ADMISSIBLE RESTRICTION TO K_2

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Abstract. Restriction to K2
Analysis of theorem 4.1 kob 1994.

1. Introduction

$G, K, T, \Psi, K_1(\Psi), K_2$ as always. Yamashita, Borho ....has shown that the associated variety for discrete series $\pi(\lambda)$ whose Harish-Chandra parameter is dominant with respect to Ψ is

$$Ass(\pi) = Ad(K_{\mathbb{C}})(\sum_{\gamma\in\Psi_n} \mathfrak{g}_{-\gamma}).$$

Since, the Lie algebra of the group $Ad(K_2)$ is generated by root vectors associated to compact simple roots, lemma ?? forces it leaves invariant $\sum_{\gamma\in\Psi_n} \mathfrak{g}_{-\gamma}$. As an application we have

Lemma 1. *For any closed subgroup L of K_2, $\mathbb{C}[\sum_{\gamma\in\Psi_n} \mathfrak{g}_{-\gamma}]^L = \mathbb{C}$ is equivalent $\pi(\lambda)$ to be admissible when restricted to L.*

Proof: We apply a theorem due to Huang and Vogan which affirms: $\mathbb{C}[Ass(\pi)]^L = \mathbb{C}$, if and only if $\pi(\lambda)_{|L}$ is admissible.

We show for any compact subgroup L of K_2 the equivalence,

$$\mathbb{C}[\textstyle\sum_{\gamma\in\Psi_n} \mathfrak{g}_{-\gamma}]^L = \mathbb{C} \text{ if and only if } \mathbb{C}[Ass(\pi)]^L = \mathbb{C}.$$

Let $f \in \mathbb{C}[Ass(\pi)]^L$. Because of the hypothesis on L for each $y \in K_{1\mathbb{C}}$ the function $f(Ad(y)?)$ restricted to $\sum_{\gamma\in\Psi_n} \mathfrak{g}_{-\gamma}$ is constant. Hence, $f(Ad(y)X) = f(Ad(y)0) = f(0)$ for any $y \in K_1, X \in \sum_{\gamma\in\Psi_n} \mathfrak{g}_{-\gamma}$. Thus, f is constant. For the converse statement we recall that when L is a subgroup of K_2, restriction map from $\mathbb{C}[Ass(\pi)]$ onto $\mathbb{C}[\sum_{\gamma\in\Psi_n} \mathfrak{g}_{-\gamma}]$ is L equivariant. □

Proposition 1. *Assume $\mathfrak{z}_{\mathfrak{k}} = 0$. For Harish-Chandra parameter dominant with respect to Ψ there is admissible restriction to $K_2(\Psi)$ if and only if $G = Sp(1,p), K_1 = Sp(1), K_2 = Sp(q)$. Moreover, there is no admissible restriction to any proper clossed subgroup of K_2.*
It is left out the case $\mathfrak{z}_{\mathfrak{k}} \neq 0$.

For this, we check for each group G case the hypothesis of lemma .... We write $\mathfrak{p}_\Psi = \sum_{\gamma\in\Psi_n} \mathfrak{g}_{-\gamma}$.

