ON THE COIFMAN TYPE INEQUALITY FOR THE OSCILLATION OF THE ONE-SIDED DISCRETE SQUARE FUNCTION

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ABSTRACT. In this paper we study the Coifman type estimate for the oscillation of the one-sided discrete square function, $S^+$. We prove that for any $A^+_\infty$ weight $w$, the $L^p(w)$-norm of this operator, and therefore the $L^p(w)$-norm of $S^+$, is dominated by a constant times the $L^p(w)$-norm of the one-sided Hardy-Littlewood maximal function iterated two times. For the $k$-th commutator with a $BMO$ function we show that $k + 2$ iterates of the one-sided Hardy-Littlewood maximal function are sufficient.

1. INTRODUCTION

In [5], Coifman and Fefferman proved that if $T$ is a Calderón-Zygmund operator, $w$ is an $A_\infty$ weight and $M$ is the Hardy-Littlewood maximal operator, then, for each $p$, $0 < p < \infty$, there exists $C$ such that

$$\int |Tf|^p w \leq C \int (Mf)^p w,$$

whenever the left-hand side is finite. Inequalities of the type

$$\int |Tf|^p w \leq C \int (M_Tf)^p w,$$

where $T$ is an operator and $M_T$ is a maximal operator which, in general, will depend on $T$, are known as Coifman type inequalities.

Recently, de la Torre and Torrea [26] and Lorente, Riveros and de la Torre [14] have studied inequalities with weights for the one-sided discrete square function defined as follows: for $f$ locally integrable in $\mathbb{R}$ and $s > 0$, let us consider the averages

$$A_s f(x) = \frac{1}{s} \int _x ^{x+s} f(y)dy.$$

The one-sided discrete square function of $f$ is given by

$$S^+ f(x) = \left( \sum _{n \in \mathbb{Z}} |A_{2^n} f(x) - A_{2^{n-1}} f(x)|^2 \right)^{1/2}.$$

We write $S^+$ instead of $S$ to emphasize that this is a one-sided operator, i.e., $S^+ f(x) = S^+(f \chi _{(x,\infty)})(x)$.

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In [14] it was shown that if $0 < p < \infty$ and $w \in A_\infty^+$, then
\[
\int_{\mathbb{R}} (S^+ f)^p \omega \leq \int_{\mathbb{R}} ((M^+)^3 f)^p \omega, \quad f \in L_c^\infty,
\]
whenever the left-hand side is finite, where $(M^+)^k$ stands for the $k$-th iteration of $M^+$, and

\[
M^+ f(x) = \sup_{h>0} \frac{1}{h} \int_x^{x+h} |f|.
\]

A natural question left open by this result is the following: can we improve the result using fewer iterates of $M^+$ in (1.1)? In this note we study a bigger operator for which two iterates are enough. Therefore the inequality (1.1) is improved in two ways: a bigger operator on the left and a smaller operator on the right. The operator that we will study is the oscillation of the averages,

\[
O^+ f(x) = \left( \sum_{n \in \mathbb{Z}} \sup_{s \in J_n} |A_{2^n} f(x) - A_s f(x)|^2 \right)^{1/2},
\]

where $J_n = [2^n, 2^{n+1}]$. It is clear that $S^+ f(x) \leq O^+ f(x)$ for all $x \in \mathbb{R}$.

If we look at the definition of $O^+ f(x)$, we see that the sequence $\{\tau_n(x)\}$, defined by $\tau_n(x) = \sup_{s \in J_n} |A_{2^n} f(x) - A_s f(x)|$, measures the oscillation of the $A_s f(x)$ in the interval $J_n$. Then we take the $L^p$ norm of the $\ell^2$ norm of this sequence. Operators of this kind are of interest in ergodic theory, [3], [8] and singular integrals. In [4] it is proved that the oscillation of a singular integral is a bounded operator in $L^p(dx)$ ($p > 1$). More recently in [9] it has been proved that the oscillation of the Hilbert Transform is bounded in $L^p(w)$ for any $w \in A_p$. Since our operator $S^+$ can be regarded as a one-sided singular integral, it is natural to try to extend their result to weights in the wider class $A_+^p$. Our main result is:

**Theorem 1.1.** Let $\omega \in A_\infty^+$ and $0 < p < \infty$. Then, there exists $C > 0$ such that

\[
\int_{\mathbb{R}} (S^+ f)^p \omega \leq \int_{\mathbb{R}} (O^+ f)^p \omega \leq C \int_{\mathbb{R}} ((M^+)^3 f)^p \omega, \quad f \in L_c^\infty,
\]

whenever the left-hand side is finite.

**Remark 1.2.** As a consequence of the above theorem, if $1 < p < \infty$ and $\omega \in A_p^+$, we obtain that $S^+$ is bounded in $L^p(\omega)$, a result that was first proved in [26].

**Remark 1.3.** As in the case of the Hilbert Transform it is an open question if (1.1) holds with $M^+$ instead of $(M^+)^3$.

