STRICT VERSUS $k_1 \subset l$

M., DUFLO, J., VARGAS

ABSTRACT. d-v means the June 1 version of what i wrote.

1. XX

I follow the notation on your writing.

Let $G \supseteq K \supseteq T$ be so that G is a simple, connected Lie group of finite center, K a maximal compact subgroup of G and T a maximal torus of G contained in K.

We fix a compact connected subgroup L of K and let Z denote the connected center of K.

We fix a maximal torus U of L contained in T.

We fix compatible systems of positive roots $\Psi_{\mathfrak{k}}, \Psi_{\mathfrak{l}}$ in $\Phi_{\mathfrak{k}}$ and $\Phi_{\mathfrak{l}}$ respectively. That is, if a root in $\Psi_{\mathfrak{k}}$ has a nonzero restriction to $\mathfrak{u}$ then it belongs to $\Psi_{\mathfrak{l}}$.

$\Phi_{\mathfrak{z}}$ roots in $\Phi_{\mathfrak{k}}$ that vanishes on $\mathfrak{u}$.

$q_{\mathfrak{u}}$ the restriction map from $\mathfrak{t}^\star$ onto $\mathfrak{u}^\star$.

We fix a system of positive roots Ψ in $\Phi_{\mathfrak{g}}$ which contains $\Psi_{\mathfrak{k}}$.

Let $k_i(\Psi), \mathfrak{t}_i = k_i(\Psi) \cap \mathfrak{t}$ as usual .

Lemma 1. *If $\mathfrak{g}$ is simple, not locally isomorphic to $\mathfrak{s}o(2, 2q+1)$ and $k_1(\Psi)$ is nontrivial, then every noncompact root has a nonzero restriction to $\mathfrak{t}_1$.*

Proof: Case $\mathfrak{z} = 0$, then, if a noncompact root had a trivial restriction to $\mathfrak{t}_1$. We would have that this noncompact root would belong to $\mathfrak{t}_2$. The fact that the simple roots in Ψ_{k_2} are simple roots for Ψ together with $\mathfrak{z} = 0$ forces that the root would be a linear combination of compact simple roots, and hence compact.

Case $\mathfrak{z} \neq 0$. We do a case by case analysis. $\mathfrak{g} = \mathfrak{s}u(p, q)$ then k_1 equals to one of $\mathfrak{s}u(p), \mathfrak{s}u(q), \mathfrak{s}u(p) \times \mathfrak{s}u(q)$ and it follows by inspection.

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$\mathfrak{g} = \mathfrak{so}(2, 2q+1)$ Here

$$\Psi_{\mathfrak{k}} = \{\epsilon_1 \pm \delta_k, \delta_k, k = 1, \cdots, q\}$$

$$\Phi_n = \{\pm\epsilon_1, \pm\epsilon_1 \pm \delta_j, 1 \leq j \leq q\}.$$

The center of $\mathfrak{k}$ is $\mathbb{C}\epsilon_1$ and $k_1(\Psi)$ is equal to $\mathfrak{so}(2q+1)$. Thus, ϵ_1 has zero restriction to $\mathfrak{t}_1$

For $\mathfrak{so}(2, 2q)$, when $2q = 4$ the computation follows from Table 2 in d-v , for the remaining cases, $\Phi_n = \{\pm\epsilon_1 \pm \delta_j, 1 \leq j \leq q\}$ and $\mathfrak{t}_1$ is equal the subspace spanned by δ_j.

For $\mathfrak{sp}(2q)$ is obvious.

For the exceptional groups we use the description of the root system in Freudenthal, Linear Lie groups,

For $\mathfrak{e}_{6(-14)}$

$\mathfrak{k} = \mathfrak{so}(10) + \mathfrak{z}$ we have, up to $\pm$,

$$\Phi_c = \{e_i - e_j, e_i + e_j + e_6, i, j \leq 5\}$$

$$\Phi_n = \{e_i - e_6, e_k + e_i + e_j, e_1 + \cdots + e_6 \ i, j, k \leq 5\}$$

Then, $(e_i - e_6) + (e_j - e_i), (e_1 + \cdots + e_6) - (e_4 + e_5 + e_6)$ and $(e_k + e_i + e_j) - (e_i + e_j + e_6)$ are roots.

