Method of straight lines for a Bingham problem *

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Abstract

In this work we develop a method of straight lines for a one-dimensional Bingham problem. A Bingham fluid has viscosity properties that produce a separation into two regions, a rigid zone and a viscous zone. We propose a method of lines with the time as a discrete variable. We prove that the method is well defined, a monotone property, and a convergence theorem. Behavior of the numerical solution and numerical experiments are presented at the end of this work.

1 Introduction

We consider a fluid between two parallel plates. Using the Navier-Stokes equation for the viscous region and Newton’s law for the rigid zone, we model the behavior of the system. The boundary that separates the two regions is an unknown that evolves in time. It is one of the most important unknown quantities of the problem. For weak formulations, in variational form, of free boundary problems like the Bingham problem the reader is referred to [6, 7, 11, 12, 13]. Moreover in [14] there is an extensive bibliography about these topics. In [1, 2, 9, 15] there are examples of the implementation of the method of straight lines for free boundary problems.

We recall that fluids in which the shear stress is a multiple of the shear strain called Newtonian fluids. The proportionality coefficient is the viscosity. Other fluids are known as non-Newtonian fluids. Examples of Newtonian fluids are: water, alcohol, benzene, kerosene and glycerine. Examples of non-newtonian fluids are: blood plasma, chocolate, tomato sauce, mustard, mayonnaise, toothpaste, asphalt, some greases and sewage.

Bingham fluids are non-Newtonian fluids and the relation between shear stress \( \tau \) and shear strain \( \sigma \) is linear. That is,

\[
\tau = \tau_0 + \eta \sigma.
\]  

where \( \eta > 0 \) is the viscosity and \( \tau_0 > 0 \) is the threshold value.

We assume that the fluid is incompressible, laminar, and with constant density \( \rho \). Fixing the \( x \) coordinate along the direction of motion, \( y \) the perpendicular
coordinate to the plates, and \( z \) the remaining coordinate, we make the following assumptions:

1. The pressure gradient, \( \nabla p \), is applied in only one direction, that is, \( \frac{\partial p}{\partial y} = \frac{\partial p}{\partial z} = 0 \).

2. The fluid is laminar, that is, the velocities \( v \) and \( w \) satisfy \( v = w = 0 \).

3. The non-zero component of the velocity \( u \) depends only on time, \( t \), and on the perpendicular position, \( y \), that is, \( \frac{\partial u}{\partial z} = \frac{\partial u}{\partial x} = 0 \).

4. There is no transport of fluid through the free boundary, \( y = s(t) \). This is a condition of no deformation, that is, \( u_y(s(t), t) = 0 \; \forall \; t > 0 \).

5. The velocity of the fluid \( u \) at the walls of the plates is zero. This is an adherence condition.

Using the above hypotheses, we obtain a system of partial differential equation, which we call problem \( (P) \). Making a change of variables, we obtain the dimensionless system.

\[
\begin{align*}
   u_t - u_{yy} & = f(t), \quad s(t) < y < 1, \; t > 0, \quad (1.2) \\
   u(1, t) & = 0, \quad t > 0, \quad (1.3) \\
   u(y, 0) & = u_0(y), \quad s(0) = s_0, \; 0 < s_0 < y < 1, \quad (1.4) \\
   u_y(s(t), t) & = 0, \quad t > 0, \quad (1.5) \\
   u_t(s(t), t) & = f(t) - \frac{\tau_0}{s(t)}, \quad t > 0. \quad (1.6)
\end{align*}
\]

The problem is similar to a problem of heat transfer, where \( f \) is the pressure gradient that, according to the hypotheses, depends only on \( t \). This system is called a free boundary problem because the function \( y = s(t) \) is the boundary that separates two regions, and is part of the unknown quantities. We suppose that the pressure gradient is greater than the threshold value \( \tau_0 \). This condition allows the movement between the layers of the fluid. That is,

\[
f(t) > \tau_0, \quad \forall \; t > 0. \quad (1.7)
\]

A more general condition can be imposed instead of a fixed boundary condition (1.3), representing that two distinct fluids are in contact. In this case we can replace (1.3) by the following equation:

\[
u(1, t) = g(t), \quad t > 0. \quad (1.8)
\]

