ADMISSIBLE RESTRICTION OF HOLOMORPHIC DISCRETE SERIES FOR EXCEPTIONAL GROUPS

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To Mischa Cotlar with respect

ABSTRACT. In this note we list the compact subgroups L of an exceptional simple Lie group such that a holomorphic discrete series for G has an admissible restriction to L.

1. Introduction

A basic problem in representation theory of Lie groups is to derive "branching laws". By this we mean, for a given unitary irreducible representation of an ambient group G, consider its restriction to a fixed subgroup H and find the decomposition as a direct integral, and in particular compute the multiplicity of each irreducible factor of the restriction. There is a vast literature on this subject, and here we just direct the reader's attention to the extensive reviews of [13], [14] and references therein. In this note, we consider Holomorphic Discrete Series and we split the problem in two subproblems, namely, to determine whether or not the given representation has an admissible restriction to a subgroup H and secondly, for an admissible representation compute the multiplicity of each irreducible factor. Let us recall that a unitary representation of a topological group is *admissible* if it is a discrete Hilbert sum of irreducible unitary subrepresentations and each irreducible summand occurs with finite multiplicity.

Let G be a connected reductive Lie group, a unitary irreducible representation (π, V) of G is a *discrete series* representation when each matrix coefficient is square integrable with respect to Haar measure in G. We fix H a reductive subgroup of G, L a maximal compact subgroup of H and (π, V) a discrete series representation of G. In [6] we

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have shown,
Fact 1: the restriction of (π, V) to H is admissible if and only if the restriction of (π, V) to L is admissible.
We also have shown a formula that allows to compute $H-$multiplicities from the knowledge of $L-$multiplicites and conversely. Thus, in order to understand admissible restriction of discrete series to reductive subgroups of G we are left to consider compact subgroups.
For a Lie group we denote its Lie algebra by the corresponding German lower case letter, to denote its complexification we add the subscript $\mathbb{C}$.
Next, we fix a maximal compact K subgroup of G. This choice gives rise to a Cartan decomposition of the Lie algebra of G, $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{s}$. It is known that the homogeneous space G/K carries a $G-$invariant complex structure if and only if $\mathfrak{s}_{\mathbb{C}}$ splits us the sum of two $K-$subrepresentations $\mathfrak{s}^{\pm}$ such that $\mathfrak{s}^{+}$ is dual to $\mathfrak{s}^{-}$. Assume G/K admits a $G-$invariant complex structure, a representation of G is *holomorphic* when its underlying Harish-Chandra module has a non zero vector so that $\mathfrak{s}^{-}$ is included in its annihilator.
From now on, we fix an exceptional simple Lie group G so that G/K admits an invariant complex structure and (π, V) and a holomorphic discrete series for G. The aim of this note is to determine the closed subgroups L of K so that (π, V) is admissible when restricted to L.
As another application of our technique we obtain noncompact semisimple subgroups H of $E_{6(-14)}$ such that π has admissible restriction to H and condition (C) does not hold for the system of positive roots associated to π. In turn, [6], this implies that to actually compute $H-$multiplicities, for these examples, by means of a Blattner-Kostant type formula we need two partition functions, whereas when condition (C) holds one partition function suffices. These are the first known examples of noncompact subgroups H such that a discrete series representation has admissible restriction to H and condition (C) does not holds. In [6] we present discrete series of real rank one Lie groups whose restriction to a compact subgroup L is admissible, condition (C) does not hold and L is not contained in any noncompact subgroup of H.

The exceptional connected simple Lie groups whose quotient by a maximal compact subgroup carries an invariant complex structure has been classified by E. Cartan. They are, up to covering, the groups $E_{6(-14)}, E_{7(-25)}$. The respective Cartan decompositions are

$$\mathfrak{e}_{6(-14)} = \mathfrak{so}(10)_{\mathbb{C}} + \mathbb{R} + (\mathfrak{s}^{+} \oplus \mathfrak{s}^{-})$$

Here, $\mathfrak{s}^{\pm}$ are the half spin representations.

$$\mathfrak{e}_{7(-25)} = \mathfrak{e}_6 + \mathbb{R} + (\varpi_1 \oplus \varpi_6).$$

Here, $\varpi_\star$ are the two fundamental representations of dimension twenty seven.

We would like to point out that in his Ph.D. thesis [17] has obtained results which follow from Theorem 1. His technique is different from the one considered in this note.

2. Admissible restriction of Holomorphic Discrete Series for $E_{6(-14)}, E_{7(-25)}$.

As usual, we denote by $E_{6(-14)}$ the analytic subgroup of the simple connected complex Lie group of type E_6 associated to the real form $\mathfrak{e}_{6(-14)}$ of the complex simple Lie algebra $\mathfrak{e}_6$. The semisimple factor K_{ss} of a maximal compact subgroup K of $E_{6(-14)}$ is isomorphic to $Spin(10)$ and K is isomorphic to $Spin(10) \times SO(2)$. We show

Proposition 1. *Let (π, V) be a holomorphic discrete series representation for $E_{6(-14)}$. Then,*
i) (π, V) restricted to $K_{ss} = Spin(10)$ is admissible.
ii) Let $U(5) \to SO(10)$ denote one of two usual imbedding and let $\widehat{U(5)}$ denote the analytic subgroup of $Spin(10)$ associated to $\mathfrak{u}(5)$, then the restriction of π to $\widehat{U(5)}$ is admissible.
iii) For any other maximal subgroup L of K_{ss}, (π, V) restricted to L is not admissible.
iv) Let $\widehat{SU(5)}$ denote the simple factor of $\widehat{U(5)}$. Then, π restricted to $\widehat{SU(5)}$ is not admissible.

