ON THE BEST RANGES FOR $A_p^+$ AND $RH_r^+$

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Abstract. In this paper we study the relationship between one-sided reverse Hölder classes $RH_r^+$ and the $A_p^+$ classes. We find the best possible range of $RH_r^+$ to which an $A_1^+$ weight belongs, in terms of the $A_1^+$ constant. Conversely we also find the best range of $A_p^+$ to which a $RH_\infty^+$ weight belongs, in terms of the $RH_\infty^+$ constant. Similar problems for $A_p^+$, $1 < p < \infty$ and $RH_r^+$, $1 < r < \infty$ are solved using factorization.

1. Introduction

It is well known that there is a relationship between the $A_p$ classes and the so-called reverse Hölder classes $RH_r$. C. J. Neugebauer [8] has studied the following problems:

(1) For $w \in A_p$, find the precise range of $r$'s such that $w \in RH_r$, the precise range of $q < p$ for which $w \in A_q$, and the precise range of $s > 1$ so that $w^s \in A_p$.

(2) Conversely, for a fixed $w \in RH_r$, find the precise range of $p$'s such that $w \in A_p$, and the precise range of $q > r$ for which $w \in RH_q$.

For the one-sided Hardy-Littlewood maximal operator,

$$M^+ f(x) = \sup_{h>0} \frac{1}{h} \int_x^{x+h} |f|, $$

the $A_p^+$ classes were introduced by E. Sawyer [9]. He proved that $M^+$ is bounded in $L^p(w)$ ($p > 1$) if, and only if, the weight satisfies $A_p^+$ i.e., there exists a constant $C$ such that for any three points $a < b < c$

$$\int_a^b w \left( \int_b^c w^{1-p} \right)^{p-1} \leq C(c-a)^p$$

The smallest constant for which this is satisfied will be called the $A_p^+$ constant of $w$ and will be denoted by $A_p^+(w)$. For $p = 1$ the weak type of the operator holds if, and only if, the weight $w$ satisfies $A_1^+$ i.e. there exists $C$ so that for any $a$ and almost every $b > a$,

$$\int_a^b w \leq C(b-a)w(b).$$

1991 Mathematics Subject Classification. Primary 42B25.

Key words and phrases. One-sided weights, one-sided reverse Hölder, factorization.

Research supported by D.G.I.Y.T. (PB94-1496), Junta de Andalucía and Universidad Nacional de Córdoba.

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The smallest such constant will be called the $A^+_1$ constant of $w$ and will be denoted by $A^+_1(w)$. For later reference we point out that it is an easy consequence of Lebesgue’s differentiation theorem that the constant in the definition of $A^+_1$ is always greater than, or equal to, one.

These classes are of interest, not only because they control the boundedness of the one-sided Hardy-Littlewood maximal operator, but they are the right classes for the weighted estimates for one-sided singular integrals [1] and they also appear in PDE [4]. In contrast to the Muckenhoupt weights, the one-sided weights are not doubling, but they satisfy a one-sided doubling property, namely if $w \in A^+_p$ then there exists $C$ such that for any $a \in \mathbb{R}$ and $h > 0$, $\int_a^{a+2h} w \leq C \int_{a+h}^{a+2h} w$. The reverse Hölder property is not satisfied by these weights either but nevertheless Martín-Reyes [5] proved that there is a weak substitute of this notion, that we will denote by $RH^+_r$, which is good enough to prove the “$p - \epsilon$” property. In [7] the class $A^+_\infty$ was introduced and it was proved that $A^+_\infty = \bigcup_{p < \infty} A^+_p = \bigcup_{1 < r} RH^+_r$.

In this note we solve the problems of the Neugebauer paper in this context. In the proofs we will make essential use of the one-sided minimal operator introduced by Cruz-Uribe, Neugebauer and Olesen [3]. It is defined as $m^+(x) = \inf_{c > x} \frac{1}{c-x} \int_x^c |f|$. We will also use the fact that for any positive function $g$, the maximal operator $M_g f(x) = \sup_{x \in I} \frac{1}{|I|} \int_I |f| g dx$ is of weak type one-one with respect to the measure $g dx$. Note that for $g = 1$, we have the classical Hardy-Littlewood maximal operator, which is denoted by $M f$.

The paper is organized as follows: in section 2 we give definitions and characterizations of $RH^+_r$, $1 < r < \infty$. In section 3 we prove two theorems of best range for the extreme classes $A^+_1$ and $RH^+_\infty$. In section 4 we give a factorization theorem for weights in $RH^+_r$, and finally in section 5 we extend the theorems of section 3 for $A^+_p$ and $RH^+_r$, using the factorization proven in section 4. We shall see, that the index range depends on the factorization of the weight.

We end this introduction with some notation: for a given interval $I = (a, a+h)$ we denote by $I^-$ the interval $(a-h, a)$, $I^+$ the interval $(a+h, a+2h)$, and $I^{++}$ the interval $(a+2h, a+3h)$. For any $1 < p < \infty$, $p'$ will be its conjugate exponent, if $g$ is locally integrable and $E$ is a measurable set, $g(E)$ will stand for $\int_E g$ and $C$ will represent a constant that may change from time to time. Finally we remark that we can change the orientation on the real line obtaining similar results for classes $RH^-_r, A^-_p$, $1 < r \leq \infty$ and $1 \leq p \leq \infty$.

2. Definition, Characterization of $RH^+_r$ for $1 < r < \infty$

We start this section with the definition of $RH^+_r$, $1 < r < \infty$.

**Definition 2.1.** A weight $w$ satisfies the one-sided reverse Hölder $RH^+_r$ condition, if there exists $C$ such that for any $a < b$

$$\int_a^b w^r \leq C \left(M(w \chi_{(a,b)}(b))\right)^{(r-1)} \int_a^b w.$$

The smallest such constant will be called the $RH^+_r$ constant of $w$ and will be denoted by $RH^+_r(w)$. 
Definition 2.3. A weight satisfies the one-sided reverse Hölder RH\(_{\infty}^t\) condition, if there exists \(C\) such that
\[
(2.4) \quad w(x) \leq Cm^+ w(x),
\]
for almost all \(x \in \mathbb{R}\).