The paper is organized as follows: In Section 2 we introduce notation and recall some basic results about one-sided weights and maximal operators associated to Young functions. In section 3 we prove Theorem 1.1 and in section 4 we study the commutators of $S^+$ and $O^+$ with a BMO function $b$. For $p > 1$ we prove Coifman type inequalities that imply the boundedness of the commutators in $L^p(w)$ whenever $w \in A_p^+$. For $p = 1$ we also obtain a weak type inequality for the $k$-th order commutator.
2. Definitions and basic facts about one-sided operators

**Definition 2.1.** The one-sided Hardy-Littlewood maximal operators $M^+$ and $M^-$ are defined for locally integrable functions $f$ by

\[ M^+ f(x) = \sup_{h>0} \frac{1}{h} \int_{x}^{x+h} |f| \quad \text{and} \quad M^- f(x) = \sup_{h>0} \frac{1}{h} \int_{x-h}^{x} |f|. \]

The one-sided weights are defined as follows,

\[ \sup_{a<b<c} \frac{1}{(c-a)^p} \int_{a}^{b} \omega \left( \int_{b}^{c} \omega^{1-p'} \right)^{p-1} < \infty, \quad 1 < p < \infty, \quad (A^+_p) \]

\[ M^- \omega(x) \leq C \omega(x) \quad \text{a.e.} \quad (A^+_1) \]

$A^+_\infty$ is defined as the union of the $A^+_p$ classes,

\[ A^+_\infty = \cup_{p \geq 1} A^+_p. \]

The $A^-_p$ classes are defined reversing the orientation of $\mathbb{R}$. It is interesting to note that $A_p = A^+_p \cap A^-_p$, $A_p \subset A^+_p$ and $A_p \subset A^-_p$. (See [23], [15], [16], [17] for more definitions and results.)

It was proved in [26], that $\omega \in A^+_1$, $1 < p < \infty$, if, and only if, $S^+$ is bounded from $L^p(\omega)$ to $L^p(\omega)$, and that $\omega \in A^+_1$, if, and only if, $S^+$ is of weak-type $(1,1)$ with respect to $\omega$.

**Definition 2.2.** Let $b$ be a locally integrable function. We say that $b \in BMO$ if

\[ ||b||_{BMO} = \sup_I \frac{1}{|I|} \int_I |b - b_I| < \infty, \]

where $I$ denotes any bounded interval and $b_I = \frac{1}{|I|} \int_I b$.

**Definition 2.3.** Let $f$ be a locally integrable function. The one-sided sharp maximal function is defined by

\[ M^{+,\#}(f)(x) = \sup_{h>0} \frac{1}{h} \int_{x}^{x+h} \left( f(y) - \frac{1}{h} \int_{x}^{x+h} f \right)^+ dy. \]

For $\delta > 0$ we define

\[ M^{+,\#}_\delta f(x) = \left( M^{+,\#} |f|^\delta(x) \right)^{1/\delta}. \]

It is proved in [18] that

\[ M^{+,\#}(f)(x) \leq \sup_{h>0} \inf_{a \in \mathbb{R}} \frac{1}{h} \int_{x}^{x+h} (f(y) - a)^+ dy + \frac{1}{h} \int_{x}^{x+h} (a - f(y))^+ dy \]

\[ \leq C ||f||_{BMO}. \]

Now we give definitions and results about Young functions. A function $B : [0, \infty) \to [0, \infty)$ is a Young function if it is continuous, convex, increasing and satisfies $B(0) = 0$ and $B(t) \to \infty$ as $t \to \infty$. The Luxemburg norm of a function $f$, given by $B$ is

\[ ||f||_B = \inf \left\{ \lambda > 0 : \int B \left( \frac{|f|}{\lambda} \right) \leq 1 \right\}, \]
and so the B-average of $f$ over $I$ is

$$\|f\|_{B,I} = \inf \left\{ \lambda > 0 : \frac{1}{|I|} \int_I B \left( \frac{|f|}{\lambda} \right) \leq 1 \right\}.$$ 

We will denote by $B$ the complementary function associated to $B$ (see [1]). The following version of H"older’s inequality holds,

$$\frac{1}{|I|} \int_I |fg| \leq 2\|f\|_{B,I} \|g\|_{B,I}.$$ 

This inequality can be extended to three functions (see [19]). If $A, B, C$ are Young functions such that

$$A^{-1}(t)B^{-1}(t) \leq C^{-1}(t),$$

then

$$\|fg\|_{C,I} \leq 2\|f\|_{A,I} \|g\|_{B,I}.$$ 

(2.1)

**Definition 2.4.** For each locally integrable function $f$, the one-sided maximal operators associated to the Young function $B$ are defined by

$$M^+_B f(x) = \sup_{x < b} \|f\|_{B,(x,b)} \quad \text{and} \quad M^-_B f(x) = \sup_{a < x} \|f\|_{B,(a,x)}.$$