It can be seen that the function \( g \) does not cause problems, because we can rewrite the system considering a new function for the pressure gradient.
We transform the problem \((P)\) using the function \(w = u_y\). The new problem \((P_y)\) satisfies the following equations:

\[
\begin{align*}
w_t - w_{yy} &= 0, & s(t) < y < 1, & t > 0, \\
w_y(1, t) &= -f(t), & t > 0, \\
w(y, 0) &= u'_0(y), & s(0) = s_0, & 0 < s_0 < y < 1, \\
w(s(t), t) &= 0, & t > 0, \\
w_y(s(t), t) &= -\frac{\tau_0}{s(t)}, & t > 0.
\end{align*}
\]

\[\text{(1.9)}\]

\[\text{(1.10)}\]

\[\text{(1.11)}\]

\[\text{(1.12)}\]

\[\text{(1.13)}\]

\section{Straight Lines Method}

We discretize the time and choose a fixed time step \(\Delta t > 0\). We define:

\[t_n = (n - 1)\Delta t, \quad n \in \mathbb{N}.
\]

\[\text{(2.1)}\]

Denoting \(w_n(y) = w(y, t_n), f_n = f(t_n), s_n = s(t_n)\) for \(n \geq 1\), \(q^2 = \frac{1}{\Delta t}\), and approximating time derivatives with the incremental quotient, the \((P_y)\) system is transformed in the \((Pd_y)\) system.

\[
\begin{align*}
w''_{n+1}(y) - q^2w_{n+1} &= -q^2w_n, & s_{n+1} < y < 1, & n \geq 1, \\
w_{n+1}(s_{n+1}) &= 0, & n \geq 1, \\
w'_{n+1}(s_{n+1}) &= -\frac{\tau_0}{s_{n+1}}, & n \geq 1, \\
w'_{n+1}(1) &= -f_{n+1}, & n \geq 1, \\
w_1(y) &= u'_0(y), & 0 < s_1 < y < 1.
\end{align*}
\]

\[\text{(2.2)}\]

\[\text{(2.3)}\]

\[\text{(2.4)}\]

\[\text{(2.5)}\]

\[\text{(2.6)}\]

Observe that with this notation, \(s_1 = s(t_1) = s(0) = s_0\) and \(w_1(y) = w(y, t_1) = w(y, 0) = u'_0(y)\). It is easy to see that (2.2)-(2.4) is a second order differential equation of the form:

\[
\begin{align*}
w'' - q^2w &= g, & s < y < 1, \\
w(s) &= 0, \\
w'(s) &= -\frac{\tau_0}{s}.
\end{align*}
\]

\[\text{(2.7)}\]

\begin{lemma}
The above system is satisfied by

\[
w(y) = -\frac{\tau_0}{qs} \sinh(q(y - s)) + \int_s^y \frac{g(\xi)}{q} \sinh(q(y - \xi))d\xi, & s < y < 1.
\]

\[\text{(2.8)}\]

\end{lemma}

\begin{proof}
We transform this second order differential equation into a first order differential equation system, taking an auxiliary variable \(v = w'\). In this way,
we obtain:
\[
\begin{pmatrix}
    w' \\
    v'
\end{pmatrix} = A \begin{pmatrix}
    w \\
    v
\end{pmatrix} + \begin{pmatrix}
    0 \\
    g
\end{pmatrix}, \quad s < y < 1,
\]
\[
\begin{pmatrix}
    w \\
    v
\end{pmatrix}(s) = \begin{pmatrix}
    0 \\
    -\frac{\tau_0}{s}
\end{pmatrix}.
\]

where \(A = \begin{pmatrix}
    0 & 1 \\
    q^2 & 0
\end{pmatrix}\). The matrix \(A\) is diagonalizable. In fact, if \(P = \begin{pmatrix}
    1 & 1 \\
    q & -q
\end{pmatrix}\) then \(PAP^{-1} = \begin{pmatrix}
    q & 0 \\
    0 & -q
\end{pmatrix}\). Thus if \(V = P^{-1} \begin{pmatrix}
    w \\
    v
\end{pmatrix}\) the we have the uncoupled system
\[
\begin{pmatrix}
    q & 0 \\
    0 & -q
\end{pmatrix} V + \begin{pmatrix}
    \frac{2}{q} \\
    -\frac{2}{q}
\end{pmatrix}, \quad s < y < 1,
\]
\[
V(s) = \begin{pmatrix}
    \frac{-\tau_0}{2qs} \\
    \frac{2\tau_0}{2q}\end{pmatrix}.
\]

