ON THE COIFMAN TYPE INEQUALITY FOR THE OSCILLATION OF THE ONE-SIDED DISCRETE SQUARE FUNCTION

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ABSTRACT. In this paper we study the Coifman type estimate for the oscillation of the one-sided discrete square function, S^+. We prove that for any A_∞^+ weight w, the $L^p(w)$-norm of this operator, and therefore the $L^p(w)$-norm of S^+, is dominated by a constant times the $L^p(w)$-norm of the one-sided Hardy-Littlewood maximal function iterated two times. For the k-th commutator with a BMO function we show that $k+2$ iterates of the one-sided Hardy-Littlewood maximal function are sufficient.

1. INTRODUCTION

In [5], Coifman and Fefferman proved that if T is a Calderón-Zygmund operator, w is an A_∞ weight and M is the Hardy-Littlewood maximal operator, then, for each p, $0 < p < \infty$, there exists C such that

$$\int |Tf|^p w \le C \int (Mf)^p w \,,$$

whenever the left-hand side is finite. Inequalities of the type

$$\int |Tf|^p w \le C \int (M_T f)^p w \,,$$

where T is an operator and M_T is a maximal operator which, in general, will depend on T, are known as Coifman type inequalities.

Recently, de la Torre and Torrea [26] and Lorente, Riveros and de la Torre [14] have studied inequalities with weights for the one-sided discrete square function defined as follows: for f locally integrable in $\mathbb{R}$ and $s > 0$, let us consider the averages

$$A_s f(x) = \frac{1}{s}\int_x^{x+s} f(y)dy.$$

The one-sided discrete square function of f is given by

$$S^+f(x) = \left(\sum_{n\in\mathbb{Z}} |A_{2^n}f(x) - A_{2^{n-1}}f(x)|^2\right)^{1/2}.$$

We write S^+ instead of S to emphasize that this is a one-sided operator, i.e., $S^+f(x) = S^+(f\chi_{(x,\infty)})(x)$.

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In [14] it was shown that if $0 < p < \infty$ and $w \in A_\infty^+$, then

$$\int_{\mathbb{R}} (S^+f)^p\omega \leq \int_{\mathbb{R}} ((M^+)^3f)^p\omega, \qquad f \in L_c^\infty\,, \tag{1.1}$$

whenever the left-hand side is finite, where $(M^+)^k$ stands for the k-th iteration of M^+, and

$$M^+f(x) = \sup_{h>0} \frac{1}{h}\int_x^{x+h} |f|.$$

A natural question left open by this result is the following: can we improve the result using fewer iterates of M^+ in (1.1)? In this note we study a bigger operator for which two iterates are enough. Therefore the inequality (1.1) is improved in two ways: a bigger operator on the left and a smaller operator on the right. The operator that we will study is the oscillation of the averages,

$$\mathcal{O}^+f(x) = \left(\sum_{n\in\mathbb{Z}} \sup_{s\in J_n} |A_{2^n}f(x) - A_sf(x)|^2\right)^{1/2},$$

where $J_n = [2^n, 2^{n+1})$. It is clear that $S^+f(x) \leq \mathcal{O}^+f(x)$ for all $x \in \mathbb{R}$.

If we look at the definition of $\mathcal{O}^+f(x)$, we see that the sequence $\{\tau_n(x)\}$, defined by $\tau_n(x) = \sup_{s\in J_n} |A_{2^n}f(x) - A_sf(x)|$, measures the oscillation of the $A_sf(x)$ in the interval J_n. Then we take the L^p norm of the ℓ^2 norm of this sequence. Operators of this kind are of interest in ergodic theory, [3], [8] and singular integrals. In [4] it is proved that the oscillation of a singular integral is a bounded operator in $L^p(dx)$ ($p > 1$). More recently in [9] it has been proved that the oscillation of the Hilbert Transform is bounded in $L^p(w)$ for any $w \in A_p$. Since our operator S^+ can be regarded as a one-sided singular integral, it is natural to try to extend their result to weights in the wider class A_p^+. Our main result is:

Theorem 1.1. *Let $\omega \in A_\infty^+$ and $0 < p < \infty$. Then, there exists $C > 0$ such that*

$$\int_{\mathbb{R}} (S^+f)^p\omega \leq \int_{\mathbb{R}} (\mathcal{O}^+f)^p\omega \leq C\int_{\mathbb{R}} ((M^+)^2f)^p\omega, \qquad f \in L_c^\infty,$$

whenever the left-hand side is finite.

Remark 1.2. As a consequence of the above theorem, if $1 < p < \infty$ and $\omega \in A_p^+$, we obtain that S^+ is bounded in $L^p(\omega)$, a result that was first proved in [26].

Remark 1.3. As in the case of the Hilbert Transform it is an open question if (1.1) holds with M^+ instead of $(M^+)^2$.

The paper is organized as follows: In Section 2 we introduce notation and recall some basic results about one-sided weights and maximal operators associated to Young functions. In section 3 we prove Theorem 1.1 and in section 4 we study the commutators of S^+ and $\mathcal{O}^+$ with a BMO function b. For $p > 1$ we prove Coifman type inequalities that imply the boundedness of the commutators in $L^p(w)$ whenever $w \in A_p^+$. For $p = 1$ we we also obtain a weak type inequality for the k-th order commutator.