Fact 0: Let $L \subset H$ reductive subgroups of G. Then, π restricted to L is admissible implies π restricted to H is admissible. For a proof [12]. Therefore, we must search for the subgroups L of K so that L is minimal among the subgroups on which π restricted to them is admissible. Proposition 1, provides un answer when we restrict ourselves to consider subgroups of K_{ss}. Our next goal is to understand admissible restriction to other subgroups of K.

In the following paragraph we construct a list of connected Lie subgroups of $E_{6(-14)}$. For root tables and fundamental weights for $\mathfrak{e}_6$ we follow [7]. We fix a compact Cartan subalgebra $\mathfrak{t}$ of $\mathfrak{k}$. We denote by Φ the root system for the pair $(\mathfrak{g}_\mathbb{C}, \mathfrak{t}_\mathbb{C})$. The system of positive roots associated to the holomorphic discrete series is denoted by Ψ. We denote the simple roots for Ψ as

$$\alpha_1, \cdots, \alpha_5, \beta$$

Here, β is the noncompact simple root, α_j is adjacent to $\alpha_{j+1}, j = 1,2,3$; α_4 is adjacent to β, α_5 is adjacent to α_3 and the maximal root is $\beta_M = \alpha_1 + 2\alpha_2 + 3\alpha_3 + 2\alpha_4 + \beta + 2\alpha_5$, β_M is adjacent to α_5. Besides, β_M is the fundamental weight associated to α_5 and is the highest weight of the representation $\mathfrak{s}^+$. Let H_M denote the vector in $\mathfrak{t}$ corresponding to the root $i\beta_M$. The center of $\mathfrak{k}$ is spanned by the vector H_0 corresponding to the fundamental weight associated to β. Thus, H_0 corresponds to $i\frac{1}{3}(2\alpha_1 + 4\alpha_2 + 6\alpha_3 + 5\alpha_4 + 4\beta + 3\alpha_5)$.

As usual, we write $\mathfrak{t}_{\mathbb{C}} = \mathbb{C}H_0 \oplus \mathfrak{t}_s$, where $\mathfrak{t}_s$ is the toric subalgebra of $\mathfrak{k}_{ss}$ spanned by the vectors $H_{\alpha_1}, \cdots, H_{\alpha_5}$. Let ρ_j denote the fundamental weight of $spin(10)$ associated to α_j. Since the root system of the centralizer of ρ_j in $spin(10)$ is spanned by the simple roots different from α_j, we obtain that the centralizer of ρ_j is equal to a semisimple Lie algebra $\mathfrak{r}_j$ plus the line spanned by H_{ρ_j}.

We fix a, b real numbers, j runs from 1 to 5
We define $\mathfrak{l}_{j,a,b}$ to be the subalgebra spanned by $\mathfrak{r}_j$ together with the vector $aH_0 + bH_{\rho_j}$. We only consider a, b such that the analytic subgroup associated to $\mathfrak{l}_{j,a,b}$ is compact. Either $\mathfrak{l}_{4,a,b}$, or $l_{5,a,b}$ is isomorphic to $\mathfrak{u}(5)$. $\mathfrak{l}_{4,0,1}, \mathfrak{l}_{5,0,1}$ are the usual two immersions of $\mathfrak{u}(5)$ in $spin(10)$. The subalgebra $\mathfrak{l}_{4,a,b}$ is not conjugated to $\mathfrak{l}_{5,a,b}$.

To define subalgebras $\mathfrak{h}_j$ we mark suitable subroot systems Φ_j of Φ as well as vectors in $\mathfrak{t}$ and then consider the real form $\mathfrak{h}_j$ of the subalgebra spanned by the root vectors corresponding to the roots in Φ_j together with the choice of vectors in $\mathfrak{t}$.

$\Phi_1 := \Phi \cap < \alpha_1, \alpha_2, \alpha_3, \alpha_4 >_{\mathbb{Z}} \cup \{\pm\beta_M\}$.
Hence, $\mathfrak{h}_1$ is isomorphic to $\mathfrak{su}(5) + \mathfrak{sl}(2, \mathbb{R})$. Since $H_M = \frac{3}{4}H_0 + H_{\rho_5}$, $\mathfrak{l}_{5,\frac{3}{4},1}$ is a maximally compact subalgebra of $\mathfrak{h}_1$.

$\Phi_2 := \Phi \cap < \alpha_1, \alpha_2, \alpha_3, \alpha_5, \beta + \alpha_4 >_{\mathbb{Z}}$.
Hence, $\mathfrak{h}_2$ is isomorphic to $\mathfrak{so}^\star(10)$. The center of $\mathfrak{k} \cap \mathfrak{h}_2$ is spanned by the vector $H_2 = \frac{15}{8}H_0 - \frac{1}{2}H_{\rho_4}$ corresponding to $i(\alpha_1 + 2\alpha_2 + 3\alpha_3 + \frac{5}{2}(\beta + \alpha_4) + \frac{3}{2}\alpha_5)$. It follows $\mathfrak{l}_{4,\frac{15}{8},-\frac{1}{2}}$ is a maximally compact subalgebra of $\mathfrak{h}_2$.