For $\mathfrak{e}_{7(-25)}$, $\mathfrak{k} = \mathfrak{e}_6 + \mathfrak{z}$

$$\Phi_c = \{e_i - e_j, e_i + e_j + e_k, 2 \leq i, j, k, e_2 + \cdots + e_7\}$$

and

$$\Phi_n = \{e_1 - e_j, e_1 + e_i + e_j, e_1 + \cdots e_{j-1} + e_{j+1} + \cdots + e_7, 2 \leq i, j, \}$$

then $(e_1 - e_i) - (e_2 - e_i), (e_1 + e_i + e_j) + (e_k - e_i), (e_1 + \cdots e_{j-1} + e_{j+1} + \cdots + e_7) - (e_1 - e_j)$ are roots.

End of the proof.

We now consider the problem of restriction to K_1 of a discrete series whose Harish-Chandra parameter, Λ, is dominant with respect to Ψ.

Lemma 2. *Whenever the set $q_{\mathfrak{t}_1}(\Psi_n)$ is strict we have that π_Λ has an admissible restriction to $K_1(\Psi)$.*

Proof:

The hypothesis implies that there exists $v \in i\mathfrak{t}_1$ so that $\beta(v) > 0$ for every noncompact root in Ψ. The computation now follows as in the proof in Prop 2 of d-v.

End of the proof.

Note 0. We do not know whether or not the converse statement holds. In *Note 3* we show: $p_{\mathfrak{k}_1}(\Psi_n)$ is strict iff $\mathbb{R}^+ \cap i\mathfrak{z}^\star = 0$

For the next statement we keep the hypothesis and notation from the beginning of this section.

Proposition 1. *If $\mathfrak{k}_1(\Psi) + \mathfrak{z}$ is contained in $\mathfrak{l}$, then, for each w in the compact Weyl group of G, the multiset*

$$q_\mathfrak{u}(w\Psi_n) \cup q_\mathfrak{u}(\Psi_\mathfrak{k} - \Psi_\mathfrak{z}) - \Psi_\mathfrak{l}$$

is strict.

For the converse statement we further assume $K_1(\Psi) \neq 0$. If for each w in the compact Weyl group of G, the multiset

$$q_\mathfrak{u}(w\Psi_n) \cup q_\mathfrak{u}(\Psi_\mathfrak{k} - \Psi_\mathfrak{z}) - \Psi_\mathfrak{l}$$

is strict, then $\mathfrak{k}_1(\Psi)$ is contained in $\mathfrak{l}$.

Proof: For the first statement. Our assumption on the compatibility between $\Psi_\mathfrak{k}$ and $\Psi_\mathfrak{l}$ implies that there exists v_1 in $i\mathfrak{u}$ so that $\alpha(v_1) > 0$ for every $\alpha \in \Psi_\mathfrak{k} - \Psi_\mathfrak{z}$. We fix C an upper bound for the numbers $|\gamma(v_1)|$, $\gamma \in \Psi$. We now recall that $\mathfrak{k}_2$ is equal to the center of $\mathfrak{k}$ plus a semisimple Lie algebra. Hence, we may write $\mathfrak{t}_2 = \mathfrak{z} \oplus \mathfrak{t}_{2,ss}$ and want to show that $(\mathbb{R}^+\Psi_n + \mathbb{R}^+\Psi_{\mathfrak{k}_1}) \cap i\mathfrak{t}^\star_{2,ss} = 0$. This follows from Remark 2 in d-v. Therefore, Lemma 3 in d-v forces that $q_{\mathfrak{k}_1+\mathfrak{z}}(\mathbb{R}^+\Psi_n + \mathbb{R}^+\Psi_{\mathfrak{t}_1})$ is a strict cone in $i\mathfrak{t}_1^\star + i\mathfrak{z}^\star$. Hence, we may choose $v_0 \in i\mathfrak{t}_1 + i\mathfrak{z}$ so that $\gamma(v_0) \geq 2C$ for every root in $\Psi_n \cup \Psi_{\mathfrak{t}_1}$. We define $v = v_0 + v_1$. Thus, for either $\gamma \in \Psi_\mathfrak{k} - \Psi_\mathfrak{z}$ or $\gamma \in \Psi_n$ we have that $\gamma(v)$ is a positive number. Thus, since $\mathfrak{k}_1 + \mathfrak{z}$ is contained in $\mathfrak{u}$, the multiset $q_\mathfrak{u}(\Psi_n) \cup q_\mathfrak{u}(\Psi_\mathfrak{k} - \Psi_\mathfrak{z}) - \Psi_\mathfrak{l}$ is strict. We write $w = w_1 w_2$ with w_j in the Weyl group of $K_j(\Psi)$. Because of Lemma 2 in d-v, w_2 is a product of reflections about compact simple roots in Ψ and we have that $w_2\Psi_n = \Psi_n$. Since $v \in i\mathfrak{u}$ and $\mathfrak{k}_1$ is contained in $\mathfrak{l}$ we have that w_1 lies in the Weyl group of L. Also $w_1(q_\mathfrak{u}(\Psi_{\mathfrak{k}_2} - \Psi_\mathfrak{z}) = q_\mathfrak{u}(\Psi_{\mathfrak{k}_2} - \Psi_\mathfrak{z})$. Therefore, the vector $w_1 v$ makes the multiset $q_\mathfrak{u}(w\Psi_n) \cup q_\mathfrak{u}(\Psi_\mathfrak{k} - \Psi_\mathfrak{z}) - \Psi_\mathfrak{l}$ strict.