2. Definitions and basic facts about one-sided operators

Definition 2.1. *The one-sided Hardy-Littlewood maximal operators M^+ and M^- are defined for locally integrable functions f by*

$$M^+f(x) = \sup_{h>0} \frac{1}{h} \int_x^{x+h} |f| \quad \text{and} \quad M^-f(x) = \sup_{h>0} \frac{1}{h} \int_{x-h}^{x} |f|.$$

The one-sided weights are defined as follows,

$$\sup_{a<b<c} \frac{1}{(c-a)^p} \int_a^b \omega \left(\int_b^c \omega^{1-p'} \right)^{p-1} < \infty, \quad 1 < p < \infty, \qquad (A_p^+)$$

$$M^-\omega(x) \leq C\omega(x) \quad \text{a.e.} \qquad (A_1^+)$$

A_∞^+ is defined as the union of the A_p^+ classes,

$$A_\infty^+ = \cup_{p\geq 1} A_p^+. \qquad (A_\infty^+)$$

The A_p^- classes are defined reversing the orientation of $\mathbb{R}$. It is interesting to note that $A_p = A_p^+ \cap A_p^-$, $A_p \subsetneq A_p^+$ and $A_p \subsetneq A_p^-$. (See [23], [15], [16], [17] for more definitions and results.)

It was proved in [26], that $\omega \in A_p^+$, $1 < p < \infty$, if, and only if, S^+ is bounded from $L^p(\omega)$ to $L^p(\omega)$, and that $\omega \in A_1^+$, if, and only if, S^+ is of weak-type (1,1) with respect to ω.

Definition 2.2. *Let b be a locally integrable function. We say that $b \in BMO$ if*

$$||b||_{BMO} = \sup_I \frac{1}{|I|} \int_I |b - b_I| < \infty,$$

where I denotes any bounded interval and $b_I = \frac{1}{|I|} \int_I b$.

Definition 2.3. *Let f be a locally integrable function. The one-sided sharp maximal function is defined by*

$$M^{+,\#}(f)(x) = \sup_{h>0} \frac{1}{h} \int_x^{x+h} \left(f(y) - \frac{1}{h} \int_{x+h}^{x+2h} f \right)^+ dy.$$

For $\delta > 0$ we define

$$M_\delta^{+,\#} f(x) = \left(M^{+,\#} |f|^\delta (x) \right)^{1/\delta}.$$

It is proved in [18] that

$$\begin{aligned} M^{+,\#}(f)(x) &\leq \sup_{h>0} \inf_{a\in\mathbb{R}} \frac{1}{h} \int_x^{x+h} (f(y) - a)^+ dy + \frac{1}{h} \int_{x+h}^{x+2h} (a - f(y))^+ dy \\ &\leq C||f||_{BMO}. \end{aligned}$$

Now we give definitions and results about Young functions. A function $B : [0, \infty) \to [0, \infty)$ is a Young function if it is continuous, convex, increasing and satisfies $B(0) = 0$ and $B(t) \to \infty$ as $t \to \infty$. The Luxemburg norm of a function f, given by B is

$$||f||_B = \inf \left\{ \lambda > 0 : \int B\left(\frac{|f|}{\lambda} \right) \leq 1 \right\},$$

and so the B-average of f over I is

$$||f||_{B,I} = \inf\left\{\lambda > 0 : \frac{1}{|I|}\int_I B\left(\frac{|f|}{\lambda}\right) \leq 1\right\}.$$

We will denote by $\overline{B}$ the complementary function associated to B (see [1]). The following version of Hölder's inequality holds,

$$\frac{1}{|I|}\int_I |f\, g| \leq 2||f||_{B,I}||g||_{\overline{B},I}.$$

This inequality can be extended to three functions (see [19]). If A, B, C are Young functions such that

$$A^{-1}(t)B^{-1}(t) \leq C^{-1}(t),$$

then

$$||fg||_{C,I} \leq 2||f||_{A,I}||g||_{B,I}. \tag{2.1}$$

Definition 2.4. *For each locally integrable function f, the one-sided maximal operators associated to the Young function B are defined by*

$$M_B^+ f(x) = \sup_{x<b} \|f\|_{B,(x,b)} \quad \textit{and} \quad M_B^- f(x) = \sup_{a<x} \|f\|_{B,(a,x)}.$$

We will be dealing with the Young functions $B_k(t) = e^{t^{1/k}} - 1$ and $\overline{B_k}(t) = t(1 + \log^+(t))^k$, $k \in \mathbb{N}$. The maximal operator associated to $\overline{B_k}$, $M^+_{\overline{B_k}}$ will be denoted by $M^+_{L(1+\log^+ L)^k}$. It is proved in [22] that $M^+_{L(1+\log^+ L)^k}$ is pointwise equivalent to $(M^+)^{k+1}$.

