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Restriction of holomorphic Discrete Series to real forms

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Abstract

Let G be a connected linear semisimple Lie group having a Holomorphic Discrete Series representation π. Let H be a connected reductive subgroup of G so that the global symmetric space attached to H is a real form of the Hermitian symmetric space associated to G. Fix a maximal compact subgroup K of G so that $H \cap K$ is a maximal compact subgroup for H. Let τ be the lowest $K-$type for π and let $\tau_\star$ denote the restriction of τ to $H \cap K$. In this note we prove that the restriction of π to H is unitarily equivalent to the unitary representation of H induced by $\tau_\star$.

Introduction

For any Lie group, we denote its Lie algebra by the corresponding German lower case letter. In order to denote complexification of either a real Lie group or a real Lie algebra we add the subindex c. Let G be a connected matrix

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semisimple Lie group. Henceforth, we assume that the homogeneous space G/K is Hermitian symmetric. Let H be a connected semisimple subgroup of G and fix a maximal compact subgroup K of G such that $K_1 := H \cap K$ is a maximal compact subgroup of H. From now on we assume that H/K_1 is a real form of the complex manifold G/K. Let (π, V) be a Holomorphic Discrete Series representation for G. Let (τ, W) be the lowest $K-$type for (π, V). For the definition and properties of lowest $K-$type of a Discrete Series representation we refer to [K]. Let $(\tau_\star, W)$ denote the restriction of τ to K_1. We then have:

Theorem 1 *The restriction of* (π, V) *to* H *is unitarily equivalent to the unitary representation of* H *induced by* $(\tau_\star, W)$.

Thus, after the work of Harish-Chandra and Camporesi [Ca] we have that the restriction of π to H is unitarily equivalent to

$$\sum_{j=1}^{r} \int_{\nu \in \mathfrak{a}^\star} Ind_{MAN}^{H}(\sigma_j \otimes e^{i\nu} \otimes 1) d\nu.$$

Here, MAN is a minimal parabolic subgroup of H so that $M \subset K_1$, and $\sigma_1, \cdots, \sigma_r$ are the irreducible factors of τ restricted to M. Whenever, τ is a one dimensional representation, the sum is unitarily equivalent to to

$$\int_{\nu \in \mathfrak{a}^\star / W(H,A)} Ind_{MAN}^{H}(1 \otimes e^{i\nu} \otimes 1) d\nu$$

as it follows from the computation in [OO], and, hence, our result agrees with the one obtained by Olafsson and Orsted in [OO].

The symmetric pairs (G, H) that satisfy the above hypothesis have been classified by A. Jaffee in [J1, J2], A very good source about the subject is by Olafsson in [Ol], they are: $(su(p,q), so(p,q))$; $(su(n,n), sl(n,\mathbb{C}) + \mathbb{R})$; $(su(2p,2q), sp(p,q))$; $(so^\star(2n), so(n,\mathbb{C}))$; $(so^\star(4n), su^\star(2n) + \mathbb{R})$; $(so(2,p+q), so(p,1)+so(p,1))$; $(sp(n,\mathbb{R}), sl(n,R)+\mathbb{R}))$; $(sp(2n,R), sp(n,\mathbb{C}))$; $(e_{6(-14)}, sp(2,2))$; $(e_{6(-14)}, f_{4(-20)})$; $(e_{7(-25)}, e_{6(-26)} + \mathbb{R})$; $(e_{7(-25)}, su^\star(8))$; $(su(p,q) \times su(p,q), sl(p+q,\mathbb{C}))$; $(so^\star(2n) \times so^\star(2n), so(2n,\mathbb{C}))$; $(so(2,n) \times so(2,n), so(n+2,\mathbb{C}))$; $(sp(n,\mathbb{R}) \times sp(2n,\mathbb{R}), sp(n,\mathbb{C}))$; $(e_{6(-14)} \times e_{6(-14)}, e_6)$; $(e_{7(-25)} \times e_{7(-25)}, e_7)$.

For classical groups we can compute specific examples of the decomposition of τ restricted to M by means of the results of Koike and other authors as stated in [Koi].
For un update of results on restriction of unitary irreducible representations we refer to the excellent announcement, survey of T. Kobayashi [Ko] and references therein.

Proof of the Theorem

In order to prove the Theorem we need to recall some Theorems and prove a few Lemmas. For this end, we fix compatible Iwasawa decompositions $G = KAN, H = K_1A_1N_1$ with $K_1 = H \cap K, A_1 \subset A, N_1 \subset N$. We denote by $\|X\| = \sqrt{-B(X, \theta X)}$ the norm of $\mathfrak{g}$ determinated by the Killing form B and the Cartan involution θ.