The smallest such constant will be called the RH\(_{\infty}^t\) constant of \(w\) and will be denoted by \(RH_{\infty}^t(w)\). It is clear that \(C \geq 1\).

The following lemma gives several characterizations of RH\(_{r}^t\). The constants are not necessarily the same.

Lemma 2.5. Let \(a < b < c < d\), \(1 < r < \infty\), and \(w \geq 0\) locally integrable, then the following statements are equivalent

i) \(\int_a^b w^r \leq C \left(M(w\chi_{(a,b)}(b))\right)^{(r-1)} \int_a^b w\).

ii) \(\frac{1}{b-a} \int_a^b w^r \leq C \left(\frac{1}{c-b} \int_b^c w\right)^r\), with \(b - a = 2(c-b)\).

iii) \(\frac{1}{b-a} \int_a^b w^r \leq C \left(\frac{1}{d-c} \int_c^d w\right)^r\), with \(b - a = d - b = 2(d-c)\).

iv) \(\frac{1}{b-a} \int_a^b w^r \leq C \left(\frac{1}{c-b} \int_b^c w\right)^r\), with \(b - a = c - b\).

v) \(\frac{1}{b-a} \int_a^b w^r \leq C \left(\frac{1}{d-c} \int_c^d w\right)^r\), with \(b - a = d - c = \gamma (d-a)\), \(0 < \gamma \leq \frac{1}{2}\).

Proof. To see i) \(\implies\) ii), we fix \(a < b < c\), \(b - a = 2(c-b)\) and take any \(x \in (b,c)\). Then
\[
\int_a^b w^r \leq \int_a^x w^r \leq C \left(M(w\chi_{(a,x)}(x))\right)^{(r-1)} \int_a^x w \leq C \left(M(w\chi_{(a,c)}(x))\right)^{(r-1)} \int_a^x w.
\]
Therefore \((b,c) \subseteq \{x : (M(w\chi_{(a,c)}(x))\right)^{(r-1)} \geq \frac{1}{C} \int_a^x w^r\}\). The weak type \((1,1)\) of the Hardy-Littlewood maximal operator yields,
\[
(c-b) \left(\int_a^b w^r\right)^\frac{1}{r-1} \leq C \left(\int_a^c w\right)^\frac{1}{r-1},
\]
which implies
\[
\frac{1}{b-a} \int_a^b w^r \leq C \left(\frac{1}{c-b} \int_b^c w\right)^r \leq C \left(\frac{1}{c-b} \int_b^c w\right)^r,
\]
the last inequality follows from the fact, proved in [7], that a weight satisfying i) satisfies \(A_p^+\) for some \(p\) and thus it satisfies the one-sided doubling condition.

We will prove now that ii) \(\implies\) i). Let us fix \(a < b\) and define a sequence \((x_k)\) as follows: \(x_0 = a\) and \(b-x_k = 2(b-x_{k+1})\). In particular \(x_{k+1} - x_k = 2(x_{k+2} - x_{k+1}) = (b - x_{k+1})\). Using condition ii) for the points \(x_k, x_{k+1}, x_{k+2}\), we have
\[
\int_a^b w^r = \sum_{k=0}^{\infty} \int_{x_k}^{x_{k+1}} w^r \leq C \sum_{k=0}^{\infty} (x_{k+1} - x_k)^{1-r} \left(\int_{x_{k+1}}^{x_{k+2}} w\right)^r
\]
\[
\leq C \sum_{k=0}^{\infty} \int_{x_{k+1}}^{x_{k+2}} w \left(\frac{1}{b-x_{k+1}} \int_{x_{k+1}}^{b} w\right)^{r-1} \leq (M(w\chi_{(a,b)}(b))^{r-1} C \int_a^b w.
\]
To see ii) \(\implies\) iii) let \(a < b < c < d\) with \(b - a = d - b = 2(d - c)\). Using that \(w\) satisfies the one-sided doubling condition, we have

\[
\frac{1}{b-a} \int_a^b w^r \leq C \left( \frac{1}{c-b} \int_b^c w \right)^r \leq C \left( \frac{1}{c-b} \frac{1}{d-c} \int_b^d w \right)^r \\
\leq C \left( \frac{1}{d-c} \int_c^d w \right)^r.
\]

iii) \(\implies\) iv) is immediate.

First of all we observe that iv) easily implies that the weight \(w\) satisfies the one-sided doubling condition. To see that iv) \(\implies\) v), let \(0 < \gamma < \frac{1}{2}\) and \(a < b < c < d\), \(b - a = d - c = \gamma(d - a)\) then if \(x\) is the mid point between \(a\) and \(d\) we have

\[
\frac{1}{b-a} \int_a^b w^r \leq \frac{1}{2\gamma} \frac{1}{x-a} \int_a^x w^r \leq C \frac{1}{2\gamma} \left( \frac{1}{d-x} \int_x^d w \right)^r,
\]

but it follows from the one-sided doubling condition that \(\int_x^d w \leq C\gamma \int_c^d w\).

Suppose v) holds, let \(a < b < c\), \(b - a = c - b = h\), we define for \(k = 0, 1, \ldots, N\) \(x_k = a+ksh\) and \(y_k = b + ksh\) where \(s = \gamma\) and \(N\) is the first integer such that \((N+1)s > 1\). We observe that the choice of \(x_k, y_k\) has been made so that for any \(0 \leq k \leq (N - 1)\) we have \(x_{k+1} - x_k = y_{k+1} - y_k = \gamma(y_{k+1} - x_k)\). Applying v), using that \(r > 1\) and the fact that the intervals \((y_k, y_{k+1})\) are disjoint, we have

\[
\int_a^b w^r \leq \sum_{k=0}^{N-1} \int_{x_k}^{x_{k+1}} w^r + \int_{b-sh}^b w^r
\]
\[
\leq C(sh)^{1-r} \sum_{k=0}^{N-1} \left( \int_{y_k}^{y_{k+1}} w \right)^r + C(sh)^{1-r} \left( \int_{c-sh}^c w \right)^r
\]
\[
\leq C\gamma(c-a)^{1-r} \left( \int_b^c w \right)^r.
\]

So we have proved that v) \(\implies\) iv).

