PROPER MAP AND MULTIPLICITIES

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Abstract. Let G be a connected reductive real Lie group and H a closed connected reductive subgroup. Let Ω be a strongly elliptic coadjoint orbit of G, corresponding to a square integrable (modulo the center) representation π of G. When the restriction map $p_{\mathfrak{h}}$ from Ω into the dual of the Lie algebra $\mathfrak{h}$ of H is proper, the restriction of π to H is a direct sum of discrete series representations of H occurring with finite multiplicities (we say then that the restriction of π to H is admissible). We give a natural simple condition on Ω which implies the properness of $p_{\mathfrak{h}}$. Moreover, this condition allows us to give a formula in Blattner's style for the multiplicities of the restriction of π to H. Our condition is not necessary, but we classify when it is satisfied, and see that it covers many of the known cases where the restriction of π to H is admissible.

Introduction

In this article, G is a real connected reductive Lie group, and H a closed connected reductive subgroup of G. We say that an irreducible unitary representation π of G is a *discrete series representation* if it is square integrable modulo the center of G. Consider a discrete series representation π of G whose restriction$\pi|_H$ to H is admissible. It means that the restriction $\pi|_H$ of π to H is isomorphic to an Hilbertian direct sum of irreducible unitary representations σ of H occurring with finite multiplicities $m(\sigma)$. The representations σ which occur (i.e. those for which $m(\sigma)$ is > 0) are also discrete series for H. So both π and the σ's are parameterized by their Harish-Chandra parameters [**?**]. In this paper *we give a simple condition (***condition (C)***, definitions 2 and* **??** *below) on π and H which insures that $\pi|_H$ is admissible, and that the multiplicities $m(\sigma)$ can be computed in a simple explicit manner by sums of Kostant's partition functions involving the roots of $\mathfrak{g}_{\mathbb{C}}$ (the complexified Lie algebra of G) which are not roots of $\mathfrak{h}_{\mathbb{C}}$.*

Two important particular cases are well known.

The first one is the case of a compact (modulo center) connected Lie group K and of a connected closed subgroup L of K. Then, an irreducible unitary representation π of K is finite dimensional, and the restriction $\pi|_L$ can be described by Kostant-Heckman's [**?**] formula in terms of partition functions.

The second one is when $H = K$, where K is a maximal compact subgroup (modulo the center of G) of G. It is a theorem of Harish-Chandra that $\pi|_K$ is admissible, and a theorem of Hecht-Schmid [**?**] that the $m(\sigma)$ can be computed in term of partition functions.

However, in general, even if several interesting conditions are known to imply, or to be equivalent to the admissibility of $\pi|_H$ (see in particular [**?**], [**?**], and below), they are not always easy to check, and they do not imply that the multiplicities $m(\sigma)$ can be easily computed in term of partition functions.

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Condition (C) allows us to prove the admissibility of $\pi|_H$, and to give a simple formula for the multiplicities $m(\sigma)$ in terms of partition functions. It is not necessary for the admissibility of $\pi|_H$ (it is a kind of *generic* condition, so one might argue that its main interest is to pinpoint the more interesting cases when $\pi|_H$ is admissible and condition (C) is not satisfied). However, simple examples show that it is unlikely to expect simple formulas in term of partition functions when condition (C) is not satisfied.

Condition (C) is expressed in term of the roots of $\mathfrak{g}_C$, the roots of $\mathfrak{h}_{\mathbb{C}}$, and the Harish-Chandra parameter of π. We explore condition (C). An interesting fact is the role of a particular invariant connected subgroup of K attached to π (see section **??**). We classify when condition (C) is satisfied, in terms of the classification of semi-simple Lie algebras. We compare our results in term with previous works, notably of Gross-Wallach [**?**] on quaternionic homogeneous spaces and (IS IT NECESSARY ? Kobayashi [**?**] on symmetric spaces —PLEASE CHECK these references....)

All the previous statements have *semi-classical* equivalents in terms of the (so called coadjoint) representation of G in the dual $\mathfrak{g}^*$ of the Lie algebra $\mathfrak{g}$ of G, more precisely in terms of the Liouville measure of the coadjoint elliptic orbits of G in $\mathfrak{g}^*$. Essentially, we treat these results in a parallel separate manner. However, they are not independent, and interaction between them is useful.

Condition (C) allows to reduce to the previously known cases: restriction from K to L, restriction from G to K. So it is not surprising that the references [**?**, **?**, **?**] play an important role in this article.

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We describe with more details condition (C) and the formulas we obtain for multiplicities. To make statements simpler, in this introduction we assume that $\mathfrak{g}$ and $\mathfrak{h}$ have the same rank (see below (**??**) for the general case). We choose a fundamental Cartan algebra $\mathfrak{t} \subset \mathfrak{h}$. It is also a fundamental Cartan subalgebra of $\mathfrak{g}$. We denote by $\boldsymbol{i}\mathfrak{t}^*$ the real vector space of linear maps from $\mathfrak{t}$ to $\boldsymbol{i}\mathbb{R}$. We denote by $\mathfrak{k} \subset \mathfrak{g}$ the maximal compact subalgebra containing $\mathfrak{t}$, and by $\mathfrak{l}$ the corresponding subalgebra $\mathfrak{l} = \mathfrak{k} \cap \mathfrak{h}$ of $\mathfrak{h}$. We denote by $\Phi_{\mathfrak{g}} \subset \boldsymbol{i}\mathfrak{t}^*$ (resp. $\Phi_{\mathfrak{k}}$, etc...) the corresponding system of roots, and $W_{\mathfrak{g}}$ (resp. $W_{\mathfrak{k}}$, etc...) the corresponding Weyl groups.

Let $\lambda \in \boldsymbol{i}\mathfrak{t}^*$ be a linear form which is a Harish-Chandra parameter for a discrete series representation $\pi^G(\lambda)$ of G —we say simply that λ is a Harish-Chandra parameter for G—. Recall (see [**?**]) that a Harish-Chandra parameter λ is $\mathfrak{g}$-regular, that it satisfies a certain integrability condition, and that, if λ' is another Harish-Chandra parameter, then $\pi^G(\lambda) = \pi^G(\lambda')$ if and only if $\lambda' \in W_{\mathfrak{k}}\lambda$. In particular, λ determines a system of positive roots $\Psi_{\mathfrak{g}}(\lambda) \subset \Phi_{\mathfrak{g}}$. We denote by $\Psi^n_{\mathfrak{g}}(\lambda) \subset \Psi_{\mathfrak{g}}(\lambda)$ the subset of positive non compact roots.

If $\Psi_{\mathfrak{k}}$ is a positive system of roots for $\Phi_{\mathfrak{k}}$, we denote by $\Psi_{\mathfrak{k}/\mathfrak{l}}$ the subset of positive roots which are not roots of $\mathfrak{l}$.

Definition 1. *We say that a finite sequence $E = \{\alpha_1, \alpha_2, \dots, \alpha_n\}$ of elements of $\boldsymbol{i}\mathfrak{t}^*$ is* strict *if it is contained in an open half-space of $\boldsymbol{i}\mathfrak{t}^*$.*

Definition 2. *We say that $\mathfrak{g}$, $\mathfrak{h}$, λ satisfy* condition (C) *if there exists a positive system of roots $\Psi_{\mathfrak{k}}$ for $\Phi_{\mathfrak{k}}$ such that, for all $w \in W_{\mathfrak{k}}$, the set $w(\Psi^n_{\mathfrak{g}}(\lambda)) \cup \Psi_{\mathfrak{k}/\mathfrak{l}}$ is strict.*

We remark that this definition depends on $\mathfrak{h}$ only through $\mathfrak{l} = \mathfrak{h} \cap \mathfrak{k}$. This is not so surprising, since for any irreducible unitary representation π of G, square integrable modulo the center of G, the restriction $\pi|_H$ to H is admissible if and only if the restriction $\pi|_L$ to L is admissible (see (**??**) below). We prove that condition (C) implies that $\pi^G(\lambda)|_L$ (or equivalently $\pi^G(\lambda)|_H$) is admissible.

To describe our multiplicity formulas, we need some more notations. We use Schwartz' distributions on $i\mathfrak{t}^*$, denoting by $*$ the convolution product when it is defined. Let $\alpha \in i\mathfrak{t}^*$. We denote by δ_α the Dirac distribution which is the evaluation at α. If $\alpha \neq 0$, we denote by y_α the distribution

$$y_\alpha = \delta_{\frac{1}{2}\alpha} * (\sum_{n\in\mathbb{N}} (\delta_\alpha)^{*n})) = \delta_{\frac{1}{2}\alpha} + \delta_{\frac{3}{2}\alpha} + \delta_{\frac{5}{2}\alpha} + \cdots . \tag{1}$$

Very informally, we may write

$$y_\alpha = \delta_{\frac{1}{2}\alpha} * (1 - \delta_\alpha)^{-1} = \frac{-1}{\delta_{\frac{1}{2}\alpha} - \delta_{-\frac{1}{2}\alpha}}, \tag{2}$$

because y_α is a particular inverse of $-(\delta_{\frac{1}{2}\alpha} - \delta_{-\frac{1}{2}\alpha})$. If $E = \{\alpha_1, \alpha_2, \dots, \alpha_n\}$ is a strict sequence of elements of $i\mathfrak{t}^*$, the convolution product

$$y_E = y_{\alpha_1} * y_{\alpha_2} * \cdots * y_{\alpha_n} \tag{3}$$

is well defined. We use the notation

$$\rho_E = \frac{1}{2}(\alpha_1 + \alpha_2 + \cdots + \alpha_n). \tag{4}$$

Then,

$$y_E = \sum_\nu K_E(\nu - \rho_E)\, \delta_\nu, \tag{5}$$

where $K_E(\nu)$ is (by definition) the Kostant partition function defined by E, that is the number of sequences $(p_1, \dots, p_n) \in \mathbb{N}^n$ such that $\nu = p_1\alpha_1 + p_2\alpha_2 + \cdots + p_n\alpha_n$.

Consider a Harish-Chandra parameter λ for a representation $\pi^G(\lambda)$, such that condition (C) is satisfied.

For each $w \in W_\mathfrak{k}$, let E_w be the subset of $w(\Psi^n_\mathfrak{g}(\lambda)) \cup \Psi_{\mathfrak{k}/\mathfrak{l}}$ consisting of the elements which are not roots of $\mathfrak{h}$. Condition (C) implies that the distributions y_{E_w} are well defined. Moreover, we define (see formula (**??**) below) a constant $\theta_w \in \{1, -1\}$. For instance, when $H = L$, θ_w is equal to the usual signature $\epsilon(w)$ of w. In general, it involves also the number of positive non compact roots of $\mathfrak{h}$ which are in $w(\Psi^n_\mathfrak{g}(\lambda))$. VERIFIER

We prove the following formula :

$$\sum_{w\in W_\mathfrak{k}} \theta_w\, \delta_{w\lambda} * y_{E_w} = \sum_\nu k(\nu)\, \delta_\nu, \tag{6}$$

then $k(\nu)$ is 0 if ν is not an Harish-Chandra parameter for H, and

$$m(\pi^H(\nu)) = |k(\nu)| \tag{7}$$

if ν is an Harish-Chandra parameter for H, and $m(\pi^H(\nu))$ the multiplicity of $\pi^H(\nu)$ in $\pi^G(\lambda)$. More explicitly, it gives the formula

$$m(\pi^H(\nu)) = \left| \sum_{w \in W_{\mathfrak{k}}} \theta_w \, K_{E_w}(\nu - w\lambda - \rho_{E_w}) \right| . \tag{8}$$

Remark 1. *If $\nu' \in W_{\mathfrak{l}}\nu$, the formula (8) for $m(\pi^H(\nu))$ and $m(\pi^H(\nu'))$ are essentially the same (see* **??** *below). VERIFIER !*

However, different choices of $\Psi_{\mathfrak{k}}$ (if they are available subject to condition (C)) may give genuinely different formulas for the multiplicities $m(\pi^H(\nu))$, that is, involving the computation of partition functions which are not naturally equivalent to each other. Going on, appropriate linear combinations of such formulas will give also new (more complicated) formulas in terms of partition functions.

Remark 2. *When $\mathfrak{g}$ and $\mathfrak{h}$ are not supposed to have the same rank, we still define condition (C), and obtain a formula similar to (8). The main difference is that the constant θ_w may take value in $\mathbb{Z}$ rather than in $\{1, -1\}$.*

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Condition (C) is not necessary for the admissibility of $\pi^G(\lambda)|_L$, but is necessary (at least when $H = L$) to write the simple formula (8). Assume that $\pi^G(\lambda)|_L$ is admissible. Considering the structure of formula (8) and remark 1, we will say that $\pi^G(\lambda)|_L$ is computable in terms of partition functions if there exists, for each $w \in W_{\mathfrak{k}}$, a family $(\Psi_{w,i})_{i=1,\dots,n_w}$ of positive root systems for $\Phi_{\mathfrak{g}}$ and for each $i = 1, \dots, n_w$, a constant $\theta_{w,i} \in \mathbb{C}$, such that, denoting by $E_{w,i}$ the subset of elements of $\Psi_{w,i}$ which are not roots of $\mathfrak{h}$, we have the following formula for the Harish-Chandra parameters ν for H :

$$m(\pi^H(\nu)) = \left| \sum_{w \in W_{\mathfrak{k}}} \sum_{i=1,\dots,n_w} \theta_{w,i} \, K_{E_{w,i}}(\nu - w\lambda - \rho_{E_{w,i}}) \right| . \tag{9}$$

We will give (see **??** below) an example where condition (C) is not satisfied, but where a formula of type (9) is still valid. I hope to find an example where this is impossible ...

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Why do we care about formulas for multiplicities in terms of partition functions? It is well known that this type of formulas, because they involve signs, are not easily used to answer questions like "does a specific representation $\pi^H(\nu)$ occurs in $\pi^G(\lambda)$?".

Our point of view is that this question is a natural manner of analyzing the complexity of $\pi^G(\lambda)|_H$: we classify the simplest cases, and leave the more interesting more singular cases for further studies.

It could also be useful for computational purposes : very efficient computer programs exist for partition functions (see [**?**, **?**]), and in some cases, the combinatorics of Weyl groups involved in formula (8) have also been handled efficiently (see [**?**]).