It is convenient to look at our operators as vector valued. Let us consider the sequence

$$H(x) = \left\{\frac{1}{2^n}\chi_{(-2^n,0)}(x) - \frac{1}{2^{n-1}}\chi_{(-2^{n-1},0)}(x)\right\}_{n\in\mathbb{Z}}$$

and let us define the operator $U : f \to Uf$ by

$$Uf(x) = \int_{\mathbb{R}} H(x-y)f(y)dy.$$

Then it is clear that $S^+f(x) = ||Uf(x)||_{\ell^2}$. If instead of the sequence H we consider for each $s > 0$ the sequence $K(x) = \{K_{n,s}(x)\}_{n\in\mathbb{Z},}$, where

$$K_{n,s}(x) = \left(\frac{1}{2^n}\chi_{(-2^n,0)}(x) - \frac{1}{s}\chi_{(-s,0)}(x)\right)\chi_{J_n}(s)\,,$$

we can define the operator V acting on locally integrable functions f, as $Vf(x) = \int_{\mathbb{R}} K(x-y)f(y)\,dy$.

If for functions $h : \mathbb{R}\times\mathbb{Z} \to \mathbb{R}$ we define the norm

$$||h||_E = \left(\sum_{n\in\mathbb{Z}} \sup_{s\in J_n} |h(s,n)|^2\right)^{1/2},$$

then $\mathcal{O}^+f(z) = ||Vf(z)||_E$.

3. Proof of Theorem 1.1

The key point in the proof of Theorem 1.1 is the following pointwise estimate: for each $0<\delta<1$, there exists C so that for any locally integrable function,

$$M_\delta^{+,\#}(\mathcal{O}^+f)(x)\le CM^+f(x)+CM^+_{L(1+\log^+L)}f(x). \tag{3.1}$$

Let us prove inequality (3.1). For $0<\delta<1$ we have

$$M_\delta^{+,\#}(\mathcal{O}^+f)(x)\le C\sup_{h>0}\inf_{c\in\mathbb{R}}\left(\frac{1}{h}\int_x^{x+2h}\big||\mathcal{O}^+f(y)|^\delta-|c|^\delta\big|\,dy\right)^{1/\delta}. \tag{3.2}$$

Let $x\in\mathbb{R}$ and $h>0$. Let us consider the unique $i\in\mathbb{Z}$ such that $2^i\le h<2^{i+1}$ and let denote by J the interval $J=[x,x+2^{i+3})$. If we write $f=f_1+f_2$, where $f_1=f\chi_J$, and choose $c=\mathcal{O}^+(f_2)(x)$, we have

$$\begin{aligned}\left(\frac{1}{h}\int_x^{x+2h}\big||\mathcal{O}^+f(y)|^\delta-|\mathcal{O}^+f_2(x)|^\delta\big|\,dy\right)^{1/\delta}&\\ \le C\left(\frac{1}{2^i}\int_x^{x+2^{i+2}}\big|\mathcal{O}^+f(y)-\mathcal{O}^+f_2(x)\big|^\delta\,dy\right)^{1/\delta}&\\ \le C\left(\frac{1}{2^i}\int_x^{x+2^{i+2}}\big|\mathcal{O}^+f_1(y)\big|^\delta\,dy\right)^{1/\delta}+C\left(\frac{1}{2^i}\int_x^{x+2^{i+2}}\big|\mathcal{O}^+f_2(y)-\mathcal{O}^+f_2(x)\big|^\delta\,dy\right)^{1/\delta}&\\ =I+II.&\end{aligned} \tag{3.3}$$

Using Lemma 2.1 (1) in [4], that is, the fact that the oscillation of S^+ is of weak type (1,1) with respect to the Lebesgue's measure, and Kolmogorov's inequality, we get

$$I\le C\frac{1}{2^i}\int_x^{x+2^{i+3}}|f(y)|dy\le CM^+f(x). \tag{3.4}$$

In order to estimate II, we first use Jensen's inequality and obtain

$$\begin{aligned}II&=C\left(\frac{1}{2^i}\int_x^{x+2^{i+2}}\big|||Vf_2(y)||_E-||Vf_2(x)||_E\big|^\delta\,dy\right)^{1/\delta}\\ &\le C\frac{1}{2^i}\int_x^{x+2^{i+2}}||Vf_2(y)-Vf_2(x)||_E\,dy.\end{aligned} \tag{3.5}$$

Let us estimate $||Vf_2(y) - Vf_2(x)||_E$.

$$
\begin{aligned}
&||Vf_2(y) - Vf_2(x)||_E \\
&= \left|\left| \left\{ \left(\frac{1}{2^n}\int_y^{y+2^n} f_2 - \frac{1}{s}\int_y^{y+s} f_2 \right) \chi_{J_n}(s) \right\}_{n\in\mathbb{Z},s\in\mathbb{R}} \right.\right. \\
&\qquad \left.\left. - \left\{ \left(\frac{1}{2^n}\int_x^{x+2^n} f_2 - \frac{1}{s}\int_x^{x+s} f_2 \right) \chi_{J_n}(s) \right\}_{n\in\mathbb{Z},s\in\mathbb{R}} \right|\right|_E \\
&\le \left|\left| \left\{ \left(\frac{1}{2^n}\int_y^{y+2^n} f_2 - \frac{1}{2^n}\int_x^{x+2^n} f_2 \right) \chi_{J_n}(s) \right\}_{n\in\mathbb{Z},s\in\mathbb{R}} \right|\right|_E \\
&\qquad + \left|\left| \left\{ \left(\frac{1}{s}\int_x^{x+s} f_2 - \frac{1}{s}\int_y^{y+s} f_2 \right) \chi_{J_n}(s) \right\}_{n\in\mathbb{Z},s\in\mathbb{R}} \right|\right|_E \\
&= III + IV. \qquad (3.6)
\end{aligned}
$$