Finally we will show that iv) \(\implies\) ii). Let \(a < b < c\) with \(b - a = 2(c - b)\). Let \(x\) be the mid point between \(a, b\), using the one-sided doubling property we have

\[
\frac{1}{b-a} \int_a^b w^r = \frac{1}{b-a} \left( \int_a^x w^r + \int_x^b w^r \right)
\]
\[
= \frac{1}{2} \left( \frac{1}{x-a} \int_a^x w^r + \frac{1}{b-x} \int_x^b w^r \right)
\]
\[
\leq C \frac{1}{2} \left( \frac{1}{b-x} \int_a^b w \right)^r + \left( \frac{1}{c-b} \int_b^c w \right)^r
\]
\[
\leq C \frac{1}{2} \left( \frac{1}{c-b} \int_a^c w \right)^r + \left( \frac{1}{c-b} \int_b^c w \right)^r
\]
\[
\leq C \left( \frac{1}{c-b} \int_b^c w \right)^r. \square
\]
Remark. The equivalence of i) and iv) was first proved in [3]. The following lemma tells us that in the definition of $A^+_p$ we can take two intervals that are not contiguous. Note that in the case of $RH^+_r$ we have seen this in the previous lemma.

Lemma 2.6. A weight $w \in A^+_p$, $p > 1$ if, and only if, there exists $0 < \gamma \leq \frac{1}{2}$ and a constant $C_\gamma$ such that for any $a < b < c < d$, $b - a = d - c = \gamma(d - a)$ then

$$\int_a^b w \left( \int_c^d w^{1-p'} \right)^{p-1} \leq C_\gamma (b-a)^p$$

Proof. If $w \in A^+_p$, $0 < \gamma \leq \frac{1}{2}$ and $a < b < c < d$, $b - a = d - c = \gamma(d - a)$ then

$$\int_a^b w \left( \int_c^d w^{1-p'} \right)^{p-1} \leq \int_a^c w \left( \int_c^d w^{1-p'} \right)^{p-1} \leq C(d-a)^p = C_\gamma (b-a)^p.$$

To prove that (2.7) implies $A^+_p$ we will show that (2.7) implies that for $\gamma$ and $a, b, c, d$ as above we have

$$\frac{1}{b-a} \int_a^b w \exp \left( \frac{1}{d-c} \int_c^d - \log(w) \right) \leq C.$$

Indeed

$$\frac{1}{b-a} \int_a^b w \exp \left( \frac{1}{d-c} \int_c^d - \log(w) \right) = \frac{1}{b-a} \int_a^b \left[ w \exp \left( \frac{1}{d-c} \int_c^d \log(w)^{1-p'} \right) \right]^{p-1} \leq \frac{1}{b-a} \int_a^b w \left( \frac{1}{d-c} \int_c^d w^{1-p'} \right)^{p-1} \leq C.$$

In the same way we prove that $w^{1-p'}$ satisfies

$$\exp \left( \frac{1}{b-a} \int_a^b \log(w)^{p'-1} \right) \frac{1}{d-c} \int_c^d w^{1-p'} \leq C.$$

But according to part j) of Theorem 1 in [7], (2.8) is equivalent to saying that $w \in A^+_\infty$, while (2.9) means that $w^{1-p'} \in A^-_\infty$ and according to Theorem 2 in [7] these two conditions imply $w \in A^+_p$.

Remark 2.10. We can easily see that $w \in A^+_1$ if, and only if, there exists $C > 0$ such that $\frac{1}{h} \int_{a-h}^a w \leq Cw(a+h)$ for almost every $a \in \mathbb{R}$ and $h > 0$. 

□
3. The extreme cases: $A_1^+$ and $RH_\infty^+$.

**Theorem 3.1.** Let $w \in A_1^+$ with $A_1^+$ constant $C > 1$, then $w \in RH_r^+$ for any $1 < r < \frac{C}{C-1}$, and this is the best possible range.

**Proof.** Let us fix the interval $I = (a, b)$. We consider the truncation of $w$ at height $N$ defined by $w_N = \min(w, N)$, which also satisfies $A_1^+$ with constant $C_N \leq C$. We claim that if $\lambda_I = M(w_N \chi_I)(b)$, and $E_\lambda = \{x \in I : w_N(x) > \lambda\}$ then

$$\int_{E_\lambda} w_N \leq C_N \lambda |E_\lambda| \quad \forall \lambda \geq \lambda_I.$$  

Indeed if $E_\lambda = I$ we do not even need the $A_1^+$ condition, since

$$w_N(E_\lambda) = \int_a^b w_N \leq M(w_N \chi_I)(b)(b - a) = \lambda_I(b - a) \leq C_N \lambda |E_\lambda|.$$  

If $E_\lambda \neq I$ we fix $\epsilon > 0$ and an open set $O$ such that $E_\lambda \subset O \subset I$ and $|O| \leq \epsilon + |E_\lambda|$. Let $J_k = (c, d)$, be one of the connected components of $O$. There are two cases

1. $a \leq c < d < b$,
2. $a \leq c < d = b$.

In the first case $d \notin E_\lambda$ and then $w_N(d) \leq \lambda$. Now $A_1^+$ gives $\int_c^d w_N \leq C_N w_N(d)(d - c) \leq C_N \lambda(d - c)$. The second case is handled as the case $E_\lambda = I$, since $\int_c^b w_N \leq M(w_N \chi_I)(b)(b - c) \leq C\lambda(b - c)$. In any case $w_N(J_k) \leq C_N \lambda |J_k|$. Adding up we get

$$w_N(E_\lambda) \leq w_N(O) \leq C_N \lambda |O| \leq C_N \lambda(\epsilon + |E_\lambda|).$$  

Since $\epsilon$ was arbitrary we are done. Now we proceed in the standard way i.e., we fix $s > -1$, multiply both sides of (3.2) by $\lambda^s$ and integrate from $\lambda_I$ to infinity to obtain,

$$\frac{1}{s + 1} \int_I (w_N^{s+2} - \lambda_I^{s+2} w_N) \leq \frac{C_N}{s + 2} \int_I w_N^{s+2}.$$  