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In the rest of this article, we investigate condition (C). Recall that in this introduction we assume that $\mathfrak{t} \subset \mathfrak{l}$. Let $\tilde{\mathfrak{k}}$ be the largest ideal of $\mathfrak{k}$ contained in $\mathfrak{l}$ and $\tilde{K}$ be the corresponding subgroup of K. Then we prove that $\pi^G(\lambda)|_{\tilde{K}}$ is still admissible (in fact, condition (C) —which we did not yet state in the case of unequal rank—is still satisfied up to $\tilde{\mathfrak{k}}$). It allow us to classify when condition (C) is satisfied in terms of the classification of real semi-simple Lie algebras and of Harish-Chandra parameters.

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Finally, we compare this classification with previous works, notably of Gross-Wallach on quaternionic representations, and COMPLETE....

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TENTATIVE PLAN
Mostly notations
Partition functions
Blattner's formula
Kostant-Heckman formulas
Condition C
Our formula
Necessary and sufficient condition for admissibility : asymptotic cones, etc... Some examples showing what can happen when there is admissibility, but condition (C) fails.
Implications of condition (C) : the ideal $\mathfrak{k}_1$.
Classification.
Comparison with results of Gross-Wallach, and others.

1. Mostly notations

1.1. **Roots.** For any Lie group A, we denote by the corresponding German letter $\mathfrak{a}$ its Lie algebra, and by $Z(A)$ the center of A.

We choose a subgroup K of G which contains $Z(G)$ and such that $K/Z(G)$ is a maximal compact subgroup of $G/Z(G)$. We choose a Cartan subalgebra $\mathfrak{t}$ of $\mathfrak{k}$, and we denote by T the corresponding Cartan subgroup of K (it is the connected subgroup of K with Lie algebra $\mathfrak{t}$).

We denote by V^* the dual of a (real or complex) vector space V. If V is real, let $V_{\mathbb{C}}$ the its complexification; we identify $(V_{\mathbb{C}})^*$ and $(V^*)_{\mathbb{C}}$ in the usual way. In particular, $\boldsymbol{i}V^*$ (where $\boldsymbol{i} \in \mathbb{C}$ is a fixed square root of -1) is the real vector space of linear forms on $V_{\mathbb{C}}$ which are purely imaginary on V. If $\lambda \in \boldsymbol{i}V^*$, then e^λ is a unitary character of the group V, and we identify in this way $\boldsymbol{i}V^*$ and the group of unitary characters of V.

We denote by $X(T)$ the group of unitary characters of T. The differential of such a character belongs to $\boldsymbol{i}\mathfrak{t}^*$. We denote by P the set of $\lambda \in \boldsymbol{i}\mathfrak{t}^*$ which exponentiate to a character e^λ of T. It is a closed subgroup of $\boldsymbol{i}\mathfrak{t}^*$. If G is semi-simple with finite center, it is a lattice (the so called lattice of weights) in $\boldsymbol{i}\mathfrak{t}^*$.

Let $\mathfrak{u}$ be a subspace of $\mathfrak{t}$. If V and W are $\mathfrak{u}_{\mathbb{C}}$-invariant subspaces of $\mathfrak{g}_{\mathbb{C}}$ such that $W \subset V$, we denote by $\Phi(\mathfrak{u}, V/W)$ the *multiset* of non zero roots of $\mathfrak{u}_{\mathbb{C}}$ in V/W : formally it is the function from $\boldsymbol{i}\mathfrak{u}^*$ to $\mathbb{N}$ which associate 0 to 0, and the multiplicity of α in V/W to a non zero element α of $\boldsymbol{i}\mathfrak{u}^*$. Informally, we consider it as the set of non zero roots of $\mathfrak{u}$ in V/W, where each root is repeated as many time as its multiplicity.

We use the notation $\Phi_{\mathfrak{g}}$ for $\Phi(\mathfrak{t}, \mathfrak{g}_{\mathbb{C}})$, $\Phi_{\mathfrak{k}}$ for $\Phi(\mathfrak{t}, \mathfrak{k}_{\mathbb{C}})$, Φ_n for $\Phi(\mathfrak{t}, \mathfrak{g}_{\mathbb{C}}/\mathfrak{k}_{\mathbb{C}})$. For example, $\Phi_{\mathfrak{k}}$ is the root system of $\mathfrak{k}$.

We denote by $\mathfrak{p}$ the unique K-stable complement of $\mathfrak{k}$ in $\mathfrak{g}$. We use the decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ to identify $\mathfrak{k}^*$ and the orthogonal of $\mathfrak{p}$ in $\mathfrak{g}^*$. In a similar manner, $\mathfrak{t}$ has a unique T-stable complement in $\mathfrak{k}$, and we identify $\mathfrak{t}^*$ with the appropriate subspace of $\mathfrak{k}^*$. Thus we identify $\mathfrak{t}^*$ with a specific subspace of $\mathfrak{g}^*$.

We fix a bilinear symmetric G-invariant form $(.,.)$ on $\mathfrak{g}$ which is non degenerate, positive definite on $\mathfrak{p}$, and negative definite on $\mathfrak{k}$. We use the same notation for the form deduced on $\mathfrak{g}^*$, and on the various subspaces and complexifications. Then it is positive definite on $\boldsymbol{i}\mathfrak{t}$, which is then an Euclidian real vector space.

1.2. **Strongly elliptic orbits.** Let $f \in \mathfrak{g}^*$ (in fact we will use more often $f \in \boldsymbol{i}\mathfrak{g}^*$). The centralizer $G(f)$ of f in G contains $Z(G)$. We say that f is *strongly elliptic* (or strongly $\mathfrak{g}$-elliptic if there is ambiguity) if the group $G(f)/Z(G)$ is compact. Strongly elliptic elements are studied by Weinstein in [**?**] under the name of strongly stable elements. He proves that there exists strongly elliptic elements if and only if $\mathfrak{t}$ is a Cartan subalgebra of $\mathfrak{g}$.

Note that this condition is the same than Harish-Chandra's condition for the existence of square integrable (modulo $Z(G)$) irreducible representations of G (the so called discrete series representations). discrete series representations and strongly elliptic coadjoint orbits are the main object of this paper. So, for the rest of the article, **we assume that $\mathfrak{t}$ is a Cartan subalgebra of $\mathfrak{g}$**.

Then $\Phi_\mathfrak{g}$ is the root system of $\mathfrak{g}$. For each $\alpha \in \Phi_\mathfrak{g}$, we denote by $\mathfrak{g}_\alpha \subset \mathfrak{g}_\mathbb{C}$ the corresponding root space, and by $h_\alpha \in \boldsymbol{i}\mathfrak{t}$ the corresponding coroot (so that $h_\alpha \in [\mathfrak{g}_\alpha, \mathfrak{g}_{-\alpha}]$ and $\alpha(h_\alpha) = 2$). We denote by W_G the group $N_G(T)/T$ where $N_G(T)$ is the normalizer of T in G. Considered as a subgroup of linear transformations of $\boldsymbol{i}\mathfrak{t}^*$, it is a subgroup of $W_\mathfrak{g}$, the *complex* Weyl group. Similarly the groups W_K and $W_\mathfrak{k}$ are defined, and it is well known that $W_G = W_K = W_\mathfrak{k}$.

An element $\lambda \in \mathfrak{t}^*_\mathbb{C}$ is said to be $\mathfrak{g}$-regular if $\lambda(h_\alpha) \neq 0$ for all $\alpha \in \Phi_\mathfrak{g}$. In a similar manner, using $\Phi_\mathfrak{k}$ and Φ_n, we define $\mathfrak{k}$-regular elements, and n-regular elements of $\mathfrak{t}^*_\mathbb{C}$.

For the rest of the article, we fix a positive system of roots $\tilde{\Psi} \subset \Phi_\mathfrak{g}$ and we put $\tilde{\Psi}_\mathfrak{k} = \tilde{\Psi}_\mathfrak{g} \cap \Phi_\mathfrak{k}$.

We denote by $\mathcal{C}_\mathfrak{k} \subset \boldsymbol{i}\mathfrak{t}^*$ the corresponding open positive chamber, i. e. the set of $\lambda \in \boldsymbol{i}\mathfrak{t}^*$ such that $\lambda(h_\alpha) > 0$ for all $\alpha \in \tilde{\Psi}_\mathfrak{k}$. The closure $cl(\mathcal{C}_\mathfrak{k})$ of $\mathcal{C}_\mathfrak{k}$ in $\boldsymbol{i}\mathfrak{t}^*$ is a system of representatives for the elliptic orbits of G in $\boldsymbol{i}\mathfrak{g}^*$. It is also a system of representatives for the orbits of K in $\boldsymbol{i}\mathfrak{k}^*$, and for the orbits of W_K in $\boldsymbol{i}\mathfrak{t}^*$.

We denote by $\mathcal{C}^\mathfrak{g}_\mathfrak{k}$ the set of $\mathfrak{g}$-regular elements of $\mathcal{C}_\mathfrak{k}$. So, we have $\mathcal{C}^\mathfrak{k}_\mathfrak{k} = \mathcal{C}_\mathfrak{k}$. For each system of positive roots Ψ for $\Phi_\mathfrak{g}$, we denote by $\mathcal{C}(\Psi) \subset \boldsymbol{i}\mathfrak{t}^*$ the corresponding open positive chamber. Then $\mathcal{C}^\mathfrak{g}_\mathfrak{k}$ is the disjoint union of the $\mathcal{C}(\Psi)$ for the set of Ψ such that $\tilde{\Psi}_\mathfrak{k} \subset \Psi$, and it is a set of representatives for the $\mathfrak{g}$-regular elliptic orbits of G in $\boldsymbol{i}\mathfrak{g}^*$.

We denote by $cl_n(\mathcal{C}_\mathfrak{k})$ the set of n regular elements of $cl(\mathcal{C}_\mathfrak{k})$. It is the set of $\lambda \in \boldsymbol{i}\mathfrak{t}^*$ such that $\lambda(h_\alpha) \geq 0$ for all $\alpha \in \Psi_\mathfrak{k}$, and $\lambda(h_\alpha) \neq 0$ if $\alpha \in \Phi_n$. It is the disjoint union of the $cl_n\mathcal{C}(\Psi)$ for the set of Ψ such that $\Psi_\mathfrak{k} \subset \Psi$, where $cl_n\mathcal{C}(\Psi)$ is the set of $\lambda \in \boldsymbol{i}\mathfrak{t}^*$ such that $\lambda(h_\alpha) \geq 0$ for all $\alpha \in \Psi_\mathfrak{k}$, and $\lambda(h_\alpha) > 0$ if $\alpha \in \Psi_n$. The set $cl_n(\mathcal{C}_\mathfrak{k})$ is a set of representatives for the strongly elliptic orbits of G in $\boldsymbol{i}\mathfrak{g}^*$.

Let $\lambda \in cl_n(\mathcal{C}_\mathfrak{k})$ be a strongly elliptic element. We denote by $\Psi_n(\lambda)$ the set of $\alpha \in \Phi_n$ such that $\lambda(h_\alpha) > 0$. Then we have $\Psi(\lambda) = \Psi_n(\lambda) \cup \tilde{\Psi}_\mathfrak{k}$.

1.3. **Discretely admissible representations of G.** We denote by $\hat{G}$ the unitary dual of G, and by $\hat{G}_d$ the subset of classes of square integrable (modulo $Z(G)$) irreducible unitary representations. Since Bargmann who discovered these representations for $SL(2,\mathbb{R})$, the elements of $\hat{G}_d$ are called discrete series representations of G. This name is still a good one because the subset of $\hat{G}_d$ which have a given restriction to $Z(G)$ is parameterized by a discrete subset of $\boldsymbol{i}\mathfrak{t}^*$ (see below).

A unitary representation Π of G is said to be admissible if it is an Hilbertian[1] direct sum of irreducible representations of G which occur with finite multiplicities. It is said *discretely admissible* if moreover all irreducible representations which occur are discrete series. Thus a discretely admissible unitary representation Π of G can be written

$$\Pi = \bigoplus_{\pi \in \hat{G}_d} m(\Pi, \pi)\pi, \tag{10}$$

where $m(\Pi,\pi) \in \mathbb{N}$ is the multiplicity of π in Π. For example, finite dimensional representations of compact groups are obviously discretely admissible, and (in this paper) we consider discretely admissible representations of G as the simplest generalization of finite dimensional representations of compact groups. Other representations (e. g. the regular representation) are interesting too, but the goal of this paper is modest, and we shall be interested mostly by this "simple" case.

To express collective properties of the numbers $m(\Pi,\pi)$ for all π's , it will be convenient to associate a measure $m_G(\Pi)$ on $\boldsymbol{i}\mathfrak{t}^*$, which somehow plays the role of a generating function. Its definition requires some notations.

First we recall Harish-Chandra's parametrization of $\hat{G}_d$.

Let $E = \{\alpha_1, \dots, \alpha_n\}$ be a finite multiset of elements of $\boldsymbol{i}\mathfrak{t}^*$, where the α_i are repeated according to their multiplicity. We use the notation $\#E = n$, and

$$\rho_E = \frac{1}{2}\sum_{i=1}^{n} \alpha_i. \tag{11}$$

Let $\Psi \subset \Phi_{\mathfrak{g}}$ be a positive system of roots. Then the subset $\rho_\Psi + P \subset \boldsymbol{i}\mathfrak{t}^*$ does not depend on Ψ. We denote it by $P_{\mathfrak{g}}$. Remark that $P_{\mathfrak{g}}$ is W_K-invariant. Harish-Chandra defined a bijection

$$\lambda \to \pi^G(\lambda) \quad \text{from} \quad P_{\mathfrak{g}} \cap \mathcal{C}_k^{\mathfrak{g}} \quad \text{to} \quad \hat{G}_d. \tag{12}$$

We shall call an element of $P_{\mathfrak{g}} \cap \mathcal{C}_k^{\mathfrak{g}}$ a Harish-Chandra parameter.