Solving directly the latter system, the lemma follows. □

Suppose that \(w_n\) and \(s_n\) are known. We extend \(w_n\) by zero in the interval \([0, s_n]\). In this way \(w_n\) is continuous in the interval [0, 1]. The solution of (2.2)-(2.4) for \(s_{n+1} < y < 1\) is
\[
w_{n+1}(y) = -\frac{\tau_0}{qs_{n+1}} \sinh(q(y - s_{n+1})) - \int_{s_{n+1}}^{y} qw_n(\xi) \sinh(q(y - \xi))d\xi. \tag{2.9}
\]

Up to now, the value of \(s_{n+1}\) is unknown. We can get \(s_{n+1}\) from the equation (2.5). Replacing in (2.9) we obtain:
\[-f_{n+1} = w'_{n+1}(1) = -\frac{\tau_0}{s_{n+1}} \cosh(q(1 - s_{n+1})) - \int_{s_{n+1}}^{1} q^2w_n(\xi) \cosh(q(1 - \xi))d\xi.\]

Now we define
\[
F_{n+1}(s) = f_{n+1} - \frac{\tau_0}{s} \cosh(q(1 - s)) - \int_{s}^{1} q^2w_n(\xi) \cosh(q(1 - \xi))d\xi. \tag{2.10}
\]

So, \(s_{n+1}\) has to be a root of \(F_{n+1}\) in the interval (0, 1).

**Proposition 2.2** If \(f_{n+1} > \tau_0\) then \(F_{n+1}\) has at least a root in the interval (0, 1).

**Proof** It is clear that \(F_{n+1}(1) = f_{n+1} - \tau_0 > 0\) and also that \(F_{n+1}(s) \to -\infty\) when \(s \to 0\). So there exists a root in the interval (0, 1) because \(F_{n+1}\) is continuous. □

**Proposition 2.3** Suppose that \(f_{n+1} > 0, w_n \leq 0, \) and that we have defined \(s_{n+1} \in (0, 1)\) that satisfies (2.2)-(2.6). Then \(w_{n+1} \leq 0\).
Proof If \( y \in [0, s_{n+1}] \), then \( w_{n+1}(y) = 0 \). Let us think in the interval \([s_{n+1}, 1]\).

It can be seen that \( w_{n+1} \) decreases locally around \( s_{n+1} \), taking negative values, because of (2.3) and (2.4).

Suppose that \( w_{n+1} \) takes positive values in \([s_{n+1}, 1]\). Let \( y_0 \) be the first root such that there is a change of sign. Let us take \( y_1 \) a point to the right of \( y_0 \) such that \( w'_{n+1}(y_1) > 0 \). The hypothesis says that \( w'_{n+1}(1) < 0 \). So, there exists a root \( y_2 \) of \( w'_{n+1} \) in \((y_1, 1)\), such that \( w_{n+1}(y_2) > 0 \) and \( w''_{n+1}(y_2) \leq 0 \).

Now \( q^2 w_n(y_2) = q^2 w_{n+1}(y_2) - w''_{n+1}(y_2) > 0 \), that is a contradiction. This concludes the proof. \( \square \)

Lemma 2.4 If \( A \) is solution of

\[
A'' - q^2 A \geq 0, \quad 0 < s < y < 1, \\
A(s) < 0, \quad A(1) < 0,
\]

then \( A \leq 0 \) in \([s, 1]\).

Proof Suppose that there exists \( y_0 \) in \((s, 1)\) such that \( A(y_0) > 0 \). We can choose \( y_0 \) such that \( A'(y_0) = 0, A(y_0) > 0 \) and \( A''(y_0) \leq 0 \). This is a contradiction because \( A''(y_0) - q^2 A(y_0) < 0 \). This concludes the proof. \( \square \)

Proposition 2.5 Suppose that \( f_{n+1} > \tau_0 \) and that \( w'_n \leq 0 \). Then \( w'_{n+1} \leq 0 \).