Lemma 1 *The restriction to H of any $K-$finite matrix coefficient of* (π, V) *is in* $L^2(H)$.

Proof: We first consider the case that the real rank of H is equal to the real rank of G. Let f be a $K-$finite matrix coefficient of (π, V). For $X \in \mathfrak{a}$, we set $\rho_H(X) = \frac{1}{2}trace(ad_H(X)|_{\mathfrak{n}_1})$. For an $ad(\mathfrak{a})-$invariant subspace R of $\mathfrak{g}$, let $\Psi(\mathfrak{a}, R)$ denote the roots of $\mathfrak{a}$ in R. Let A_G^+, A_H^+ be the positive closed Weyl chambers for $\Psi(\mathfrak{a}, \mathfrak{n})$, $\Psi(\mathfrak{a}, \mathfrak{n}_1)$ respectively. Then $A_G^+ \subset A_H^+$. Let $\Psi_1 := \Psi(\mathfrak{a}, \mathfrak{n}), \ldots, \Psi_s$ be the positive root systems in $\Psi(\mathfrak{a}, \mathfrak{g})$ such that $\Psi_i \supset \Psi(\mathfrak{a}, \mathfrak{n}_1)$. Let A_i^+ denote the positive closed Weyl chamber associated to Ψ_i. Thus, $A_H^+ = A_1^+ \cup \ldots \cup A_s^+$. For each i, let $\rho_i(X) = \frac{1}{2}trace(ad(X)|_{\sum_{\alpha \in \Psi_i} \mathfrak{g}_\alpha})$. For $X \in A_i^+$ we have that $\rho_i(X) \geq \rho_H(X)$. Indeed, for $\alpha \in \Psi_i$, if $\alpha \in \Psi_i \cap \Psi(\mathfrak{a}, \mathfrak{n}_1) = \Psi(\mathfrak{a}, \mathfrak{n}_1)$, then the multiplicity of α as a $\mathfrak{g}-$ root is equal to or bigger than the multiplicity of α as a $\mathfrak{h}-$root, if $\alpha \in \Psi_i - \Psi(\mathfrak{a}, \mathfrak{n}_1)$, then $\alpha_i(X) \geq 0$. Thus,

$$\rho_i(X) \geq \rho_H(X) \text{ for every } X \in A_i^+.$$

We now recall the Ξ and σ functions for G and H and the usual estimates for Ξ. (cf [K] page 188). For $Y \in \mathfrak{a}, x \in G$ put $\rho_G(Y) = \frac{1}{2}trace(ad_{|\mathfrak{n}}(Y)$, and

$$\Xi_G(x) = \int_K e^{-\rho_G(H(xk))} dk.$$

Here, $H(x)$ is uniquely defined by the equation $x = kexp(H(x))n, (k \in K, H(x) \in \mathfrak{a}, n \in N)$. If $x = kexp(X), (k \in K, X \in \mathfrak{s}, \mathfrak{g} = \mathfrak{k} \oplus \mathfrak{s}$, Cartan decomposition for $\mathfrak{g}$), we put $\sigma_G(x) = \|X\|$. Since the group H might be reductive we follow [HC] page 106, 129 in order to define σ_H. Now, all the norms in a finite dimensional vector space are equivalent. Thus, have that $\sigma_G << \sigma_H << \sigma_G$. The estimates are:

$$\Xi_G(exp(X)) \leq c_G e^{-\rho_i(X)}(1+\sigma_G(exp(X)))^r$$
$$\text{with } r > 0, 0 < c_G < \infty,\ X \in A_i^+,\ i = 1, \cdots, s, \text{and}$$
$$e^{-\rho_H(X)} \leq \Xi_H(exp(X)) \leq c_H e^{-\rho_H(X)}(1+\sigma_H(exp(X)))^{r_1}$$

Therefore, for $X \in A_i^+$ we have that

$$\begin{aligned}\Xi_G(expX) &\leq c_G(1+\sigma_G(expX))^r e^{-\rho_i(X)} \\ &= e^{-\rho_H(X)} c_G(1+\sigma_G(expX))^r e^{\rho_H(X)-\rho_i(X)} \\ &\leq \Xi_H(expX) c_G(1+\sigma_G(expX))^r e^{\rho_H(X)-\rho_i(X)}.\end{aligned}$$