For $\lambda \in P_{\mathfrak{g}} \cap \mathcal{C}_k^{\mathfrak{g}}$ and $f \in G\lambda$, we also put $\pi^G(f) = \pi^G(\lambda)$. Harish-Chandra's parametrization is then a bijection between a certain set of elliptic regular coadjoint orbits, those orbits Ω which satisfy the "translated by ρ_Ψ" integrality condition, and $\hat{G}_d$. We shall recall below (see remark 4) a consequence of Hecht-Schmid's theorem on the restriction $\pi^G(\lambda)|_K$ of $\pi^G(\lambda)$ to K which (among other things) is sufficient to specify uniquely $\pi^G(\lambda)$. For the time being, we recall a few properties of this bijection.

Notice that everything we said applies to the group K. We already choose $\mathcal{C}_k$, $\Psi_{\mathfrak{k}}$, so we have at our disposal $\rho_{\Psi_{\mathfrak{k}}}$, $P_{\mathfrak{k}} = \rho_{\Psi_{\mathfrak{k}}} + P$ (in general $P_{\mathfrak{k}}$ and $P_{\mathfrak{g}}$ are not equal). For $\mu \in P_{\mathfrak{k}} \cap \mathcal{C}_k$, Harish-Chandra's parametrization gives an irreducible unitary representation $\pi^K(\mu)$ of K. As $K/Z(G)$ is compact, this is a finite irreducible representation of K,

[1] In this article we consider only separable Hilbert spaces, so the there is at most a countable sum involved.

which we identify to an irreducible representation of $\mathfrak{k}_{\mathbb{C}}$. Then $\pi^K(\mu)$ is the irreducible representation of $\mathfrak{k}_{\mathbb{C}}$ with highest weight $\mu - \rho_{\Psi_{\mathfrak{k}}}$ with respect to the positive system $\Psi_{\mathfrak{k}} \subset \Phi_{\mathfrak{k}}$.

Recall the Harish-Chandra isomorphism γ between the algebra $S(\mathfrak{t}_C)^{W_{\mathfrak{g}}}$ of $W_{\mathfrak{g}}$ invariant elements of $S(\mathfrak{t}_C)$ and the center $Z(\mathfrak{g}_{\mathbb{C}})$ of the enveloping algebra of $\mathfrak{g}_C$. The action of an element $z \in Z(\mathfrak{g}_{\mathbb{C}})$ in the set of smooth vectors of $\pi^G(\lambda)$ is a multiple of the identity which we denote by $\pi^G(\lambda)(z)\mathrm{Id}$. It defines a character (the infinitesimal character of $\pi^G(\lambda)$) of $Z(\mathfrak{g}_{\mathbb{C}})$. Harish-Chandra's formula for the infinitesimal character is :

$$\pi^G(\lambda)(\gamma(p)) = p(\lambda)\mathrm{Id} \quad \text{ for all } p \in S(\mathfrak{t}_C)^{W_{\mathfrak{g}}} . \tag{13}$$

For the group K, this property of the infinitesimal character implies that (as we stated above) $\pi^K(\mu)$ is the irreducible representation of $\mathfrak{k}_{\mathbb{C}}$ with highest weight $\mu - \rho_{\Psi_{\mathfrak{k}}}$. For the group G, this property is usually not sufficient to determine $\pi^K(\lambda)$. Still, we shall also call Harish-Chandra's parametrization "parametrization by infinitesimal character" (versus, in the case of compact groups, "parametrization by highest weight"). When working exclusively with compact groups, parametrization by highest weights and parametrization by infinitesimal characters do have respective merits. The goal is to relate properties of an irreducible representation of K to properties (e.g. in algebraic geometry, in symplectic geometry, in Kähler geometry, etc...) of a suitable coadjoint orbit, usually either the orbit of the highest weight $\lambda - \rho_{\Psi_{\mathfrak{k}}}$, or the orbit of the infinitesimal character λ. These orbits might be very different (e. g. of different dimensions, as when $\lambda = \rho_{\Psi_{\mathfrak{k}}}$), and so one gets many genuinely different (and eventually interesting) theorems. The kind of properties of the irreducible representation of K which can be considered has no limit : dimension, character, restriction to subgroups, concrete realizations, basis, explicit formulas for the coefficients in a specific basis, etc ... A simple concrete example is given below (22) for the second parametrization.

However, for non compact groups, there is no clear alternative to Harish-Chandra's parametrization, so we will stick to it.

Let $\lambda \in P_{\mathfrak{g}} \cap \mathcal{C}_k^{\mathfrak{g}}$ be a Harish-Chandra parameter. Recall the set $\Psi_n(\lambda)$, and let $\rho_n(\lambda)$ the corresponding half sum. It is obvious that we have $\lambda + \rho_n(\lambda) \in P_{\mathfrak{k}}$. It is a classical fact [?] that $\lambda + \rho_n(\lambda) \in P_{\mathfrak{k}} \cap \mathcal{C}_k$ and that $\pi^K(\lambda + \rho_n)$ occurs with multiplicity one in $\pi^G(\lambda)$. It is the "minimal K-type of $\pi^G(\lambda)$" in the sense of Vogan.

Let $\Psi \subset \Phi_{\mathfrak{g}}$ be a positive system of roots. We denote by $\epsilon_{\mathfrak{g}}^{\Psi}$ the corresponding sign function : it is the function on $\boldsymbol{i}\mathfrak{t}^*$ which is the sign of $\prod_{\alpha \in \Psi}$, where the sign of a real number x is -1, 0, or 1 depending on the position of x with respect to 0.

If $\Psi \subset \Phi_{\mathfrak{g}}$ and $\Psi' \subset \Phi_{\mathfrak{g}}$ are two positive sets of roots, then there exists a constant $\epsilon(\Psi, \Psi')$ (equals to 1 or -1) such that

$$\epsilon_{\mathfrak{g}}^{\Psi} = \epsilon(\Psi, \Psi')\epsilon_{\mathfrak{g}}^{\Psi'}. \tag{14}$$

We write

$$\epsilon_{\mathfrak{g}} = \epsilon_{\mathfrak{g}}^{\tilde{\Psi}}. \tag{15}$$

Similarly, we define $\epsilon_{\mathfrak{k}}$ as the sign of $\prod_{\alpha \in \tilde{\Psi}_{\mathfrak{k}}}$. Thus $\epsilon_{\mathfrak{k}}(\lambda) = 0$ if λ is not $\mathfrak{k}$-regular, and if $\lambda \in \boldsymbol{i}\mathfrak{t}^*$ is $\mathfrak{k}$-regular, we have

$$\epsilon_{\mathfrak{k}}(\lambda) = \epsilon(w), \tag{16}$$

where $w \in W_K$ is the unique element such that $w\lambda \in \mathcal{C}_k$, and $\epsilon(w)$ is the usual signature of w.

An element of a vector space in which W_K acts is said to be W_K-skew-invariant if it is an eigenvalue for the character $w \to \epsilon(w)$. For instance, the functions $\epsilon_{\mathfrak{k}}$ and $\epsilon_{\mathfrak{g}}$ are skew invariant.

Let us go back to a discretely admissible representation Π of G. We consider the measure $m_G(\Pi)$ on $\boldsymbol{i}\mathfrak{t}^*$ such that, for all $\lambda \in \boldsymbol{i}\mathfrak{t}^*$, we have

$$m_G(\Pi)(\{\lambda\}) = 0 \quad \text{if } \lambda \text{ is not } \mathfrak{g}\text{-regular, or not in } P_{\mathfrak{g}} \tag{17}$$

$$m_G(\Pi)(\{\lambda\}) = \epsilon_{\mathfrak{g}}(\lambda) m(\Pi, \pi^G(\lambda)) \quad \text{otherwise.} \tag{18}$$

It is a W_K-skew-invariant measure. It is determined by its restriction to $\mathcal{C}_k^{\mathfrak{g}}$. Of course, the measure $m_G(\Pi)$ determines the multiplicities $m(\Pi, \pi)$ for $\pi \in \hat{G}_d$, and the equivalence class of Π. For all the representations Π which will occur in this article, the support of Π is discrete in $\boldsymbol{i}\mathfrak{t}^*$, so that $m_G(\Pi)$ is a Radon measure.

Example 1. *Let $\lambda \in \mathcal{C}_{\mathfrak{k}}^{\mathfrak{g}} \cap P_{\mathfrak{g}}$. Then*

$$m_G(\pi^G(\lambda)) = \epsilon_{\mathfrak{g}}(\lambda) \sum_{w \in W_K} \epsilon(w) \delta_{w\lambda}. \tag{19}$$

Similarly, for $\mu \in \mathcal{C}_{\mathfrak{k}} \cap P_{\mathfrak{k}}$, we have (remind that $\epsilon_{\mathfrak{k}}(\mu) = 1$ for $\mu \in \mathcal{C}_{\mathfrak{k}}$) :

$$m_K(\pi^K(\mu)) = \sum_{w \in W_K} \epsilon(w) \delta_{w\mu}. \tag{20}$$

1.4. **Invariant measures on $\boldsymbol{i}\mathfrak{g}^*$.** In the method of orbits of Kirillov, irreducible unitary representations of G correspond[2] to coadjoint orbits of G in[3] $\boldsymbol{i}\mathfrak{g}^*$. Recall that each coadjoint orbit Ω is a symplectic manifold, and thus is provided with a Liouville measure β_Ω (see details below), which can be considered as an invariant positive measure on $\boldsymbol{i}\mathfrak{g}^*$. In this setting, it is natural to consider that the analog of an unitary representation of G is a positive invariant measures on $\boldsymbol{i}\mathfrak{g}^*$. Considering Harish-Chandra's bijection, it is also natural to consider that the analog of discrete series representations of G are $\mathfrak{g}$-regular elliptic orbits of G in $\mathfrak{g}^*$ (that is, we forget about integrability conditions, but try to keep everything else). So we will mimic the previous subsection by considering the problem of describing G-invariant measures on $\boldsymbol{i}\mathfrak{g}^*$ for which the set of elements which are not regular elliptic is of measure 0.

We provide a coadjoint orbit $\Omega \subset \boldsymbol{i}\mathfrak{g}^*$ with the (real) symplectic structure ω_Ω such that $\omega_{\Omega,f}(Xf, Yf) = \boldsymbol{i}f([X,Y])$ for any $f \in \Omega$, $X \in \mathfrak{g}$, $Y \in \mathfrak{g}$, where Xf (resp. Yf) is the tangent vector to Ω at f corresponding to the infinitesimal action of X (resp. Y). The dimension of Ω is even, let it be $2d$. Then

$$\tilde{\beta}_\Omega = \frac{1}{(2\pi)^d} \omega_\Omega^d \tag{21}$$

is an invariant form of maximal degree on Ω. We denote by β_Ω the corresponding positive invariant measure[4]. When Ω is compact, the volume of Ω (with respect to the measure

[2]This correspondence is very far from a well defined and universally accepted correspondence in the set theoretical sense.

[3]Considering orbits in $\boldsymbol{i}\mathfrak{g}^*$ avoids a choice of a square root of -1.

[4]It is well known [**?**] to be a tempered Radon measure.

β_Ω) is called the symplectic volume, and is denoted by $\mathrm{vol}(\Omega)$. The definition of ω_Ω depends on various rather arbitrary choices of signs or of powers of $\boldsymbol{i}$, but the definition of β_Ω does not depend on them. The inclusion of the factor $\frac{1}{(2\pi)^d}$ in the definition of β_Ω is justified by the fact that it simplifies many formulas. We give a well known example, which will be used later on. Let $\mu \in P_{\mathfrak{k}} \cap \mathcal{C}_k$. Then we have :

$$\dim(\pi^K(\mu)) = \mathrm{vol}(K\mu). \tag{22}$$

This is the evaluation at the origin of Kirillov's character formula for compact connected Lie groups (see [?], theorem 8.4).

It is convenient to give a definition. We denote by $\mathcal{C}_c(\boldsymbol{i}\mathfrak{g}^*)$ the space of continuous functions with compact support on $\boldsymbol{i}\mathfrak{g}^*$, and by $\mathcal{C}_c^\infty(\boldsymbol{i}\mathfrak{g}^*)$ the subspace of smooth functions.

Let $\phi \in \mathcal{C}_c^\infty(\boldsymbol{i}\mathfrak{g}^*)$. It is known (see below) that the function, defined on the set of $\mathfrak{g}$-regular elements λ of $\boldsymbol{i}\mathfrak{t}^*$ by

$$\lambda \to \int_{G\lambda} d\beta_{G\lambda}(f)\phi(f), \tag{23}$$

extends to a smooth function with compact support on each of the set $cl(\mathcal{C}(\Psi)) \subset \boldsymbol{i}\mathfrak{t}^*$, where $\Psi \subset \Phi_{\mathfrak{g}}$ is a positive system of roots. However the limiting values of this function and of its derivatives on a singular element in the boundary of $\mathcal{C}(\Psi)$ depends on Ψ, and it is in general not possible to extend the function (23) to a smooth function on $\boldsymbol{i}\mathfrak{t}^*$ —and not even to a continuous function if G is not compact—. Harish-Chandra described a family of relations between these limiting values, and Bouaziz [?] proved that this family gives a complete set of relations.

I DO NOT KNOW IF WE WILL HAVE TO USE THAT.

From their work, it follows that it simplifies statements to use the function

$$I_G(f)(\lambda) = \epsilon_{\mathfrak{g}}(\lambda) \int_{G\lambda} d\beta_{G\lambda}(f)\phi(f), \tag{24}$$

defined for $\mathfrak{g}$-regular elements of $\boldsymbol{i}\mathfrak{t}^*$: it extends to a smooth function on the set of strongly regular elements.

Consider for instance the group K. In this case

$$\lambda \to \int_{K\lambda} d\beta_{K\lambda}(f)\phi(f) \tag{25}$$

extends to a W_K-invariant continuous function with compact support on $\boldsymbol{i}\mathfrak{t}^*$ which is zero on k-singular elements. It is usually not smooth (locally it might look like $|t|$ for $t \in \mathbb{R}$). However, the function

$$\lambda \to I_K(f)(\lambda) = \epsilon_{\mathfrak{k}}(\lambda) \int_{K\lambda} d\beta_{K\lambda}(f)\phi(f) \tag{26}$$

extends to a smooth function with compact support $I(f) \in \mathcal{C}_c^\infty(\boldsymbol{i}\mathfrak{t}^*)$, and $f \to I_K(f)$ is a surjection of $\mathcal{C}_c^\infty(\boldsymbol{i}\mathfrak{k}^*)$ to the space of W_K-skew-invariant smooth functions on $\boldsymbol{i}\mathfrak{t}^*$.