We will be dealing with the Young functions $B_k(t) = e^{t^{1/k}} - 1$ and $B_k(t) = t(1 + \log^+(t))^k$, $k \in \mathbb{N}$. The maximal operator associated to $B_k$, $M^+_B$, will be denoted by $M^+_{L(1+\log^+)L_k}$. It is proved in [22] that $M^+_{L(1+\log^+)L_k}$ is pointwise equivalent to $(M^+)^k$. It is convenient to look at our operators as vector valued. Let us consider the sequence

$$H(x) = \left\{ \frac{1}{2^n} \chi((-2^n,0)) - \frac{1}{2^{n-1}} \chi((-2^{n-1},0)) \right\}_{n \in \mathbb{Z}}$$

and let us define the operator $U : f \to Uf$ by

$$Uf(x) = \int_{\mathbb{R}} H(x-y) f(y) dy.$$ 

Then it is clear that $S^+ f(x) = \|Uf(x)\|_{L^2}$. If instead of the sequence $H$ we consider for each $s > 0$ the sequence $K(x) = \{K_{n,s}(x)\}_{n \in \mathbb{Z}}$, where

$$K_{n,s}(x) = \left( \frac{1}{2^n} \chi((-2^n,0)) - \frac{1}{s} \chi((-s,0)) \right) \chi_{J_n}(s),$$

we can define the operator $V$ acting on locally integrable functions $f$, as $Vf(x) = \int_{\mathbb{R}} K(x-y) f(y) dy$.

If for functions $h : \mathbb{R} \times \mathbb{Z} \to \mathbb{R}$ we define the norm

$$\|h\|_E = \left( \sum_{n \in \mathbb{Z}} \sup_{s \in J_n} |h(s, n)|^2 \right)^{1/2},$$

then $O^+ f(z) = \|Vf(z)\|_E$. 

3. Proof of Theorem 1.1

The key point in the proof of Theorem 1.1 is the following pointwise estimate: for each $0 < \delta < 1$, there exists $C$ so that for any locally integrable function,

$$M_{\delta}^+(O^+ f)(x) \leq CM^+ f(x) + CM_{L(1+\log^+ L)}^+ f(x). \quad (3.1)$$

Let us prove inequality (3.1). For $0 < \delta < 1$ we have

$$M_{\delta}^+(O^+ f)(x) \leq C \sup_{h>0} \inf_{c \in \mathbb{R}} \left( \frac{1}{h} \int_{x}^{x+2h} ||O^+ f(y)\delta - |c|\delta|| \, dy \right)^{1/\delta}. \quad (3.2)$$

Let $x \in \mathbb{R}$ and $h > 0$. Let us consider the unique $i \in \mathbb{Z}$ such that $2^i \leq h < 2^{i+1}$ and let denote by $J$ the interval $J = [x, x+2^{i+3})$. If we write $f = f_1 + f_2$, where $f_1 = f_{\chi_J}$, and choose $c = O^+(f_2)(x)$, we have

$$\left( \frac{1}{h} \int_{x}^{x+2h} ||O^+ f(y)\delta - |O^+ f_2(x)\delta|| \, dy \right)^{1/\delta} \leq C \left( \frac{1}{2^i} \int_{x}^{x+2^i+2} ||O^+ f(y) - O^+ f_2(x)\delta|| \, dy \right)^{1/\delta} \leq C \left( \frac{1}{2^i} \int_{x}^{x+2^i+2} |O^+ f_1(y)\delta\, dy \right)^{1/\delta} + C \left( \frac{1}{2^i} \int_{x}^{x+2^i+2} |O^+ f_2(y) - O^+ f_2(x)\delta\, dy \right)^{1/\delta} = I + II. \quad (3.3)$$

Using Lemma 2.1 (1) in [4], that is, the fact that the oscillation of $S^+$ is of weak type $(1,1)$ with respect to the Lebesgue’s measure, and Kolmogorov’s inequality, we get

$$I \leq C \frac{1}{2^i} \int_{x}^{x+2^{i+3}} |f(y)| \, dy \leq CM^+ f(x). \quad (3.4)$$

In order to estimate $II$, we first use Jensen’s inequality and obtain

$$II = C \left( \frac{1}{2^i} \int_{x}^{x+2^i+2} \|[Vf_2(y)]_E - [Vf_2(x)]_E\delta\| \, dy \right)^{1/\delta} \leq C \frac{1}{2^i} \int_{x}^{x+2^i+2} \|[Vf_2(y) - Vf_2(x)]_E\| \, dy. \quad (3.5)$$
Let us estimate $||V f_2(y) - V f_2(x)||_E$.

\[
||V f_2(y) - V f_2(x)||_E = \left\| \left\{ \left( \frac{1}{2^n} \int_y^{y+2^n} f_2 - \frac{1}{s} \int_y^{y+s} f_2 \right) \chi_{J_n}(s) \right\} \right\|_{E}
\]

\[
= \left\{ \left( \frac{1}{2^n} \int_x^{x+2^n} f_2 - \frac{1}{s} \int_x^{x+s} f_2 \right) \chi_{J_n}(s) \right\} \right\|_{E}
\]

\[
\leq \left\{ \left( \frac{1}{2^n} \int_y^{y+2^n} f_2 - \frac{1}{s} \int_y^{y+s} f_2 \right) \chi_{J_n}(s) \right\} \right\|_{E}
\]

\[
= III + IV.
\]