Observe that since $y \in (x, x+2^{i+2})$ and f_2 has support in $(x+2^{i+3}, \infty)$, it follows that, if $n \le i+2$, then $x+2^n \le x+2^{i+2}$ and $y+2^n \le x+2^{i+2}+2^n \le x+2^{i+2}+2^{i+2} = x+2^{i+3}$. As a consequence the only non-zero terms in III are those with $n > i+2$. Therefore

$$
III \le \left(\sum_{n=i+3}^{\infty} \left| \frac{1}{2^n}\int_{x+2^n}^{y+2^n} f \right|^2 \right)^{1/2}. \qquad (3.7)
$$

Let us consider the Young function $B_1(t) = \mathrm{e}^t - 1$. Then $\overline{B_1}(t) = t(1+\log^+ t)$ and $B_1^{-1}(t) = \log^+(1+t)$. Using the generalized Hölder's inequality, we obtain that, for $n \ge i+3$,

$$
\begin{aligned}
\left| \frac{1}{2^n}\int_{x+2^n}^{y+2^n} f \right| &\le \frac{1}{2^n}\int_{x+2^n}^{x+2^{n+1}} |f|\chi_{[x+2^n,y+2^n)} \\
&\le CM^+_{\overline{B_1}}f(x)\, \left|\left|\chi_{[x+2^n,y+2^n)}\right|\right|_{B_1,[x+2^n,x+2^{n+1})} \\
&= CM^+_{\overline{B_1}}f(x)\frac{1}{B_1^{-1}\left(\frac{2^n}{y-x}\right)} \le CM^+_{\overline{B_1}}f(x)\frac{1}{B_1^{-1}\left(2^{n-i-2}\right)}, \qquad (3.8)
\end{aligned}
$$

where in the last inequality we have used that $y - x \le 2^{i+2}$ and that B_1^{-1} is nondecreasing.

Putting together (3.7) and (3.8) we obtain

$$
\begin{aligned}
III &\le CM^+_{\overline{B_1}}f(x)\left(\sum_{n=i+3}^{\infty} \frac{1}{\left(B_1^{-1}\left(2^{n-i-2}\right)\right)^2} \right)^{1/2} \\
&\le CM^+_{\overline{B_1}}f(x)\left(\sum_{n=i+3}^{\infty} \frac{1}{(n-i-2)^2} \right)^{1/2} = CM^+_{\overline{B_1}}f(x). \qquad (3.9)
\end{aligned}
$$

Let us estimate IV. For $n \in \mathbb{Z}$, set

$$\beta_n = \sup_{s \in J_n} \left| \frac{1}{s} \int_x^{x+s} f_2 - \frac{1}{s} \int_y^{y+s} f_2 \right|.$$

Then, if $\beta_n \neq 0$, we have that there exists $s_n \in J_n$ such that

$$\left| \frac{1}{s_n} \int_x^{x+s_n} f_2 - \frac{1}{s_n} \int_y^{y+s_n} f_2 \right| > \frac{1}{2} \beta_n.$$

If $n \leq i+1$ then $y + s_n \leq y + 2^{n+1} \leq x + 2^{i+2} + 2^{n+1} \leq x + 2^{i+3}$. Therefore in IV we may assume $n \geq i+2$.

Using again generalized Hölder's inequality, we get that, for $n \geq i+2$,

$$\begin{aligned} \beta_n &\leq C \frac{1}{s_n} \int_{x+s_n}^{y+s_n} |f_2| \leq C \frac{1}{s_n} \int_x^{x+2^{n+2}} |f| \chi_{[x+s_n, y+s_n)} \\ &\leq C \frac{2^{n+2}}{s_n} M_{B_1}^+ f(x) \left|\left| \chi_{[x+s_n, y+s_n)} \right|\right|_{B_1, (x, x+2^{n+2})} \\ &= C M_{B_1}^+ f(x) \frac{1}{B_1^{-1}\left(\frac{2^{n+2}}{y-x}\right)} \leq C M_{B_1}^+ f(x) \frac{1}{B_1^{-1}(2^{n-i})}. \end{aligned} \tag{3.10}$$

Then,

$$IV \leq C M_{B_1}^+ f(x) \left(\sum_{n=i+2}^{\infty} \frac{1}{\left(B_1^{-1}(2^{n-i})\right)^2} \right)^{1/2} = C M_{B_1}^+ f(x). \tag{3.11}$$

Collecting inequalities (3.2)–(3.6), (3.9) and (3.11), we obtain (3.1). On the other hand, we have that $M_{B_1}^+ f = M_{L(1+\log^+ L)}^+ f$ is pointwise equivalent to $(M^+)^2 f$ (see [22]). As a consequence, (3.1) gives

$$M_\delta^{+,\#}(\mathcal{O}^+ f)(x) \leq C (M^+)^2 f(x), \quad \text{a.e. } x \in \mathbb{R}.$$