Definition 3. *A Radon G-invariant measure ξ on $\boldsymbol{i}\mathfrak{g}^*$ is said to be* regular elliptic *if there exists a W_G-skew-invariant Radon measure $M_{\mathfrak{g}}(\xi)$ on $\boldsymbol{i}\mathfrak{t}^*$ such that the set of $\mathfrak{g}$-singular elements of $\boldsymbol{i}\mathfrak{t}^*$ is of measure 0, and such that for all $\phi \in \mathcal{C}_c(\boldsymbol{i}\mathfrak{g}^*)$, we have*

$$\int_{\boldsymbol{i}\mathfrak{g}^*} d\xi(f)\phi(f) = \frac{1}{\# W_G} \int_{\boldsymbol{i}\mathfrak{t}^*} dM_{\mathfrak{g}}(\xi)(\lambda) \left(\epsilon_{\mathfrak{g}}(\lambda) \int_{G\lambda} d\beta_{G\lambda}(f)\phi(f) \right). \tag{27}$$

It follows that the set of elements of $\boldsymbol{i}\mathfrak{g}^*$ which are not regular elliptic is of measure zero for ξ. Formula (27) can be written

$$\int_{\boldsymbol{i}\mathfrak{g}^*} d\xi(f)\phi(f) = \int_{\mathcal{C}_k^{\mathfrak{g}}} dM_{\mathfrak{g}}(\xi)(\lambda)\epsilon_{\mathfrak{g}}(\lambda) \int_{G\lambda} d\beta_{G\lambda}(f)\phi(f). \tag{28}$$

So the restriction of the measure $\epsilon_{\mathfrak{g}}(\lambda)M_{\mathfrak{g}}(\xi)(\lambda)$ to $\mathcal{C}_{\mathfrak{k}}^{\mathfrak{g}}$ can be considered as the quotient of ξ by the invariant measures $\beta_{G\lambda}$. In particular, it is positive if ξ is positive.

Remark 3. *In the case of K, we can consider in an analog manner invariant distributions on $\boldsymbol{i}\mathfrak{k}^*$ which are of the form*

$$\int_{\boldsymbol{i}\mathfrak{k}^*} d\xi(f)\phi(f) = \frac{1}{\# W_K} \int_{\boldsymbol{i}\mathfrak{t}^*} dM_{\mathfrak{k}}(\xi)(\lambda) \left(\epsilon_{\mathfrak{k}}(\lambda) \int_{K\lambda} d\beta_{K\lambda}(f)\phi(f) \right) \tag{29}$$

for $\phi \in \mathcal{C}_c^\infty(\boldsymbol{i}\mathfrak{k}^*)$, for a suitable W_K-skew invariant distribution $M_{\mathfrak{k}}(\xi)$ on $\boldsymbol{i}\mathfrak{t}^*$. In fact, it is known [?] *that the map $\xi \to M_{\mathfrak{k}}(\xi)$ is a bijection.*

For instance, consider the Dirac measure at the origin of $\boldsymbol{i}\mathfrak{k}^*$. For $\phi \in \mathcal{C}_c^\infty(\boldsymbol{i}\mathfrak{k}^*)$ we have

$$\phi(0) =< D, I_K(\phi) > \tag{30}$$

where, for $\psi \in \mathcal{C}_c^\infty(\boldsymbol{i}\mathfrak{t}^*)$, we have

$$< D, \psi >= \frac{1}{\# W_K} \left((\prod_{\alpha \in \tilde{\Psi}_{\mathfrak{k}}} \partial_\alpha)\psi \right)(0). \tag{31}$$

Example 2. *Let $\lambda \in \boldsymbol{i}\mathfrak{t}^*$. Then the measure $\beta_{G\lambda}$ is regular elliptic if and only if λ is $\mathfrak{g}$-regular. If $\lambda \in \mathcal{C}_{\mathfrak{k}}^{\mathfrak{g}}$, we have*

$$M_{\mathfrak{g}}(\beta_{G\lambda}) = \epsilon_{\mathfrak{g}}(\lambda) \sum_{w \in W_K} \epsilon(w)\delta_{w\lambda}. \tag{32}$$

Similarly, if $\lambda \in \mathcal{C}_{\mathfrak{k}}$, we have

$$M_{\mathfrak{k}}(\beta_{K\lambda}) = \sum_{w \in W_K} \epsilon(w)\delta_{w\lambda}. \tag{33}$$

2. Partition functions

Let α be a non zero vector in $\boldsymbol{i}\mathfrak{t}^*$. We introduce the following Radon measures, which, on a function $\phi \in \mathcal{C}_c(\boldsymbol{i}\mathfrak{t}^*)$ are defined by the following formulas.

$$Y_\alpha(\phi) = \int_0^\infty \phi(t\alpha)dt, \tag{34}$$

$$y_\alpha(\phi) = \sum_{n\in\mathbb{N}} \phi((n+\frac{1}{2})\alpha)dt, \tag{35}$$

$$t_\alpha(\phi) = \int_{-\frac{1}{2}}^{\frac{1}{2}} \phi(t\alpha)dt, \tag{36}$$

$$d_\alpha(\phi) = \phi(-\frac{1}{2}\alpha) - \phi(\frac{1}{2}\alpha). \tag{37}$$

We use the symbol $*$ for the convolution product. Note that

$$d_\alpha * y_\alpha = \delta_0, \tag{38}$$

so that y_α is an inverse of d_α. It is often useful to write y_α as

$$y_\alpha = \delta_{\frac{1}{2}\alpha} * (\sum_{n\in\mathbb{N}} (\delta_\alpha)^{*n})) = \delta_{\frac{1}{2}\alpha} * (1-\delta_\alpha)^{-1}), \tag{39}$$

where $(1-\delta_\alpha)^{-1}$ is the inverse obtained by formally writing the geometrical series. We have :

$$t_\alpha * y_\alpha = Y_\alpha. \tag{40}$$

We use also distributions. We define, for a function $\phi \in \mathcal{C}_c^\infty(\boldsymbol{i}\mathfrak{t}^*)$,

$$D_\alpha(\phi) = -\frac{d}{dt}\phi(t\alpha)|_{t=0}. \tag{41}$$

We have :

$$D_\alpha * Y_\alpha = \delta_0, \tag{42}$$

so that Y_α is an inverse of D_α.

Let $E\{\alpha_1, \ldots, \alpha_n\}$ be a finite multiset set of elements of $\boldsymbol{i}\mathfrak{t}^*$. We will say that E is *strict* if there exists some $x \in \boldsymbol{i}\mathfrak{t}$ such that $\alpha(x) > 0$ for all $\alpha \in E$. This condition allows to define the measure

$$Y_E = Y_{\alpha_1} * Y_{\alpha_2} * \cdots * Y_{\alpha_n}. \tag{43}$$

We will also write this as

$$Y_E = \bigstar_{\alpha\in E} Y_\alpha. \tag{44}$$

Similarly we define

$$y_E = \bigstar_{\alpha\in E} y_\alpha. \tag{45}$$

We define in a similar manner (the "strict" condition is not needed there) the measures or distributions t_E, d_E, and D_E.

The Kostant's partition function associated to E is the function K_E on $\boldsymbol{i}\mathfrak{t}^*$ defined by the formula

$$\bigstar_{\alpha\in E}(1-\delta_\alpha)^{-1} = \sum_\mu K_E(\mu)\delta_\mu, \tag{46}$$

that is, $K_E(\mu)$ is the cardinal number of the set of $(n_\alpha) \in \mathbb{N}^E$ such that $\mu = \sum n_\alpha \alpha$. In particular the support of K_E is the set $\mathbb{N}E$. The definition of y_E encodes the function K_E and an appropriate shift of ρ_E :

$$y_E = \delta_{\rho_E} * \bigstar_{\alpha \in E}(1-\delta_\alpha)^{-1} = \sum_\mu K_E(\mu)\delta_{\mu+\rho_E} = \sum_\mu K_E(\mu-\rho_E)\delta_\mu. \tag{47}$$

The support of y_E is $\rho_E + \mathbb{N}E$.

Corresponding facts for Y_E are as follows. The support of Y_E is the cone $\mathbb{R}^{\geq 0}E$ which is strictly convex, and has a non empty interior relative to the subspace $\mathbb{R}E$. With respect to a standard Lebesgue measure on $\mathbb{R}E$, it has a density which is a function homogeneous of degree $\#E - \dim(\mathbb{R}E)$, and continuous on $\mathbb{R}^{\geq 0}E$.

For instance, suppose that $\#E = \dim(\mathbb{R}E)$. This situation produces "multiplicity one" theorems because then K_E is the characteristic function of $\mathbb{N}E$, and Y_E is the standard Lebesgue measure in the basis E multiplied by the characteristic function of $\mathbb{R}^{\geq 0}E$ (see [**?**]).

Let us record some of the formulas relating these distributions which will be needed in the sequel. They follow immediately of the formulas (40, 38, 42) above.

$$Y_E = t_E * y_E, \tag{48}$$

$$D_E * Y_E = \delta_0, \tag{49}$$

$$d_E * y_E = \delta_0. \tag{50}$$

Let F be a multiset of non zero elements of $\boldsymbol{i}\mathfrak{t}^*$, and suppose that F is symmetric (that is the multiplicity of α is equal to the multiplicity of $-\alpha$). Let E be a positive system in F (we leave to the reader to define it). Then t_E does not depend on the choice of E. It will be denoted by

$$r_F = t_E, \tag{51}$$

where r_F is for "square root", since $r_F * r_F = t_F$.

3. Restriction to K.

3.1. **Blattner's formula.** Let $\lambda \in P_{\mathfrak{g}} \cap \mathcal{C}_k^{\mathfrak{g}}$ be an Harish-Chandra's parameter. It is a basic theorem of Harish-Chandra that any $\pi \in \hat{G}$ has an admissible restriction $\pi|_K$ to K. This is in particular true of the discrete series $\pi^G(\lambda)$. Moreover, Harish-Chandra's description of $\pi^G(\lambda)$, through its distribution character, impose some constraints on its restriction to K. Blattner's formula is the most natural solution to these constraints, and it very remarkable and pleasant that it is the correct answer. However it was difficult to prove, and this theorem is due (at least for linear groups) to Hecht and Schmid [**?**]. Since, other proofs have been provided, taking care also of remaining cases. We quote [**?**], because it is used in the present paper. Blattner's formula can be stated as follows :

$$m_K(\pi^G(\lambda)|_K) = \sum_{w \in W_K} \epsilon(w) w(\delta_\lambda * y_{\Psi_n(\lambda)}). \tag{52}$$

In particular, the parameter λ can be recovered from $\pi^G(\lambda)$. We state it in a lemma, since it is heavily used later on.

Lemma 1. *We have :*

$$d_{\Psi_n(\lambda)} * m_K(\pi^G(\lambda)|_K) = \sum_{w\in W_K} \epsilon(w)w(\delta_\lambda) = \sum_{w\in W_K} \epsilon(w)\delta_{w\lambda}. \tag{53}$$

Proof. It follows from (50) and (52). The main point is that d_{Ψ_n} is invariant under W_K for any positive system of roots $\Psi \subset \Phi_{\mathfrak{g}}$. It is a consequence from the fact that the sign character of $W_{\mathfrak{g}}$, when restricted to W_K, coincides with that of W_K. □

Remark 4. *May be, a more elegant way of writing this formula is as follows:*

$$d_{\tilde{\Psi}_n} * m_K(\pi^G(\lambda)|_K) = m_G(\pi^G(\lambda)). \tag{54}$$

In particular, we recover λ from $m_K(\pi^G(\lambda)|_K)$.

VERIFY SIGNS !!!

3.2. **DHV's formula.** We denote by $p_{\mathfrak{g},\mathfrak{k}}$ the projection map (i. e. the restriction map) from $\boldsymbol{i}\mathfrak{g}^*$ to $\boldsymbol{i}\mathfrak{k}^*$. If there is no ambiguity, we shall simply denote it by $p_{\mathfrak{k}}$. Thus $p_{\mathfrak{k}}$ means *restriction to* $\mathfrak{k}$. It is a simple and well known fact that the restriction of $p_{\mathfrak{k}}$ to any closed coadjoint G-orbit $\Omega \subset \boldsymbol{i}\mathfrak{g}^*$ is proper. The direct image of the Liouville measure is well defined. We shall denote it by $p_{\mathfrak{k}}(\beta_\Omega)$. So, $p_{\mathfrak{k}}(\beta_\Omega)$ is a positive K-invariant measure on $\boldsymbol{i}\mathfrak{k}^*$ with support $p_{\mathfrak{k}}(\Omega)$.

When $\lambda \in \mathcal{C}_k^{\mathfrak{g}}$ is a $\mathfrak{g}$-regular element, [?] gives the following formula (which says in particular that $p_{\mathfrak{k}}(\beta_{G\lambda})$ is $\mathfrak{k}$-regular),

$$M_{\mathfrak{k}}(p_{\mathfrak{k}}(\beta_{G\lambda})) = \sum_{w\in W_K} \epsilon(w)w(\delta_\lambda * Y_{\Psi_n(\lambda)}). \tag{55}$$

Remark 5. *The support of the measure $M_{\mathfrak{k}}(p_{\mathfrak{k}}(\beta_{G\lambda}))$ is the set $p_{\mathfrak{k}}(G\lambda) \cap \boldsymbol{i}\mathfrak{t}^*$.*

As stated in [?], (55) and (49) implies:

$$D_{\Psi_n(\lambda)} * M_{\mathfrak{k}}(p_{\mathfrak{k}}(\beta_{G\lambda})) = \sum_{w\in W_K} \epsilon(w)\delta_{w\lambda}, \tag{56}$$

or,

$$D_{\tilde{\Psi}_n} * M_{\mathfrak{k}}(p_{\mathfrak{k}}(\beta_{G\lambda})) = M_{\mathfrak{g}}(\beta_{G\lambda}). \tag{57}$$

VERIFY SIGNS !!!

