Sixth Workshop on Lie Theory and Geometry

Branching laws for square integrable representations

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FAMAF-CIEM
Córdoba, Argentine
November 2007

$H \subset G$ connected reductive matrix Lie groups.

(π, V) irreducible, square integrable Rep. of G.

Assume (π, V) has an admissible restriction to H. That is, $res_H(\pi)$ is a discrete Hilbert sum of irreducible representations of H and each irreducible factor has finite multiplicity. Hence,

$$res_H(\pi) = \sum_{\sigma \in \widehat{H}_{dis}} m(\pi, \sigma)\sigma$$

Want σ such that $m(\pi, \sigma) > 0$ and formulae for $m(\pi, \sigma)$.

Plan of the talk

1) Description of Harish Chandra for the set of irreducible square integrable reps.

2) Criterium for admissible restriction.

- Via moment map
- Via root systems

3) For $res_H(\pi)$ admissible, Kostant-Blattner type multiplicity formulae for irred. $H-$factors.

Irreducible square integrable representations

K maximal compact subgroup of G.

T maximal torus for K. W Weyl group of T.

Theorem (Harish Chandra) G admits irreducible square integrable representations if and only if T is a maximal abelian subgroup of G.

Moreover, he provided a parametrization of the set of equivalence classes of irreducible square integrable representations of G, when $G(\mathbb{C})$ is simply connected is by the set

$$\{e^{\Lambda} \text{ character of } T, \ \Lambda \text{ regular}\}/W$$

For $\Lambda \in i\mathfrak{t}^{\star}$ regular $\rightsquigarrow \pi(\Lambda)$ squa. int. rep. of G asoc to Λ.

Moment map and admissibility

$\lambda \in \mathfrak{t}^\star$ such that $\Lambda := i\lambda$,

Think of λ in $\mathfrak{g}^\star$.

$\Omega_\lambda := Ad(G)\lambda =$coadjoint orbit defined by λ.

$$p_{\mathfrak{h}} : \Omega_\lambda \to \mathfrak{h}^\star \text{ restriction map.}$$

Theorem *$p_{\mathfrak{h}}$ is a proper map if and only if $res_H \pi(\Lambda)$ is admissible.*

Description of $p_{\mathfrak{h}}(\Omega_\lambda)$

Definition

An element X of $\mathfrak{h}^\star$ is *strongly elliptic* if its centralizer in H is a compact subgroup.

Theorem

If $p_{\mathfrak{h}} : \Omega_\lambda \to \mathfrak{h}^\star$ *a proper map, then,*
$p_{\mathfrak{h}}(\Omega_\lambda)$ *is contained in the set of strongly elliptic elements of* $\mathfrak{h}^\star$.

L maximal compact subgroup of H

U maximal torus in L

so that $U := T \cap H, \quad L := K \cap H$.

Assume $p_{\mathfrak{h}} : \Omega_\lambda \to \mathfrak{h}^\star$ is proper.

Hence, every element of $p_{\mathfrak{h}}(\Omega_\lambda)$ is conjugated to an element of $\mathfrak{u}$.

Fix $C_{\lambda,\mathfrak{u}^\star,+}$ closed Weyl chamber for $\Phi(\mathfrak{l},\mathfrak{u})$ s.t.
$p_{\mathfrak{h}}(\Lambda)$ is dominant.

Thus,

$$p_{\mathfrak{h}}(\Omega_\lambda) = Ad(H)(p_{\mathfrak{h}}(\Omega_\lambda) \cap C_{\lambda,\mathfrak{u}^\star,+}).$$

Theorem *There exists a unique open Weyl chamber* $C_{\lambda,+}$ *for* $\Phi(\mathfrak{h},\mathfrak{u})$ *so that*

i) $C_{\lambda,+}$ *is contained in* $C_{\lambda,\mathfrak{u}^\star,+}$.

ii) $p_{\mathfrak{h}}(\Omega_\lambda)\cap C_{\lambda,\mathfrak{u}^\star,+}$ *is contained in the relative closure of* $C_{\lambda,+}$ *in* $C_{\lambda,\mathfrak{u}^\star,+}$.

Proposition

When $p_{\mathfrak{h}}$ *is a proper map,*

$p_{\mathfrak{h}}(\Omega_\lambda) \cap C_{\lambda,\mathfrak{u}^\star,+}$ *is a convex polyhedron,*

$p_{\mathfrak{h}}^{-1}(\mu)$ *is connected.*

aqui van figuras

(G, H, σ) symmetric pair

Assume set of square int. rep. for G is non empty.

T "maximally split" and $\sigma-$invariant Cartan subgroup.