To finish the proof of Theorem 1.1 we use theorem 4 in [18]: since $w \in A_\infty^+$, there exists $r > 1$, such that $w \in A_r^+$. Then, for δ small enough, we get that $r < p/\delta$ and thus, $w \in A_{p/\delta}^+$. Therefore,

$$\begin{aligned} \int_{\mathbb{R}} \left|\mathcal{O}^+ f\right|^p \omega &\leq \int_{\mathbb{R}} \left(M_\delta^+ \left(\mathcal{O}^+ f\right)\right)^p \omega \\ &\leq C \int_{\mathbb{R}} \left(M_\delta^{+,\#} \left(\mathcal{O}^+ f\right)\right)^p \omega \leq C \int_{\mathbb{R}} \left((M^+)^2 f\right)^p \omega, \end{aligned} \tag{3.12}$$

whenever the left hand side is finite. □

4. Commutators

The commutators of singular integrals with BMO functions have been extensively studied (see [2],[6],[24],[25],[20],[21],[11],[12],[13]). Since S^+ can be considered as a singular integral whose kernel satisfies a weaker condition (see [14]), it is interesting to know if the results about commutators of singular integrals can be extended to S^+. In [14] we have proved that the classical results about boundedness with weights can be extended to S^+ and, furthermore, can be improved allowing a wider class of weights, since S^+ is a one-sided operator. The results in [20] and [21] have been improved in [11] for one-sided singular integrals. Observe that for singular integrals satisfying the usual Lipschitz condition one obtains Mf instead of M^2f in Theorem 1.1. Therefore we can not expect to obtain the same results for the commutator of S^+ as we obtained in [11] for one-sided singular integrals. However, we can give estimates of the same kind, increasing in one the iterations of M^+. Concretely, for the k-th order commutator of S^+ we have:

Theorem 4.1. *Let $b \in BMO$ and $k = 0, 1, 2, \ldots$. Let us define the k-th order commutator of S^+ and $\mathcal{O}^+$ by*

$$S_b^{+,k} f(x) = \left\| \int_{\mathbb{R}} (b(x) - b(y))^k H(x-y) f(y) dy \right\|_{\ell^2},$$

and

$$\mathcal{O}_b^{+,k} f(x) = \left\| \int_{\mathbb{R}} (b(x) - b(y))^k K(x-y) f(y) dy \right\|_{E}.$$

(Observe that for $k = 0$ we obtain S^+ and $\mathcal{O}^+$.) Then, for $0 < p < \infty$ and $w \in A_\infty^+$, there exists $C > 0$ such that,

$$\int_{\mathbb{R}} (S_b^{+,k} f)^p w \leq \int_{\mathbb{R}} (\mathcal{O}_b^{+,k} f)^p w \leq C \int_{\mathbb{R}} ((M^+)^{k+2} f)^p w, \qquad f \in L_c^\infty,$$

whenever the left-hand side is finite.

Remark 4.2. In [10], the $L^{\mathcal{A},k}$-Hörmander condition was introduced. If we just use that H, the vector valued kernel of S^+, satisfies the $L^{\mathcal{A},k}$-Hörmander condition for $\mathcal{A}(t) = e^{\frac{1}{1+k+\epsilon}}$, then theorem 3.3 in [10] gives $(M^+)^{k+3}$ instead of $(M^+)^{k+2}$ in the previous inequality for $S_b^{+,k}$.

Remark 4.3. In particular, we have that for $1 < p < \infty$ and $w \in A_p^+$, $S_b^{+,k}$ is bounded in $L^p(\omega)$ which was proved using a different approach in [12].

In [26] it was proved that S^+ is of weak type (1,1) with respect to w, iff $w \in A_1^+$. The commutator is more singular than the operator. A fact that is not apparent in the $L^p(w)$ norm but it makes a difference near $L^1(w)$.

Theorem 4.4. *Let $b \in BMO$, $w \in A_\infty^+$ and $k = 0, 1, 2....$ Then, there exists $C > 0$ such that*

$$\begin{aligned} w(\{x \in \mathbb{R} : S_b^{+,k} f(x) > \lambda\}) &\leq w(\{x \in \mathbb{R} : \mathcal{O}_b^{+,k} f(x) > \lambda\}) \\ &\leq C \int_{\mathbb{R}} \frac{|f(x)|}{\lambda} \log^+ \left(1 + \frac{|f(x)|}{\lambda}\right)^{k+1} M^- w(x)\, dx, \quad f \in L_c^\infty, \end{aligned}$$

whenever the left-hand side is finite.

Remark 4.5. If $w \in A_1^+$, we can put w instead of $M^- w$ in the right hand side.

The following lemma will allow us to use induction in the proof of Theorem 4.1.