$\mathfrak{su}(5,1)$ is the subalgebra obtained from the subroot system spanned by $\alpha_1, \alpha_2, \alpha_3, \alpha_4, \beta$. Here, $\mathfrak{r}_5$ is the semisimple factor of a maximal compact subalgebra. The center is spanned by $\frac{15}{4}H_0 - 3H_{\rho_5}$ and corresponds to $\alpha_1 + 2\alpha_2 + 3\alpha_3 + 4\alpha_4 + 5\beta$.

$\mathfrak{su}(4,1) + \mathfrak{su}(2)$ is the subalgebra constructed from the subroot system spanned by $\alpha_1, \alpha_3, \alpha_4, \alpha_5, \beta$. Here, $\mathfrak{r}_2$ is the semisimple factor of a maximally compact subalgebra. The center is spanned by $5H_0 - 2H_{\rho_2}$ corresponds to $\alpha_5 + 2\alpha_3 + 3\alpha_4 + 4\beta$.

From now on, we write $\mathbb{C}^\star(aH_0+bH_{\rho_j})$ for the group $K\cap exp(\mathbb{C}(aH_0+bH_{\rho_j})$.

Proposition 2. *A holomorphic discrete series for $E_{6(-14)}$ has an admissible restriction to the subgroups:*

$$\mathbb{C}^\star(aH_0+bH_{\rho_1}) \text{ iff } |a|>|\frac{b}{2}|\,;\ \mathbb{C}^\star(aH_0+bH_{\rho_2}) \text{ iff } |a|>|b|;$$

$$\mathbb{C}^\star(aH_0+bH_{\rho_3}) \text{ iff } |a|>|\frac{3b}{2}|\,;\ \mathbb{C}^\star(aH_0+bH_{\rho_4}) \text{ iff } (a-\frac{5b}{4})(a+\frac{3b}{4})>0;$$

$$\mathbb{C}^\star(aH_0+bH_{\rho_5}) \text{ iff } (a+\frac{5b}{4})(a-\frac{3b}{4})>0;$$

$$L_{4,a,b} \text{ iff } (a-\frac{5b}{4})\neq 0\,;$$

$$L_{5,a,b} \text{ iff } (a+\frac{5b}{4})\neq 0\,;$$

$$H_1 \simeq SU(5)\times SL(2,\mathbb{R});\ H_2\simeq SO^\star(10);\ SU(4,1)\times SU(2).$$

The restriction of a holomorphic discrete series to any of the groups

$$SU(5,1);\ F_{4(-20)};\ SO(8,2);\ R_j,\ j=1 \text{ to } 5$$

is discretely decomposable and not admissible.

The subgroups of $E_{6(-14)}$ considered in Proposition 1 are related to the list of semisimple symmetric spaces $(E_{6(-14)},H)$.

We are now ready to state

Theorem 1. *Let (π,V) be a holomorphic discrete series for $E_{6(-14)}$ and L a closed subgroup of K so that π restricted to L is admissible. Then, L is one of the following*

a) The center of K is contained in L

b) L contains one of $\mathbb{C}^\star(aH_0+bH_{\rho_1})$ j,a,b as in Proposition 2.

Proposition 3. *Let (π,V) be a holomorphic discrete series for $E_{7(-25)}$. Then, π restricted to the semisimple factor of K, E_6, is not admissible.*

For (π,V) a holomorphic discrete series representation, a result of Harish-Chandra states that the $K-$module structure of the underlying Harish-Chandra module for π is equivalent to $S[\mathfrak{s}^+]\otimes V$. Here, $S[\mathfrak{s}^+]$ denotes the symmetric algebra of $\mathfrak{s}^+$ and V is the lowest $K-$type of π.

Fact 2: Let L be a closed subgroup of K. In [12], [6] it is shown: (π,V) has an admissible restriction to L if and only if the $L-$module $S[\mathfrak{s}^+]$ is admissible if and only if $S[\mathfrak{s}^+]^L=\mathbb{C}$.

Since all noncompact roots in a holomorphic chamber takes on the same value on a generator of the center of K it follows that $S[\mathfrak{s}^+]^{Z_K} = \mathbb{C}$. Hence, a holomorphic discrete series has an admissible restriction to the center of K. Fact 0 yields that a holomorphic discrete series has an admissible restriction to any subgroup which contains the center of K. In particular, we obtain admissible restriction to the subgroups $L_{j,b,0}$ for b not equal to zero.

Proposition 3 follows from Fact 2 and the table in [15] which states $S[\varpi_1]^{E_6}$ is a polynomial algebra in one generator of degree three.

Proof of Proposition 1. We recall the Cartan decomposition of $\mathfrak{e}_{6(-14)} = \mathfrak{so}(10) + \mathbb{R} \oplus \mathfrak{s}^+ + \mathfrak{s}^-$ where $\mathfrak{s}^+, \mathfrak{s}^-$ are the two spin representations of $Spin(10)$.

