case $l(Q) \geqslant 2|x|$. In this situation $\bigcup\limits_{1 \leqslant k \leqslant m} \overline{Q}_k \supset \{y \colon |y| < 3D|x|\}$. Indeed, if $z \in \left(\bigcup_{1 \leqslant k \leqslant m} \overline{Q}_k\right)^c$, then $|z| \geqslant |z - a_1^{-1}x| - |a_1^{-1}x| \geqslant l(\overline{Q_1}) - D|x| \geqslant dl(\overline{Q}) - D|x| \geqslant 3D|x|$. Moreover, in this case $|x - a_1 z| \leqslant |x| + |a_1 z| \leqslant \big(|a_1| + \frac{1}{3D}\big)|z|$ then

$$\begin{aligned}
|x - a_i z| &\geqslant |a_i z| - |x| \geqslant \Big(|a_i| - \frac{1}{3d}\Big)|z| \\
&\geqslant \Big(\frac{3|a_i|D - 1}{3|a_1|D + 1}\Big)\frac{1}{2}|x - a_1 z|.
\end{aligned} \tag{1.4}$$

Thus we apply the mean value theorem to obtain, for $x, y \in Q$ and $z \in \Big(\bigcup_{1\leqslant k\leqslant m} \overline{Q}_k\Big)^c$,

$$|K(y,z) - K(x,z)| \leqslant |x-y| \sum_{i=1}^{m} \frac{\alpha_i}{|\xi - a_i z|^{\alpha_i+1} \prod_{l\neq i} |\xi - a_l z|^{\alpha_l}}$$

for some ξ between x and y. But $|a_i^{-1}\xi - z| \geqslant |a_i^{-1}x - z| - |a_i^{-1}\xi - a_i^{-1}x| \geqslant \frac{1}{2}|a_i^{-1}x - z|$, so (1.4) implies

$$|K(y,z) - K(x,z)| \leqslant c\, \frac{|x-y|}{|x - a_1 z|^{n+1}}. \tag{1.5}$$

Thus

$$\begin{aligned}
&\frac{1}{|Q|}\int_Q \bigg| \int_{\left(\bigcup_{1\leqslant k\leqslant m} \overline{Q}_k\right)^c} (K(y,z) - K(x,z)) f(z)\,\mathrm{d}z \bigg| \,\mathrm{d}y \\
&\quad\leqslant \frac{c}{|Q|}\int_Q \sum_{k=1}^{\infty} \int_{2^k Dl(Q) \leqslant |a_1^{-1}x - z| < 2^{k+1}Dl(Q)} \frac{|x-y|}{|a_1^{-1}x - z|^{n+1}} |f(z)|\,\mathrm{d}z\,\mathrm{d}y \\
&\quad\leqslant cl(Q) \sum_{k=1}^{\infty} \frac{1}{2^k Dl(Q)} \frac{1}{(2^k Dl(Q))^n} \int_{|a_1^{-1}x - z| < 2^{k+1}Dl(Q)} |f(z)|\,\mathrm{d}z \\
&\quad\leqslant cMf(a_1^{-1}x) \leqslant c(Mf^s(a_1^{-1}x))^{1/s}.
\end{aligned}$$

Case $l(Q) < 2|x|$. We decompose

$$\int_{\left(\bigcup_{1\leqslant k\leqslant m} \overline{Q}_k\right)^c} (K(y,z) - K(x,z)) f(z)\,\mathrm{d}z = \int_{|z|\geqslant 3D|x|} + \int_{\{|z|<3D|x|\}\cap\left(\bigcup_{1\leqslant k\leqslant m} \overline{Q}_k\right)^c}.$$

To estimate the first integral, we proceed as before and we obtain (1.5) for $x, y \in Q$ and $|z| \geqslant 3D|x|$, then

$$\frac{1}{|Q|}\int_Q \bigg| \int_{|z|\geqslant 3D|x|} (K(y,z) - K(x,z)) f(z)\,\mathrm{d}z \bigg| \,\mathrm{d}y \leqslant c(Mf^s(a_1^{-1}x))^{1/s}.$$