Lemma 4.6. *Let $0<\delta<\gamma<1$, $b\in BMO$ and $k\in\mathbb{N}\cup\{0\}$. Then there exists $C>0$ such that for any locally integrable f,*

$$M_\delta^{+,\#}\left(\mathcal{O}_b^{+,k}f\right)(x)\le C\sum_{j=0}^{k-1}M_\gamma^{+}(\mathcal{O}_b^{+,j}f)(x)+CM^{+}_{L(1+\log^+L)^{1+k}}f(x)$$
$$\le C\sum_{j=0}^{k-1}M_\gamma^{+}(\mathcal{O}_b^{+,j}f)(x)+C(M^+)^{k+2}f(x)\ a.e.$$

Proof. The case $k=0$ follows from inequality (3.1) in the proof of Theorem (1.1). Let us prove the case $k\ge 1$. Let λ be an arbitrary constant. Then, we proceed as in inequality (3.2) in [12], and get

$$\mathcal{O}_b^{+,k}f(x)=\left|\left|\int_{\mathbb{R}}(b(x)-b(y))^kK(x-y)f(y)dy\right|\right|_E$$
$$\le\mathcal{O}^+((b-\lambda)^kf)(x)+\sum_{m=0}^{k-1}C_{k,m}|b(x)-\lambda|^{k-m}\mathcal{O}_b^{+,m}f(x).\qquad(4.1)$$

Let $x\in\mathbb{R}$ and $h>0$. Let $i\in\mathbb{Z}$ be such that $2^i\le h<2^{i+1}$ and set $J=[x,x+2^{i+3})$. Then, write $f=f_1+f_2$, where $f_1=f\chi_J$ and set $\lambda=b_J$. Then (see (3.2) in [11]), for any $a\in\mathbb{R}$ we have

$$\left(\frac{1}{h}\int_x^{x+h}\left|(\mathcal{O}_b^{+,k}f(y))^\delta-|a|^\delta\right|dy\right)^{\frac{1}{\delta}}+\left(\frac{1}{h}\int_{x+h}^{x+2h}\left|(\mathcal{O}_b^{+,k}f(y))^\delta-|a|^\delta\right|dy\right)^{\frac{1}{\delta}}$$
$$\le C\left[\sum_{m=0}^{k-1}\left(\frac{1}{h}\int_x^{x+2h}|b(y)-b_J|^{(k-m)\delta}(\mathcal{O}_b^{+,m}f(y))^\delta dy\right)^{\frac{1}{\delta}}\right.$$
$$+\left(\frac{1}{h}\int_x^{x+2h}|\mathcal{O}^+((b-b_J)^kf_1)(y)|^\delta dy\right)^{\frac{1}{\delta}}$$
$$\left.+\left(\frac{1}{h}\int_x^{x+2h}|\mathcal{O}^+((b-b_J)^kf_2)(y)-a|^\delta dy\right)^{\frac{1}{\delta}}\right]$$
$$=(I)+(II)+(III).\qquad(4.2)$$

(I) is estimated exactly as in inequality (3.3) of [11],

$$(I)\le C\sum_{m=0}^{k-1}M_\gamma^{+}(\mathcal{O}_b^{+,m}f)(x).\qquad(4.3)$$

Kolmogorov's inequality plus the fact that $\mathcal{O}^+$ is of weak type $(1,1)$ with respect to the Lebesgue measure imply

$$(II)\le C\frac{1}{h}\int_x^{x+2^{i+3}}|b(y)-b_J|^k|f(y)|dy.$$

Using now the generalized Hölder's inequality with $B_{k+1}(t)=e^{t^{1/(k+1)}}-1$ and $\overline{B_{k+1}}(t)=t(1+\log^+t)^{k+1}$ we get,

$$(II)\le C|||b-b_J|^k||_{B_{k+1},J}||f||_{\overline{B_{k+1}},J}.$$

It follows from John-Nirenberg's inequality that

$$
\begin{aligned}
(II) &\leq C||b-b_J||^{k+1}_{B_1,J}||f||_{\overline{B_{k+1}},J} \leq C||b||^{k+1}_{BMO}M^+_{\overline{B_{k+1}}}f(x)\\
&\leq C(M^+)^{k+2}f(x). \qquad (4.4)
\end{aligned}
$$

For (III) we take $a=\mathcal{O}^+((b-b_J)^kf_2)(x)$. Then, by Jensen's inequality,

$$
\begin{aligned}
(III) &\leq C\frac{1}{2^i}\int_x^{x+2^{i+3}} |\mathcal{O}^+((b-b_J)^kf_2)(y)-\mathcal{O}^+((b-b_J)^kf_2)(x)|\,dy\\
&\leq C\frac{1}{2^i}\int_x^{x+2^{i+3}} ||V((b-b_J)^kf_2)(y)-V((b-b_J)^kf_2)(x)||_E\,dy. \qquad (4.5)
\end{aligned}
$$

For $j\geq 3$, let $I_j=[x+2^j,x+2^{j+1})$ and $\tilde{I}_j=[x,x+2^{j+1})$. As in inequality (3.6) we have

$$
\begin{aligned}
&||V((b-b_J)^kf_2)(y)-V((b-b_J)^kf_2)(x)||_E\\
&\leq \left|\left|\left\{\left(\frac{1}{2^n}\int_y^{y+2^n}(b-b_J)^kf_2-\frac{1}{2^n}\int_x^{x+2^n}(b-b_J)^kf_2\right)\chi_{J_n}(s)\right\}_{n\in\mathbb{Z},s\in\mathbb{R}}\right|\right|_E\\
&\quad+\left|\left|\left\{\left(\frac{1}{s}\int_x^{x+s}(b-b_J)^kf_2-\frac{1}{s}\int_y^{y+s}(b-b_J)^kf_2\right)\chi_{J_n}(s)\right\}_{n\in\mathbb{Z},s\in\mathbb{R}}\right|\right|_E\\
&=(III_n)+(III_s). \qquad (4.6)
\end{aligned}
$$