$\mathfrak{t} = \mathfrak{t}_+ + \mathfrak{t}_-$, $\mathfrak{t}_\pm = \pm 1-$eigenspace of σ.

Ψ =roots in $\Phi(\mathfrak{g}, \mathfrak{t})$ has positive inner product with Λ.

$\mathbb{R}^+\Psi_n$ cone spanned by noncompact roots in Ψ.

Theorem $p_\mathfrak{h} : \Omega_\lambda \to \mathfrak{h}^\star$ *is a proper map if and only if* $\mathbb{R}^+\Psi_n \cap i\mathfrak{t}_-^\star = \{0\}$.

Theorem Assume (G, H, σ) symmetric pair *If* $\mathfrak{k}$ *is a simple Lie algebra, then no* $\pi(\Lambda)$ *has admissible restriction to* H.

General semisimple $H \subseteq G$.

L maximal compact subgroup of H

π irreducible square integrable rep. of G

Fact: π restricted to H is admissible if and only if π restricted to L is admissible.

This has an interpretation in the orbit language

Has two moment maps, one with values in $\mathfrak{h}^\star$ and the other in $\mathfrak{l}^\star$

Proposition

$p_{\mathfrak{h}} : \Omega_\lambda \to \mathfrak{h}^\star$ is proper iff $p_{\mathfrak{l}} : \Omega_\lambda \to \mathfrak{l}^\star$ is proper

Let V be an inner product space, a multisubset S of V is *strict* when there exists a non zero vector having positive inner product with each element of S.

Back to arbitrary H, G!
Fix system of positive roots Ψ in $\Phi(\mathfrak{g}, \mathfrak{t})$.
$\Phi_{\mathfrak{z}}$ =roots in $\Phi(\mathfrak{k}, \mathfrak{t})$ that vanishes on $\mathfrak{u}$.
$p_{\mathfrak{u}} : \mathfrak{t}^\star \longrightarrow \mathfrak{u}^\star$ restriction map.

Definition *Condition (C) holds for Ψ and L if there exists a system of positive roots Δ_1 in $\Phi(\mathfrak{k}, \mathfrak{t})$ so that $p_{\mathfrak{u}}(\Delta_1 - \Phi_{\mathfrak{z}})$ is strict, and for each w in the compact Weyl group of G the multiset*

$$p_{\mathfrak{u}}(w\Psi_n) \cup q_{\mathfrak{u}}(\Delta_1 - \Phi_{\mathfrak{z}}) - \Phi(\mathfrak{l}, \mathfrak{u})$$

is strict.

Fix Λ reg. dominant w.r.t Ψ
Theorem If the center of $\mathfrak{k}$ is contained in $\mathfrak{l}$ and condition (C) holds for Ψ and L, then $\pi(\Lambda)$ restricted to H (or L) is admissible.

Technical result

Ψ system of positive roots in $\Phi(\mathfrak{g}, \mathfrak{t})$.

Δ_1, Δ_2 systems of positive roots for $\Phi(\mathfrak{k}, \mathfrak{t})$.

Assume, each $p_{\mathfrak{u}}(\Delta_j - \Phi_{\mathfrak{z}})$ is strict multiset and $\mathfrak{z}_{\mathfrak{k}} \subseteq \mathfrak{l}$

Proposition If condition (C) holds for Ψ, L by mean of Δ_1, then it holds for Ψ, L by mean of Δ_2.

Example $G =$*noncpact real form of* G_2,
$L = SU_2(short root)$, Ψ *s.t. short simple is ncp, long simple cpct, then condition (C) holds and* $\pi(\Lambda)$ *restricted to* L *is admissible.*
$L = SU_2(long root)$, Ψ *s.t. short simple is cp, long simple noncpct, then condition (C) holds and* $\pi(\Lambda)$ *restricted to* L *is admissible*

When Ψ is small, then condition (C) holds for the group K_1 defined by Gross-Wallach, and we do not need the hypothesis center of $\mathfrak{k}$...

Assume L is a a proper subgroup of $SO(2n)$ which acts transitively on the unit sphere of $\mathbb{R}^{2n}$

Any admissible irreducible representation for $SO(2n,1)$ has admissible restriction to L. This is due to the fact that $L\backslash SO(2n)/M$ is a point, now apply Mackey restriction theorem. In particular, square integrable representations. However, condition (C) for Ψ, L is not valid.

Multiplicity formulae

To $\Lambda \rightsquigarrow \Psi$ and ideal of $\mathfrak{k}$

$\mathfrak{k}_1(\Psi) :=$ ideal spanned by $\sum_{\alpha,\beta\in\Psi_n} \mathbb{C}X_{\alpha+\beta}$

Proposition *$K_1(\Psi) \subseteq L$ iff condition (C) holds for Ψ and L.*

Hence, whenever $Z_K K_1(\Psi) \subseteq L$, $res_H(\pi_\Lambda)$ is admissible.