(3.6)

Observe that since $y \in (x, x + 2^{i+2})$ and $f_2$ has support in $(x + 2^{i+3}, \infty)$, it follows that, if $n \leq i + 2$, then $x + 2^n \leq x + 2^{i+2}$ and $y + 2^n \leq x + 2^{i+2} + 2^n \leq x + 2^{i+2} + 2^{i+2} = x + 2^{i+3}$. As a consequence the only non-zero terms in $III$ are those with $n > i + 2$. Therefore

\[
III \leq \left( \sum_{n=i+3}^{\infty} \left\| \frac{1}{2^n} \int_x^{y+2^n} f \right\|^2 \right)^{1/2}.
\]

(3.7)

Let us consider the Young function $B_1(t) = e^t - 1$. Then $B_1^{-1}(t) = t(1 + \log^+ t)$ and $B_1^{-1}(t) = \log^+(1 + t)$. Using the generalized Hölder’s inequality, we obtain that, for $n \geq i + 3$,

\[
\left| \frac{1}{2^n} \int_x^{y+2^n} f \right| \leq \frac{1}{2^n} \int_x^{x+2^n} |f| \chi_{[x+2^n, y+2^n)}
\]

\[
\leq CM^{-1}_{B_1} f(x) \left\| \chi_{[x+2^n, y+2^n)} \right\|_{B_1, [x+2^n, y+2^n+)}
\]

\[
= CM^{-1}_{B_1} f(x) \frac{1}{B_1^{-1} \left( \frac{2^n}{y-x} \right)} \leq CM^{-1}_{B_1} f(x) \frac{1}{B_1^{-1} \left( \frac{2^n}{2^{i+2}} \right)},
\]

(3.8)

where in the last inequality we have used that $y - x \leq 2^{i+2}$ and that $B_1^{-1}$ is nondecreasing.

Putting together (3.7) and (3.8) we obtain

\[
III \leq CM^{-1}_{B_1} f(x) \left( \sum_{n=i+3}^{\infty} \frac{1}{(B_1^{-1} \left( \frac{2^n}{2^{i+2}} \right))^2} \right)^{1/2}
\]

\[
\leq CM^{-1}_{B_1} f(x) \left( \sum_{n=i+3}^{\infty} \frac{1}{(n - i - 2)^2} \right)^{1/2} = CM^{-1}_{B_1} f(x).
\]

(3.9)
Let us estimate IV. For \( n \in \mathbb{Z} \), set
\[
\beta_n = \sup_{s \in \mathcal{J}_n} \left| \frac{1}{s} \int_x^{x+s} f_2 - \frac{1}{s} \int_y^{y+s} f_2 \right|.
\]
Then, if \( \beta_n \neq 0 \), we have that there exists \( s_n \in \mathcal{J}_n \) such that
\[
\left| \frac{1}{s_n} \int_x^{x+s_n} f_2 - \frac{1}{s_n} \int_y^{y+s_n} f_2 \right| > \frac{1}{2} \beta_n.
\]
If \( n \leq i + 1 \) then \( y + s_n \leq y + 2^{n+1} \leq x + 2^{i+2} + 2^{n+1} \leq x + 2^{i+3} \). Therefore in IV we may assume \( n \geq i + 2 \).

Using again generalized Hölder’s inequality, we get that, for \( n \geq i + 2 \),
\[
\beta_n \leq C \frac{1}{s_n} \int_{x+s_n}^{y+s_n} |f_2| \leq C \frac{1}{s_n} \int_x^{x+2^{n+2}} |f| \chi_{(x+s_n,y+s_n)}
\]
\[
\leq C \frac{2^{n+2}}{s_n} M_{B_1}^+ f(x) \Vert \chi_{(x+s_n,y+s_n)} \Vert_{B_1(x,x+2^{n+2})}
\]
\[
= CM_{B_1}^+ f(x) \frac{1}{B_1^{-1} (2^n - i)} \leq CM_{B_1}^+ f(x) \frac{1}{B_1^{-1} (2^n - i)}. \quad (3.10)
\]

Then,
\[
IV \leq CM_{B_1}^+ f(x) \left( \sum_{n=i+2}^{\infty} \frac{1}{B_1^{-1} (2^n - i)^2} \right)^{1/2} = CM_{B_1}^+ f(x). \quad (3.11)
\]

Collecting inequalities (3.2)–(3.6), (3.9) and (3.11), we obtain (3.1). On the other hand, we have that \( M_{B_1}^+ f = M_{L(1+\log^+ L)}^+ f \) is pointwise equivalent to \( (M^+)^2 f \) (see [22]). As a consequence, (3.1) gives
\[
M_{B_1}^+ (O^+ f)(x) \leq C (M^+)^2 f(x), \quad \text{a.e. } x \in \mathbb{R}.
\]