Proof If we define \( A = w'_{n+1}, s = s_{n+1} \), and deriving (2.2)-(2.6), we conclude that \( A \) satisfies (2.11), that implies

\[
A'' - q^2 A \geq 0, \quad 0 < s < y < 1, \\
A(s) = -\frac{\tau_0}{s} < 0, \quad A(1) = -f_{n+1} < 0.
\]

By Lemma (2.4) we obtain that \( w'_{n+1} \leq 0 \). This concludes the proof. \( \square \)

Observation 2.6 The function \( h(s) = 1 + sq \tanh(q(1 - s)) \) is concave and strictly positive in \([0, 1]\) since

\[
h''(s) = -2q^2 \left[1 - \tanh^2(q(1 - s))\right] \\
- 2sq^3 \tanh(q(1 - s)) \left[1 - \tanh^2(q(1 - s))\right] < 0,
\]

and \( h(0) = 1 = h(1) \).

Proposition 2.7 If \( w_n \leq 0 \) and \( w'_n \leq 0 \), the function \( F_{n+1} \) has at most a critical point, that is, there exists at most a point \( x_0 \) such that \( F'_{n+1}(x_0) = 0 \).
Proof From (2.10) we obtain:

\[ F'_{n+1}(s) = \frac{\tau_0}{s^2} \cosh(q(1-s)) + \frac{\tau_0 q}{s} \sinh(q(1-s)) + q^2 w_n(s) \cosh(q(1-s)). \]  

(2.14)

If \( F_{n+1} \) has no critical points, the proposition is proved. Suppose now that there exists at least \( s^* \) such that \( F'_{n+1}(s^*) = 0 \). Clearly \( s^* \neq 0 \) because \( w_n \equiv 0 \) in \([0, s_n]\). We multiply by \( s^2 \tau_0 \) and we get

\[ h(s^*) + \frac{s^2 q^2 w_n(s^*)}{\tau_0} = 0. \]  

(2.15)

So, the critical point \( s^* \) satisfies (2.15). Clearly the function above has a unique root because \( s^2 q^2 w_n(s^*)/\tau_0 \) is negative (and decreasing in \((s_n, 1)\)). Since \( h \) is concave and positive, we deduce that there exists a unique \( s^* \) that holds \( F'_{n+1}(s^*) = 0 \). This concludes the proof. \( \square \)

Observation 2.8 From Proposition (2.2) and (2.7), we conclude that \( F_{n+1} \) has a unique root in \((0,1)\) if in each step of time holds \( w_n \leq 0 \) and \( w'_n \leq 0 \).

Observation 2.9 The process of the algorithm is as follows: since \( w_0 = u'_0 \leq 0 \) and \( w'_0 = u''_0 \leq 0 \), using Proposition (2.3) and Proposition (2.5) we get that \( w_1 \leq 0 \) and \( w'_1 \leq 0 \); then we have a unique solution of \( F'_2(s) = 0 \), and besides that, \( w_2 \leq 0 \) and \( w'_2 \leq 0 \); following inductively, we obtain the movement of the free boundary.

3 Properties of the Straight Lines Method

Proposition 3.1 Suppose that \( f_n > \tau_0 \) for all \( n \) and that \( u'_0 \leq 0 \). Then \( w_n \leq 0 \) for all \( n \geq 1 \).

Proof The hypothesis says that \( w_1 = u'_0 \leq 0 \). Let us assume that \( w_n \leq 0 \). By Proposition 2.3 we get \( w_{n+1} \leq 0 \). By induction the proof is concluded. \( \square \)

Proposition 3.2 Suppose that \( f_n > \tau_0 \) for all \( n \) and that \( u''_0 \leq 0 \). Then \( w'_n \leq 0 \) for all \( n \).

Proof We know that \( w'_1 = u''_0 \leq 0 \). Suppose that \( w'_{n-1} \leq 0 \). By Proposition 2.5 we get that \( w'_{n+1} \leq 0 \). This inductive step concludes the proof. \( \square \)

Proposition 3.3 Suppose that \( f_n > \tau_0 \) for all \( n \) and that \( u''_0 \leq 0 \). Then \( w_n < 0 \) for all \( y \in (s_n, 1] \) for all \( n > 1 \).
Proof By Proposition 3.2 we know that \( w_n' \leq 0 \forall n \). Because of \( w_n(s_n) = 0 \) and \( w_n'(s_n) = -\frac{\tau_0}{s_n} < 0 \). Then \( w_n(y) \) is strictly decreasing in a neighborhood of \( s_n \), then \( w_n < 0 \) in \( (s_n, 1] \). This concludes the proof. \( \square \)