Since on A_i^+ we have the inequality $\rho_H(X) - \rho_i(X) \leq 0$, and i is arbitrary from $1, \cdots, s$, we obtain

$$\Xi_G(k_1 a k_2) = \Xi_G(a) \leq \Xi_H(a) c_G(1+\sigma_G(a))^r$$
$$\text{for } a \in exp(A_H^+),\ k_1, k_2 \in K_1.$$

Now, Trombi and Varadarajan [T-V], have proven that for any $K-$finite matrix coefficient of a Discrete Series representation of the group G the following estimate holds,

$$|f(x)| \leq c_f \Xi_G^{1+\gamma}(x)(1+\sigma_G(x))^q$$
$$\forall\ x \in G, \text{with } 0 < c_f < \infty, \gamma > 0,\ q \geq 0.$$

Hence, for $a \in exp(A_H^+), k_1, k_2 \in K_1$, we have:

$$\begin{aligned}|f(k_1 a k_2)|^2 &\leq C\Xi_H(a)^{2+2\gamma}(1+\sigma_G(a))^{2(q+r(\gamma+1))} \\ &\leq Ce^{(-2-2\gamma)\rho_H(loga)}(1+\sigma_G(a))^{2(q+\gamma r+r)}(1+\sigma_H(a))^{r_1(1+\gamma)}.\end{aligned}$$

We set $R = 2(q+\gamma r + r) + 2r_1(1+\gamma)$, since $\sigma_G(expY) = \sigma_H(expY)$. The integration formula for the decomposition $H = K_1 exp(A_H^+) K_1$ yields:

$$\begin{aligned}\int_H |f(x)|^2 dx &= \int_{A_H^+} \Delta(Y) \int_{K_1 \times K_1} |f(k_1 exp(Y) k_2)|^2 dk_1 dk_2 dY \\ &\leq C \int_{A_H^+} \Delta(Y) e^{(-2-2\gamma)\rho_H(Y)}(1+\sigma_G(expY))^R dY\end{aligned}$$

Since $\Delta(Y) \leq C_H e^{2\rho_H(Y)}$ on A_H^+, $(C_H < \infty)$ and $\sigma_G(expY)$ is of polynomial growth on Y. We may conclude that the restriction to H of f is square integrable in H, proving Lemma 1 for the equal rank case.
For the nonequal rank case let A_H^+ be the closed Weyl chamber in $\mathfrak{a}_1$ corresponding to N_1. Let $C_1, \cdots C_s$ be the closed Weyl chambers in $\mathfrak{a}$ so that $interior(A_H^+) \cap C_j \neq \emptyset$, $j = 1, \cdots s$. Thus, $A_H^+ = \cup_j (A_H^+ \cap C_j)$ and

$$\int_{A_H^+} |f(expY)|^2 \Delta(Y) dY \leq \sum_j \int_{C_j \cap A_H^+} |f(expY)|^2 \Delta(Y) dY.$$

Let $\rho_j(Y) = \frac{1}{2} trace(ad(Y)|_{\sum_{\alpha:\alpha(C_j)>0} \mathfrak{g}_\alpha})$. Then, as before, on $C_j \cap A_H^+$ we have

$$|f(expY)|^2 << e^{2(\rho_H(Y) - \rho_j(Y))}(1 + \|Y\|^2)^R e^{-2\gamma\rho_j(Y)}.$$

If $\alpha \in \Phi(\mathfrak{a}, \mathfrak{n}(C_j))$, the restriction β of α to $\mathfrak{a}_H$ is either zero, or a restricted root for $(\mathfrak{a}_H, \mathfrak{n}_1)$, or a nonzero linear functional on $\mathfrak{a}_H$. In the last two cases we have that $\beta(C_j \cap A_H^+) \geq 0$, and if β is a restricted root, the multiplicity of β is less or equal than the multiplicity of α. Finally, we recall that any $\beta \in \Psi(\mathfrak{a}_H, \mathfrak{n}_1)$ is the restriction of a positive root for C_j. Thus, $e^{2(\rho_H(Y) - \rho_j(Y))} \leq 1$, and $\rho_j(Y) \geq 0$ for every $Y \in A_H^+$. Hence, $|f(exp(Y))|^2 \Delta(Y)$ is dominated by an exponential whose integral is convergent. This concludes the proof of Lemma 1.