3.3. **Comparison.** We shall use the measure[5]

$$r_{\mathfrak{g}/\mathfrak{k}} = r_{\Phi_n}. \tag{58}$$

The measure $r_{\mathfrak{g}/\mathfrak{k}}$ is a W_K-invariant probability measure with compact support. We denote its support by $c_{\mathfrak{g}/\mathfrak{k}}$. The set $c_{\mathfrak{g}/\mathfrak{k}}$ is the convex hull of the ρ_{Ψ_n}, for all positive systems $\Psi \subset \Phi_{\mathfrak{g}}$. If $\mathfrak{g}$ is semi-simple without simple compact factors, it has a non empty interior.

Let $\lambda \in \mathcal{C}_{\mathfrak{k}}^{\mathfrak{g}} \cap P_{\mathfrak{g}}$ be an Harish-Chandra's parameter. From (52) and (55), we obtain

$$r_{\mathfrak{g}/\mathfrak{k}} * m_K(\pi^G(\lambda)|_K) = M_{\mathfrak{k}}(p_{\mathfrak{k}}(\beta_{G\lambda})). \tag{59}$$

Formula (59) allows to compare the supports of the measures $m_K(\pi^G(\lambda)|_K)$ and of $M_{\mathfrak{k}}(p_{\mathfrak{k}}(\beta_{G\lambda}))$, and so between $\pi^G(\lambda)|_K$ and $p_{\mathfrak{k}}(G\lambda)$.

One type of inclusion is easy :

[5] We thank David Vogan for this idea.

Lemma 2. *Let $\nu \in p_{\mathfrak{k}}(G\lambda) \cap \boldsymbol{i}\mathfrak{t}^*$. Then there exists a $\mathfrak{k}$-regular $\mu \in P_{\mathfrak{k}}$ such that $\pi^K(\mu)$ occurs in $\pi^G(\lambda)|_K$ and $\nu \in \mu + c_{\mathfrak{g}/\mathfrak{k}}$.*

Proof. Let S be the support of $m_K(\pi^G(\lambda)|_K)$. It is the set of $\mathfrak{k}$-regular $\mu \in P_{\mathfrak{k}}$ such that $\pi^K(\mu)$ occurs in $\pi^G(\lambda)|_K$. The set $p_{\mathfrak{k}}(G\lambda) \cap \boldsymbol{i}\mathfrak{t}^*$ is the support of the measure $M_{\mathfrak{k}}(p_{\mathfrak{k}}(\beta_{G\lambda}))$. By equation (59), it is contained in $S + c_{\mathfrak{g}/\mathfrak{k}}$. □

If we prefer to use parameters in $\mathcal{C}_k$, it gives :

Lemma 3. *Let $\nu \in p_{\mathfrak{k}}(G\lambda) \cap cl(\mathcal{C}_{\mathfrak{k}})$. Then there exists $\mu \in P_{\mathfrak{k}} \cap \mathcal{C}_{\mathfrak{k}}$ and $w \in W_K$ such that $\pi^K(\mu)$ occurs in $\pi^G(\lambda)|_K$ and $\nu \in w\mu + c_{\mathfrak{g}/\mathfrak{k}}$, or equivalently, $\mu \in w^{-1}\nu + c_{\mathfrak{g}/\mathfrak{k}}$.*

Proof. The distribution $r_{\mathfrak{g}/\mathfrak{k}}$ is W_K-invariant and symmetric. □

COMMENT Reverse kind of inclusion (according to Vogan) should follow from the formula, but I do not know how. I think that the formula is sufficient to get the same asymptotic support for $m_K(\pi^G(\lambda)|_K)$ and $M_{\mathfrak{k}}(p_{\mathfrak{k}}(\beta_{G\lambda}))$. Is it true ? Is it interesting ?

We get some kind of reverse inclusion for “large” parameters.

Remark 6. *Suppose that $\mu \in P_k \cap \mathcal{C}_{\mathfrak{k}}$ is far from the walls, more precisely suppose that the $\mathfrak{k}$-regular elements of $P_{\mathfrak{k}} \cap (\mu + c_{\mathfrak{g}/\mathfrak{k}})$ are contained in $\mathcal{C}_{\mathfrak{k}}$, and that the representation $\pi^K(\mu)$ occurs in $\pi^G(\lambda)|_K$. Then μ belongs to $p_{\mathfrak{k}}(G\lambda) \cap \mathcal{C}_{\mathfrak{k}}$ (and even to the relative interior with respect to the affine subspace $\lambda + \mathbb{R}\Psi_n$).*

Proof. Consider $M = r_{\mathfrak{g}/\mathfrak{k}} * m_K(\pi^G(\lambda)|_K)$. If we write $m_K(\pi^G(\lambda)|_K) = \sum_\nu c_\nu \delta_\nu$, then $M = \sum_\nu c_\nu r_{\mathfrak{g}/\mathfrak{k}} * \delta_\nu$. Let V be a small neighborhood compact convex symmetric of 0 in $\boldsymbol{i}\mathfrak{t}^*$. In $\mu + V$, the ν which will contribute to this sum are such that $\nu + c_{\mathfrak{g}/\mathfrak{k}} \cap \mu + V \neq \emptyset$, that is $\nu \in \mu + c_{\mathfrak{g}/\mathfrak{k}} + V$. The hypothesis implies that, for V small enough, these ν's will be in $\mathcal{C}_{\mathfrak{k}}$, so that c_ν is ≥ 0. Since $c_\mu > 0$ by hypothesis, we find that in a neighborhood of μ, the measure M verifies $M \geq c_\mu r_{\mathfrak{g}/\mathfrak{k}} * \delta_\mu$. □

Remark 7. *With a similar proof, one obtains the following result : Suppose that $\mu \in P_k \cap \mathcal{C}_{\mathfrak{k}}$ is very far from the walls, more precisely suppose that the $\mathfrak{k}$-regular elements of $P_{\mathfrak{k}} \cap (\mu + 2c_{\mathfrak{g}/\mathfrak{k}})$ are contained in $\mathcal{C}_{\mathfrak{k}}$, and that the representation $\pi^K(\mu)$ occurs in $\pi^G(\lambda)|_K$. Then $\mu + c_{\mathfrak{g}/\mathfrak{k}}$ is contained in $p_{\mathfrak{k}}(G\lambda) \cap \mathcal{C}_{\mathfrak{k}}$. In particular, for all positive systems of roots $\Psi \subset \Phi_{\mathfrak{g}}$, $\mu + \rho_{\Psi_n}$ is contained in $p_{\mathfrak{k}}(G\lambda) \cap \mathcal{C}_{\mathfrak{k}}$.*

However, Paradan [**?**] obtained by more involved method the following better result :

Proposition 1. *Let such that the representation $\pi^K(\mu)$ occurs in $\pi^G(\lambda)|_K$. Then $\mu + \rho_n(\lambda) \in p_{\mathfrak{k}}(G\lambda) \cap \mathcal{C}_{\mathfrak{k}}$.*

Paradan proves even more : $\mu + \rho_n(\lambda)$ belongs to the interior of $p_{\mathfrak{k}}(G\lambda) \cap \mathcal{C}_{\mathfrak{k}}$ in the affine space generated by $p_{\mathfrak{k}}(G\lambda) \cap \mathcal{C}_{\mathfrak{k}}$.

3.4. **Asymptotic cones.** Let $X \subset V$ be a non empty subset of a finite dimensional real vector space V. The *asymptotic cone* $C(X)$ of X is the set of $v \in V$ such that there exists a sequence $(\epsilon_n)_{n>0}$ of real numbers $\epsilon_n > 0$ with limit 0, and a sequence of $(x_n)_{n>0}$ of $x_n \in X$, such that v is the limit of the sequence $(\epsilon_n x_n)_{n>0}$. The asymptotic cone $C(X)$ is a cone (that is stable by multiplication by $t \in [0, \infty)$), and closed in V. The asymptotic cone is reduced to $\{0\}$ if and only if X is bounded.

Let W be a vector space, and $p : V \to W$ a linear map.

Lemma 4. *The restriction $p|_{C(X)}$ is proper if and only if $C(X) \cap \ker(p) = \{0\}$.*

Proof. This is true for any closed cone $C \subset V$. $\square$

Lemma 5. *If $p|_{C(X)}$ is proper and X closed in V, then $p|_X$ is proper.*

Proof. We can assume that p is surjective. We can assume that V is a direct sum $V = U \oplus W$ and that p is the projection on W with kernel U. We choose norms on U and W. Then there exists $c > 0$ such that $||w|| \geq c||u||$ for all $v = w + u \in C(X)$.

Let $0 < c' < c$. We prove that *there exists $N > 0$ such that if $v = w + u \in X$ with $w \in W$ and $u \in U$, $||w|| + ||u|| \geq N$, then $||w|| > c'||u||$.*

Suppose that it is not the case. Then there exists a sequence $(v_n = w_n + u_n)_{n>0}$ of elements of X such that $||w_n|| + ||u_n|| \geq n$ and $||w_n|| \leq c'||u_n||$ for all $n > 0$. Let $\epsilon_n = (||w_n|| + ||u_n||)^{-1}$. By extracting a subsequence, we can assume that the limit $v = \lim \epsilon_n v_n$ exists. Then v belongs to $C(X)$, and $v \neq 0$. Write $v = w + u$. Then $||w|| \leq c'||u||$. This contradicts the assertion $||w|| \geq c||u||$.

The end of the proof is clear. $\square$

Remark 8. *The converse is not always true. Consider the parabola X with equation $y = x^2$ in $V = \mathbb{R}^2$. The asymptotic cone $C(X)$ is the positive y-axis, and the natural projection p on the x-axis is proper on X but not on $C(X)$.*

A simple way to avoid these parabolic branches is the following :

Lemma 6. *Let $Y \subset V$ be a closed subset. Suppose that there exists a compact set $B \subset V$ such that $Y \subset B + X$. If $p|_X$ is proper then $p|_Y$ is proper.*

Proof. Let $y_{n n>0}$ be a sequence of elements of Y such that $p(y_n)$ is bounded. Write in some manner $y_n = b_n + x_n$ with $b_n \in B$ and $x_n \in X$. Then $p(x_n)$ is bounded. Since $p|_X$ is proper, x_n is bounded. Then y_n is bounded. $\square$

Lemma 7. *a) We have*

$$p(C(X)) \subset C(p(X)). \tag{60}$$

b) If moreover $p|_{C(X)}$ is proper, we have

$$p(C(X)) = C(p(X)). \tag{61}$$

Proof. a) It follows from the continuity of p.

b) We use the notations of the proof of lemma 5. Let $w \in C(p(X))$. If $w = 0$, then we have $w \in p(C(X))$.

Suppose that $w \neq 0$. We write $w = \lim \epsilon_n w_n$ with $\epsilon_n > 0$, $\lim \epsilon_n = 0$, and $v_n = w_n + u_n \in X$. For n large enough, we have (from the proof of lemma 5) that $||w_n|| \geq c'||u_n||$. It follows that the sequence $\epsilon_n u_n$ is bounded. By extracting a subsequence, we can assume that the limit $\lim \epsilon_n u_n$ exists. Then the limit $v = \lim \epsilon_n(w_n + u_n)$ exists. We have $v \in C(X)$ and $p(v) = w$. $\square$

The example in remark 8 shows that it is not sufficient, in the lemma b) above, to assume that $p|_X$ is proper. The purpose of the next subsection is to make sure that in the situations of interest for this paper, this kind of parabolic behavior do not occur.

3.5. **Asymptotic cones : restriction to $\mathfrak{k}$.** Let X be a coadjoint orbit in $\mathfrak{g}^*$. Then (see [?]) $C(X)$ is contained in the nilpotent cone $\mathcal{N} \subset \mathfrak{g}^*$, which implies that the projection $p_{\mathfrak{k}}|_{C(X)}$ is proper. It follows from lemma 7 that

$$p(C(X)) = C(p(X)). \tag{62}$$

Recall that a *polyedron* in a finite dimensional real vector space V is a finite intersection of closed half affine subspaces. A subset $\mathcal{P} \subset V$ is *polyhedral* if it is a polyhedron.

Let X be a closed coadjoint orbit in $\boldsymbol{i}\mathfrak{g}^*$. Since the projection $p_{\mathfrak{k}}|_X$ is proper (see [?]), it follows from [?], theorem XXX, that $p_{\mathfrak{k}}(X) \cap cl(\mathcal{C}_k) \subset \boldsymbol{i}\mathfrak{t}^*$ is a closed convex set, locally polyhedral. Consider now an elliptic regular coadjoint orbit $G\lambda$ with $\lambda \in \mathcal{C}^{\mathfrak{g}}_{\mathfrak{k}}$. It follows from formula (55) that $p_{\mathfrak{k}}(G\lambda) \cap \boldsymbol{i}\mathfrak{t}^*$ is a finite union of polyhedrons (in which the measure $M_{\mathfrak{k}}(p_{\mathfrak{k}}(\beta_{G\lambda}))$ has a non zero polynomial density with respect to an invariant measure on the affine hull). This implies the following lemma :

Lemma 8. *Let $\lambda \in \mathcal{C}^{\mathfrak{g}}_{\mathfrak{k}}$. Then $p_{\mathfrak{k}}(G\lambda) \cap cl(\mathcal{C}_k) \subset \boldsymbol{i}\mathfrak{t}^*$ is a polyhedron.*

Remark 9. *Let $\lambda \in \mathcal{C}^{\mathfrak{g}}_{\mathfrak{k}}$. Then λ is a vertex of $p_{\mathfrak{k}}(G\lambda) \cap cl(\mathcal{C}_k)$, and it follows from (55) that $\lambda + \mathbb{R}^{\geq 0}\Psi_n(\lambda)$ is the local cone at λ. It follows that the affine hull of $p_{\mathfrak{k}}(G\lambda) \cap cl(\mathcal{C}_k)$ is $\lambda + \mathbb{R}\Psi_n(\lambda)$, and that*

$$p_{\mathfrak{k}}(G\lambda) \cap cl(\mathcal{C}_k) \subset (\lambda + \mathbb{R}^{\geq 0}\Psi_n(\lambda)) \cap cl(\mathcal{C}_k). \tag{63}$$

This is of course well known to experts : the local structure of $p_{\mathfrak{k}}(G\lambda) \cap cl(\mathcal{C}_k)$ can be computed from local data on $G\lambda$ (see [?]).