Therefore, in order to conclude the proof of Proposition 1 we are to show:

j) $S[\mathfrak{s}^+]^{Spin(10)} = \mathbb{C}$.

jj) $S[\mathfrak{s}^+]^{\widehat{U(5)}} = \mathbb{C}$.

jjj) $S[\mathfrak{s}^+]^L \neq \mathbb{C}$ for a maximal subgroup of K_{ss} not locally isomorphic to $U(5)$.

jv) $S[\mathfrak{s}^+]^{\widehat{SU(5)}} \neq \mathbb{C}$

In [2], [15] we find a proof of $S[\mathfrak{s}^+]^{Spin(10)} = \mathbb{C}$. Thus j) follows.
To verify jj), we consider the maximal tori subalgebra $\mathfrak{t}_s$ of $\mathfrak{so}(10)$ and a basis $\epsilon_1, \cdots, \epsilon_5$ of $i\mathfrak{t}_s^\star$ so that a system of positive compact roots is $\epsilon_i \pm \epsilon_j, i < j$ and the weights of the representation $\mathfrak{s}^+$ are $\frac{1}{2}(\pm\epsilon_1 \pm \epsilon_2 \pm \epsilon_3 \pm \epsilon_4 \pm \epsilon_5)$ with and odd number of $+$. The positive roots of $\mathfrak{u}(5)$ are $\epsilon_i - \epsilon_j, i < j$. As $\mathfrak{u}(5)-$module $\mathfrak{s}^+$ decomposes as

$$V_{\frac{1}{2}(\epsilon_1+\epsilon_2+\epsilon_3+\epsilon_4+\epsilon_5)} \oplus V_{\epsilon_1-\frac{1}{2}(\epsilon_1+\epsilon_2+\epsilon_3+\epsilon_4+\epsilon_5)} \oplus V_{\epsilon_1+\epsilon_2+\epsilon_3-\frac{1}{2}(\epsilon_1+\epsilon_2+\epsilon_3+\epsilon_4+\epsilon_5)}$$

In [10] page 98 we find a proof of

$$(SL(2m+1), \Lambda^2(\mathbb{C}^{2m+1}) \oplus \Lambda^1(\mathbb{C}^{2m+1})^\star)$$

is a prehomogeneous space. Hence, $SU(5)_{\mathbb{C}}$ has an open orbit in

$$V_{\epsilon_1} \oplus V_{\epsilon_1+\epsilon_2+\epsilon_3}$$

Then, $\widehat{U(5)_{\mathbb{C}}}$ has an open orbit in $\mathfrak{s}^+$. Therefore, $S[\mathfrak{s}^+]^{\widehat{U(5)}} = \mathbb{C}$.
In order to show jjj) we recall the list of maximal subgroups of $SO(10)$.

These subgroups have been classified by Dynkin in [5] They are:
Reducible subgroups: $SO(r) \times SO(s)$ for $r + s = 10$.
Subgroups of equal rank not listed above: $U(5)$.
Irreducible, non-simple subgroups: none.

Irreducible simple subgroups: $L \subseteq SO(10)$, where L is a simple, connected subgroup so that the representation in $\mathbb{R}^{10}$ is irreducible.

To continue, we assume for each maximal subgroup L of $Spin(10)$ not locally isomorphic to $U(5)$ the restriction of π to L is admissible, equivalently, $S[\mathfrak{s}^+]^L = \mathbb{C}$, from this we derive a contradiction.
Let $\mathfrak{s}^+ : Spin(10) \longrightarrow Gl(\mathfrak{s}^+)$ denote the half spin representation. To begin with we consider the case $\tau : L \longrightarrow Spin(10)$ is an irreducible, simple, maximal subgroup. $Then(\mathfrak{s}^+ \circ \tau, \mathfrak{s}^+)$ decomposes as the sum irreducible $L-$modules

$$V_1 \oplus \cdots \oplus V_r.$$

We set τ_j equal to the projection onto V_j followed by $\mathfrak{s}^+ \circ \tau$. Owing to our hypothesis, it follows $S[V_j]^{\tau_j(L)} = \mathbb{C}$ for $j = 1, \cdots, r$. In [10], [11] we find the list of triple (L, τ_j, V_j) satisfying: L is a simple algebraic group, τ_j is an irreducible representation and $S[V_j]^{\tau_j(L)} = \mathbb{C}$. The list is:

$$(A_n, \Lambda_1, \mathbb{C}^{n+1});\ (A_{2n}, \Lambda_2, \mathbb{C}^{n(2n+1)});\ (C_n, \Lambda_1, \mathbb{C}^{2n}).$$

We verify none of the V_j is equivalent to (C_n, Λ_1). L of type C_1 cannot be because the ten dimensional irreducible representation of C_1 is symplectic. For L of type $C_n, n \geq 2$ and $r = 1$ we obtain $n = 8$, a contradiction. For L of type $C_n, n \geq 2$ and $r \geq 2$ the symplectic form lead us to $S[\mathfrak{s}^+]^L \neq \mathbb{C}$, another contradiction.