For (III_n), we proceed as in the estimate of (III) in Theorem 1.1. Since $y\in(x,x+2^{i+2})$ and f_2 has support in $(x+2^{i+3},\infty)$, it follows that, if $n\leq i+2$, then $x+2^n\leq x+2^{i+2}$ and $y+2^n\leq x+2^{i+2}+2^n\leq x+2^{i+2}+2^{i+2}=x+2^{i+3}$. As a consequence, we only have to take into account $n>i+2$. Therefore

$$
\begin{aligned}
(III_n) &= \left(\sum_{n=i+3}^{\infty}\left|\frac{1}{2^n}\int_{x+2^n}^{y+2^n}f(b-b_J)^k\right|^2\right)^{1/2}\\
&\leq C\left(\sum_{n=i+3}^{\infty}\left|\frac{1}{2^n}\int_{x+2^n}^{y+2^n}f(b-b_{I_n})^k\right|^2\right)^{1/2}\\
&\qquad\qquad+C\left(\sum_{n=i+3}^{\infty}\left|\frac{1}{2^n}\int_{x+2^n}^{y+2^n}f(b_{I_n}-b_J)^k\right|^2\right)^{1/2}\\
&= C\left(\sum_{n=i+3}^{\infty}|(IV_n)|^2\right)^{1/2}+C\left(\sum_{n=i+3}^{\infty}|(V_n)|^2\right)^{1/2}. \qquad (4.7)
\end{aligned}
$$

Using the generalized Hölder's inequality (2.1) with $A = B_1$, $B = \overline{B_{k+1}}$ and $C = \overline{B_k}$, followed by John-Nirenberg's inequality we get

$$\begin{aligned}(IV_n) &\leq C\frac{\sqrt{2}}{2^n}\int_{I_n}|b(t)-b_{I_n}|^k|f(t)|\chi_{(x+2^n,y+2^n)}(t)\,dt\\ &\leq C||(b-b_{I_n})^k||_{B_k,\tilde{I_n}}||f\chi_{(x+2^n,y+2^n)}||_{\overline{B_k},\tilde{I_n}}\\ &\leq C||b||^k_{BMO}||f||_{\overline{B_{k+1}},\tilde{I_n}}||\chi_{(x+2^n,y+2^n)}||_{B_1,\tilde{I_n}}\\ &\leq CM^+_{\overline{B_{k+1}}}f(x)\frac{1}{B_1^{-1}(2^{n-i-2})}. \qquad (4.8)\end{aligned}$$

For (V_n) again the generalized Hölder's inequality is used to obtain

$$\begin{aligned}(V_n) &\leq C(n-i-1)^k||f||_{\overline{B_{k+1}},\tilde{I_n}}||\chi_{(x+2^n,y+2^n)}||_{B_{k+1},\tilde{I_n}}\\ &\leq C(n-i-1)^kM^+_{\overline{B_{k+1}}}f(x)\frac{1}{B_{k+1}^{-1}(2^{n-i-2})}. \qquad (4.9)\end{aligned}$$

Putting together inequalities (4.8) and (4.9) we get

$$\begin{aligned}(III_n) \leq\ & CM^+_{\overline{B_{k+1}}}f(x)\left(\sum_{n\geq i+3}\frac{1}{(B_1^{-1}(2^{n-i-2}))^2}\right)^{1/2}\\ &+CM^+_{\overline{B_{k+1}}}f(x)\left(\sum_{n\geq i+3}(n-i-1)^{2k}\frac{1}{(B_{k+1}^{-1}(2^{n-i-2}))^2}\right)^{1/2}\\ \leq\ & CM^+_{\overline{B_{k+1}}}f(x)\leq C(M^+)^{k+2}f(x). \qquad (4.10)\end{aligned}$$

Let us estimate (III_s). As in Theorem 1.1, for $n \in \mathbb{Z}$, set

$$\beta_n = \sup_{s\in J_n}\left|\frac{1}{s}\int_x^{x+s}(b-b_J)^kf_2-\frac{1}{s}\int_y^{y+s}(b-b_J)^kf_2\right|.$$

Then, if $\beta_n \neq 0$ there exists $s_n \in J_n$, such that

$$\left|\frac{1}{s_n}\int_x^{x+s_n}(b-b_J)^kf_2-\frac{1}{s_n}\int_y^{y+s_n}(b-b_J)^kf_2\right|>\frac{1}{2}\beta_n.$$

If $n \leq i+1$ then $y+s_n \leq y+2^{n+1} \leq x+2^{i+2}+2^{n+1} \leq x+2^{i+3}$. Therefore we only have to consider $n \geq i+2$ in the estimate of III_s. Then

$$\begin{aligned}\beta_n &\leq C\frac{1}{s_n}\int_{x+s_n}^{y+s_n}|(b-b_J)^kf_2|\\ &\leq C\frac{2^{n+2}}{s_n}\frac{1}{2^{n+2}}\int_x^{x+2^{n+2}}|(b(t)-b_J)^kf(t)\chi_{[x+s_n,y+s_n)}(t)|dt\\ &\leq C\frac{1}{2^{n+2}}\int_{\tilde{I}_{n+1}}|(b(t)-b_{I_{n+1}})^kf(t)\chi_{[x+s_n,y+s_n)}(t)|dt\\ &\quad+C\frac{1}{2^{n+2}}\int_{\tilde{I}_{n+1}}|(b_{I_{n+1}}-b_J)^kf(t)\chi_{[x+s_n,y+s_n)}(t)|dt.\end{aligned}$$