Proposition 3.4 The stationary solution of Problem (Py) (2.2)-(2.6) is

\[
\begin{align*}
s_\infty &= \frac{\tau_0}{f_\infty}, \\
w_\infty(y) &= \begin{cases} -f_\infty(y - s_\infty) & \text{if } y \in [s_\infty, 1] \\ 0 & \text{if } y \in [0, s_\infty) \end{cases}
\end{align*}
\]

where \( s_\infty = \lim_{n \to \infty} s_n \), \( w_\infty(y) = \lim_{n \to \infty} w_n(y) \) and \( f_\infty = \lim_{n \to \infty} f_n \), if the limits exist.

Proof If we take \( \lim_{n \to \infty} \) in (2.2)-(2.6) we can get the system

\[
\begin{align*}
w''_\infty &= 0, \quad \text{in } [s_\infty, 1], \\
w_\infty(s_\infty) &= 0, \\
w'_\infty(s_\infty) &= -\frac{\tau_0}{s_\infty}, \\
w'_\infty(1) &= -f_\infty.
\end{align*}
\]

Since \( w'_\infty \) is constant, then \( w'_\infty(s_\infty) = w'_\infty(1) \). That implies \( s_\infty = \frac{\tau_0}{f_\infty} \). Since \( w_\infty \) is a straight line with slope \( -f_\infty \) and root \( s_\infty \), then we have \( w_\infty(y) = -f_\infty(y - s_\infty) \) in the interval \( [s_\infty, 1] \). If \( y \in [0, s_\infty) \), then there exists \( N \in \mathbb{N} \) such that \( y \in [0, s_n] \) for all \( n \geq N \). This said \( w_n(y) = 0 \) for all \( n \geq N \). This is equivalent to \( w_\infty(y) = 0 \) in \( [0, s_\infty] \). This concludes the proof. \( \square \)

Lemma 3.5 If \( W \) is a solution of

\[
\begin{align*}
W'' - q^2 W &\geq 0, \quad 0 < s < y < 1, \\
W(s) &< 0, \quad W'(1) < 0,
\end{align*}
\]

then \( W \leq 0 \) on \( [s, 1] \).

Proof Suppose that \( W \) takes positive values. Since \( W(s) < 0 \) there exists \( y_0 \in (s, 1] \) such that \( W(y_0) = 0 \). Now we can choose \( y_1 \in (y_0, 1) \) such that \( W(y_1) > 0 \) and \( W'(y_1) > 0 \). Since \( W'(1) < 0 \) there exists \( y_2 \in (y_1, 1) \) such that \( W'(y_2) = 0 \). We can choose \( y_2 \) such that \( W(y_2) > 0 \) and \( W''(y_2) \leq 0 \). This is a contradiction because \( W''(y_2) \geq q^2 W(y_2) > 0 \). This concludes the proof. \( \square \)

Lemma 3.6 If \( V \) is solution of

\[
\begin{align*}
V''(y) &\leq 0, \quad 0 < s < y < 1, \\
V(s) &\geq 0, \quad V'(1) \geq 0,
\end{align*}
\]

then \( V \geq 0 \) in \( [s, 1] \).
Proof Suppose that $V$ assumes negative values in $(s, 1]$. There exists $y_0 \in (s, 1]$ such that $V''(y_0) < 0$. Since $V'' \leq 0$ we obtain $V'(1) < 0$, and this is a contradiction. The proof is finished.

**Theorem 3.7** Suppose $f_{n+1} > 0$ for all $n$ and $w''_0 \leq 0$. Then:

(A) If $f_{n+1} > f_n$ and $w_n \leq w_{n-1}$ then $s_{n+1} < s_n$ and $w_{n+1} \leq w_n$. Moreover, if $\{f_{n+1}\}$ is a strictly increasing sequence convergent to $f_\infty$ then $s_{n+1} > \frac{n}{f_{n+1}}$, $s_n < s_{n+1}$ and $w_\infty \leq w_{n+1}$.