For L of type $A_n, n \geq 2$, if at least one V_j is equivalent to (A_{2n}, Λ_2), then $n = 2k$ and $k(2k+1) \leq 16$, hence, L is one of A_2, A_4. The ten dimensional irreducible representations of $SL(3)$ has highest weight $(3,0,0) = 3\Lambda_1$ or is the dual representation, neither of these two representations are orthogonal [3]. The ten dimensional representations of $SL(5)$ have highest weight Λ_2 or Λ_3 which are not orthogonal.
We are left with the case all V_j are equivalent to (A_n, Λ_1). For L of type $A_n, n \geq 2$ since L is a subgroup of $Spin(10)$ we have $n \leq 5$, hence, we are left with $n = 3, 5$. The ten dimensional irreducible representations of $SL(4)$ have highest weight $2\Lambda_1$ or $2\Lambda_3$ they are not orthogonal. $SL(6)$ has no irreducible representation of dimension ten.

To conclude the proof of jjj) we show

$$S[\mathfrak{s}^+]^{\mathfrak{so}(p)\oplus\mathfrak{so}(q)} \neq \mathbb{C} \text{ for } p \geq 1, q \geq 1, p + q = 10.$$

We recall the following facts, for a proof, [3] Table 1,

a) A half spin representations $(s^{\pm})$ for $Spin(2k)$ restricted to $Spin(2k-1)$ is equivalent to the spin representation (s).

b) The spin representation for $Spin(2k+1)$ restricted to $Spin(2k)$ is equivalent to the sum of the two half spin representations.

c) Any irreducible spin representation for $Spin(9), Spin(8), Spin(7)$ is orthogonal.

d) Any irreducible spin representation for $Spin(5), Spin(4)$ is symplectic

For $p = 9, q = 1$, $\mathfrak{s}^+$ restricted to $Spin(9)$ is equivalent to the spin representation of $Spin(9)$. Since the spin representation of $Spin(9)$ is orthogonal, we obtain $S[\mathfrak{s}^+]^{\mathfrak{so}(1)\oplus\mathfrak{so}(9)} \neq \mathbb{C}$.

For $p = 8, q = 2$ $\mathfrak{s}^+_{|_{Spin(8)}} = s^+ \oplus\ s^-$, besides $spin(2)$ acts on $s^\pm$ by $\pm\frac{1}{2}$. Let $b_\pm$ denote a $Spin(8)$ invariant quadratic form in $\mathfrak{s}^\pm$. Then b_+b_- is invariant under $Spin(8) \times Spin(2)$.

For $p = 7, q = 3$, $\mathfrak{s}^+_{|_{Spin(7)}} = s \oplus s$. Hence, $\mathfrak{s}^+_{|_{Spin(7)\times Spin(3)}} = s \boxtimes \mathbb{C}^2$. In [15] it is shown there is an invariant of degree four. Else, in [11], it is shown it is not an irreducible prehomogeneous vector space. Since $Spin(7) \times Spin(3)$ is a semisimple Lie group, there are polynomials invariant under its action.

For $p = 6, q = 4$, $\mathfrak{s}^+_{|_{Spin(6)}} = (s_+ \oplus s_-) \oplus (s_+ \oplus s_-)$.

It follows $Spin(4)\mathfrak{s}^\pm = \mathfrak{s}^\pm$.

Here, $L = SL(4) \times SL(2)_+ \times SL(2)_-$ and the restriction of $\mathfrak{s}^+$ to L is equivalent to

$$\mathbb{C}^4 \boxtimes \mathbb{C}^2 \boxtimes \mathbb{C} \oplus (\mathbb{C}^4)^\star \boxtimes \mathbb{C} \boxtimes \mathbb{C}^2$$

Hence, the restriction of $\mathfrak{s}^+$ to L is equivalent to

$$\mathbb{C}^{4\times 2} \oplus \mathbb{C}^{4\times 2}$$

with action

$$(T, A, B)^{-1}(X, Y) = (T^{-1}XA, T^tYB)$$
$$T \in SL(4), A, B \in SL(2), X, Y \in \mathbb{C}^{4\times 2}$$

We claim this action is not a prehomogeneous vector space. Therefore, since L is semisimple it admits invariant polynomials. To verify it is not a prehomogeneous space we apply [11] Prop. 7.52. The statement of the proposition is: $(G, \rho_1 \oplus \rho_2)$ is a prehomogeneous space if and only if (G, ρ_1) is a prehomogeneous space and $(G_v, \rho_{2_{|_{G_v}}}, V_2)$ is a prehomogeneous vector space. Here, $G_v =$ generic stabilizer of ρ_1.

In our case, ρ_1 is the first summand, for

$$v = \begin{pmatrix} I \\ 0 \end{pmatrix},$$

$$L_v = \{(\begin{pmatrix} A & R \\ 0 & B \end{pmatrix}, A^{-1}, B), A, B \in SL(2), R \in \mathbb{C}^{2\times 2}\}$$

It readely follows dimension of a generic orbit for $\rho_2(G_v)$ is seven. Actually, each generic orbit is the set (X, Y) satisfying $detX = c$, hence, $(G_v, \rho_{2_{|G_v}}, V_2)$ is not a prehomogeneous vector space.