$\square$

Remark 1 *Under our hypothesis we have the inequality*

$$\Xi_G(k_1 a k_2) = \Xi_G(a) \leq \Xi_H(a) c_G (1 + \sigma_G(a))^r$$
$$\text{for } a \in exp(A_H^+),\ k_1, k_2 \in K_1.$$

Let (π, V) be a Holomorphic Discrete Series representation for G and let (τ, W) denote the lowest $K-$type for π. Let E be the homogeneous vector bundle over G/K attached to (τ, W). G acts on the sections of E by left translation. We fix a $G-$invariant inner product on sections of E. The corresponding space of square integrable sections is denoted by $L^2(E)$. Since (π, V) is a holomorphic representation we may choose a $G-$invariant holomorphic structure on G/K such that the L^2-kernel of $\bar{\partial}$ is a realization of (π, V). That is, $V := Ker(\bar{\partial} : L^2(E) \to \mathcal{C}^\infty(E \otimes T^\star(G/K)^{0,1})$. (cf. [K], [N-O], [Sc]). Since $H \subset G$ and $K_1 = H \cap K$ we have that $H/K_1 \subset G/K$ and the

$H-$homogeneous vector bundle $E_\star$ over H/K_1, determined by $\tau_\star$ is contained in E. Thus, we may restrict smooth sections of E to $E_\star$. From now on, we think of (π, V) as the L^2-kernel of the $\bar\partial$ operator.

Lemma 2 *Let f be a holomorphic square integrable section of E and assume that f is left $K-$finite. Then the restriction of f to H/K_1 is also square integrable.*

Proof: Since the $\bar\partial$ operator is elliptic, the L^2-topology on its kernel V is stronger than the topology of uniform convergence on compact subsets. Therefore, the evaluation map at a point in G/K is a continuous map from V to W in the L^2-topology on V. We denote by λ evaluation at the coset eK. Fix an orthonormal basis $v_1, \dots, v_m$ for W. Thus $\lambda = \sum_{i=1}^m \lambda_i v_i$ where the λ_i are in the topological dual to V. We claim that the λ_i are $K-$finite. In fact: if $k \in K,\ v \in V$, $(L_k\lambda)(f) = \sum_i [(L_k\lambda_i)(f)] \otimes v_i = f(k^{-1}) = \tau(k)f(e) = \sum_i \lambda_i(f)\tau(k)v_i = \sum_i\sum_j c_{i\,j}(k)\lambda_i(f)v_i = \sum_i[\sum_j c_{j\,i}\lambda_j(f)] \otimes v_i$. Thus $L_k(\lambda_i)$ belongs to the subspace spanned by $\lambda_1, \cdots, \lambda_m$. Now, $f(x) = \lambda(L_x f) = \sum_i \lambda_i(L_x f)v_i = \sum_i < L_x f, \lambda_i > v_i$. Here,$<,>$ denotes the $G-$invariant inner product on V and λ_i the vector in V that represents the linear functional λ_i. Since f and λ_i are $K-$finite, Lemma 1 says that the functions $x \to < L_x f, \lambda_i >$ are in $L^2(E_\star)$.

$\square$

Therefore the restriction map from V to $L^2(E_\star)$ is well defined on the subspace of $K-$finite vectors in V. Let D be the subspace of functions on V such that their restriction to H is square integrable. Lemma 2 implies that D is a dense subspace in V.

We claim that the restriction map $r : D \to L^2(E_\star)$ is a closed linear transformation. In fact, if f_n is a sequence in D that converges in L^2 to $f \in V$ and such that $r(f_n)$ converges to $g \in L^2(E_\star)$, then, since f_n converges uniformly on compacts to f, g is equal to $r(f)$ almost everywhere. That is, $f \in D$.

Since r is a closed linear transformation, it is equal to the product

$$r = UP \tag{1}$$

of a positive semidefinite linear operator P on V times a unitary linear map U from V to $L^2(E_\star)$. Moreover, if X is the closure of the image of r in

$L^2(E_\star)$, then the image of U is X. Besides, whenever r is injective, U is an isometry of V onto X ([F], 13.9). Since r is $H-$equivariant we have that U is $H-$equivariant ([F], 13.13).