We now study the second integral. For $1 \leqslant i \leqslant m$, $x, y \in Q$ and $z \in \{z\colon |z| < 3D|x|\} \cap \Big(\bigcup_{1\leqslant k\leqslant m} \overline{Q}_k\Big)^c$, we have

$$|a_i^{-1}y - z| \geqslant |a_i^{-1}x - z| - |a_i^{-1}y - a_i^{-1}x| \geqslant \frac{|a_i^{-1}x - z|}{2},$$

hence

$$|K(y,z) - K(x,z)| \leqslant c|K(x,z)|.$$

So

$$\int_{\{|z|<3D|x|\}\cap\left(\bigcup_{1\leqslant k\leqslant m}\overline{Q}_k\right)^c}(K(y,z)-K(x,z))f(z)\,\mathrm{d}z$$
$$\leqslant c\int_{\{z\colon\,|z|<3D|x|\}}\frac{|f(z)|}{|x-a_1z|^{\alpha_1}\ldots|x-a_mz|^{\alpha_m}}\,\mathrm{d}z.$$

We define $b=\frac{1}{2}\min\limits_{1\leqslant l,\,j\leqslant m}(|a_l^{-1}-a_j^{-1}|)$. We set $A_i=\{z\colon\,|a_i^{-1}x-z|\leqslant b|x|\}$, $1\leqslant i\leqslant m$, and $A_{m+1}=\left(\bigcup\limits_{i=1}^{m}A_i\right)^c$ and decompose

$$\int_{\{z\colon\,|z|<3D|x|\}}\frac{|f(z)|}{|x-a_1z|^{\alpha_1}\ldots|x-a_mz|^{\alpha_m}}\,\mathrm{d}z$$
$$=\int_{A_1}+\ldots+\int_{A_m}+\int_{A_{m+1}\cap\{z\colon\,|z|<3D|x|\}}.$$

For $z\in A_i$ and $l\neq i$ we have $|a_l^{-1}x-z|\geqslant b|x|$, hence

$$\int_{A_i}\frac{|f(z)|}{|x-a_1z|^{\alpha_1}\ldots|x-a_mz|^{\alpha_m}}\,\mathrm{d}z$$
$$\leqslant\frac{c}{|x|^{n-\alpha_i}}\sum_{j=0}^{\infty}\int_{2^{-j-1}b|x|\leqslant|a_i^{-1}x-z|\leqslant 2^{-j}b|x|}\frac{|f(z)|}{|a_i^{-1}x-z|^{\alpha_i}}\,\mathrm{d}z$$
$$\leqslant c\sum_{j=1}^{\infty}2^{j(\alpha_i-n)}\frac{1}{(2^{-j}b|x|)^n}$$
$$\times\int_{|z-a_i^{-1}x|\leqslant 2^{-j}b|x|}|f(z)|\,\mathrm{d}z\leqslant cMf(a_i^{-1}x)\leqslant c(Mf^s(a_i^{-1}x))^{1/s}.$$

Now

$$\int_{A_{m+1}\cap\{z\colon\,|z|<3D|x|\}}\frac{|f(z)|}{|x-a_1z|^{\alpha_1}\ldots|x-a_mz|^{\alpha_m}}\,\mathrm{d}z\leqslant c|x|^{-n}\int_{\{z\colon\,|z|<3D|x|\}}|f(z)|\,\mathrm{d}z$$
$$\leqslant cMf(a_1^{-1}x)\leqslant c(Mf^s(a_1^{-1}x))^{1/s},$$

and the lemma follows. □

Lemma 1.6. *Let T be defined by* (1.1), $1<p<\infty$, $w\in A_p$ *and* $f\in L^p(w)$. *Then* $Tf\in L^p(w)$.