(B) If $f_{n+1} < f_n$ and $w_n \geq w_{n-1}$ then $s_{n+1} > s_n$ and $w_{n+1} \geq w_n$. Moreover, if $\{f_{n+1}\}$ is a strictly decreasing sequence convergent to $f_\infty$ then $s_{n+1} < \frac{n}{f_{n+1}}$, $s_n > s_{n+1}$ and $w_\infty \geq w_{n+1}$.

Proof From the expression of $F_n$ in (2.10) we have

$$F_{n+1}(s) - F_n(s) = -\int_0^1 q^2 (w_n(\xi) - w_{n-1}(\xi)) \cosh(q(1 - \xi)) d\xi + (f_{n+1} - f_n).$$

(3.5)

For part (A), since $f_{n+1} > f_n$ and $w_n \leq w_{n-1}$, we see from (3.5) that $F_{n+1}(s) - F_n(s) > 0$ for all $s$. From this we deduce that $s_{n+1} < s_n$.

Let $W = w_{n+1} - w_n$. Then from (2.2)-(2.6), we see that on $[s_n, 1]$, $W$ satisfies

$$W'' - q^2 W = -q^2(w_n - w_{n-1}) \geq 0,$$

$$W(s_n) = w_{n+1}(s_n) < 0,$$

$$W'(1) = -(f_{n+1} - f_n) < 0.$$  

Therefore, on the interval $[s_n, 1]$, $W$ satisfies

$$W'' - q^2 W \geq 0, \quad W(s_n) < 0, \quad W'(1) < 0.$$  

(3.6)

and out this interval, $W$ satisfies

$$W(y) = \begin{cases} 
0, & \text{if } y \in [0, s_{n+1}], \\
W_{n+1}(y), & \text{if } y \in [s_{n+1}, s_n]. 
\end{cases}$$

Knowing that the $w_{n+1}$ are negative functions on $(s_{n+1}, s_n)$ (Proposition 3.3), and using the Lemma 3.5, we observe that $W \leq 0$ on $[s_{n+1}, 1]$ finally $W \leq 0$ on $[0, 1]$, and this is equivalent to $w_{n+1} \leq w_n$.

When we integrate the equation (2.2) from $s_{n+1}$ to 1, using (2.4),(2.5) we obtain:

\[
\int_{s_{n+1}}^{1} w''_{n+1}(\xi) d\xi = \int_{s_{n+1}}^{1} q^2 (w_{n+1}(\xi) - w_n(\xi)) d\xi, \\
w'_{n+1}(1) - w'_{n+1}(s_{n+1}) = q^2 \int_{s_{n+1}}^{s_n} w_{n+1}(\xi) d\xi + \int_{s_n}^{1} q^2 (w_{n+1} - w_n)(\xi) d\xi < 0,
\]
From this we deduce that \( s_{n+1} > \frac{\tau_0}{f_{n+1}} \). Since \( f_{n+1} < f_\infty \) and \( s_{n+1} > \frac{\tau_0}{f_n+1} > \frac{\tau_0}{f_\infty} = s_\infty \), it follows that \( s_{n+1} > s_\infty \).

Let \( V_{n+1} = w_{n+1} - w_\infty \) be. Using (2.2) and the Proposition (3.4), the following equation holds on \((s_{n+1}, 1)\),

\[
V''_{n+1} - q^2 V_{n+1} = -q^2 V_n. \tag{3.7}
\]

Then it is clear that

\[
V''_{n+1} = q^2 (w_{n+1} - w_n) \leq 0, \quad \text{in} \quad (s_{n+1}, 1),
\]

\[
V_{n+1}(s_{n+1}) = w_{n+1}(s_{n+1}) - w_\infty(s_{n+1}) > 0,
\]

\[
V''_{n+1}(s_{n+1}) = \frac{-\tau_0}{s_{n+1}} + \frac{\tau_0}{s_\infty} > 0,
\]

\[
V'_{n+1}(1) = -f_{n+1} + f_\infty > 0.
\]

Therefore, on the interval \([0, s_\infty]\), \( V \) satisfies

\[
V''_{n+1} \leq 0, \quad V_{n+1}(s_{n+1}) > 0, \quad V'_{n+1}(s_{n+1}) > 0, \quad V'_{n+1}(1) > 0. \tag{3.8}
\]

and out of this interval, \( V \) satisfies

\[
V_{n+1}(y) = \begin{cases} 
0, & \text{if } y \in [0, s_\infty], \\
-w_\infty(y), & \text{if } y \in [s_\infty, s_{n+1}]. 
\end{cases}
\]

Because of Lemma 3.6 we get that \( V_{n+1} \geq 0 \) on \([0, 1]\), and this is equivalent to \( w_{n+1} \geq w_\infty \).