In order to continue we need to recall the Borel embedding of a bounded symmetric domain and to make more precise the realization of the holomorphic Discrete Series (π, V) as the square integrable holomorphic sections of a holomorphic vector bundle. Since G is a linear Lie group, G is the identity connected component of the set of real points of a complex connected semisimple Lie group G_c. The $G-$invariant holomorphic structure on G/K determines an splitting $\mathfrak{g} = \mathfrak{p}_- \oplus \mathfrak{k} \oplus \mathfrak{p}_+$ so that $\mathfrak{p}_-$ becomes isomorphic to the holomorphic tangent space of G/K at the identity coset. Let $P_-, K_{\mathbb{C}}, P_+$ be the associated complex analytic subgroups of G_c Then, the map $P_- \times K_{\mathbb{C}} \times P_+ \longrightarrow G_c$ defined by multiplication is a diffeomorphism onto an open dense subset in $G_{\mathbb{C}}$. Hence, for each $g \in G$ we may write $g = p_-(g)k(g)p_+(g) = p_-k(g)p_+$ with $p_- \in P_-, k(g) \in K_{\mathbb{C}}, p_+ \in P_+$. Moreover, there exists a connected, open and bounded domain $\mathcal{D} \subset \mathfrak{p}_-$ such that $G \subset exp(\mathcal{D})K_{\mathbb{C}}P_+$ and such that the map

$$g \longrightarrow p_-(g)k(g)p_+(g) \longrightarrow log(p_-(g)) \in \mathfrak{p}_- \qquad (2)$$

gives rise to a byholomorphism between G/K and $\mathcal{D}$. The identity coset corresponds to 0. Now we consider the embedding of H into G. Our hypothesis on H implies that there exists a real linear subspace $\mathfrak{q}_0$ of $\mathfrak{p}_-$ so that $dim_{\mathbb{R}}\mathfrak{q}_0 = dim_{\mathbb{C}}\mathfrak{p}_-$ and $H \cdot 0 = \mathcal{D} \cap \mathfrak{q}_0$. In fact, let J denote complex multiplication on the tangent space of G/K, then $\mathfrak{q}_0$ is the subspace $\{X - iJX\}$ where X runs over the tangent space of H/K_1 at the identity coset. Let E be the holomorphic vector bundle over G/K attached to (τ, W). As it was pointed out we assume that (π, V) is the space of square integrable holomorphic sections for E. We consider the real analytic vector bundle $E_\star$ over H/K_1 attached to $(\tau_\star, W)$. Thus $E_\star \subset E$ The restriction map $r : \mathcal{C}^\infty(E) \longrightarrow \mathcal{C}^\infty(E_\star)$ maps the $K-$finite vectors V_F of V into $L^2(E_\star)$. Because we are in the situation $H/K_1 = \mathcal{D} \cap \mathfrak{q}_0 \subset \mathcal{D} \subset \mathfrak{p}_-$ and H/K_1 is a real form of G/K, r is one to one when restricted to the subspace of holomorphic sections of E. Thus, $r : V \longrightarrow \mathcal{C}^\infty(E_\star)$ is one to one. Hence, U gives rise to a unitary equivalence (as $H-$module) from V to a subrepresentation of $L^2(E_\star)$. We need to show that the map U, defined in (1), is onto, equivalently to show that the image of r is dense. To this end, we use the fact that the holomorphic vector bundle

E is holomorphically trivial. We now follow [J-V]. We recall that

$$\mathcal{C}^\infty(E) = \{F : G \longrightarrow W,\ F(gk) = \tau(k)^{-1}F(g)\,\text{and smooth}\}.$$

$$\mathcal{O}(E) = \{F : G \to W,\ F(gk) = \tau(k)^{-1}F(g)\,\text{smooth and} R_Y f = 0\,\forall Y \in \mathfrak{p}_+\}.$$

We also recall that (τ, W) extends to a holomorphic representation of $K_{\mathbb{C}}$ in W and to $K_{\mathbb{C}}P_+$ as the trivial representation of P_+. We denote this extension by τ. Let $\mathcal{C}^\infty(\mathcal{D}, W) = \{f : \mathcal{D} \longrightarrow W,\ f\,\text{is smooth}\}$. Then, the following correspondence defines a linear bijection from $\mathcal{C}^\infty(E)$ to $\mathcal{C}^\infty(\mathcal{D}, W)$:

$$\begin{gathered}\mathcal{C}^\infty(E) \ni F \leftrightarrow f \in \mathcal{C}^\infty(\mathcal{D}, W)\\ F(g) = \tau(k(g))^{-1}f(g\cdot 0),\ f(z) = \tau(k(g))F(g),\ z = g\cdot 0\end{gathered} \tag{3}$$

Here, $k(g)$ is as in (2). Note that $\tau(k(gk)) = \tau(k(g))\tau(k)$. Moreover, the map (3) takes holomorphic sections onto holomorphic functions. The action of G in E by left translation, corresponds to the following

$$(g\cdot f)(z) = \tau(k(x))\tau(k(g^{-1}x))^{-1}f(g^{-1}\cdot z)\quad for\ z = x\,.\,0 \tag{4}$$