Write $\mathfrak{k} = \mathfrak{k}_1(\Psi) + \mathfrak{k}_2(\Psi)$ (sum of ideals)

W_j = Weyl group of $\mathfrak{k}_j(\Psi)$

Assume $K_1(\Psi)T \subseteq L$.

Q shifted partition function associated to

$$\Psi - \Phi(\mathfrak{h}, \mathfrak{t}) = \{\gamma_1, \cdots, \gamma_q\}$$

$Q(\nu) := card\{(n_j) : \nu = \sum_{j=1}^{q} (\frac{1}{2} + n_j)\gamma_j\}$

$$\Delta_L := \Phi(\mathfrak{h}, \mathfrak{t}) \cap \Psi$$

μ analytically integral form dominant for Δ_L, regular $\rightsquigarrow \sigma(\mu)$ squa. int. rep. of H asoc to μ.

Theorem

Multiplicity of $\sigma(\mu)$ in $\pi(\Lambda)$ restricted to H is

$$m(\pi(\Lambda), \sigma(\mu)) = \sum_{w \in W_1,\, s \in W_2} \epsilon(ws) Q(w\mu - s\Lambda).$$

Partition functions via discrete Heaviside distributions

For $\gamma \in i\mathfrak{u}^\star \rightsquigarrow \delta_\gamma$ denote Dirac delta function attached to γ.

Define

$$y_\gamma = \sum_{n \geq 0} \delta_{\frac{\gamma}{2}+n\gamma} = \delta_{\frac{\gamma}{2}} + \delta_{\frac{\gamma}{2}+\gamma} + \delta_{\frac{\gamma}{2}+2\gamma} + \cdots$$

Fix $S = \{\gamma_1, \cdots, \gamma_q\}$ strict multiset,

Fact: there exists the convolution

$$y_{\gamma_1} \star \cdots \star y_{\gamma_q}$$

Definition Shifted partition function associated to S
$Q_S(\nu) := card\{(n_j) : \nu = \sum_{j=1}^q (\frac{1}{2} + n_j)\gamma_j\}$

Fact:

$$\sum_{\nu \in i\mathfrak{u}^\star} Q_S(\nu)\delta_\nu = y_{\gamma_1} \star \cdots \star y_{\gamma_q}$$

Assume condition (C) for Ψ and L, and $\mathfrak{z}_{\mathfrak{k}} \subseteq \mathfrak{l}$, hence, for Λ dom. w.r.to. Ψ, $\pi(\Lambda)$ has admissible restriction to L

For irred rep. $\tau(\nu)$ of $L \rightsquigarrow m(\Lambda, \nu)$:=multiplicity in $\pi(\Lambda)$ restricted to L. Kobayashi has shown $m(\Lambda, \nu)$ is of polynomial growth in ν hence

$$\sum_{\nu \, inf.cha} m(\Lambda, \nu)\delta_\nu$$

converges as a distribution

$$\sum_{\nu \in P_L} m(\Lambda, \nu) \delta_\nu$$

$$= \sum_{w \in W} \epsilon(w) \, \varpi(w\lambda)$$

$$\delta_{p_\mathfrak{u}(w\Lambda)} \star \star_{\beta \in p_\mathfrak{u}(w\Psi_n) \cup (-1) p_\mathfrak{u}(\Delta - \Phi_\mathfrak{z}) - \Phi_\mathfrak{l}} y_\beta$$

Δ compact roots in Ψ, we may arrange $p_\mathfrak{u}(\Delta - \Phi_\mathfrak{z})$ is strict

ν is the infinitesimal character of $\tau(\nu)$, dom. w.r.t. Δ

$m(\Lambda, \nu)$ has been extended to $\nu \in P_L$ to be antisymmetric w.r.t. W_L,

$$\varpi(\gamma) = \frac{\prod_{\alpha \in \Psi_\mathfrak{z}} \gamma(h_\alpha)}{\prod_{\alpha \in \Psi_\mathfrak{z}} \rho_{\Psi_\mathfrak{z}}(h_\alpha)}$$

The proof of the formula is as follows

- Case L contains T

- Case L is normalized by T (here shows up the ϖ factor)

- Hence the formula holds for $K_1(\Psi)Z_K U$ ($U = L \cap T$ max. torus of L)

- If L_1 subgroup L of the same rank, and the formula holds for L_1 then it holds for L.