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G	$K_1(\Psi)$	$(K_2)_{ss}$	$\mathfrak{p}_\Psi$	A
$SU(p,q)$	$Sl(p)$, $1 \leq a < p$	$Sl(q)$	$\mathbb{C}^{(p-a)q} \boxtimes (\mathbb{C}^{qa})^*$	N
$SO(4,1)$	$SO(\epsilon_1 \pm \epsilon_2)$	$S0(\epsilon_1 \mp \epsilon_2)$	$\mathbb{C}^2$	Y
$SO(4,2q+1)$	$SO(\epsilon_1 \pm \epsilon_2)$	$S0(\epsilon_1 \mp \epsilon_2) \times SO(2q+1)$	$\mathbb{C}^2 \oplus \mathbb{C}^{2q+1}$	N
$SO(4,2q+1)$	$SO(\epsilon_1 \pm \epsilon_2) \times SO(2q+1)$	$S0(\epsilon_1 \mp \epsilon_2)$	$\mathbb{C}^2 \boxtimes 1^{2q+1}$	N
$SO(2p,2q+1)$	$SO(2p)$	$SO(2q+1)$	$1^p \boxtimes \mathbb{C}^{2q+1}$	N
$Sp(1,q)$	$Sp(1)$	$Sp(q)$	$1^1 \boxtimes \mathbb{C}^{2q}$	Y
$Sp(1,q)$	$Sp(q), 2 \leq q$	$Sp(1)$	$\mathbb{C}^2 \boxtimes 1^q$	N
$Sp(p,q)$	$Sp(p), 2 \leq p \leq q$	$Sp(q)$	$1^p \boxtimes \mathbb{C}^{2q}$	N
$SO(4,2q)$	$SO(\epsilon \pm \epsilon_2)$	$SO(\epsilon_1 \mp \epsilon_2) \times SO(2q)$	$\mathbb{C}^2 \boxtimes \mathbb{C}^{2q}$	N
$SO(4,2q)$	$SO(\epsilon \pm \epsilon_2) \times SO(2q)$	$SO(\epsilon_1 \mp \epsilon_2)$	$\mathbb{C}^2 \boxtimes 1^{2q}$	N
$SO(4,2q)$	$SO(2q)$	$SO(4)$	$\mathbb{C}^4 \boxtimes 1^q$	N
$SO(2p,2q)$	$SO(2q)$	$SO(2p)$	$\mathbb{C}^{2p} \boxtimes 1^q$	N
$SO(2p,2q)$	$SO(2p)$	$SO(2q)$	$1^p \boxtimes \mathbb{C}^{2q}$	N
$G_{2(2)}$	$SU_2(\alpha_{long})$	$SU_2(\alpha_{short})$	$1 \boxtimes S^3(\mathbb{C}^2)$	N
$G_{2(2)}$	$SU_2(\alpha_{short})$	$SU_2(\alpha_{long})$	$\mathbb{C}^2 \boxtimes 1^2$	N
$F_{4(4)}$	SU_2	$SP(3)$	$1 \boxtimes \Lambda^3(\mathbb{C}^6)_0$	N
$F_{4(4)}$	$SP(3)$	SU_2	$\mathbb{C}^2 \boxtimes 1^7$	N
$E_{6(2)}$	SU_2	$SU(6)$	$1 \boxtimes \Lambda^3(\mathbb{C}^6)$	N
$E_{6(2)}$	$SU(6)$	$SU(2)$	$\mathbb{C}^2 \boxtimes 1^{10}$	N
$E_{7(-5)}$	SU_2	$Spin(12)$	$1 \boxtimes \mathbb{C}^{32}$	N
$E_{7(-5)}$	$Spin(12)$	$SU(2)$	$\mathbb{C}^2 \boxtimes 1^{16}$	N
$E_{8(-24)}$	SU_2	E_7	$1 \boxtimes \mathbb{C}^{56}$	N
$E_{8(-24)}$	E_7	$SU(2)$	$\mathbb{C}^2 \boxtimes 1^{28}$	N

Analysis Thm 4.1 kob 1994

We follow the notation in Thm 4.1 of kob 94, and we show that when we assume $L := L_{sch}$, for Ψ, then hypothesis in Thm. 4.1 of kob 1994 imply $K_1(\Psi)$ is a subgroup of K'.

We notice hypothesis 4.1(a) is equivalent to there exists $H_0 \in i\mathfrak{k}'$ so that $\mathfrak{l}$ is the centralizer in $\mathfrak{g}$ of H_0 and $\Phi_\mathfrak{u}$ is the set of roots who takes on a positive value on H_0.

Hypothesis 4.1(b): $S(\mathfrak{u}\cap\mathfrak{p}) \otimes S(\bar{\mathfrak{u}}\cap\mathfrak{k}/\bar{\mathfrak{u}}\cap\mathfrak{k}')$ is an admissible representations for $\mathfrak{l}\cap\mathfrak{k}'$.