Recall (see [?], 1.5) that to a polyhedron $\mathcal{P} \subset V$ is associated a polyhedral cone $\mathrm{Rec}(\mathcal{P}) \subset V$, the recession cone. It has the property that there exists a compact polyhedron $\mathcal{Q}$ such that $\mathcal{P}$ is the Minkowski's sum $\mathcal{P} = \mathcal{Q} + \mathrm{Rec}(\mathcal{P})$. It follows that $\mathrm{Rec}(\mathcal{P}) = C(\mathcal{P})$, the asymptotic cone, and that $x + \mathrm{Rec}(\mathcal{P}) \subset \mathcal{P}$ for any $x \in P$.

Lemma 9. *Let $\lambda \in \mathcal{C}^{\mathfrak{g}}_{\mathfrak{k}}$. Then $p_{\mathfrak{k}}(C(G\lambda)) \cap cl(\mathcal{C}_{\mathfrak{k}})$ is equal to the the polyhedral cone $C(p_{\mathfrak{k}}(G\lambda) \cap cl(\mathcal{C}_k))$.*

Proof. From (62) we obtain the equality $p_{\mathfrak{k}}(C(G\lambda)) = C(p_{\mathfrak{k}}(G\lambda))$. Let $X = p_{\mathfrak{k}}(G\lambda)$. It is a K-invariant closed subset of $\boldsymbol{i}\mathfrak{k}^*$. We want to prove that $C(X) \cap cl(\mathcal{C}_{\mathfrak{k}}) = C(X \cap cl(\mathcal{C}_{\mathfrak{k}}))$. We prove that it is true for any invariant closed subset $X \subset \boldsymbol{i}\mathfrak{k}^*$.

Let X be an invariant closed subset $X \subset \boldsymbol{i}\mathfrak{k}^*$. It is obvious that $C(X \cap cl(\mathcal{C}_{\mathfrak{k}})) \subset C(X) \cap cl(\mathcal{C}_{\mathfrak{k}})$. Conversely, let $\mu \in C(X) \cap cl(\mathcal{C}_{\mathfrak{k}})$. We write $\mu = \lim \epsilon_n k_n \mu_n$ with $\epsilon_n > 0$, $\lim \epsilon_n = 0$, $k_n \in K$, $\mu_n \in X \cap cl(\mathcal{C}_{\mathfrak{k}})$. By extracting a subsequence, we can assume that the limit $k = \lim k_n$ exists. Then the limit $\nu = \lim \epsilon_n \mu_n$ exists and is equal to $k^{-1}\mu$. Since ν and μ are in $cl(\mathcal{C}_{\mathfrak{k}})$, we obtain that $\mu = \nu$. We have $\mu = \nu \in C(X \cap cl(\mathcal{C}_{\mathfrak{k}}))$. □

Lemma 10. *Let $\lambda \in \mathcal{C}^{\mathfrak{g}}_{\mathfrak{k}}$. Let $B = -K\lambda \subset \boldsymbol{i}\mathfrak{k}^*$. Then we have $C(p_{\mathfrak{k}}(G\lambda)) \subset B + p_{\mathfrak{k}}(G\lambda)$.*

Proof. We have $\lambda + C(p_{\mathfrak{k}}(G\lambda) \cap cl(\mathcal{C}_k)) \subset p_{\mathfrak{k}}(G\lambda) \cap cl(\mathcal{C}_{\mathfrak{k}})$, so $C(p_{\mathfrak{k}}(G\lambda) \cap cl(\mathcal{C}_k)) \subset -\lambda + p_{\mathfrak{k}}(G\lambda) \cap cl(\mathcal{C}_{\mathfrak{k}})$. The result follows. □

QUESTIONS.

1) Everything there should be true for any closed coadjoint orbit.

2) Everything there should be true for the closure of a coadjoint orbit, in particular the closure of a nilpotent orbit.

3) In general, for a closed coadjoint orbit X, $C(X)$ is not the closure of a coadjoint orbit (for example, for hyperbolic regular orbits of $SL(2,\mathbb{R})$, $C(X)$ is the nilpotent cone,

the closure of the union of two nilpotent orbits. The projection on $\mathfrak{k}^* = \mathfrak{t}^*$ of the closure of each one of these orbits is half a line. The union is a line (still a polyhedral cone !).

4) Is $C(G\lambda)$ for $\lambda \in \mathcal{C}_k^{\mathfrak{g}}$ the closure of a nilpotent orbit X? I think yes. Moreover, the image of β_X can be computed (see Barbasch, Fourier transform of nilpotent orbits) by applying a suitable differential operator in λ to the corresponding image of $\beta_{G\lambda}$.

5) There is a finite number of nilpotent orbits X. For each of them, one should compute $p_{\mathfrak{k}}(X)$ and the corresponding measure.

3.6. **Asymptotic cones: restriction to $\mathfrak{l}$.** Let L be a connected closed subgroup of K. We choose a Cartan subgroup U of L. We assume that T contains U. Let $\lambda \in \mathcal{C}_{\mathfrak{k}}^{\mathfrak{g}}$. We consider the projection p_l on $\mathfrak{l}^*$.

Proposition 2. *Let $\lambda \in \mathcal{C}_{\mathfrak{k}}^{\mathfrak{g}}$. Then $p_{\mathfrak{l}}|_{G\lambda}$ is proper if and only if $p_{\mathfrak{l}}|_{C(G\lambda)}$ is proper.*

Proof. Recall the notation $p_{\mathfrak{k},\mathfrak{l}}$ for the projection from $\mathfrak{k}$ to $\mathfrak{l}$, and recall that we used the simplified notations $p_{\mathfrak{l}} = p_{\mathfrak{g},\mathfrak{l}}$, $p_{\mathfrak{k}} = p_{\mathfrak{g},\mathfrak{k}}$. Since $p_{\mathfrak{l}} = p_{\mathfrak{k},\mathfrak{l}} p_{\mathfrak{k}}$, then, for any closed subset $E \subset \boldsymbol{i}\mathfrak{g}^*$ such that $p_{\mathfrak{k}}|_E$ is proper, $p_{\mathfrak{l}}|_E$ is proper if and only if $p_{\mathfrak{k},\mathfrak{l}}|_{p_{\mathfrak{k}}(E)}$ is proper. Thus we have to prove that $p_{\mathfrak{k},\mathfrak{l}}$ is proper on $p_{\mathfrak{k}}(G\lambda)$ if and only if it is proper on $p_{\mathfrak{k}}(C(G\lambda))$.

Since $p_{\mathfrak{k}}(C(G\lambda)) = C(p_{\mathfrak{k}}(G\lambda))$ by (62), it follows from lemma 7 that $p_{\mathfrak{k},\mathfrak{l}}$ is proper on $p_{\mathfrak{k}}(G\lambda)$ if it is proper on $p_{\mathfrak{k}}(C(G\lambda))$.

Assume conversely that $p_{\mathfrak{k},\mathfrak{l}}$ is proper on $p_{\mathfrak{k}}(G\lambda)$. Then it is proper on $p_{\mathfrak{k}}(C(G\lambda))$ by lemmas 6 and 10. □

Proposition 2 is very useful, because numbers of results are available about $C(G\lambda)$, notably in [?]. For instance, $C(G\lambda)$ is a finite union of nilpotent coadjoint orbits, so there is a finite number of possibilities for $C(G\lambda)$ when λ varies. We give two corollaries.

Corollary 1. *Let $\lambda \in \mathcal{C}_{\mathfrak{k}}^{\mathfrak{g}}$ and $\lambda' \in \mathcal{C}_{\mathfrak{k}}^{\mathfrak{g}}$. Suppose that $\Psi_n(\lambda) = \Psi_n(\lambda')$. Then $p_{\mathfrak{l}}$ is proper on $G\lambda$ if and only if it is proper on $G\lambda'$.*

Proof. It is known ([?], Proposition 3.7) that $C(G\lambda) = C(G\lambda')$.

I AM AFRAID THAT IT PROVIDES A PROOF ONLY FOR parameters of discrete series, ie for "integral" λ.

Even if we are not able to prove (or to find a reference) that $C(G\lambda) = C(G\lambda')$, then DHV formula should prove the weaker result that $C(p_{\mathfrak{k}}(G\lambda)) = C(p_{\mathfrak{k}}(G\lambda'))$? □

Corollary 2. *Let $\lambda \in \mathcal{C}_{\mathfrak{k}}^{\mathfrak{g}}$. Let $C = C(p_{\mathfrak{k}}(G\lambda) \cap cl(\mathcal{C}_{\mathfrak{k}})) \subset cl(\mathcal{C}_{\mathfrak{k}})$ be the recession cone of $p_{\mathfrak{k}}(G\lambda) \cap cl(\mathcal{C}_{\mathfrak{k}})$. Let $\mathfrak{l}^{\perp}$ be the orthogonal of $\mathfrak{l}$ in $\mathfrak{k}$. Then $p_{\mathfrak{l}}$ is proper on $G\lambda$ if and only if $KC \cap \mathfrak{l}^{\perp} = \{0\}$.*

4. Restriction from K to T.

Let $\mu \in P_k \cap \mathcal{C}_{\mathfrak{k}}$. Recall that we fixed a positive system $\Psi_{\mathfrak{k}}$ for $\Phi_{\mathfrak{k}}$. Pick another (possibly the same) positive system $\Psi'_{\mathfrak{k}}$ for $\Phi_{\mathfrak{k}}$. Kostant's formula for the restriction $\pi^K(\mu)|_T$ can be written:

$$m_T(\pi^K(\mu)|_T) = (-1)^{\#\Psi_{\mathfrak{k}} \cup \Psi'_{\mathfrak{k}}} \left(\sum_{w \in W_K} \epsilon(w) \delta_{w\mu} \right) * y_{\Psi'_{\mathfrak{k}}}. \tag{64}$$

We obtain (see [?])

$$d_{\Psi'_{\mathfrak{k}}} * m_T(\pi^K(\mu))|_T) = (-1)^{\#\Psi_{\mathfrak{k}}\cup\Psi'_{\mathfrak{k}}}\left(\sum_{w\in W_K}\epsilon(w)\delta_{w\mu}\right) \tag{65}$$

$$= (-1)^{\#\Psi_{\mathfrak{k}}\cup\Psi'_{\mathfrak{k}}} m_K(\pi^K(\mu)) \tag{66}$$

Decomposing a finite dimensional representation R of K in irreducible components, we obtain:

Lemma 11. *Let R be a finite dimensional representation of K. Then :*

$$d_{\Psi'_{\mathfrak{k}}} * m_T(R|_T) = (-1)^{\#\Psi_{\mathfrak{k}}\cup\Psi'_{\mathfrak{k}}} m_K(R) \tag{67}$$

$$m_T(R|_T) = (-1)^{\#\Psi_{\mathfrak{k}}\cup\Psi'_{\mathfrak{k}}} m_K(R) * y_{\Psi'_{\mathfrak{k}}}. \tag{68}$$

Similarly (see [?]), let $\mu \in \mathcal{C}_k$. Then

$$M_T(p_{\mathfrak{t}}(\beta_{K\mu})) = (-1)^{\#\Psi_{\mathfrak{k}}\cup\Psi'_{\mathfrak{k}}}\left(\sum_{w\in W_K}\epsilon(w)\delta_{w\mu}\right) * Y_{\Psi'_{\mathfrak{k}}} \tag{69}$$

and

$$D_{\Psi"_{\mathfrak{k}}} * M_T(p_{\mathfrak{t}}(\beta_{K\mu})) = (-1)^{\#\Psi_{\mathfrak{k}}\cup\Psi'_{\mathfrak{k}}}\left(\sum_{w\in W_K}\epsilon(w)\delta_{w\mu}\right) \tag{70}$$

$$= (-1)^{\#\Psi_{\mathfrak{k}}\cup\Psi'_{\mathfrak{k}}} M_K(\beta_{K\mu}). \tag{71}$$

5. Restriction from K to U.

Let U be a closed connected subgroup of T. We consider the centralizer $Z = Z_K(U)$ of U in K and its Lie algebra $\mathfrak{z}$. Then T is a Cartan subgroup of Z. We denote by q the projection $p_{\mathfrak{t},\mathfrak{u}}$. We denote by $Q \subset \boldsymbol{i}\mathfrak{u}^*$ the subgroup of differentials of characters of U. We have $Q = q(P)$. We have $\Phi_{\mathfrak{z}} = \Phi_{\mathfrak{k}} \cap \ker(q)$.

Let $\Psi'_k \subset \Phi_{\mathfrak{k}}$ be a positive system. Then $\Psi'_{\mathfrak{z}} := \Psi'_k \cap \Phi_{\mathfrak{z}}$ is a positive system for $\Phi_{\mathfrak{z}}$. We write $\Psi'_{\mathfrak{k}/\mathfrak{z}} = \Psi'_{\mathfrak{k}} \backslash \Psi'_{\mathfrak{z}}$. A positive system $\Psi'_{\mathfrak{k}}$ of $\Phi_{\mathfrak{k}}$ is said to be $\mathfrak{u}$-admissible if the image $q(\Psi'_{\mathfrak{k}/\mathfrak{z}})$ is strict in $\boldsymbol{i}\mathfrak{u}^*$. Recall that we consider the Φ's as multisets, so that we take into account the multiplicities in $q(\Psi'_{\mathfrak{k}/\mathfrak{z}})$ if several roots have the same restriction to $\mathfrak{u}$.

Lemma 12. *a) A positive system $\Psi'_{\mathfrak{k}}$ of $\Phi_{\mathfrak{k}}$ is $\mathfrak{u}$-admissible if and only if the simple roots of $\Psi'_{\mathfrak{z}}$ are simple in $\Psi'_{\mathfrak{k}}$.*

b) Let $\Psi'_k \subset \Phi_{\mathfrak{k}}$ be a positive $\mathfrak{u}$-admissible system. Then $\rho_{\Psi'_{\mathfrak{k}}}(h_\alpha) = \rho_{\Psi'_{\mathfrak{z}}}(h_\alpha)$ for all $\alpha \in \Phi_{\mathfrak{z}}$.