By the generalized Hölder's inequality (2.1) with the Young functions used in (4.8) and (4.9), we get

$$\begin{aligned}\beta_n &\le C\|(b-b_{I_{n+1}})^k\|_{B_k,\tilde{I}_{n+1}}\|f\|_{\overline{B_{k+1}},\tilde{I}_{n+1}}\|\chi_{(x+s_n,y+s_n)}\|_{B_1,\tilde{I}_{n+1}}\\ &\quad + C(n-i)^k\|f\|_{\overline{B_{k+1}},\tilde{I}_{n+1}}\|\chi_{(x+s_n,y+s_n)}\|_{B_{k+1},\tilde{I}_{n+1}}\\ &\le CM^+_{\overline{B_{k+1}}}f(x)\left(\frac{1}{B_1^{-1}(\frac{2^{n+2}}{y-x})}+(n-i)^k\frac{1}{B_{k+1}^{-1}(\frac{2^{n+2}}{y-x})}\right)\\ &\le CM^+_{\overline{B_{k+1}}}f(x)\left(\frac{1}{B_1^{-1}(2^{n-i})}+\frac{(n-i)^k}{B_{k+1}^{-1}(2^{n-i})}\right).\end{aligned}$$

Then,

$$\begin{aligned}(III_s) &\le CM^+_{\overline{B_{k+1}}}f(x)\left[\left(\sum_{n=i+2}^{\infty}\left(\frac{1}{B_1^{-1}(2^{n-i})}\right)^2\right)^{1/2}+\left(\sum_{n=i+2}^{\infty}\left(\frac{(n-i)^k}{B_{k+1}^{-1}(2^{n-i})}\right)^2\right)^{1/2}\right]\\ &\le CM^+_{\overline{B_{k+1}}}f(x)\le C(M^+)^{k+2}f(x). \qquad (4.11)\end{aligned}$$

Collecting now inequalities (4.2)–(4.6), (4.10) and (4.11) we finish the proof of Lemma 4.6.

□

Proof of Theorem 4.1. Let us observe that from the definition of $\|\cdot\|_E$, it follows that $S_b^{+,k}f \le \mathcal{O}_b^{+,k}f$, therefore the first inequality in Theorem 4.1 holds trivially. For the second one, we will proceed by induction on k. The case $k=0$ is Theorem 1.1. Let now $k\in\mathbb{N}$ and suppose that Theorem 4.1 holds for $j=1,...,k-1$. In order to prove the case $j=k$ we proceed as in (3.12): since $w\in A_\infty^+$, there exists $r>1$, such that $w\in A_r^+$. Then, for δ small enough, we get that $r<p/\delta$ and thus, $w\in A^+_{p/\delta}$. If γ is such that $\delta<\gamma<1$, then by Theorem 4 in [18] and Lemma 4.6 we have

$$\begin{aligned}\|\mathcal{O}_b^{+,k}f\|_{L^p(w)} &\le \|M_\delta^+(\mathcal{O}_b^{+,k}f)\|_{L^p(w)}\\ &\le C\|M_\delta^{+,\#}(\mathcal{O}_b^{+,k}f)\|_{L^p(w)}\\ &\le C\sum_{j=0}^{k-1}\|M_\gamma^+(\mathcal{O}_b^{+,j}f)\|_{L^p(w)}\\ &\qquad + C\|(M^+)^{k+2}f\|_{L^p(w)}. \qquad (4.12)\end{aligned}$$

Then, by recurrence, we can continue the chain of inequalities in (4.12) by

$$\le C\sum_{j=0}^{k-1}\|(M^+)^{j+2}f\|_{L^p(w)}+C\|(M^+)^{k+2}f\|_{L^p(w)}\le C\|(M^+)^{k+2}f\|_{L^p(w)}.$$

□

Proof of Theorem 4.4. First of all we observe that, by theorem 3 in [22] we have that

$$w(\{x\in\mathbb{R}:(M^+)^{k+2}f(x)>\lambda\})\le C\int_{\mathbb{R}}\frac{|f|}{\lambda}\log^+\left(1+\frac{|f|}{\lambda}\right)^{k+1}M^-w.$$

On the other hand, let us notice that theorem 3.1 in [7] holds for one-sided weights with minor changes in the proof. Therefore, using the Coifman type estimate in Theorem 4.1 and the fact that $\overline{B_{k+1}}(t) = t(1+\log^+ t)^{k+1}$ is submultiplicative, we get

$$\sup_{t>0} \frac{1}{\overline{B_{k+1}}(1/t)} w(\{x : |\mathcal{O}_b^{+,k} f(x)| > t\}) \leq C \sup_{t>0} \frac{1}{\overline{B_{k+1}}(1/t)} w(\{x : (M^+)^{k+2} f(x) > t\}).$$

Now, following the same argument used in the proof of theorem 3.3, part (a) in [10], we get the desired result. □

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