Finally we examine $p = q = 5$. Here, the restriction of $\mathfrak{s}^+$ to $Spin(5) \times Spin(5)$ is equivalent to $s \boxtimes s$. In [11] Apendix, it is shown that this representation is not a prehomogeneous vector space. Else, the spin representation of $Spin(5)$ is a symplectic representation. Hence, $S[\mathfrak{s}^+]^{Spin(5)\times Spin(5)} \neq \mathbb{C}$ and we have concluded the proof of Proposition 1.

Proof of Proposition 2: For a root γ let Y_γ denote one of its nonzero root vectors. From the root tables for $\mathfrak{e}_6$, [7], we find that restriction of $\mathfrak{s}^+$ to the semisimple factor of $\mathfrak{l}_{5,a,b} \simeq \mathfrak{su}(5)$ decomposes as

$$V_0 + V_{\Lambda_1} + V_{\Lambda_3}.$$

Here Λ_j is the fundamental weight of $\mathfrak{l}_{5,0,0} \backsimeq \mathfrak{su}(5)$ attached to α_j. A highest weight vector for each submodule is respectively:

$$Y_{\beta_M},\ Y_{\alpha_1+\alpha_2+\alpha_3+\alpha_4+\beta},\ Y_{\alpha_1+2\alpha_2+3\alpha_3+2\alpha_4+\beta+\alpha_5}$$

In [11] we find a proof $SL(5)$ has an open orbit in $V_{\Lambda_1} + V_{\Lambda_3}$. Since $\rho_5 = \frac{1}{2}\alpha_1 + \frac{2}{2}\alpha_2 + \frac{3}{2}\alpha_3 + \frac{3}{4}\alpha_4 + \frac{5}{4}\alpha_5$, we obtain $\beta_M(aH_0 + bH_{\rho_5}) = (\frac{a}{2} + \frac{13b}{8})(\beta, \beta)$. Thus, the complexification of $L_{5,a,b}$ has an open orbit in $\mathfrak{s}^+$ if and only if $\frac{a}{2} + \frac{13b}{8} \neq 0$. Therefore, $\frac{a}{2} + \frac{13b}{8} \neq 0$ if and only if there is no invariant polynomial by $L_{5,a,b}$. Thus, the claim for $L_{5,a,b}, H_1$ follows.
We now show the statement on $\mathbb{C}^\star(aH_0 + bH_{\rho_5})$. For this, after a computation follows

$$(aH_0 + bH_{\rho_5})(\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 + \beta) = (\frac{a}{2} - \frac{3b}{8})(\beta, \beta)$$

$$(aH_0 + bH_{\rho_5})(\alpha_1 + 2\alpha_2 + 3\alpha_3 + 2\alpha_4 + \beta + \alpha_5) = (\frac{a}{2} + \frac{b}{8})(\beta, \beta)$$

We recall
Fact 3 Let $\mathbb{C}^\star$ acting on vector space V. Then the representation of $\mathbb{C}^\star$ in $S[V]$ is admissible iff the weights of $\mathbb{C}^\star$ in V lies in an open half space. In our case, this condition for admissibility is equivalent to the real numbers $\frac{a}{2} + \frac{13b}{8}, \frac{a}{2} - \frac{3b}{8}, \frac{a}{2} + \frac{b}{8}$ have the same positivity. Thus, the statement on $\mathbb{C}^\star(aH_0 + bH_{\rho_5})$ follows.

To show the admissibility statement for $\mathfrak{l}_{4,a,b}$ and $\mathbb{C}^\star(aH_0 + bH_{\rho_4})$ the proof goes as in the previous case, we now have the same decomposition for the restriction of $\mathfrak{s}^+$ to the semisimple factor of $L_{4,a,b}$ and highest weight vectors are:

$$Y_\beta,\ Y_{\beta_M},\ Y_{\alpha_1+2\alpha_2+2\alpha_3+\alpha_4+\beta+\alpha_5}$$

Since $\rho_4 = \frac{1}{2}\alpha_1 + \frac{2}{2}\alpha_2 + \frac{3}{2}\alpha_3 + \frac{5}{4}\alpha_4 + \frac{3}{4}\alpha_5$, we get $\beta(aH_0 + bH_{\rho_4}) = (\frac{a}{2} - \frac{5b}{8})(\beta,\beta)$. The claim on $L_{4,a,b}, H_2$ follows.

$$(aH_0 + bH_{\rho_4})(\beta_M) = (\frac{a}{2} + \frac{3b}{8})(\beta,\beta)$$

$$(aH_0 + bH_{\rho_4})(\alpha_1 + 2\alpha_2 + 2\alpha_3 + \alpha_4 + \beta + \alpha_5) = (\frac{a}{2} - \frac{b}{8})(\beta,\beta)$$

Owing to Fact 3, the condition for admissibility to $\mathbb{C}^\star(aH_0 + bH_{\rho_4})$ is equivalent to the real numbers $\frac{a}{2} - \frac{5b}{8}, \frac{a}{2} + \frac{3b}{8}, \frac{a}{2} - \frac{b}{8}$ have the same positivity. Thus, the statement on $\mathbb{C}^\star(aH_0 + bH_{\rho_4})$ follows.