Proof. If $\operatorname{supp}f\subset B(0,R)$ and $|x|>2R$ then $|K(x,y)|\leqslant c/|x|^n$ and so in this case $|Tf(x)|\leqslant c_R/|x|^n$. The proof follows as in Theorem 7.18 in [2], since T is a bounded operator on $L^p(\mathbb{R}^n,\mathrm{d}x)$ (see [4]). □

P r o o f of Theorem 1. a) Taking account of Lemmas 1.3 and 1.6, we proceed as in the proof of Theorem 7.18 in [2] to obtain, for $f \in S(\mathbb{R}^n)$,

$$\begin{aligned}\int |Tf(x)|^p w(x)\,\mathrm{d}x& \\ &\leqslant c\int |(Mf^s(a_1^{-1}x))^{1/s}+\ldots+(Mf^s(a_m^{-1}x))^{1/s}|^p w(x)\,\mathrm{d}x\\ &\leqslant c\int |Mf^s(x)|^{p/s}w(a_1x)\,\mathrm{d}x+\ldots+\int |Mf^s(x)|^{p/s}w(a_mx)\,\mathrm{d}x\\ &\leqslant c\int |Mf^s(x)|^{p/s}w(x)\,\mathrm{d}x.\end{aligned}$$

The last inequality follows from the hypothesis about the weight w. The rest of the proof is as in Theorem 7.18 in [2].

b) For $\lambda > 0$ we perform the Calderón-Zygmund decomposition for f to obtain a sequence of disjoint $\{Q_j\}_{j\in\mathbb{N}}$ such that $f(x) \leqslant \lambda$ for almost every $x \notin \bigcup\limits_{j\in\mathbb{N}} Q_j$. We take

$$g(x) = \begin{cases} f(x) & \text{if } x \notin \bigcup\limits_{j\in\mathbb{N}} Q_j,\\ \dfrac{1}{|Q_j|}\displaystyle\int_{Q_j} f & \text{if } x \in Q_j\end{cases}$$

and write $f = g + b$.

As usual, from a), we obtain

$$w\{x\colon\ |Tg(x)| > \lambda\} \leqslant \frac{c}{\lambda}\int |f(x)|w(x)\,\mathrm{d}x.$$

For each $i = 1,\ldots,m$ and $j \in \mathbb{N}$ we denote by $\overline{Q_j}$ the cube with the same center as Q_j and such that $l(\overline{Q_j}) \geqslant 2D/d\cdot l(Q_j)$, and $\overline{Q_{j,i}} = a_i\overline{Q_j}$. We obtain

$$\begin{aligned}w\Big(\bigcup_{j\in\mathbb{N}}\overline{Q_{j,i}}\Big) &\leqslant \sum_{j\in\mathbb{N}} w(\overline{Q_{j,i}}) \leqslant c\sum_{j\in\mathbb{N}}\frac{w(\overline{Q_{j,i}})}{|\overline{Q_{j,i}}|}|\overline{Q_{j,i}}|\\ &\leqslant c\sum_{j\in\mathbb{N}}|Q_j|\frac{w(\overline{Q_{j,i}})}{|\overline{Q_{j,i}}|} \leqslant \sum_{j\in\mathbb{N}}\frac{c}{\lambda}\int_{Q_j}|f|\frac{w(\overline{Q_{j,i}})}{|\overline{Q_{j,i}}|}\\ &\leqslant \frac{c}{\lambda}\sum_{j\in\mathbb{N}}\int_{Q_j}|f(y)|Mw(a_iy)\,\mathrm{d}y\\ &\leqslant \frac{c}{\lambda}\int |f(y)|w(a_iy)\,\mathrm{d}y \leqslant \frac{c}{\lambda}\int |f(y)|w(y)\,\mathrm{d}y.\end{aligned}$$

Then

$$w\Big(\bigcup_{j\in\mathbb{N},\,i=1,\ldots,m}\overline{Q_{j,i}}\Big) \leqslant \frac{c}{\lambda}\int |f(y)|w(y)\,\mathrm{d}y.$$