To finish the proof of Theorem 1.1 we use theorem 4 in [18]: since \( w \in A^+_\infty \), there exists \( r > 1 \), such that \( w \in A^+_r \). Then, for \( \delta \) small enough, we get that \( r < p/\delta \) and thus, \( w \in A_{p/\delta}^+ \). Therefore,
\[
\int_{\mathbb{R}} |O^+ f|^p \omega \leq \int_{\mathbb{R}} (M_{B_1}^+ (O^+ f))^p \omega
\]
\[
\leq C \int_{\mathbb{R}} (M_{B_1}^{+\#} (O^+ f))^p \omega \leq C \int_{\mathbb{R}} ((M^+)^2 f)^p \omega, \quad (3.12)
\]
whenever the left hand side is finite. \( \square \)

4. Commutators

The commutators of singular integrals with BMO functions have been extensively studied (see [2],[6],[24],[25],[20],[21],[11],[12],[13]). Since \( S^+ \) can be considered as a singular integral whose kernel satisfies a weaker condition (see [14]), it is interesting to know if the results about commutators of singular integrals can be extended to \( S^+ \). In [14] we have proved that the classical results about boundedness with weights can be extended to \( S^+ \) and, furthermore, can be improved allowing a wider class of weights, since \( S^+ \) is a one-sided operator. The results in [20] and [21] have been
improved in [11] for one-sided singular integrals. Observe that for singular integrals satisfying the usual Lipschitz condition one obtains $Mf$ instead of $M^2f$ in Theorem 1.1. Therefore we can not expect to obtain the same results for the commutator of $S^+$ as we obtained in [11] for one-sided singular integrals. However, we can give estimates of the same kind, increasing in one the iterations of $M^+$. Concretely, for the $k$-th order commutator of $S^+$ we have:

**Theorem 4.1.** Let $b \in BMO$ and $k = 0, 1, 2, \ldots$. Let us define the $k$-th order commutator of $S^+$ and $O^+$ by

$$S^+_b f(x) = \left\| \int_R (b(x) - b(y))^k H(x - y)f(y)dy \right\|_{L^2},$$

and

$$O^+_b f(x) = \left\| \int_R (b(x) - b(y))^k K(x - y)f(y)dy \right\|_{L^p}.$$  

(Observe that for $k = 0$ we obtain $S^+$ and $O^+$.) Then, for $0 < p < \infty$ and $w \in A^{+\infty}$, there exists $C > 0$ such that,

$$\int_R (S^+_b f)^p w \leq \int_R (O^+_b f)^p w \leq C \int_R ((M^+)_k f)^p w,$$

whenever the left-hand side is finite.

**Remark 4.2.** In [10], the $L^{A,k}$-Hörmander condition was introduced. If we just use that $H$, the vector valued kernel of $S^+$, satisfies the $L^{A,k}$-Hörmander condition for $A(t) = e^{\frac{t}{1+\epsilon}}$, then theorem 3.3 in [10] gives $(M^+)^{k+3}$ instead of $(M^+)^{k+2}$ in the previous inequality for $S^+_b$.

**Remark 4.3.** In particular, we have that for $1 < p < \infty$ and $w \in A^+_p$, $S^+_b$ is bounded in $L^p(\omega)$ which was proved using a different approach in [12].

In [26] it was proved that $S^+$ is of weak type $(1, 1)$ with respect to $w$, iff $w \in A^+_1$. The commutator is more singular than the operator. A fact that is not apparent in the $L^p(w)$ norm but it makes a difference near $L^1(w)$.

**Theorem 4.4.** Let $b \in BMO$, $w \in A^{+\infty}$ and $k = 0, 1, 2,...$. Then, there exists $C > 0$ such that

$$w\{x \in \mathbb{R} : S^+_b f(x) > \lambda\} \leq w\{x \in \mathbb{R} : O^+_b f(x) > \lambda\} \leq C \int_R \frac{|f(x)|}{\lambda} \log^+ \left(1 + \frac{|f(x)|}{\lambda}\right)^{k+1} M^- w(x) dx, \quad f \in L^\infty_c,$$

whenever the left-hand side is finite.

**Remark 4.5.** If $w \in A^+_1$, we can put $w$ instead of $M^- w$ in the right hand side.

The following lemma will allow us to use induction in the proof of Theorem 4.1.
Lemma 4.6. Let $0 < \delta < \gamma < 1$, $b \in BMO$ and $k \in \mathbb{N} \cup \{0\}$. Then there exists $C > 0$ such that for any locally integrable $f$,

$$M_\delta^{+,k} \left( \mathcal{O}_b^{+,k} f \right) (x) \leq C \sum_{j=0}^{k-1} M_\gamma \left( \mathcal{O}_b^{+,j} f \right) (x) + C M_L^{+,1} (1 + \log^+ L)^{1+k} f(x)$$

$$\leq C \sum_{j=0}^{k-1} M_\gamma \left( \mathcal{O}_b^{+,j} f \right) (x) + C (M^+)^{k+2} f(x) \ a.e.$$  