We now consider the statement related to ρ_1. $\mathfrak{l}_{1,a,b} = \mathbb{C}(aH_0 + bH_{\rho_1}) + \mathfrak{so}(8)$. $\mathfrak{s}^+_{|\mathfrak{r}_1} = s^+ s^-$. Respective highest weight vectors are

$$Y_{\beta_M}, Y_{\alpha_2+2\alpha_3+2\alpha_4+\beta+\alpha_5}.$$

Both, representations $s^\pm$ are orthogonal representations, we denote by $b_\pm$ respective nonzero invariant quadratic forms. Then, $S[s^\pm] = \mathbb{C}[b_\pm]$. Hence, Fact 1 implies there is no admissible restriction to $\mathfrak{r}_1$. Since $\rho_1 = \alpha_1 + \alpha_2 + \alpha_3 + \frac{1}{2}(\alpha_4 + \alpha_5)$ we obtain

$$(aH_0 + bH_{\rho_1})(\beta_M) = (\frac{a}{2} + \frac{b}{4})(\beta,\beta).$$

$$(aH_0 + bH_{\rho_1})(\alpha_2 + 2\alpha_3 + 2\alpha_4 + \beta + \alpha_5) = (\frac{a}{2} - \frac{b}{4})(\beta,\beta).$$

Therefore, Fact 3, leads to the statement on $\mathbb{C}^\star(aH_0 + bH_{\rho_1})$

We now show the statement on $\mathfrak{l}_{2,a,b} = \mathbb{C}(aH_0 + bH_{\rho_2}) \oplus \mathfrak{r}_2, \mathbb{C}^\star(aH_0 + bH_{\rho_2})$ where, $\mathfrak{r}_2 = \mathfrak{su}_2(\alpha_1) \oplus \mathfrak{su}_4(\alpha_5,\alpha_3,\alpha_4)$. $\mathfrak{s}^+$ restricted to $SU(2) \times SU(4)$ is equivalent to

$$\mathbb{C} \boxtimes \Lambda_3 + \Lambda_1 \boxtimes \Lambda_3 + \mathbb{C} \boxtimes \Lambda_1$$

Highest weight vectors are

$$Y_{\beta_M},\ Y_{\alpha_1+\alpha_2+2\alpha_3+2\alpha_4+\beta+\alpha_5},\ Y_{\alpha_3+\alpha_4+\beta+\alpha_5}$$

Since for the group $SU(4)$, Λ_3 is dual to Λ_1, we obtain a polynomial invariant under $SU(2)\times SU(4)$. Thus, there is no admissible restriction to $SU(2)\times SU(4)$. Since $\rho_2 = \alpha_1+2\alpha_2+2\alpha_3+\alpha_4+\alpha_5$, the next equalities readily follows

$$(aH_0 + bH_{\rho_2})(\beta_M) = (\frac{a}{2} + \frac{b}{2})(\beta,\beta)$$

$$(aH_0 + bH_{\rho_2})(\alpha_1 + \alpha_2 + 2\alpha_3 + 2\alpha_4 + \beta + \alpha_5) = \frac{a}{2}(\beta, \beta)$$

$$(aH_0 + bH_{\rho_2})(\alpha_3 + \alpha_4 + \beta + \alpha_5) = (\frac{a}{2} - \frac{b}{2})(\beta, \beta).$$

Next we apply Fact 3. Thus, π restricted to $\mathbb{C}(aH_0+bH_{\rho_2})$ is admissible iff $a, a \pm b$ lies in an open half space in $\mathbb{C}$. This is equivalent to $|a| \geq |b|$. To show holomorphic discrete series has an admissible restriction to $SU(2) \times SU(4,1)$ we actually show that holomorphic discrete series is admissible when restricted to the center of $K \cap (SU(2) \times SU(4,1))$. A generator of the Lie algebra of the center is $5H_0 - 2H_{\rho_2}$ which corresponds to $\alpha_5+2\alpha_3+3\alpha_4+4\beta$. It readily follows that $\alpha_5+2\alpha_3+3\alpha_4+4\beta$ has positive inner product with any noncompact positive root. Hence, there is no polynomial invariant under the center unless is a constant polynomial.

Finally we examine

$$\mathfrak{l}_{3,a,b} = \mathbb{C}(aH_0 + bH_{\rho_3}) \oplus \mathfrak{su}_3(\alpha_1, \alpha_2) \oplus \mathfrak{su}_2(\alpha_4) \oplus \mathfrak{su}_2(\alpha_5).$$

The restriction of $\mathfrak{s}^+$ to $\mathfrak{r}_3 + \mathfrak{t}$ is equal to

$$\mathbb{C} \boxtimes \mathbb{C} \boxtimes \mathbb{C}^2 \oplus \mathbb{C}^3 \boxtimes \mathbb{C}^2 \boxtimes \mathbb{C} \oplus \mathbb{C}^3 \boxtimes \mathbb{C} \boxtimes \mathbb{C}^2 \oplus \mathbb{C} \boxtimes \mathbb{C}^2 \boxtimes \mathbb{C}$$

Highest weight vectors respectively are,

$$Y_{\beta_M},\ Y_{\alpha_1+2\alpha_2+2\alpha_3+2\alpha_4+\beta+\alpha_5},\ Y_{\alpha_1+\alpha_2+\alpha_3+\alpha_4+\beta+\alpha_5}, Y_{\alpha_4+\beta}$$