Now if $r = s + 2 < \frac{C_N}{C_N - 1}$ then $\frac{1}{s + 1} - \frac{C_N}{s + 2} > 0$, and we get

$$\int_I w_N^r \leq C_N \lambda_I^{r-1} \int_I w_N = C_N(M(w_N \chi_I)(b))^{r-1} \int_I w_N.$$  

Now $C_N \leq C$ implies $\frac{C_N}{C_N - 1} \geq \frac{C}{C-1}$, and therefore if $r \leq \frac{C}{C-1}$ then

$$\int_a^b w_N^r \leq C_N(M(w_N \chi_{(a,b)})(b))^{r-1} \int_a^b w_N \leq C(M(w \chi_{(a,b)})(b))^{r-1} \int_a^b w$$  

and the monotone convergence theorem gives $w \in RH_r^+$. To see that this is the best possible range we consider the function

$$w(x) = x^{\frac{1}{r}-1} \chi_{(0, \infty)}(x).$$  

It is clear that does not satisfy $RH_r^+$, because $w^{\frac{C}{C-1}}(x) = \frac{1}{x}$ for $x > 0$. To see that it satisfies $A_1^+$ with constant $C$, we consider three cases

1. $a < b \leq 0$
2. $a \leq 0 < b$
3. $0 < a < b$
In the first case there is nothing to check. In the second case \( \frac{1}{b-a} \int_a^b w < \frac{1}{ \beta } \int_0^b w(x) = \frac{C}{b} b^{\beta} = Cw(b) \). Finally if \( 0 < a < b \), \( \int_a^b w = C(b^{\beta} - a^{\beta}) \leq C(b-a)w(b) \). 

**Remark.** Note that if \( C = 1 \), then \( w(x) = M^{-w(x)} \), and this implies that \( w \) is non-decreasing. This tells us that \( w \in RH_{\infty}^+ \).

**Theorem 3.3.** If \( w \) satisfies \( RH_{\infty}^+ \) with constant \( C > 1 \), then \( w \in A_p^+ \) for all \( p > C \), and this is the best possible range.

**Proof.** A truncation argument as in Theorem 3.1 allows us to suppose that \( w \) is bounded away from zero, i.e. there exists \( \beta > 0 \) so that \( w(x) \geq \beta \) for all \( x \). Let us fix \( I = (a, b) \) and consider \( \lambda_I = m^+(w \chi_I)(a) \). We claim that if \( \lambda < \lambda_I \) and \( E_\lambda = \{ x \in I : w(x) < \lambda \} \), then

\[
\lambda |E_\lambda| \leq C \int_{E_\lambda} w.
\]

As before if \( E_\lambda = I \) then \( \lambda |E_\lambda| = \lambda(b-a) < \lambda_I(b-a) = \int_0^b w \leq w(E_\lambda) \). If \( E_\lambda \neq I \) then we approximate it by an open set \( O = \bigcup J_k \) where \( E_\lambda \subset O \subset I \) and \( w(O) < \epsilon + w(E_\lambda) \). Let us fix \( J_k = (c, d) \). There are two cases

1. \( a < c \)
2. \( a = c \).

In the first case \( c \notin E_\lambda \) and then \( \lambda(d-c) \leq w(c)(d-c) \leq Cm^+ w(c)(d-c) \leq C \int_c^d w \).

In the second case \( \lambda(d-c) \leq \lambda_I(d-a) \leq \int_a^d w \), and (3.4) follows. If we multiply both sides of (3.4) by \( \lambda^{-r} \) with \( r > 2 \) and integrate we have

\[
\int_0^{\lambda_I} \lambda^{1-r} \int \chi_{E_\lambda}(x) dx d\lambda \leq C \int_0^{\infty} \lambda^{-r} \int_{E_\lambda} w(x) dx d\lambda.
\]

For the left hand side we obtain,

\[
\int_0^{\lambda_I} \lambda^{1-r} \int \chi_{E_\lambda}(x) dx d\lambda = \frac{1}{2-r} \int_{\{ x \in I : w(x) < \lambda_I \}} \lambda_I^{2-r} - w^{2-r} dx
\]

\[
\geq \frac{1}{2-r} \int \lambda_I^{2-r} - w^{2-r} dx = \frac{1}{r-2} \int w^{2-r} - \frac{|I|}{r-2} \lambda_I^{2-r},
\]

while the right hand side is equal to \( \frac{C}{r-1} \int_I w^{2-r} \). Therefore

\[
\frac{1}{r-2} \int_I w^{2-r} \leq \frac{C}{r-1} \int_I w^{2-r} + \frac{|I|}{r-2} \lambda_I^{2-r}.
\]

If we choose \( r > 2 \) such that \( C(r-2) < (r-1) \), we obtain that there exists \( C \) so that

\[
\frac{1}{|I|} \int_I w^{2-r} \leq C \left( m^+(w \chi_I)(a) \right)^{2-r}.
\]

We now claim that (3.5) implies that \( w \in A_p^+ \) with \( p = \frac{r-1}{r-2} \). Let us fix \( a < b < c \) and choose \( x \in (a, b) \). If we keep in mind that \( 1 - p' = 2 - r \) we may write

\[
\left( \frac{1}{c-a} \int_b^c w^{1-p'} \right)^{p-1} \leq \left( \frac{1}{c-x} \int_x^c w^{1-p'} \right)^{p-1} \leq C \left( m^+(w \chi_{(x,c)})(x) \right)^{1-p},
\]
but
\[
\left( m^+ \left( \frac{w}{\chi(x,c)} \right)(x) \right)^{-1} = \left( \inf_{x<d<c} \frac{1}{d-x} \int_x^d w \right)^{-1} = \sup_{x<d<c} \frac{d-x}{\int_x^d w} = M_w \left( \frac{\chi(a,c)}{w} \right)(x).
\]

We have thus proved that if
\[
\lambda = \left( \frac{1}{c-a} \int_b^c w^{1-p'} \right)^{p-1}
\]
and the weak type of \( M_w \) with respect to the measure \( wdx \) yields \( \int_a^b w \leq C(c-a)^p \left( \int_b^c w^{1-p'} \right)^{1-p} \) which is \( A_p^+ \). Finally it can be checked that the function \( w(x) \) which is 0 for \( x < -1 \), identically one for \( x > 0 \) and \( |x|^{C-1} \) between \(-1\) and \(0\), satisfies \( RH_{\infty}^+ \) with constant \( C \), but is not in \( A_C^+ \). \( \square \)

Remark. Note that if \( C = 1 \), then \( w(x) = m^+ w(x) \), and this implies that \( w \) is non-decreasing. This tells us that \( w \in A_1^+ \).