Proof. The case $k = 0$ follows from inequality (3.1) in the proof of Theorem (1.1). Let us prove the case $k \geq 1$. Let $\lambda$ be an arbitrary constant. Then, we proceed as in inequality (3.2) in [12], and get

$$\mathcal{O}_b^{+,k} f(x) = \left\| \int_R (b(x) - b(y))^k K(x-y) f(y)dy \right\|_E$$

$$\leq \mathcal{O}^+ ((b - \lambda)^k)(x) + \sum_{m=0}^{k-1} C_{k,m} |b(x) - \lambda|^{k-m} \mathcal{O}_b^{+,m} f(x). \quad (4.1)$$

Let $x \in \mathbb{R}$ and $h > 0$. Let $i \in \mathbb{Z}$ be such that $2^i \leq h < 2^{i+1}$ and set $J = [x, x + 2^{i+3})$. Then, write $f = f_1 + f_2$, where $f_1 = f \chi_J$ and set $\lambda = b_J$. Then (see (3.2) in [11]), for any $a \in \mathbb{R}$ we have

$$\left( \frac{1}{h} \int_x^{x+h} |(\mathcal{O}_b^{+,k} f(y))^{\delta} - |a^{\delta}| dy \right)^{\frac{1}{\delta}} + \left( \frac{1}{h} \int_{x+h}^{x+2h} |(\mathcal{O}_b^{+,k} f(y))^{\delta} - |a^{\delta}| dy \right)^{\frac{1}{\delta}}$$

$$\leq C \sum_{m=0}^{k-1} \left( \frac{1}{h} \int_x^{x+h} |b(y) - b_J|^{(k-m)\delta} (\mathcal{O}_b^{+,m} f(y))^{\delta} dy \right)^{\frac{1}{\delta}}$$

$$+ \left( \frac{1}{h} \int_x^{x+2h} |\mathcal{O}^+ ((b - b_J)^k f_1)(y)|^{\delta} dy \right)^{\frac{1}{\delta}}$$

$$+ \left( \frac{1}{h} \int_x^{x+2h} |\mathcal{O}^+ ((b - b_J)^k f_2)(y) - |a|^{\delta} dy \right)^{\frac{1}{\delta}}$$

$$= (I) + (II) + (III). \quad (4.2)$$

$(I)$ is estimated exactly as in inequality (3.3) of [11],

$$(I) \leq C \sum_{m=0}^{k-1} M_\gamma^+ (\mathcal{O}_b^{+,m} f)(x). \quad (4.3)$$

Kolmogorov’s inequality plus the fact that $\mathcal{O}^+$ is of weak type $(1,1)$ with respect to the Lebesgue measure imply

$$(II) \leq C \int_x^{x+2^{i+3}} |b(y) - b_J|^k |f(y)| dy.$$  

Using now the generalized Hölder’s inequality with $B_{k+1}(t) = e^{t/(k+1)} - 1$ and $\overline{B_{k+1}}(t) = t(1 + \log^+ t)^{k+1}$ we get,

$$(II) \leq C||b - b_J||_{B_{k+1},J} ||f||_{\overline{B_{k+1}},J}.$$
It follows from John-Nirenberg’s inequality that
\[
(II) \leq C\|b - b_j\|_{B^{k+1}_{k+1}}\|f\|_{B^{k+1}_{k+1}} \leq C\|b\|_{BMO}M^{+}_{k+1}f(x) \\
\leq C(M^+)^{k+2}f(x). \tag{4.4}
\]

For (III) we take \(a = O^+((b - b_j)^k f_2)(x)\). Then, by Jensen’s inequality,
\[
(III) \leq C \frac{1}{2^n} \int_{x}^{x + 2^{i+3}} |O^+((b - b_j)^k f_2)(y) - O^+((b - b_j)^k f_2)(x)| \, dy \\
\leq C \frac{1}{2^n} \int_{x}^{x + 2^{i+3}} \|V((b - b_j)^k f_2)(y) - V((b - b_j)^k f_2)(x)\|_E \, dy. \tag{4.5}
\]

For \(j \geq 3\), let \(I_j = [x + 2^j, x + 2^{j+1})\) and \(\tilde{I}_j = [x, x + 2^{j+1})\). As in inequality (3.6) we have
\[
\|V((b - b_j)^k f_2)(y) - V((b - b_j)^k f_2)(x)\|_E \\
\leq \left| \left\{ \left( \int_{y}^{y + 2^n} (b - b_j)^k f_2 \, dy - \frac{1}{2^n} \int_{x}^{x + 2^n} (b - b_j)^k f_2 \right) \chi_{J_n}(s) \right\}_{n \in \mathbb{Z}, s \in \mathbb{R}} \right|_E \\
+ \left| \left\{ \left( \frac{1}{s} \int_{x}^{x + s} (b - b_j)^k f_2 - \frac{1}{s} \int_{y}^{y + s} (b - b_j)^k f_2 \right) \chi_{J_n}(s) \right\}_{n \in \mathbb{Z}, s \in \mathbb{R}} \right|_E \\
= (III_n) + (III_s). \tag{4.6}
\]