The hypothesis L is compact implies $\mathfrak{u}\cap\mathfrak{p} = \mathfrak{u}_\Psi$. We now show $[\mathfrak{u}\cap\mathfrak{p}, \mathfrak{u}\cap\mathfrak{p}]$ is contained in $\mathfrak{u}\cap\mathfrak{k}'$. For this we pick $Y_{\beta_j}, \beta_j \in \Psi_n$ so that its bracket is non zero. If $Y_{-(\beta_1+\beta_2)}$ were not in $\mathfrak{u}\cap\mathfrak{k}'$. Then, the vectors $Y^r_{\beta_1}Y^r_{\beta_2} \otimes Y^r_{-(\beta_1+\beta_2)}$, $r = 1, 2, \ldots$, would be non zero in the tensor product algebra and would belong to the invariants under the center of $\mathfrak{l}\cap\mathfrak{k}'$ which would contradict hypothesis 4.1(b). Hence, we obtain the inclusion $[\mathfrak{u}_\Psi, \mathfrak{u}_\Psi] \subset \mathfrak{k}'$. Lemma.... implies $\mathfrak{k}_1(\Psi)$ is contained in $\mathfrak{k}'$. □

When $\mathfrak{z}_\mathfrak{k} \neq 0$, $\mathfrak{z}_\mathfrak{k}$ it is not always contained in $\mathfrak{k}'$ as the following shows,

Example: $G := SO(2,2q)$, $\mathfrak{k}' := SO(2q)$, $\Psi = \{\delta_1 > \cdots > \delta_q > \epsilon\}$. Then root system for $L_{sch} := U(q)Z_K$ is $\pm(\delta_j - \delta_k)$; roots for $\mathfrak{u}_{sch}$ are $\delta_j + \delta_k, \delta_j \pm \epsilon$. $H_0 = \sum_j \delta_j$. The inclusion $SO(2q)/U(q) \to SO(2,2q)/L_{sch}$ is holomorphic. Hence, there is admissible restriction to $SO(2q)$ and $\mathfrak{z}_\mathfrak{k}$ is not contained in $\mathfrak{k}'$.

Conversely, if L is compact and $K_1(\Psi)Z_K$ is contained in L then the hypothesis of Thm 4.1 readily follows.

Next, we deduce a particular case of corollary 4.4 in kob 94 under the hypothesis $\mathfrak{l} := \mathfrak{l}_{sch}$

Corollary 1. *If $\mathfrak{u}$ is an ideal of $\mathfrak{k}$ so that $\mathfrak{z}_{\mathfrak{l}_{sch}} \subseteq \mathfrak{u}$, then $\pi(\lambda)$ has admissible restriction to $\mathfrak{u}$.*

Indeed, since $\mathfrak{z}_{\mathfrak{l}_{sch}}$ is the zero set of the compact simple roots in Ψ any root in $\Phi - \Phi_{\mathfrak{l}_{sch}}$ has a nonzero restriction to $\mathfrak{z}_{\mathfrak{l}_{sch}}$. Otherwise a root in $\Phi - \Phi_{\mathfrak{l}_{sch}}$ would be a linear combination of compact simple roots for Ψ. Thus, if β_j are roots in Ψ_n whose sum is a root there exists $H \in \mathfrak{z}_{\mathfrak{l}_{sch}}$ so that $(\beta_1 + \beta_2)(H) \neq 0$. Since $\mathfrak{u}$ is an ideal we have that $(\beta_1 + \beta_2)(H)Y_{\beta_1+\beta_2} = [H, Y_{\beta_1+\beta_2}]$ is in $\mathfrak{u}$. Hence, $\mathfrak{u}_\Psi$ is contained in $\mathfrak{u}$ and therefore $\mathfrak{k}_1(\Psi)$ is contained in $\mathfrak{u}$. The hypothesis forces $\mathfrak{z}_\mathfrak{k}$ is also contained in $\mathfrak{l}$. We now apply prop..... □