We had several different characterizations of \( RH_r^+ \), one involved the maximal operator, but dealt with one interval, and the others involved two intervals but no operator. We can now prove that for \( RH_{\infty}^+ \) the situation is the same, we can characterize \( RH_{\infty}^+ \) using two intervals instead of the minimal operator.

**Corollary 3.6.** \( w \in RH_{\infty}^+ \) if, and only if, there exists \( C \) such that for any interval \( I \),

\[
\text{esssup}_{I} w \leq C \frac{1}{|I^+|} \int_{I^+} w
\]

**Proof.** It is immediate that (3.7) implies \( RH_{\infty}^+ \). Assume now that \( w \in RH_{\infty}^+ \). The preceding theorem tells us that \( w \in A_p \) for some \( p \), and therefore it satisfies the one-sided doubling condition. Therefore if \( I = (a, b) \) is any interval, \( I^+ = (b, c) \) and \( x \in I \) we have
\[
w(x) \leq \frac{C}{c-x} \int_x^c w \leq \frac{C}{c-b} \int_b^c w,
\]
which is (3.7). \( \square \)

Remark. Note that with this definition, we have that \( RH_{\infty}^+ \subset \cap_{r>1} RH_r^+ \).

**4. Factorization of weights in \( RH_r^+ \), \( 1 < r \leq \infty \).**

The theorems on the best range for weights in \( A_p^+ \) \( p > 1 \) or in \( RH_r^+ \), \( r < \infty \) will be stated in terms of factorizations of the given weight. Therefore this section will be devoted to prove a factorization of functions in \( RH_r^+ \). The bilateral case was studied in [2].

**Definition 4.1.** A function \( w \) is said to be essentially increasing if there exists \( C \) so that \( w(x) \leq C w(y) \) for any \( x < y \).
Lemma 4.2. A function belongs to $RH_{\infty}^+ \cap A_1^+$ if, and only if, it is essentially increasing.

Proof. Assume that $w \in RH_{\infty}^+ \cap A_1^+$ and $x < y$ then $w(x) \leq C \frac{1}{y-x} \int_x^y w \leq C w(y)$ and $w$ is essentially increasing. Conversely, if $w$ is essentially increasing then for any $x$ and $h > 0$ we have $w(x) \leq C \frac{1}{h} \int_x^{x+h} w$, then $w \in RH_{\infty}^+$. On the other hand \( \frac{1}{h} \int_x^{x-h} w \leq C w(x) \), so $w \in A_1^+$. □

Lemma 4.3. Let $1 < r \leq \infty$ and $1 \leq p < \infty$.

1. If $u$ is essentially increasing and $v \in RH_r^+$ then $w \in RH_r^+$.
2. If $u$ is essentially increasing and $v \in A_p^+$ then $w \in A_p^+$.

Proof. This proof follows immediately from Definition 4.1. □

Lemma 4.4. Let $1 < r \leq \infty$ and $1 \leq p < \infty$. $w \in RH_r^+ \cap A_p^+$ if, and only if, $w^r \in A_q^+$, with $q = r(p-1) + 1$.

Proof. Let $C_1 = RH_r^+(w)$, and $C_2 = A_p^+(w)$, $w \in RH_r^+ \cap A_p^+$, and $q = r(p-1) + 1$. Also note that $1 - q' = 1 - \frac{r(p-1)+1}{r(p-1)} = \frac{1}{r(1-p)}$,

\[
\left( \frac{1}{|I|} \int_I w^r \right) \left( \frac{1}{|I|} \int_I w^{r(1-q')} \right)^{q-1} 
\leq C_1 \left( \frac{1}{|I|} \int_I w \right)^r \left( \frac{1}{|I|} \int_I w^{1-p} \right)^{r(p-1)} 
\leq C_1 C_2^r,
\]

and by Lemma 2.6 we have that $w^r \in A_q^+$.

If $w^r \in A_q^+$, by Hölder’s inequality

\[
\left( \frac{1}{|I|} \int_I w \right) \left( \frac{1}{|I|} \int_I w^{-1/(p-1)} \right)^{p-1} 
\leq \left( \frac{1}{|I|} \int_I w^r \right)^{1/r} \left( \frac{1}{|I|} \int_I w^{-(q-1)} \right)^{(q-1)/r} 
\leq C^{1/r},
\]

obtaining in this way that $w \in A_p^+$. Now again by Hölder’s inequality

\[
1 = \frac{1}{|I|} \int_{I+} w^{-1/p} w^{1/p} \leq \left( \frac{1}{|I|} \int_{I+} w \right)^{1/p} \left( \frac{1}{|I|} \int_{I+} w^{-p'/p} \right)^{1/p'}
\]

so

\[
\left( \frac{1}{|I|} \int_{I+} w^{-1/(p-1)} \right)^{1-p} \leq \frac{1}{|I|} \int_{I+} w,
\]

and we get

\[
\left( \frac{1}{|I|} \int_I w^r \right)^{1/r} \leq C \left( \frac{1}{|I|} \int_{I+} w^{-(q-1)} \right)^{-(q-1)/r} \left( \frac{1}{|I|} \int_{I+} w^{-1/(p-1)} \right)^{1-p} 
\]

\[
\leq C \frac{1}{|I|} \int_{I+} w,
\]

proving that $w \in RH_r^+$. □
**Factorization Theorem for weights in** $RH_r^+ \cap A_p^+$. A weight $w \in RH_r^+ \cap A_p^+$ with $1 \leq p < \infty$, $1 < r \leq \infty$ if, and only if, there exists weights $w_0$ and $w_1$ such that $w_0 \in RH_r^+ \cap A_1^+$, $w_1 \in RH\infty^+ \cap A_p^+$ and $w = w_0 w_1$.