For (III\(_n\)), we proceed as in the estimate of (III) in Theorem 1.1. Since \(y \in (x,x + 2^{i+2})\) and \(f_2\) has support in \((x + 2^{i+3}, \infty)\), it follows that, if \(n \leq i + 2\), then \(x + 2^n \leq x + 2^{i+2}\) and \(y + 2^n \leq x + 2^{i+2} + 2^n \leq x + 2^{i+2} + 2^{i+2} = x + 2^{i+3}\). As a consequence, we only have to take into account \(n > i + 2\). Therefore
\[
(III_n) = \left( \sum_{n=i+3}^{\infty} \left| \frac{1}{2^n} \int_{x + 2^n}^{y + 2^n} f(b - b_j)^k \right|^{2} \right)^{1/2} \\
\leq C \left( \sum_{n=i+3}^{\infty} \left| \frac{1}{2^n} \int_{x + 2^n}^{y + 2^n} f(b - b_{I_n})^k \right|^{2} \right)^{1/2} \\
+ C \left( \sum_{n=i+3}^{\infty} \left| \frac{1}{2^n} \int_{x + 2^n}^{y + 2^n} f(b_{I_n} - b_j)^k \right|^{2} \right)^{1/2} \\
= C \left( \sum_{n=i+3}^{\infty} |IV_n|^2 \right)^{1/2} + C \left( \sum_{n=i+3}^{\infty} |V_n|^2 \right)^{1/2}. \tag{4.7}
\]
Using the generalized Hölder’s inequality (2.1) with $A = B_1$, $B = \overline{B}_{k+1}$ and $C = \overline{B}_k$, followed by John-Nirenberg’s inequality we get

$$(IV_n) \leq C \frac{\sqrt{2}}{2^n} \int_{I_n} |b(t) - b_{I_n}| |f(t)| \chi_{(x+2^n,y+2^n)}(t) \, dt$$

$$\leq C \| (b - b_{I_n}) \|_{B_{k+1},f_n} \| f \chi_{(x+2^n,y+2^n)} \|_{B_{k+1},f_n}$$

$$\leq C \| b \|_{BMO} \| f \|_{B_{k+1},f_n} \| \chi_{(x+2^n,y+2^n)} \|_{B_{k+1},f_n}$$

$$\leq CM^+_{B_{k+1}} f(x) \frac{1}{B_{k+1}^{-1}(2^{n-1}-2)}.$$  

(4.8)

For $(V_n)$ again the generalized Hölder’s inequality is used to obtain

$$(V_n) \leq C (n - i - 1)^k \| f \|_{B_{k+1},f_n} \| \chi_{(x+2^n,y+2^n)} \|_{B_{k+1},f_n}$$

$$\leq C (n - i - 1)^k M^+_{B_{k+1}} f(x) \frac{1}{B_{k+1}^{-1}(2^{n-1}-2)}.$$  

(4.9)

Putting together inequalities (4.8) and (4.9) we get

$$(III_n) \leq CM^+_{B_{k+1}} f(x) \left( \sum_{n \geq i+3} \frac{1}{(B_{1}^{-1}(2^{n-1}-2))^2} \right)^{1/2}$$

$$+ CM^+_{B_{k+1}} f(x) \left( \sum_{n \geq i+3} (n - i - 1)^{2k} \frac{1}{(B_{k+1}^{-1}(2^{n-1}-2))^2} \right)^{1/2}$$

$$\leq CM^+_{B_{k+1}} f(x) \leq C (M^+)^{k+2} f(x).$$  

(4.10)

Let us estimate $(III_n)$. As in Theorem 1.1, for $n \in \mathbb{Z}$, set

$$\beta_n = \sup_{s \in J_n} \left| \frac{1}{s} \int_x^{x+s} (b - b_j)^k f_2 - \frac{1}{s} \int_y^{y+s} (b - b_j)^k f_2 \right|.$$  

Then, if $\beta_n \neq 0$ there exists $s_n \in J_n$, such that

$$\left| \frac{1}{s_n} \int_x^{x+s_n} (b - b_j)^k f_2 - \frac{1}{s_n} \int_y^{y+s_n} (b - b_j)^k f_2 \right| > \frac{1}{2} \beta_n.$$