end of the proof of the first statement.

In order to show the converse statement, we first verify that $q_\mathfrak{u}(\gamma)$ is nonzero for every root in $\Psi_n \cup \Psi_{\mathfrak{k}_1}$ and that for $\alpha \in \Psi_{\mathfrak{k}_1}$, $q_\mathfrak{u}(\alpha)$ is root for the pair $(\mathfrak{l}, \mathfrak{u})$. To begin with, let $\beta \in \Psi_n$, we now show that $q_\mathfrak{u}(\beta)$ is not equal to $q_\mathfrak{u}(\gamma)$ for every $\gamma \in \Psi_{\mathfrak{k}_1}$. Indeed, assume $q_\mathfrak{u}(\beta) = q_\mathfrak{u}(\alpha)$, since $k_1(\Psi) \neq 0$ the chamber Ψ is non holomorphic, thus, there exists $w \in W_\mathfrak{k}$ so that $w\beta = -\beta$. Hence $\pm\alpha$ belong to the strict multiset associated to w, a contradiction. For $\alpha \in \Psi_{\mathfrak{k}_1}$, in d-v it is shown that there exists non compact roots β_1, β_2 in Ψ and w in the Weyl group

of $K_1(\Psi)$ so that $\alpha = w(\beta_1 + \beta_2)$. Thus, $q_{\mathfrak{u}}(\alpha)$ lies in $q_{\mathfrak{u}}(w\mathbb{R}^+\Psi_n)$, the hypothesis that the multiset associated to w is strict implies that $q_{\mathfrak{u}}(\alpha)$ is non zero. Hence, $q_{\mathfrak{u}}(\alpha)$ belongs to either $\Phi(\mathfrak{l}, \mathfrak{u})$ or $\Phi(\mathfrak{k}/\mathfrak{l}, \mathfrak{u})$. If $q_{\mathfrak{u}}(\alpha)$ does not belong to $\Phi(\mathfrak{l}, \mathfrak{u})$, then lies in $q_{\mathfrak{u}}(\Psi_{\mathfrak{k}} - \Psi_{\mathfrak{z}}) - \Psi_{\mathfrak{l}}$. On the other hand, in d-v it is proven that there exist w in the Weyl group of k_1 so that $-\alpha = w(\beta_1 + \beta_2)$. Hence, $\pm\alpha$ belongs to the strict multiset $q_{\mathfrak{u}}(w\Psi_n) \cup q_{\mathfrak{u}}(\Psi_{\mathfrak{k}} - \Psi_{\mathfrak{z}}) - \Psi_{\mathfrak{l}}$, contradiction and we have shown the claim. Next we show: if $\gamma \in \Psi_{\mathfrak{k}}$ and $\alpha \in \Psi_{\mathfrak{k}_1}$ has the same restriction to $\mathfrak{u}$, then they are equal. Indeed, we choose w in the Weyl group of K_1 so that $w\alpha = -\alpha$. Then, $-q_{\mathfrak{u}}(\alpha)$ will belong to the multiset associated to w, besides $q_{\mathfrak{u}}(\alpha)$ has at least multiplicity two in the multiset $q_{\mathfrak{u}}(\Psi_{\mathfrak{k}} - \Psi_{\mathfrak{z}})$. Hence, $\pm\alpha$ belongs to the multiset associated to w, a contradiction.
In order to show that K_1 is contained in L, we consider $\alpha \in \Psi_{\mathfrak{k}_1}$, then a root vector of $q_{\mathfrak{u}}(\alpha)$ is a sum of root vectors associated to roots in $\Psi_{\mathfrak{k}}$ that agree with α in $\mathfrak{u}$. Hence, a root vector of $q_{\mathfrak{u}}(\alpha)$ is equal to a root vector of α. Thus, all the root vectors of the roots in $\Psi_{\mathfrak{k}_1}$ belongs to $\mathfrak{l}_{\mathbb{C}}$. Therefore, $\mathfrak{k}_1$ is an ideal of $\mathfrak{l}$.