Observe that since $\cup p<\infty A_p^+ = \cap 1<r RH_r^*$ every weight in $RH_r^+$ is in some $A_p^+$.

**Proof.** Let us consider first the cases $p = 1$ or $r = \infty$.

If $p = 1$ and $r \leq \infty$, we put $w_1 = 1$, and $w_0 = w$, then obviously $w_0 \in RH_r^+ \cap A_1^+$, and $w_1 \in RH\infty^+ \cap A_1^+$.

If $p \geq 1$ and $r = \infty$, we put $w_0 = 1$, and $w_1 = w$, obtaining $w_0 \in RH\infty^+ \cap A_1^+$, $w_1 \in RH\infty^+ \cap A_p^+$.

Conversely, given $w_0$ and $w_1$, at least one of them belongs to $RH\infty^+ \cap A_1^+$, (because $p = 1$ or $r = \infty$), so one of them is essentially increasing, therefore $w_0 w_1 \in RH_r^+ \cap A_1^+$ (Lemma 4.3).

Let us suppose now, $p > 1$ and $r < \infty$. Let $w = w_0 w_1$, with $w_0 \in RH_r^+ \cap A_1^+$, and $w_1 \in RH\infty^+ \cap A_p^+$, we want to see that $w \in RH_r^+ \cap A_p^+$. Note that for $w_1$ the following holds

$$\frac{1}{|I|} \int_I w_1^{1-p^\prime} \leq C \left( \frac{1}{|I|} \int_I w_1 \right)^{1-p^\prime} \leq C w_1 (a-h)^{1-p^\prime},$$

this implies, $w_1^{1-p^\prime} \in A_1^-$ (Remark 2.10). Let $v = w_1^{1-p^\prime}$, then $w_1 = v^{1-p}$ with $v \in A_1^-$, so $w = w_0 w_1 = w_0 v^{1-p}$ with $w_0 \in A_1^+$ and $v \in A_1^-$ (see [7]), and this implies $w \in A_p^+$.

Now

$$\frac{1}{|I|} \int_I w^r = \frac{1}{|I|} \int_I w_0^r w_1^{r} \leq (\sup_I w_1)^r C \left( \frac{1}{|I|} \int_I w_0 \right)^r \leq C \left( \frac{1}{|I|} \int_I w_1 \right)^r \leq C \left( \frac{1}{|I|} \inf_I w_0 \right)^r,$$

by Lemma 2.5, we have $w \in RH_r^+$. Conversely let $w \in RH_r^+ \cap A_p^+$, then by Lemma 4.4 $w^r \in A_q^1$, with $q = r(p - 1) + 1$, there exists $v_0 \in A_1^+$, and $v_1 \in A_1^-$, such that $w^r = v_0 v_1^{1-q}$ (see [7]), or equivalently $w = v_0^{1/r} v_1^{(1-q)/r} = v_0^{1/r} v_1^{1-p}$. Let $w_0 = v_0^{1/r}$ and $w_1 = v_1^{1-p}$. We will see that $w_0 \in RH_r^+ \cap A_1^+$. We note,

$$\frac{1}{|I|} \int_I w_0^r = \frac{1}{|I|} \int_I v_0 \leq C \inf_{I^+} v_0 \leq C \left( \frac{1}{|I|} \int_{I^+} v_0^{1/r} \right)^r = C \left( \frac{1}{|I|} \int_{I^+} w_0 \right)^r,$$

and also,

$$\frac{1}{|I|} \int_I w_0 = \frac{1}{|I|} \int_I v_0^{1/r} \leq \left( \frac{1}{|I|} \int_I v_0 \right)^{1/r} \leq C \inf_{I^+} v_0^{1/r} = C \inf_{I^+} w_0.$$
We only have to see now, that \( w_1 \in RH^{+}_{\infty} \cap A^+_p \) and we are done.

First we claim

\[(4.5) \quad w \in A_1^- \text{ then } w^{-\gamma} \in RH^+_\infty, \text{ for all } \gamma > 0.\]

In fact by Hölder’s inequality, we have for any interval \( I = (a, b) \), \( \left( \frac{1}{|I|} \int_I w \right)^{-\gamma} \leq \frac{1}{|I|} \int_I w^{-\gamma} \) and as \( w \in A_1^- \) we have that for almost every \( x \in I^- \), \( Cw(x) \leq \frac{1}{|I|} \int_I w \), and therefore

\[
\frac{1}{|I|} \int_I w^{-\gamma} \leq C \left( \frac{1}{|I|} \int_I w \right)^{-\gamma} \leq \frac{1}{|I|} \int_I w^{-\gamma} \leq C \frac{1}{b-x} \int_x^b w^{-\gamma}.
\]

Let \( w_1 = v_1^{1-p} \). As \( v_1 \in A_1^- \), then \( w_1 \in RH^+_\infty \). Moreover

\[
\frac{1}{|I|} \int_I w_1 \left( \frac{1}{|I^+|} \int_{I^+} w_1^{1-p'} \right)^{p-1} = \frac{1}{|I|} \int_I v_1^{1-p} \left( \frac{1}{|I^+|} \int_{I^+} v_1 \right)^{p-1}
\]

\[
\leq \frac{1}{|I|} \int_I v_1^{1-p} (C \inf_I v_1)^{p-1}
\]

\[
\leq \frac{C}{|I|} \int_I v_1^{1-p} v_1^{p-1} \leq C,
\]

i.e. \( w_1 \in A_p^+ \). \( \square \)

**Factorization Theorem for weights in \( A^+_p \).** A weight \( w \in A^+_p \) if, and only if, there exists \( w_1 \in RH^+_\infty \) and \( w_0 \in A^+_1 \) such that \( w = w_0 w_1 \).

**Proof.** If \( w \in A^+_p \) then \( w \in A^+_q \) for some \( 1 < q < \infty \), so there exist \( v_0 \in A^+_1 \) and \( v_1 \in A_1^- \) such that \( w = v_0 v_1^{1-q} \). Let \( w_0 = v_0 \) and \( w_1 = v_1^{1-q} \). By (4.5) \( w_1 \in RH^+_\infty \).