If $n \leq i+1$ then $y + s_n \leq y + 2^{n+1} \leq x + 2^{i+2} + 2^{n+1} \leq x + 2^{i+3}$. Therefore we only have to consider $n \geq i+2$ in the estimate of $(III_n)$. Then

$$\beta_n \leq C \left( \frac{1}{s_n} \right) \left| \int_x^{x+s_n} (b - b_j)^k f_2 \right|$$

$$\leq C \frac{2^{n+2}}{s_n} \frac{1}{2^{n+2}} \int_x^{x+2^{n+2}} |(b(t) - b_j)^k f(t)\chi_{[x+s_n,y+s_n]}(t)| \, dt$$

$$\leq C \frac{1}{2^{n+2}} \int_{I_{n+1}} |(b(t) - b_{I_{n+1}})^k f(t)\chi_{[x+s_n,y+s_n]}(t)| \, dt$$

$$+ C \frac{1}{2^{n+2}} \int_{I_{n+1}} |(b_{I_{n+1}} - b_j)^k f(t)\chi_{[x+s_n,y+s_n]}(t)| \, dt.$$
By the generalized Hölder’s inequality (2.1) with the Young functions used in (4.8) and (4.9), we get

$$
\beta_n \leq C \|(b - b_{l_n + 1})^k \|_{B_k, I_{n+1}} \|f\|_{B_{k+1}, I_{n+1}} \|\chi(x + s_n y + s_n)\|_{B_1, I_{n+1}}
+ C(n - i)^k \|f\|_{B_k, I_{n+1}} \|\chi(x + s_n y + s_n)\|_{B_1, I_{n+1}}
\leq CM^+_n f(x) \left( \frac{1}{B_1^{-1}(2^{n-i})} + \frac{1}{B_{k+1}^{-1}(2^{n-i})} \right)
\leq CM^+_n f(x) \left( \frac{1}{B_1^{-1}(2^{n-i})} + \frac{1}{B_{k+1}^{-1}(2^{n-i})} \right).
$$

Then,

$$(III)_n \leq CM^+_n f(x) \left[ \left( \sum_{n=i+2}^{\infty} \left( \frac{1}{B_1^{-1}(2^{n-i})} \right)^2 \right)^{1/2} + \left( \sum_{n=i+2}^{\infty} \left( \frac{1}{B_{k+1}^{-1}(2^{n-i})} \right)^2 \right)^{1/2} \right]
\leq CM^+_n f(x) \leq C(M^+)^{k+2} f(x). \quad (4.11)$$

Collecting now inequalities (4.2)–(4.6), (4.10) and (4.11) we finish the proof of Lemma 4.6.

□

Proof of Theorem 4.1. Let us observe that from the definition of \( || \cdot ||_E \), it follows that \( S_{b,k}^+ f \leq O_{b,k}^+ f \), therefore the first inequality in Theorem 4.1 holds trivially. For the second one, we will proceed by induction on \( k \). The case \( k = 0 \) is Theorem 1.1. Let now \( k \in \mathbb{N} \) and suppose that Theorem 4.1 holds for \( j = 1, \ldots, k - 1 \). In order to prove the case \( j = k \) we proceed as in (3.12): since \( w \in A_\infty^{+} \), there exists \( r > 1 \), such that \( w \in A_r^{+} \). Then, for \( \delta \) small enough, we get that \( r < p/\delta \) and thus, \( w \in A_{p/\delta}^{+} \). If \( \gamma \) is such that \( \delta < \gamma < 1 \), then by Theorem 4 in [18] and Lemma 4.6 we have

$$
||O_{b,k}^+ f||_{L^p(w)} \leq ||M_{g}^+(O_{b,k}^+ f)||_{L^p(w)}
\leq C||M_{g}^+,#(O_{b,k}^+ f)||_{L^p(w)}
\leq C \sum_{j=0}^{k-1} ||M_{g}^+(O_{b,j}^+ f)||_{L^p(w)}
+ C||(M^+)^{k+2} f||_{L^p(w)}. \quad (4.12)
$$

Then, by recurrence, we can continue the chain of inequalities in (4.12) by

$$
\leq C \sum_{j=0}^{k-1} ||(M^+)^{j+2} f||_{L^p(w)} + C||(M^+)^{k+2} f||_{L^p(w)} \leq C||(M^+)^{k+2} f||_{L^p(w)}.
\quad \Box
$$

Proof of Theorem 4.4. First of all we observe that, by theorem 3 in [22] we have that

$$
w(\{x \in \mathbb{R} : (M^+)^{k+2} f(x) > \lambda \}) \leq C \int_{\mathbb{R}} \frac{|f|}{\lambda} \log^+ \left( 1 + \frac{|f|}{\lambda} \right)^{k+1} M^- w.
$$
On the other hand, let us notice that theorem 3.1 in [7] holds for one-sided weights with minor changes in the proof. Therefore, using the Coifman type estimate in Theorem 4.1 and the fact that \( B_{k+1}^+(t) = t(1 + \log^+ t)^{k+1} \) is submultiplicative, we get

\[
\sup_{t>0} \frac{1}{B_{k+1}^+(1/t)} w(\{ x : |O_b^{+,k} f(x)| > t \}) \leq C \sup_{t>0} \frac{1}{B_{k+1}^+(1/t)} w(\{ x : (M^+)^{k+2} f(x) > t \}).
\]

Now, following the same argument used in the proof of theorem 3.3, part (a) in [10], we get the desired result.

References


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