In the realm of fundamental weights of $\mathfrak{r}_3$, the highest weights are

$$\Lambda_{\frac{1}{2}\alpha_5},\ \Lambda_{\alpha_2} + \Lambda_{\frac{1}{2}\alpha_4},\ \Lambda_{\alpha_1} + \Lambda_{\frac{1}{2}\alpha_5},\ \Lambda_{\frac{1}{2}\alpha_4}$$

Since $\mathfrak{r}_3$ is a subalgebra of $\mathfrak{so}(6)+\mathfrak{so}(4)$ Proposition 1 and Fact 0 imply π restricted R_3 is not admissible. Hence, there are invariant polynomials under R_3 in $\mathfrak{s}^+$. $\rho_3 = \alpha_1+2\alpha_2+3\alpha_3+\frac{3}{2}(\alpha_4+\alpha_5)$ and $(\rho_3, \beta) = -\frac{3}{4}(\beta, \beta)$.

$$(aH_0 + bH_{\rho_3})(\beta_M) = (\frac{a}{2} + \frac{3b}{4})(\beta, \beta)$$

$$(aH_0 + bH_{\rho_3})(\alpha_1 + 2\alpha_2 + 2\alpha_3 + 2\alpha_4 + \beta + \alpha_5) = (\frac{a}{2} + \frac{b}{4})(\beta, \beta)$$

$$(aH_0 + bH_{\rho_3})(\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 + \beta + \alpha_5) = (\frac{a}{2} - \frac{b}{4})(\beta, \beta)$$

$$(aH_0 + bH_{\rho_3})(\alpha_4 + \beta) = (\frac{a}{2} - \frac{3b}{4})(\beta, \beta).$$

Thus, Fact 3 implies there is admissible restriction to $\mathbb{C}^\star(aH_0 + bH_{\rho_3})$ if the four numbers $a \pm \frac{b}{2}, a \pm \frac{3b}{2}$ have the same positivity. This is equivalent to $|a| > \frac{3}{2}|b|$.

For the second statement we notice that all these restrictions are discretely decomposable because they contain suitable holomorphic discrete series for the subgroup, hence, we may apply [12], Lemma 1.5. We now show they are not admissible. The maximal compact subgroup of $SO(8,2)$ (resp $F_{4(-20)}$) $SO(8) \times SO(2)$ (resp $Spin(9)$), is contained in maximal subgroups of $Spin(10)$ which satisfy iii). Thus, Fact 1 let us conclude the second statement for this two groups. The center of $\mathfrak{k} \cap \mathfrak{su}(5,1)$ corresponds to the vector $\alpha_1 + 2\alpha_2 + 3\alpha_3 + 4\alpha_4 + 5\beta$ which is orthogonal to β_M. Hence, Y_{β_M} is fixed by $\mathfrak{k} \cap \mathfrak{su}(5,1)$. Thus, restriction to $\mathfrak{su}(5,1)$ is not admissible. This concludes the proof of Proposition 2.

Proof of Theorem 1 Let L be a proper closed subgroup so that π restricted to L is admissible. To begin with, we assume L is contained in K_{ss} and let V be a maximal subgroup of $Spin(10)$ containing L. Owing to Fact 0, π restricted to L is admissible. Proposition 1 implies $V = L$ and L is one of the two immersions of $U(5)$ in $Spin(10)$. Next we consider the case L does not contain the center of K as well as is not contained in K_{ss}. Hence, the projection onto $Spin(10)$ yields L is isomorphic to a subgroup of $Spin(10)$ and the projection onto the center of K shows that the center of L is non trivial. Thus, there exists number a, a vector Y in $spin(10)$ and a subalgebra $\mathfrak{l}'$ of $spin(10)$ which centralizes Y so that

$$\mathfrak{l} = \mathbb{C}(aH_0 + Y) \oplus \mathfrak{l}'.$$

The structure of centralizers of a tori implies there exists a fundamental weight ρ_j so that, after conjugation by an element of K,

$$\mathfrak{l} \subseteq \mathbb{C}(aH_0 + bH_{\rho_j}) \oplus \mathfrak{r}_j.$$

Hence, Fact 0 implies π restricted to $L_{j,a,b}$ is admissible, Proposition 2 forces L to be $L_{j,a,b}$.

3. Branching laws

Let L denote a compact subgroup of K so that a holomorphic discrete series π has an admissible restriction to L. Since, in our case condition C does not hold, in order to compute multiplicities by means of Blattner-Kostant type formulas, we need to compute the restriction of π to L and to compute restriction to L of irreducible representations of K. The last computation can be done via a formula shown by Heckman and Kostant [8] and the former is done by a formula conjectured by Blattner and shown by [9]. A different presentation of the formula of Blattner is in [4], this presentation is more suitable for theoretical developments. In our setting, Paradan [16] has given a geometric formula for multiplicities. For the case considered in this note, the computation of multiplicity can actually be carried out by means of a computer, even though the complexity is quite high because the method needs of two partition functions. For an actual state of the art of computing multiplicities we refer to the work of Baldoni, Vergne [1].

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