So we are done. Conversely if \( w_1 \in RH^+_\infty \), then \( w_1 \in A^+_q \) for some \( 1 < q \), i.e., there exists \( C \) such that

\[
\left( \frac{1}{|I|} \int_I w_1 \right)^{q-1} \frac{1}{|I^+|} \int_{I^+} w_1^{1-q'} \leq C,
\]

but then

\[
\left( \sup_{I^-} w_1 \right)^{q-1} \frac{1}{|I|} \int_{I^+} w_1^{1-q'} \leq \left( \frac{1}{|I|} \int_I w_1 \right)^{q-1} \frac{1}{|I|} \int_{I^+} w_1^{1-q'} \leq C,
\]

and we get

\[
\frac{1}{|I|} \int_{I^+} w_1^{1-q'} \leq C \inf_{I^-} w_1^{1-q'},
\]

and it is easy to see that this inequality implies \( w_1^{1-q'} \in A_1^- \). Then \( v_1 = w_1^{1-q'} \in A_1^- \), so \( w = w_0 w_1 = w_0 v_1^{1-q} \in A^+_q \subset A^+_\infty \). \( \square \)
5. Classes $A_p^+$ and $RH_{r}^+$.

In this section we will use Theorems 3.1 and 3.3 and the factorization theorems to obtain the best ranges for the classes $A_p^+$ and $RH_{r}^+$. As we shall see, the range of the index will depend on the factorization of the weights.

The following theorem gives us the precise range in $A_p^+$ for weights in $RH_{r}^+$.

**Theorem 5.1.** Let $w \in RH_{r}^+$, $w = w_0^{1/r'}$ with $w_0 \in RH_{\infty}^+$ and $w_1 \in A_1^+$, then $w \in A_p^+$ for all $p > C$, where $C = RH_{\infty}^+(w_0)$ and this is the best possible range.

**Proof.** Let $w_0 \in RH_{r}^+$ and $w_1 \in A_1^+$. By Theorem 3.3 $w_0 \in A_p^+$ for all $p > C$. Let $p > C$, there exists $\epsilon > 0$ such that $w_0 \in A_{p-\epsilon}^+$, so we choose $s > 1$ satisfying $1 - (p - \epsilon)' = s(1 - p')$, and by Hölder’s inequality

$$\frac{1}{|I^+|} \int_I w_0 w_1 \left(\frac{1}{|I^+|} \int_I (w_0 w_1)^{1-p'}\right)^{p-1} \leq \left(\frac{1}{|I^+|} \int_I w_1 \left(\frac{1}{|I^+|} \int_I w_0 w_1^{s(1-p')/q} \right)^{(p-1)/s} \right)^{(p-1)/p} \leq C.$$

To see that this is the best range, we consider $w_0$ as in Theorem 3.3 and $w_1 = 1$. □

**Remark 5.2.** Given $w \in RH_{r}^+$ there exist $u \in RH_{\infty}^+$, and $v \in A_1^+$ such that $w = uv^{1/r}$. We only have to consider the factorization theorem and choose $u = w_1$ and $v = w_0$. We have to prove that $v \in A_1^+$. Keeping in mind that $w_0 \in RH_{r}^+ \cap A_1^+$ we have

$$\frac{1}{|I^+|} \int_{|I^+|} v = \frac{1}{|I^+|} \int_{|I^-|} w_0' \leq C \left(\frac{1}{|I^+|} \int_{|I^+|} w_0\right)^r \leq Cw_0^r(x) = Cv(x),$$

for almost every $x \in I^+$, i.e., $v \in A_1^+$.

The next theorem shows us the precise range of the higher integrability of $w \in RH_{r}^+$.

**Theorem 5.3.** Let $w \in RH_{r}^+$, $w = uv^{1/r}$ with $u \in RH_{\infty}^+$ and $v \in A_1^+$. If $C = A_1^+(v)$ then $w \in RH_{s}^+$ for all $s < Cr / (C-1)$. The range of $s$ is the best possible.

**Proof.** Let $r < s < \frac{Cr}{C-1}$, we choose $q > 1$ such that $s < \frac{Cr}{q(C-1)}$. As $1 < \frac{q}{r} < \frac{C}{C-1}$, by Theorem 3.1 $v \in RH_{r}^+$, using Hölder’s inequality, that $u^s \in RH_{\infty}^+$ and $v \in A_1^+$ we have,

$$\frac{1}{|I^+|} \int_I w^s = \frac{1}{|I^+|} \int_I u^{s/r} \leq \left(\frac{1}{|I^+|} \int_I u^{q^s/r}\right)^{1/q} \leq \left(\frac{1}{|I^+|} \int_I v^{q^s/r}\right)^{1/q} \leq \sup_I u^s \left(\frac{1}{|I^+|} \int_I v\right)^{s/r} \leq C \left(\frac{1}{|I^+|} \int_I u\right)^{s/r} \leq C \left(\frac{1}{|I^+|} \int_I u\right)^{s/r} \leq C \left(\frac{1}{|I^+|} \int_I v\right)^{s/r} \leq C \left(\frac{1}{|I^+|} \int_I w\right)^{s},$$
and we get that \( w \in RH^+_s \), (Lemma 2.5).

To see this is the best range possible, we choose \( v \in A^+_1 \) as in Theorem 3.1 and \( u = 1 \), then \( w = v^{1/r} \in RH^+_s \) for all \( r \leq s < \frac{C}{C-1} \) (\( C = A^+_1(v) \)). If \( s = \frac{C}{C-1} \) and \( w \in RH^+_s \) then \( v \in RH^+_s \), but we have seen (Theorem 3.1) that this can not happen. \( \Box \)

The next theorem shows us which is the best range in \( RH^+_r \) for a given weight in \( A^+_p \).

**Theorem 5.4.** Let \( w \in A^+_p \), \( w = uv^{1-r} \), with \( u \in A^+_1 \), \( v \in A^-_1 \) and \( C = A^+_1(u) \), then \( w \in RH^+_r \) for all \( 1 < r < \frac{C}{C-1} \), being this range the best possible.