Now for each fixed $i = 1, \ldots, m$, if c_j denotes the center of Q_j, we have

$$
\begin{aligned}
& w\biggl(\{x\colon\ |Tb(x)| > \lambda\} \cap \Bigl(\bigcup_{j\in\mathbb{N}} \overline{Q_{j,i}}\Bigr)^c\biggr) \\
&\quad \leqslant \frac{c}{\lambda} \sum_{j\in\mathbb{N}} \int_{(\overline{Q_{j,i}})^c} \left| \int_{Q_j} b_j(y)(K(x,y) - K(x,c_j))\,\mathrm{d}y \right| w(x)\,\mathrm{d}x \\
&\quad \leqslant \frac{c}{\lambda} \sum_{j\in\mathbb{N}} \int_{Q_j} |b_j(y)| \int_{(\overline{Q_{j,i}})^c} |K(x,y) - K(x,c_j)| w(x)\,\mathrm{d}x\,\mathrm{d}y.
\end{aligned}
$$

Now we observe that $K(x,y) = c\widetilde{K}(y,x)$ where $\widetilde{K}(x,y) = |x - a_1^{-1}y|^{-\alpha_1} \ldots |x - a_m^{-1}y|^{-\alpha_m}$. Reasoning as in a) with $\widetilde{K}$ instead of K and using the hypothesis on w, we get

$$
\int_{(\overline{Q_{j,i}})^c} |K(x,y) - K(x,c_j)| w(x)\,\mathrm{d}x \leqslant cMw(a_i y) \leqslant cw(y).
$$

So

$$
\begin{aligned}
& w\biggl(\{x\colon\ |Tb(x)| > \lambda\} \cap \Bigl(\bigcup_{j\in\mathbb{N},\, i=1,\ldots,m} \overline{Q_{j,i}}\Bigr)^c\biggr) \\
&\quad \leqslant \frac{c}{\lambda} \int |b(y)| w(y)\,\mathrm{d}y \leqslant \frac{c}{\lambda} \int |f(y)| w(y)\,\mathrm{d}y.
\end{aligned}
$$

□

P r o o f of Theorem 2. It follows straightforward from Lemma 1.3. □

P r o o f of Theorem 3. a) Let $f \in L^\infty(\mathbb{R}^n)$ and let x_0 be such that $T|f|(x_0) < \infty$. We take $R = 4D|x_0|$, denote $B = B(0,R) = \{x \in \mathbb{R}^n\colon\ |x| \leqslant R\}$, define $f_1 = |f|\chi_B$ and decompose $|f| = f_1 + f_2$. Then

$$
\begin{aligned}
Tf_1(x) &\leqslant \int_B |x - a_1 y|^{-\alpha_1} \ldots |x - a_m y|^{-\alpha_m} f(y)\,\mathrm{d}y \\
&\leqslant \|f\|_\infty \int_B |x - a_1 y|^{-\alpha_1} \ldots |x - a_m y|^{-\alpha_m}\,\mathrm{d}y.
\end{aligned}
$$

If $x \neq 0$ we choose $r > 0$ such that $r = \frac{1}{4} \min\limits_{1\leqslant i,\, k\leqslant m} |a_i^{-1} - a_k^{-1}|\,|x|$. For $1 \leqslant i \leqslant m$, we define $B_i = B(a_i^{-1}x, r)$. We have

$$
\begin{aligned}
& \int_B |x - a_1 y|^{-\alpha_1} \ldots |x - a_m y|^{-\alpha_m}\,\mathrm{d}y \\
&\quad \leqslant \sum_{1\leqslant i\leqslant m} \int_{B_i} |x - a_1 y|^{-\alpha_1} \ldots |x - a_m y|^{-\alpha_m}\,\mathrm{d}y \\
&\qquad + \int_{B\cap\left(\bigcup_{1\leqslant i\leqslant m} B_i\right)^c} |x - a_1 y|^{-\alpha_1} \ldots |x - a_m y|^{-\alpha_m}\,\mathrm{d}y.
\end{aligned}
$$

Now

$$\int_{B_i} |x-a_1y|^{-\alpha_1} \dots |x-a_my|^{-\alpha_m} \,\mathrm{d}y$$
$$\leqslant c\prod_{k\neq i} r^{-\alpha_k} \int_{B_i} |x-a_iy|^{-\alpha_i} \,\mathrm{d}y \leqslant c\prod_{k\neq i} r^{-\alpha_k} r^{-\alpha_i+n} = c.$$