Thus, $(k\cdot f)(z) = \tau(k)f(k^{-1}\cdot z), k \in K$. The $G-$invariant inner product on E corresponds to the inner product on $\mathcal{C}^\infty(\mathcal{D}, W)$ whose norm is

$$\|f\|^2 = \int_G \|\tau(k(g))^{-1}f(g\cdot 0)\,\|^2 dg \tag{5}$$

Actually, the integral is over the $G-$invariant measure on $\mathcal{D}$ because the integrand is invariant under the right action of K on G. We denote by $L^2(\tau)$ the space of square integrable functions from $\mathcal{D}$ into W with respect to the inner product (5). Now, in [Sc] it is proved that the $K-$finite holomorphic sections of E are in $L^2(E)$. Hence, Lemma 2 implies that

the $K-$finite holomorphic functions from $\mathcal{D}$ into W are in $L^2(\tau)$. (6)

Via the Killing from, $\mathfrak{p}_-, \mathfrak{p}_+$ are in duality. Thus, we identify the space of holomorphic polynomial functions from $\mathcal{D}$ into W with the space $\mathcal{S}(\mathfrak{p}_+)\otimes W$. The action (4) of K becomes the tensor product of the adjoint action on $\mathcal{S}(\mathfrak{p}_+)$ with the τ action of K in W. Thus, (6) implies that $\mathcal{S}(\mathfrak{p}_+)\otimes W$ are the $K-$finite vectors in $L^2(\tau)\cap\mathcal{O}(\mathcal{D}, W)$. In particular, the constant functions from $\mathcal{D}$ to W are in $L^2(\tau)$. The sections of the homogeneous vector bundle $E_\star$ over H/K_1 are the functions from H to W such that $f(hk) = \tau(k)^{-1}f(h),\ k \in$

K_1, $h \in H$. We identify sections of $E_\star$ with functions form $\mathcal{D} \cap \mathfrak{q}_0$ into W via the map (3). Thus, $L^2(E_\star)$ is identified with the space of functions

$$L^2(\tau_\star) := \{f : \mathcal{D} \longrightarrow W, \int_H \|\tau(k(h))^{-1} f(h \cdot 0)\|^2 dh < \infty\}$$

The action on $L^2(\tau_\star)$ is as in (4). Now, the restriction map for functions from $\mathcal{D}$ into W to functions from $\mathcal{D} \cap \mathfrak{q}_0$ into W is equal to the map (3) followed by restriction of sections from $\mathcal{D}$ to $\mathcal{D} \cap \mathfrak{q}_0$ followed by (3). Therefore, Lemma 2 together with (6) imply that the restriction to $\mathcal{D} \cap \mathfrak{q}_0$ of a $K-$finite holomorphic function from $\mathcal{D}$ to W is an element of $L^2(\tau_\star)$. Since $\mathfrak{q}_0$ is a real form of $\mathfrak{p}_-$ when we restrict holomorphic polynomials in $\mathfrak{p}_-$ to $\mathfrak{q}_0$ we obtain all the polynomial functions in $\mathfrak{q}_0$. Thus, all the polynomial functions from $\mathfrak{q}_0$ into W are in $L^2(\tau_\star)$. In particular, we have that

$$\int_H \|\tau(k(h))^{-1} v\|^2 dh < \infty, \forall\, v \in W \qquad (7).$$

Now, given $\epsilon > 0$ and a compactly supported continuous function f from $\mathcal{D} \cap \mathfrak{q}_0$ to W, the Stone-Weierstrass Theorem produces a polynomial function p from $\mathfrak{q}_0$ into W so that $\|f(x) - p(x)\| \leq \epsilon$, $x \in \overline{\mathcal{D}} \cap \mathfrak{q}_0$. Formula (7) says that $\|f - p\|_{L^2(\tau_\star)} \leq \epsilon$. Hence, the image by the restriction map of $V = \mathcal{O}(\mathcal{D}, W) \cap L^2(\tau)$ is a dense subset. Thus, the linear transformation U in (1) is a unitary equivalence from V to $L^2(\tau_\star)$. Therefore, Theorem 1 is proved.

Remark 2 *For a holomorphic unitary irreducible representations which is not necessarily square integrable, condition (7) is exactly the condition used by Olafsson in [O2] to show an equivalent statement to Theorem 1.*

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