This ends the proof of the proposition.

Note 1. For every w in the Weyl group of $\mathfrak{k}$ we have that the multiset

$$q_{\mathfrak{t}_1+\mathfrak{z}}(w\Psi_n) \cup q_{\mathfrak{t}_1+\mathfrak{z}}(\Psi_{\mathfrak{k}} - \Psi_{\mathfrak{z}}) - \Psi_{\mathfrak{l}}$$

is strict. In fact, we write $w = w_1 w_2$, with w_j in the Weyl group of K_j, then

$$q_{\mathfrak{u}}(w\Psi_n) \cup q_{\mathfrak{u}}(\Psi_{\mathfrak{k}} - \Psi_{\mathfrak{z}}) - \Psi_{\mathfrak{l}} = q_{\mathfrak{t}_1}(w_1\Psi_n) = w_1 q_{\mathfrak{t}_1}(\Psi_n)$$

Now we apply Remark 2 and Lemma 3 in d-v.

Note 2. The inclusion $\mathfrak{k}_1 \subseteqq \mathfrak{l}$ does not always imply that the multiset in $\mathfrak{u}$ associated to each w is strict . In fact, let $\mathfrak{g} = \mathfrak{so}(2,4)$ then in Table 2 in d-v we find systems of positive roots Ψ so that $\mathfrak{k}_1 = \mathfrak{so}(4)$ and the corresponding discrete series has no admissible restriction to $\mathfrak{so}(4)$. Hence, by inspection on Table 2 or by Lemma 1, we have that the multiset $q_{\mathfrak{u}}(\Psi_n)$ is not strict.

Note 3 The fact that the every multiset is strict does not always implies that the center of $\mathfrak{k}$ is contained in $\mathfrak{l}$. Indeed, for $\mathfrak{l} = \mathfrak{k}_1$ we now show: $\mathbb{R}^+\Psi_n \cap i\mathfrak{z} = \{0\}$ iff every multiset is strict. This follows because for $w = w_1 w_2$, w_j in the Weyl group of K_j we have,

$$q_{\mathfrak{u}}(w\Psi_n) \cup q_{\mathfrak{u}}(\Psi_{\mathfrak{k}} - \Psi_{\mathfrak{z}}) - \Psi_{\mathfrak{l}} = q_{\mathfrak{t}_1}(w_1\Psi_n) = w_1 q_{\mathfrak{t}_1}(\Psi_n)$$

Now, we apply Lemma 3 in d-v. Concrete examples that this hypothesis holds are certain systems of positive roots in $SU(p,q), Sp(n,\mathbb{R})$.

Note 4 We know that $q_{\mathfrak{u}}(w...)$ is strict for every w, then π_Λ has an admissible restriction to L.

The converse statement does not hold. For example, for $Spin(4,1)$ consider the system $e_1 \pm e_2, e_1, e_2$. Then, $k_1 = su(e_1+e_2)$. Set $\mathfrak{l} = su(e_1 - e_2)$. Then, we know that π_Λ restricted to L is admissible (d-v). We have $\mathfrak{k}_1$ is not contained in $\mathfrak{l}$ and $p_{\mathfrak{u}}(\Psi_n) = \pm(e_1 - e_2)$, hence, $p_{\mathfrak{u}}(\Psi_n)$ is not strict.

Note 5: For the direct statement in proposition, when $K_1 = 0$ the result still holds. For the converse statement when we consider the case $K_1 = 0$, that is, a holomorphic chamber, the statement does not make sense, because the thesis is true in spite of what ever the hypothesis is.

Note 6: Holomorphic discrete series for su(p,q) p smaller that q is admissible restricted to su(q) this is in d-v, however,

$$p_{\mathfrak{u}}(\Psi_n) = \{-\delta_j + \frac{1}{q}(\sum \delta_k)\}$$

is not strict!

Note that su(q) is the $k_(\Psi')$.