If $|a_i^{-1}x| < 2R$ for some $1 \leqslant i \leqslant m$, then, for $y \in B \cap (B_i)^c$, we have $r < |a_i^{-1}x - y| \leqslant 3R$ and so

$$\begin{aligned}\int_{B\cap\left(\bigcup_{1\leqslant i\leqslant m} B_i\right)^c} & |x-a_1y|^{-\alpha_1} \dots |x-a_my|^{-\alpha_m} \,\mathrm{d}y \\ &\leqslant c\prod_{k\neq i} r^{-\alpha_k} \int_{B\cap(B_i)^c} |x-a_iy|^{-\alpha_i} \,\mathrm{d}y \\ &\leqslant c\prod_{k\neq i} r^{-\alpha_k} \int_r^{3R} t^{-\alpha_i+n-1} \,\mathrm{d}t \\ &= c\prod_{k\neq i} r^{-\alpha_k} [(3R)^{\alpha_i+n} - r^{-\alpha_i+n}] = c\left(|x|^{\sum_{k\neq i} -\alpha_k} + 1\right),\end{aligned}$$

so for $x \neq 0$ and such that $|a_i^{-1}x| < 2R$ we obtain

$$|Tf_1(x)| \leqslant c\|f\|_\infty \left(1 + |x|^{\sum_{k\neq i} -\alpha_k}\right). \tag{1.7}$$

Now if $|a_i^{-1}x| \geqslant 2R$ for all $1 \leqslant i \leqslant m$, then $|a_i^{-1}x - y| \geqslant R$ for $y \in B(0,R)$ and so

$$|Tf_1(x)| \leqslant \|f\|_\infty.$$

So (1.7) holds for all $x \neq 0$. Then $Tf_1(x) < \infty$ for all $x \neq 0$ and it belongs to $L^1_{\mathrm{loc}}(\mathbb{R}^n)$.

Now $Tf_2(x_0) < \infty$ so we write, for $x \in \mathbb{R}^n$, $Tf_2(x) = Tf_2(x) - Tf_2(x_0) + Tf_2(x_0)$. Then we have to study

$$\int_{B^c} |K(x,y) - K(x_0,y)|\,|f|(y) \,\mathrm{d}y.$$

For $x \neq 0$ we have

$$\begin{aligned}\int_{B^c} |K(x,y) - K(x_0,y)|\,|f|(y) \,\mathrm{d}y \leqslant & \int_{B^c\cap B(0,4D|x|)^c} |K(x,y) - K(x_0,y)|\,|f|(y) \,\mathrm{d}y \\ & + \int_{B^c\cap B(0,4D|x|)} |K(x,y)|\,|f|(y) \,\mathrm{d}y + c.\end{aligned}$$

To estimate the first integral, we proceed as in the proof of Lemma 1.3 to obtain that, for $y \in B^c \cap B(0, 4D|x|)^c$,

$$|K(x, y) - K(x_0, y)| \leqslant c\frac{|x - x_0|}{|x - a_1 y|^{n+1}},$$

so

$$\begin{aligned}\int_{B^c \cap B(0,4D|x|)^c} |K(x, y) - K(x_0, y)|\, |f|(y)\, \mathrm{d}y &\leqslant c|x - x_0| \int_{B^c} \frac{|f|(y)}{|x - a_1 y|^{n+1}}\, \mathrm{d}y \\ &\leqslant c|x - x_0| \|f\|_\infty.\end{aligned}$$

To study the second integral, we observe that it appears only if $D|x| \geqslant R/4$, so we proceed as in the previous estimate for Tf_1 to obtain that, for x in this region,

$$\int_{B^c \cap B(0,4D|x|)} |K(x, y)|\, |f|(y)\, \mathrm{d}y \leqslant c\|f\|_\infty.$$

So, for $x \neq 0$, $Tf_2(x) < \infty$ and it belongs to $L^1_{\mathrm{loc}}(\mathbb{R}^n)$.

b) If f satisfies the hypothesis of a) we obtain that $M^\#(Tf)(x)$ is well defined for all $x \in \mathbb{R}^n$, so Lemma 1.3 still holds for these functions, and b) follows. □

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