- When L_1 subgroup L of the same rank, and the multiplicity formula holds for L_1 then it holds for L.

$$d_\alpha(f) = (\delta_{-\alpha/2} - \delta_{\alpha/2}) \star f, \ f \in \mathcal{D}'(i\mathfrak{u}^\star).$$

$$\bigstar_{\alpha \in \Delta_L - \Delta_{L_1}} d_\alpha\Big(\sum_{\nu_1 \in P_{L_1}} m_1(\Lambda, \nu_1)\delta_{\nu_1}\Big) = \sum_{\nu \in P_L} m(\Lambda, \nu)\delta_\nu$$

Formula to deduce $H-$multiplicity $m(\Lambda, \mu)$ from $L-$multiplicity $m(\Lambda, \nu)$.

$$d_\alpha(f) = (\delta_{-\alpha/2} - \delta_{\alpha/2}) \star f,\ \ f \in \mathcal{D}'(i\mathfrak{u}^\star).$$

$$d^{n,\mathfrak{h}} = \prod_{\alpha \in \Psi_n(p_\mathfrak{u}(\Lambda))} d_\alpha$$

$$d^{n,\mathfrak{h}}\ \ [\sum_{\nu \in P_L} m(\Lambda, \nu)\, \delta_\nu] = \sum_{\mu \in P_L} m(\Lambda, \mu) \delta_\mu$$

$\mathfrak{g} = \mathfrak{so}(4, 2q)$, $q \geq 3$,

$\Delta = \{\epsilon_1 \pm \epsilon_2, \delta_r \pm \delta_s,\ 1 \leq r < s \leq q\}, \Phi_n = \{\pm\epsilon_i \pm \delta_j\}$.

$\Psi_{\pm a} = \{\delta_1 > \cdots > \delta_a > \epsilon_1 > \pm\epsilon_2 > \delta_{a+1} > \cdots > \delta_q\}$

$K_1(\Psi_{\pm 0}) = \mathfrak{su}_2(\epsilon_1 \pm \epsilon_2)$. Both systems are small.
$1 \leq a < q$

$$\mathfrak{k}_1(\Psi_{\pm a}) = \mathfrak{su}_2(\epsilon_1 \pm \epsilon_2) \oplus \mathfrak{so}(2q)$$

$$m(\Psi_1) = m(S_{\epsilon_2+\delta_q} S_{\epsilon_2-\delta_q} \Psi_1) = 3$$

$m(\Psi_a) = m(S_{\epsilon_2+\delta_q} S_{\epsilon_2-\delta_q} \Psi_a) = 4 \ for\ 1 < a < q$

$q \geq 3$, $\mathfrak{k}_1(\Psi_{\pm q}) = \mathfrak{so}(2q)$. $\Psi \pm q$ are small.

Any other system of positive roots $\mathfrak{k}_1(\Psi) = \mathfrak{k}$.
Every ideal is equal to a $\mathfrak{k}_1(\Psi)$ but $\{0\}, \mathfrak{so}(4)$.

The subgroups $K_1(\Psi)$

The zero ideal is equal to a $\mathfrak{k}_1(\Psi)$ *if and only if* G/K *is Hermitian. Any non zero, semisimple ideal in* $\mathfrak{k}$ *is equal to an ideal* $\mathfrak{k}_1(\Psi)$ *except for: four ideals in* $\mathfrak{so}(4)\times\mathfrak{so}(4) \subset \mathfrak{so}(4,4)$; $\mathfrak{so}(2q+1)$ *in* $\mathfrak{so}(2p, 2q+1), p \geq 2$; $\mathfrak{so}(4)$ *in* $\mathfrak{so}(4,n), (n \neq 2)$.

Tensor products
Δ system of positive roots in $\Phi(\mathfrak{k}, \mathfrak{t})$.

$\Psi, \tilde{\Psi}$ system of positive roots in $\Phi(\mathfrak{g}, \mathfrak{t})$ both of them containing Δ.

λ, $\Psi-$ dominant, regular ....

μ, $\tilde{\Psi}-$ dominant regular...

$\pi(\gamma)$ squar. int. of Harish-Chandra param γ.

If $\pi(\lambda) \otimes \pi(\mu)$ is admissible under the diagonal action of G, then $\Psi = \tilde{\Psi}$ and Ψ is a holomorphic system. The converse statement is also true.

Existence of discrete factors

Arbitrary G, H.

π square int. rep. of G.

W lowest $K-$type of π. Write $W_{|_L} = Z_1 \oplus \cdots$

Assume there exists square int. irred. rep μ of H with lowest $L-$type Z_1

Then, μ is a subrepresentation of $res_H(\pi)$.