Proof. a) Suppose that $\Psi'_{\mathfrak{k}}$ is $\mathfrak{u}$-admissible. Let $\alpha \in \Psi'_{\mathfrak{z}}$. Suppose that $\alpha = \beta + \gamma$ with β and γ in $\Psi'_{\mathfrak{k}}$. Then $0 = q(\beta) + q(\gamma)$. Since $q(\Psi'_{\mathfrak{k}}\backslash\Phi_{\mathfrak{z}})$ is strict, this implies that $q(\beta) = q(\gamma) = 0$, that is β and γ are in $\Psi'_{\mathfrak{z}}$.

Conversely, let $\alpha \in \Psi'_{\mathfrak{z}}$ a root which is simple in $\Psi'_{\mathfrak{z}}$, but not in $\Psi'_{\mathfrak{k}}$. We write $\alpha = \beta+\gamma$ with β and γ in $\Psi'_{\mathfrak{k}}\backslash\Psi'_{\mathfrak{z}}$. Then $0 = q(\beta) + q(\gamma)$, and $q(\Psi'_{\mathfrak{k}}\backslash\Phi_{\mathfrak{z}})$ is not strict.

b) It is sufficient to verify the formula for the simple roots of $\Psi'_{\mathfrak{z}}$. Then it follows from a) that they are also simple in $\Psi'_{\mathfrak{k}}$. So both sides of the formula are equal to 1. ☐

There exists at least one positive $\mathfrak{u}$-admissible system. In this paper, $\mathfrak{g}$, $\mathfrak{k}$, $\mathfrak{t}$, $\mathfrak{u}$ are given. The choice of the positive system $\Psi_{\mathfrak{k}}$ is not important : its main use is to provide a specific parametrization of the set of elliptic coadjoint orbits in $\boldsymbol{i}\mathfrak{g}^*$.

For convenience, in the rest of this paper, we choose $\Psi_{\mathfrak{k}}$ $\mathfrak{u}$-admissible. We have now at our disposal $\Psi_{\mathfrak{z}}$, $\mathcal{C}_z$, $\rho_{\psi_{\mathfrak{z}}}$, W_Z, etc...

However, there may exist other positive $\mathfrak{u}$-admissible systems which are not conjugate to $\Psi_{\mathfrak{k}}$ under natural automorphisms of this setting. We shall need all of them in the sequel.

We consider Weyl's dimension polynomial for Z. It is given for μ in $\boldsymbol{i}\mathfrak{t}^*$ by

$$\varpi(\mu) = \frac{\prod_{\alpha\in\Psi_{\mathfrak{z}}} \mu(h_\alpha)}{\prod_{\alpha\in\Psi_{\mathfrak{z}}} \rho_{\Psi_{\mathfrak{z}}}(h_\alpha)}. \tag{72}$$

We recall two well known formulas.

Let $\mu \in \mathcal{C}_{\mathfrak{z}}$. Then (see (74) and (22) :

$$\operatorname{vol}(Z\mu) = \varpi(\mu). \tag{73}$$

Let $\tilde{Z}$ be the simply connected covering group of Z. Denote by $\tilde{P}$ the differential of characters of the subgroup $\tilde{T}$ of $\tilde{Z}$ with Lie algebra $\mathfrak{t}$. Then $\tilde{P}$ is the set of $\mu \in \boldsymbol{i}\mathfrak{t}^*$ which take integer values on the h_α for $\alpha \in \Phi_{\mathfrak{z}}$. It contains $\rho_{\Psi_{\mathfrak{z}}}$, so that $\tilde{P}_{\mathfrak{z}} = \tilde{P}$. Let $\mu \in \tilde{P}$ be $\mathfrak{z}$-regular. We defined the (finite dimensional) irreducible representation $\pi^{\tilde{Z}}(\mu)$ of $\tilde{Z}$. For $\mu \in P^{\tilde{Z}} \cap \mathcal{C}_{\mathfrak{z}}$, Weyl's dimension formula reeds :

$$\dim \pi^{\tilde{Z}}(\mu) = \varpi(\mu). \tag{74}$$

Let $\mu \in P_k \cap \mathcal{C}_{\mathfrak{k}}$. Following [?], we give formulas for the restriction $\pi^K(\mu)|_U$ which are half way between Kostant's formula (64) for $U = T$, and Weyl's dimension formula (73) for $U = \{1\}$. We give such a formula for each choice of a positive $\mathfrak{u}$-admissible system of $\Phi_{\mathfrak{k}}$. Let $\Psi'_{\mathfrak{k}}$ be such a system.

We denote by $W_Z\backslash W_K$ the subset of $w \in W_K$ such that $w\mathcal{C}_{\mathfrak{k}} \subset \mathcal{C}_{\mathfrak{z}}$. Every element of W_K can be written in a unique manner as vw with $v \in W_Z$ and $w \in W_Z\backslash W_K$. So $W_Z\backslash W_K$ is a set of representatives (depending on $\Psi_{\mathfrak{k}}$) for the quotient also denoted by $W_Z\backslash W_K$.

Lemma 13. *Let $\mu \in P_k \cap \mathcal{C}_{\mathfrak{k}}$. We have*

$$m_U(\pi^K(\mu)|_U) = (-1)^{\#\Psi_{\mathfrak{k}\backslash\mathfrak{z}}\cup\Psi'_{\mathfrak{k}\backslash\mathfrak{z}}} \left(\sum_{w\in W_Z\backslash W_K} \epsilon(w)\varpi(w\mu)\delta_{q(w\mu)} \right) * y_{q(\Psi'_{\mathfrak{k}/\mathfrak{z}})}. \tag{75}$$

Proof. Since T and U are commutative, the measure $m_U(\pi^K(\mu)|_U)$ is the projection on $\boldsymbol{i}\mathfrak{u}^*$ of the measure $m_T(\pi^K(\mu)|_T)$, that is $m_U(\pi^K(\mu)|_U) = q(m_T(\pi^K(\mu)|_T)$. However, we cannot directly use formula (64) because q is not proper on the support of $y_{\Psi'_{\mathfrak{k}}}$ if $\Phi_{\mathfrak{z}}$ is not empty.

We rewrite (64) as

$$m_T(\pi^K(\mu)|_T) = (-1)^{\#\Psi_{\mathfrak{k}}\cup\Psi'_{\mathfrak{k}}} \left(\sum_{w\in W_K} \epsilon(w)\delta_{w\mu} \right) * y_{\Psi'_{\mathfrak{z}}} * y_{\Psi'_{\mathfrak{k}/\mathfrak{z}}}. \tag{76}$$

Since q is proper on the support of $\Psi'_{\mathfrak{k}/\mathfrak{z}}$, we can consider the measure $q(y_{\Psi'_{\mathfrak{k}/\mathfrak{z}}})$. Is is equal to $y_{q(\Psi'_{\mathfrak{k}/\mathfrak{z}})}$. So it remains to prove :

$$q\left((\sum_{w\in W_K} \epsilon(w)\delta_{w\mu}) * y_{\Psi'_{\mathfrak{z}}} \right) = (-1)^{\#\Psi_{\mathfrak{z}}\cup\Psi'_{\mathfrak{z}}} \sum_{w\in W_Z\backslash W_K} \epsilon(w)\varpi(w\mu)\delta_{q(w\mu)}. \tag{77}$$

For a fix $w \in W_Z\backslash W_K$, we consider the measure

$$(\sum_{v\in W_Z} \epsilon(vw)\delta_{vw\mu}) * y_{\Psi'_{\mathfrak{z}}}. \tag{78}$$

Remark that, thanks to lemma 12, $w\mu$ is in $\tilde{P}\cap\mathcal{C}_{\mathfrak{z}}$ and that $W_Z = W_{\tilde{Z}}$. By formula (64) applied to $\tilde{Z}$, this is (up to the sign $(-1)^{\#\Psi_{\mathfrak{z}}\cup\Psi'_{\mathfrak{z}}}\epsilon(w)$) the measure $m_T(\pi^{\tilde{Z}}(w\mu)|_T)$. Since $\mathfrak{u}$ is central in $\mathfrak{z}$, the restriction of $\pi^{\tilde{Z}}(w\mu)$ to the subgroup with Lie algebra $\mathfrak{u}$ is a multiple of the character with differential $w\mu - \rho_{\Psi_{\mathfrak{z}}}$. The multiplicity is $\dim \pi^{\tilde{Z}}(w\mu) = \varpi(w\mu)$. Since $\rho_{\Psi_{\mathfrak{z}}}$ vanishes on $\mathfrak{u}$, we obtain :

$$q\left((\sum_{v\in W_Z} \epsilon(vw)\delta_{vw\mu}) * y_{\Psi'_{\mathfrak{z}}} \right) = (-1)^{\#\Psi_{\mathfrak{z}}\cup\Psi'_{\mathfrak{z}}}\epsilon(w)\varpi(w\mu)\delta_{q(w\mu)}. \tag{79}$$

Summing this over $W_Z\backslash W_K$ completes the proof of the lemma. □

Similarly, using the fact that the volume of $Z\mu$ is given also by Weyl's polynomial $\varpi(\mathfrak{u})$ for $\mu \in \mathcal{C}_{\mathfrak{z}}$, one proves (see [?]) the following lemma:

Lemma 14. *Let $\mu \in \mathcal{C}_{\mathfrak{k}}$. We have*

$$M_U(p_{\mathfrak{u}}(\beta_{K\mu})) = (-1)^{\#\Psi_{\mathfrak{k}\backslash\mathfrak{z}}\cup\Psi'_{\mathfrak{k}\backslash\mathfrak{z}}} \left(\sum_{w\in W_Z\backslash W_K} \epsilon(w)\varpi(w\mu)\delta_{q(w\mu)} \right) * Y_{q(\Psi'_{\mathfrak{k}/\mathfrak{z}})}. \tag{80}$$

Comparing lemmas 13 and 14, we obtain :

Lemma 15. *Let $\mu \in P_{\mathfrak{k}} \cap \mathcal{C}_{\mathfrak{k}}$. We have*

$$M_U(p_{\mathfrak{u}}(\beta_{K\mu})) = r_{q(\Phi_{\mathfrak{k}/\mathfrak{z}})} * m_U(\pi^K(\mu)|_U). \tag{81}$$

6. Restriction from K to L.

Let L be a connected closed subgroup of K. We choose a Cartan subgroup U of L. We assume that T contains U. We use the notations (Z, etc...) introduced in section 4. In particular, we fixed a positive $\mathfrak{u}$-admissible system $\Psi_{\mathfrak{k}} \subset \Phi_{\mathfrak{k}}$.

Let $\Psi'_{\mathfrak{k}} \subset \Phi_{\mathfrak{k}}$ be another positive $\mathfrak{u}$-admissible system. Then $q(\Psi'_{\mathfrak{k}/\mathfrak{z}})$ contains a positive system of roots, denoted by $\Psi'_{\mathfrak{l}}$, of $\Phi_{\mathfrak{l}}$. We denote by $\Psi'_{\mathfrak{k}/\mathfrak{l}}(\mathfrak{u})$ the multiset $q(\Psi'_{\mathfrak{k}/\mathfrak{l}})\backslash\Psi'_{\mathfrak{l}}$: it is a positive set of roots for the non zero roots of $\mathfrak{u}$ in $\mathfrak{k}/\mathfrak{l}$, counted with multiplicities.

We recall Heckman's formula for the measure $m_L(\pi^K(\mu))$.

Lemma 16. *Let $\mu \in P_{\mathfrak{k}} \cap \mathcal{C}_{\mathfrak{k}}$. We have*

$$m_L(\pi^K(\mu)|_L) = (-1)^{\#\Psi_{\mathfrak{k}\backslash\mathfrak{l}}(\mathfrak{u})\cup\Psi'_{\mathfrak{k}\backslash\mathfrak{l}}(\mathfrak{u})} \left(\sum_{w\in W_Z\backslash W_K} \epsilon(w)\ \varpi(q(w\mu))\ \delta_{q(w\mu)} \right) * y_{\Psi'_{\mathfrak{k}/\mathfrak{l}}(\mathfrak{u})}. \tag{82}$$

Proof. We apply the operator of convolution by $d_{\Psi'_{\mathfrak{l}}}$ to equation (74). The left hand side gives, by formula (68) applied to the pair (L, U)

$$d_{\Psi'_{\mathfrak{l}}} * m_U(\pi^K(\mu)|_U) = (-1)^{\#\Psi_{\mathfrak{l}} \cup \Psi'_{\mathfrak{l}}} m_L(\pi^K(\mu)|_L).$$

The right hand side gives

$$(-1)^{\#\Psi_{\mathfrak{k}\backslash\mathfrak{z}} \cup \Psi'_{\mathfrak{k}\backslash\mathfrak{z}}} \left(\sum_{w \in W_Z \backslash W_K} \epsilon(w)\ \varpi(q(w\mu))\ \delta_{q(w\mu)} \right) * y_{\Psi'_{\mathfrak{k}/\mathfrak{l}}(\mathfrak{u})}.$$

□

Similarly :

Lemma 17. *Let $\mu \in P_{\mathfrak{k}} \cap \mathcal{C}_{\mathfrak{k}}$. We have*

$$M_L(p_{\mathfrak{l}}(\beta_{K\mu})) = (-1)^{\#\Psi_{\mathfrak{k}\backslash\mathfrak{l}}(\mathfrak{u}) \cup \Psi'_{\mathfrak{k}\backslash\mathfrak{l}}(\mathfrak{u})} \left(\sum_{w \in W_Z \backslash W_K} \epsilon(w)\ \varpi(q(w\mu))\ \delta_{q(w\mu)} \right) * Y_{\Psi'_{\mathfrak{k}/\mathfrak{l}}(\mathfrak{u})}. \tag{83}$$

$$M_L(p_{\mathfrak{l}}(\beta_{K\mu})) = r_{\Phi_{\mathfrak{k}/\mathfrak{l}}(\mathfrak{u})} * m_L(\pi^K(\mu)|_L). \tag{84}$$

7. Restriction from G to L.

Let $\lambda \in P_{\mathfrak{g}} \cap \mathcal{C}^{\mathfrak{g}}_{\mathfrak{k}}$ be an Harish-Chandra's parameter. We show that the admissibility of $\pi^G(\lambda)|_L$ is equivalent to the fact that $p_{\mathfrak{l}}$ is proper on $G\lambda$. In this case, we prove the formula. WE DEFINE $r_{\mathfrak{g}/\mathfrak{l}}$, and prove again :

$$r_{\mathfrak{g}/\mathfrak{l}} * m_L(\pi^G(\lambda)|_L) = M_L(p_{\mathfrak{l}}(\beta_{G\lambda})). \tag{85}$$

8. Restriction from G to H (equal rank).

9. Restriction from G to H (general case).

This is where we need something on non compact roots ....