The proof of part (B) is similar to (A) and we omit it. \(\square\)

**Theorem 3.8** Suppose that \( \lim_{n \to \infty} f_n = f_\infty, \quad f_n > 0 \quad \forall \ n, \quad u'_0 \leq 0, \quad u''_0 \leq 0, \quad \lim_{n \to \infty} s_n = s^* \quad \text{and} \quad \lim_{n \to \infty} w_n = w^*. \) Then \( s^* = s_\infty \) and \( w^* = w_\infty \).

**Proof** From Theorem 3.7 we can take \( \lim_{n \to \infty} \) in (2.9) and we obtain:

\[
w^* = -\frac{\tau_0}{qs^*} \sinh(q(y - s^*)) - \int_{s^*}^{y} q w^*(\xi) \sinh(q(y - \xi))d\xi. \tag{3.9}
\]

Computing the derivatives of the function \( w^* \) we get that

\[
w'''^*(y) = 0, \quad w'^*(s^*) = -\frac{\tau_0}{s^*}, \quad w^*(s^*) = 0. \tag{3.10}
\]

On the other hand, if we differentiate (2.9) for \( s_{n+1} < y < 1 \), we have

\[
w'_{n+1}(y) = -\frac{\tau_0}{s_{n+1}} \cosh(q(y - s_{n+1})) - \int_{s_{n+1}}^{y} q^2 w_n(\xi) \cosh(q(y - \xi))d\xi. \tag{3.11}
\]
Taking \( \lim_{n \to \infty} \) we get:

\[
\lim_{n \to \infty} w'_{n+1}(y) = -\frac{\tau_0}{s^*} \cosh(q(y - s^*)) - \int_{s^*}^{y} q^2 w^*(\xi) \cosh(q(y - \xi)) d\xi. \tag{3.12}
\]

From (3.9) and (3.16) we obtain that \( \lim_{n \to \infty} w'_n(y) = w^*(y) \). Then \( w'(1) = \lim_{n \to \infty} w'_{n+1}(1) = -\lim_{n \to \infty} f_{n+1} = -f_{\infty} \). Since \( w^* \) is constant, we deduce that \( w'(1) = w^*(s^*) \). Then

\[
s^* = \frac{\tau_0}{f_{\infty}}, \quad w^* = -\frac{\tau_0}{s^*}(y - s^*). \]

A comparison with Proposition 3.4 completes this proof. \( \square \)

**Observation 3.9** Under the hypotheses of Theorem 3.7, the solutions \( s_n \) and \( w_n \) converge to the stationary solutions, \( s_\infty, w_\infty \).

![Figure 1: Solution with \( s_0 = 0.2, \tau_0 = 1, \Delta t = 0.05, f(t) = 2 \)](image)

**Numerical Results**

The algorithm for the following results was programmed in Fortran. First we compute the root of \( F_{n+1} \), and then we compute \( w_{n+1} \) from (2.9). The functions \( w_n \) are stored as splines functions and the integrals are computed by the Simpson's Rule. The numerical experiments are shown in Figures 1, 2, and 3. They show that the algorithm reproduces the theoretical behavior of the solution (see [3]).
Figure 2: Solution with $s_0 = 0.8$, $\tau_0 = 1$, $\Delta t = 0.05$, $f(t) = 2$.

Concluding Remarks

For the discrete solution of (2.2) – (2.6), we have obtained the same properties that the continuous solution of Problem $(Py)$ satisfies (see [3]). That is $w_n(y) < 0$, $w'_n(y) < 0$, $\{s_n\}$ is monotone if $\{f_n\}$ is monotone, the stationary solution for the discrete problem (which agrees with the stationary solution for the continuous problem) is established, and the discrete solution converges to the stationary solution. Moreover the algorithm is well defined for all $f_n$ that satisfy $f_n > \tau_0$. This condition is the corresponds to the motion between the layers of the fluid.

References


Figure 3: Solution with \( s_0 = 0.8, \tau_0 = 1, \Delta t = 0.05, f(t) = 2 + t \)


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