**Proof.** By Theorem 3.1 \( u \in RH^+_r \) for all \( 1 < r < \frac{C}{C-1} \) and we know that \( v^{1-p} \in RH^+_{\infty} \), then

\[
\frac{1}{|I|} \int_I w^r \leq \frac{1}{|I|} \int_I u^r \sup_I (v^{-r(p-1)})
\]

\[
\leq C \left( \frac{1}{|I^+|} \int_{I^+} u \right)^r \left( \frac{1}{|I^+|} \int_{I^+} v^{1-p} \right)^r \leq C \left( \inf_{I^{++}} u \right)^r \left( \sup_{I^+} v^{1-p} \right)^r
\]

\[
\leq C \left( \inf_{I^{++}} u \right)^r \left( \frac{1}{|I^{++}|} \int_{I^{++}} v^{1-p} \right)^r \leq C \left( \frac{1}{|I^{++}|} \int_{I^{++}} w \right)^r.
\]

By Lemma 2.5 \( w \in RH^+_r \).

To see this is the best range we take \( u \) as in Theorem 3.1 and \( v = 1 \). So we have \( w = u \in A^+_p \), and \( w \notin RH^+_{C/r} \). \( \Box \)

**Corollary 5.5.** Let \( w = uv^{1-p} \in A^+_p \) with \( u \in A^+_1 \), \( v \in A^-_1 \) and \( C = \max\{A^+_1(u), A^-_1(v)\} \), then \( w^r \in A^+_p \) for all \( 1 \leq \tau < \frac{C}{C-1} \) and the range is the best possible.

**Proof.** By Theorem 5.4 we have that \( w \in RH^+_r \) for all \( 1 \leq \tau < \frac{C}{C-1} \) and \( w^{1-p'} \in RH^+_r \) for all \( 1 \leq \tau' < \frac{C}{C-1} \). Let \( a < d \), we choose \( b, c \) such that \( b-a = d-c = \frac{1}{2}(d-a) \), and we also choose the point \( \frac{c+b}{2} \). Then, we have four intervals, namely, \( I^- = (a,b) \), \( I^+ = (b, \frac{c+b}{2}) \), \( I^* = (\frac{b+c}{2}, c) \), and \( I^{++} = (c,d) \). Now

\[
\frac{1}{|I^-|} \int_{I^-} w^\tau \left( \frac{1}{|I^{++}|} \right) \int_{I^{++}} w^{\tau(1-p')} \right)^{p-1} \leq \left( \frac{1}{|I|} \int_I w \right)^{\tau} \left( \frac{1}{|I^+|} \int_{I^+} w^{1-p'} \right)^{\tau(p-1)}
\]

\[
\leq C^\tau,
\]

thus \( w^r \in A^+_p \), (Lemma 2.6). Considering \( u \) as in Theorem 3.1, we see this is the best possible range. \( \Box \)

Using Theorem 5.4 we will show the exact range of \( q < p \) such that \( w \in A^+_p \) implies \( w \in A^+_q \).

**Theorem 5.6.** Let \( w = wv^{1-p} \in A^+_p \) with \( u \in A^+_1 \), \( v \in A^-_1 \) and \( C = A^-_1(v) \), then \( w \in A^+_q \) for all \( 1 + \frac{(p-1)(C-1)}{C} < q < \infty \) and this is the best range for \( q \).

**Proof.** Note that \( w^{1-p'} = vu^{1-p'} \in A^+_p \), by Theorem 5.4 \( w^{1-p'} \in RH^+_r \) for all \( 1 < r < \frac{C}{C-1} \). From lemma 4.4 for the classes \( RH^+_r \) and \( A^+_p \) we have that \( w^{(1-p')r} \in A^+_q \).
where \( q' = r(p'-1)+1 = \frac{r}{p-1} + 1 \). But this is the same as \( w^{1-q'} \in A_{q'}^+ \) i.e., \( w \in A_q^+ \)
for all \( 1 + (p - 1) \frac{C-1}{C} < q \).

To see this is the best range, let \( v(x) = x^{\frac{1-C}{C}} \) if \( x \leq 0 \) and equal to 0 if \( x > 0 \) and \( u = 1 \) for all \( x \). Note that \( v \in A_1^- \) and \( A_1^- (v) = C \). Then \( w = v^{1-p} \in A_p^+ \) and \( w \in A_q^+ \) for all \( q > 1 + (p - 1) \frac{C-1}{C} \). Observe that \( w \notin A_{1+(p-1)}^+ \). □

Finally the last theorem gives us the best possible range, for a weight in \( A_{\infty}^+ \).

**Theorem 5.7.** Let \( w \in A_{\infty}^+ \), \( w = w_0w_1 \), \( w_0 \in A_1^+, w_1 \in RH_\infty^+ \) and \( C = RH_\infty^+(w_1) \), then \( w \in A_p^+ \) for all \( p > C \). The range of \( p \)’s is the best possible.

**Proof.** Note that \( w_1 \in RH_\infty^+ \) implies \( w_1 \in A_p^+ \) for all \( p > C \), then

\[
\frac{1}{|I|} \int_I w_0w_1 \left( \frac{1}{|I|^{1+}} \int_{I^+} (w_0w_1)^{1-p'} \right)^{p-1} \\
\leq \sup_I w_1 \frac{1}{|I|} \int_I w_0 \left( \sup_{I^+} w_0^{1-p'} \right)^{p-1} \left( \frac{1}{|I|^{1+}} \int_{I^+} w_1^{1-p'} \right)^{p-1} \\
\leq C \frac{1}{|I|} \int_{I^+} w_0 \inf_{I^+} (w_0)^{p-1} \left( \frac{1}{|I|^{1+}} \int_{I^+} w_1^{1-p'} \right)^{p-1} \\
\leq C \sup_{I^+} (w_0)^{p-1} \frac{1}{|I|} \int_{I^+} w_0 \\
\leq C \frac{1}{\inf_{I^+} w_0} \leq C,
\]

by Lemma 2.6 \( w \in A_p^+ \) for all \( p > C \).

To see this is the best range, we consider \( w(x) = 0 \) if \( x \leq -1 \), \( |x|^{C-1} \) if \( -1 < x \leq 0 \) and 1 if \( x \geq 0 \) □

**References**


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