10. vargas

11. Conditions for admissibility

11.1. **Admissibility is equivalent to properness.** Let G be a connected reductive Lie group with compact center. We fix a maximal compact subgroup K for G. Let L be a connected subgroup of K, as usual, we denote by $\pi(\lambda)$ the discrete series attached to the coadjoint orbit Ω.

Proposition 3. *$\pi(\lambda)$ has admissible restriction to L if and only if the projection $p_{\mathfrak{l}}$: $\Omega \to \imath\mathfrak{l}^*$ is a proper map.*

Proof: We first show that properness implies admissibility. We are indebted to E. Paradan for this proof. Let τ_μ be an irreducible representation of L having infinitesimal character μ so that τ_μ is contained in the restriction of $\pi(\lambda)$ to L. Thus, there exists an irreducible representation ϕ_ν of K, whose infinitesimal character is ν, so that τ_μ is equivalent to a subrepresentation of ϕ_ν restricted to L. In [**?**], Proposition 5.2 Paradan shows that

$$\nu \in p_{\mathfrak{k}}(\Omega).$$

In [?] Paradan proves that

$$\mu \in p_{\mathfrak{l}}(K \cdot \nu).$$

Thus, $\mu \in p_{\mathfrak{l}}(\Omega)$. Next, we denote by $\phi_{\nu_1}, \cdots, \phi_{\nu_R}$ inequivalent $K-$irreducible submodules of $\pi(\lambda)$ so that a copy of τ_μ is contained in ϕ_{ν_j} for $j = 1, \cdots, R$. We are to show that R is a finite number. For every j, we choose $y_j \in p_{\mathfrak{k},\mathfrak{l}}^{-1}(\mu) \cap K \cdot \nu_j$. Since $p_{\mathfrak{l}}$ restricted to Ω is a proper map, $p_{\mathfrak{k},\mathfrak{l}} : p_{\mathfrak{k}}(\Omega) \to \mathfrak{l}^*$ is a proper map. Thus, $p_{\mathfrak{k},\mathfrak{l}}^{-1}(\mu) \cap p_{\mathfrak{k}}(\Omega)$ is a compact set. Since $K \cdot \nu_r \cap K \cdot \nu_s = \emptyset$ for $r \neq s$, $\{y_1, \cdots, y_R\}$ is a discrete set. Therefore R is a finite number and the multiplicity of τ_μ in the restriction of $\pi(\lambda)$ to L is finite.

For the converse statement we notice that $L-$admissibility implies that the subspace of $L-$finite vector is equal to subspace of $K-$finite vectors, [?], also, for each $L-$type of $\pi(\lambda)$ we have that there exists finitely many $K-$types whose restriction to L contains the given $L-$type, hence we have a well defined proper map from the set of $K-$types of $\pi(\lambda)$ into the set of $L-$types in such a way that fiber over a point is finite and contained in the $K-$types whose restriction to L contains the point. Because of prop ...this map gives rise to a proper map from $AS_K(\pi(\lambda))$ into $\mathfrak{l}^*$. In ??? we showed that there exists a compact convex set D_λ so that $p_{\mathfrak{k}}(G\lambda) = Ad(K)(D_\lambda + AS_K(\pi(\lambda)))$. Thus, the restriction map from $p_{\mathfrak{k}}(G\lambda)$ into $\mathfrak{l}^*$ is proper and hence $p_{\mathfrak{l}}$ restricted to $G\lambda$ is proper. □

Another proof of the converse statement

Since $\pi(\lambda)$ has an admissible restriction to L, it follows that a $L-$type is contained in finitely many $K-$types. Otherwise, the restriction to L would not be admissible. Let $Aso(\pi) \subseteq \mathfrak{p}_{\mathbb{C}}$ be the associated variety of π. In [?] we find a proof that $Aso(\pi)$ is an irreducible variety and connected in the Euclidean topology. Huang-Vogan has shown [?]

$$\mathbb{C}[Aso(\pi)] \subseteq \pi(\lambda)_{|K} \otimes W_0$$

here, W_0 is a finite dimensional $K-$module. Hence, $\mathbb{C}[Aso(\pi)]$ is an admissible $L-$module. It follows from Vergne [?] the moment map $\mu_{\mathfrak{l}} : Aso(\pi) \to \mathfrak{l}^*$ is a proper map. Next, Vergne [?] has shown there exists a $K-$ equivariant diffeomorphism between the real associated variety $Aso_{\mathbb{R}}(\pi)$ and $Aso(\pi)$ and the diagram

space

commutes. Hence, $p_{\mathfrak{l}} : Aso_{\mathbb{R}}(\pi) \to \mathfrak{l}$ is a proper map. Barbash-Vogan [?] has shown $Aso_{\mathbb{R}}(\pi(\lambda)) = C(G \cdot \lambda)$. Proposition 2 yields $p_{\mathfrak{l}} : G \cdot \lambda \to \mathfrak{l}$ is a proper map. □

Corollary 3. *$p_{\mathfrak{h}} : \Omega \to \mathfrak{h}^*$ is a proper map if and only if $\pi(\lambda)$ has an admissible restriction to H.*

The corollary follows from Corollary ... and Proposition ....

11.2. **Sufficient conditions for admissibility and properness.** In this subsection we point out quite general pairs (G, L) and positive root system Ψ in $\mathfrak{t}^*$ in order to obtain sufficient conditions for properness of the moment map $G\lambda \to \mathfrak{l}^*$. Let Ψ a system of positive root in $\Phi(\mathfrak{g}, \mathfrak{t})$ such that $\Delta = \Psi_{\mathfrak{k}}$. Let $\mathfrak{p}_\Psi$ be the linear space spanned by the root vectors corresponding to the roots in Ψ_n, and let

$$\mathfrak{k}_1 := \mathfrak{k}_1(\Psi) :=< [\mathfrak{p}_\Psi, \mathfrak{p}_\Psi] >$$

denote the ideal in $\mathfrak{k}_{\mathbb{C}}$ spanned by $[\mathfrak{p}_\Psi, \mathfrak{p}_\Psi]$. It readily follows that $\mathfrak{k}_1$ is the complexification of an ideal $\mathfrak{k}_1$ of $\mathfrak{k}$. Let $\mathfrak{k}_2 := \mathfrak{k}_2(\Psi)$ be the orthogonal complement of $\mathfrak{k}_1$ in $\mathfrak{k}$ with respect to the Killing form. Hence, $\mathfrak{k}_2$ is an ideal in $\mathfrak{k}$ so that $\mathfrak{k} = \mathfrak{k}_1 \oplus \mathfrak{k}_2$ and $\mathfrak{t} = \mathfrak{t}_1 \oplus \mathfrak{t}_2$, with $\mathfrak{t}_j \subset \mathfrak{k}_j$, $j = 1, 2$. Note that either $\mathfrak{t}_2$ or $\mathfrak{k}_2$ depends on Ψ.

Lemma 18. *Every simple root for $\Psi \cap \Phi(\mathfrak{k}_2, \mathfrak{t})$ is simple for Ψ.*

Proof: Let α be a simple root for $\Psi \cap \Phi(\mathfrak{k}_2, \mathfrak{t})$. If α were not simple for Ψ, then $\alpha = \beta_1 + \beta_2$, $\beta_1, \beta_2 \in \Psi$. It follows that β_1, β_2 are noncompact roots. Hence, the root vector of α lies in $[\mathfrak{p}_\Psi, \mathfrak{p}_\Psi] \subseteq \mathfrak{k}_1$, a contradiction. □

Now, it readily follows that $\mathfrak{k}_2(\Psi)$ is the largest ideal of $\mathfrak{k}$ contained in the Lie subalgebra spanned by the roots system generated by the compact simple roots in Ψ. This subalgebra has been defined by [?]. More precisely, [?] define a parabolic subalgebra $\mathfrak{l}+\mathfrak{u}$ as follows: $\mathfrak{l}$ is the subalgebra of $\mathfrak{k}$ spanned by $\mathfrak{t}$ and the root vectors corresponding to compact roots which are linear combination of compact simple roots in Ψ, $\mathfrak{u}$ is generated by root vectors in Ψ outside of $\mathfrak{l}$. Thus, $\mathfrak{u} = \mathfrak{p}_\Psi$.

Lemma 19.

$$\mathfrak{k}_1(\Psi) = [\mathfrak{u} \cap \mathfrak{p}, \mathfrak{u} \cap \mathfrak{p}] + [\bar{\mathfrak{u}} \cap \mathfrak{p}, \bar{\mathfrak{u}} \cap \mathfrak{p}] + [[\mathfrak{u} \cap \mathfrak{p}, \mathfrak{u} \cap \mathfrak{p}], [\bar{\mathfrak{u}} \cap \mathfrak{p}, \bar{\mathfrak{u}} \cap \mathfrak{p}]]. \tag{86}$$

Proof: We first verify that

$$\mathfrak{u} \cap \mathfrak{k} = [\mathfrak{u} \cap \mathfrak{p}, \mathfrak{u} \cap \mathfrak{p}] \tag{87}$$

We show $\mathfrak{u} \cap \mathfrak{k} \subset [\mathfrak{u} \cap \mathfrak{p}, \mathfrak{u} \cap \mathfrak{p}]$ by induction on the length of the root γ such that its root space $\mathfrak{g}_\gamma$ is contained in $\mathfrak{u} \cap \mathfrak{k}$. Let ℓ denote the length function for the root system Δ. If $\ell(\gamma) = 1$, then $\gamma = \beta_1 + \beta_2$ with roots $\beta_j \in \mathfrak{u} \cap \mathfrak{p}$, otherwise, $\gamma \in \Phi_\mathfrak{l} \cap \Psi_\mathfrak{u}$. When $\ell(\gamma) > 1$ there exists a simple root $\alpha \in \Delta$ so that $\gamma - \alpha \in \Delta$. Since $\gamma \in \Phi_{\mathfrak{u}\cap\mathfrak{k}}$ then either $\gamma - \alpha \in \Phi_{\mathfrak{u}\cap\mathfrak{k}}$ or $\alpha \in \Phi_{\mathfrak{u}\cap\mathfrak{k}}$, hence $\gamma = \beta_1 + \beta_2 + \alpha$ or $\gamma = \beta_1 + \beta_2 + \gamma - \alpha$ with $\beta_j \in \Phi_{\mathfrak{u}\cap\mathfrak{p}}$. Jacobi identity implies that either $\beta_1 + \alpha$ or $\beta_2 + \alpha$ is a root. Thus, γ is the sum of two noncompact roots.

The equality (87) yields the inclusions $\mathfrak{u} \cap \mathfrak{k} \subset \mathfrak{k}_1(\Psi), \bar{\mathfrak{u}} \cap \mathfrak{k} \subset \mathfrak{k}_1(\Psi)$. Hence the right hand side of the equality we are to show is contained in $\mathfrak{k}_1(\Psi)$. Next, we show the right hand side is an ideal in $\mathfrak{k}$. Indeed, if $X_\gamma \in \mathfrak{l}$, since $\mathfrak{l} + \mathfrak{u}$ is a parabolic subalgebra and $\mathfrak{l}$ is a subalgebra of $\mathfrak{k}$ we obtain X_γ normalizes the right hand side. When X_γ lies in $\mathfrak{u} \cap \mathfrak{k}$, for Y_j root vectors in $\mathfrak{u} \cap \mathfrak{k}$, it readily follows $[X_\gamma, [Y_1, \bar{Y_2}]]$ belongs to the right hand side. The definition of $\mathfrak{k}_1$ concludes the proof of the lemma. □

Corollary 4. $[\mathfrak{p}_\Psi, \mathfrak{p}_\Psi]$ *contains root spaces corresponding to roots of both lengths in each simple factor of* $\mathfrak{k}_1(\Psi)$.

The decomposition (86) yields the subalgebra $[\mathfrak{p}_\Psi, \mathfrak{p}_\Psi] + [[\mathfrak{p}_\Psi, \mathfrak{p}_\Psi], [\bar{\mathfrak{p}_\Psi}, \bar{\mathfrak{p}_\Psi}]]$ is a parabolic subalgebra of $\mathfrak{k}_1(\Psi)$. For a simple Lie algebra either root vectors for the maximal root or for the short largest root belong to the nilpotent radical of any parabolic subalgebra and the corollary follows.

It follows from Lemma .... and its corollary the following

Corollary 5. *a) Any root in $\mathfrak{k}_1(\Psi)$ is up to sign either the sum of two noncompact roots or the difference of two roots which are the sum of two noncompact roots.*

b) Any root in $\mathfrak{k}_1(\Psi)$ is conjugated to a root in $\Psi_{\mathfrak{u}\cap\mathfrak{k}}$.

Lemma 20. *Let V be a finite dimensional inner product space. Assume $V = V_1 \oplus V_2$ is a nontrivial orthogonal decomposition of V. Let C be a strict cone in V so that $C \cap V_2 = 0$. Then the image of C by the orthogonal projection of V onto V_1 is a strict cone.*

Proof: Let us recall that a cone C is strict if there exists a nonzero vector v in V so that $(v, C-\{0\})>0$. Equivalently, C is a strict cone iff C does not contain a line ([?], Lemma A.5). Let $p: V \to V_1$ be the orthogonal projection. Hence, $p(C)$ is a cone and if $p(z)=0$ with $z \in C$ then the hypothesis implies $z=0$. Thus, if $p(C)$ contained a line there would be $w \neq 0, w$ and $-w$ in $p(C)$. Hence, $w=p(x), x\in C, -w=p(y), y\in C$. Thus, $0=p(x+y)$, with $x+y\in C\cap V_2$. This contradicts that C is strict unless $x=y=0$. □

It obviously follows that if $p(C)$ is a strict cone and $C\cap V_2=0$, then C is a strict cone.

Remark 10. Let W_0 be the subgroup of W spanned by the reflections about some compact simple roots $\{\alpha_1,\dots,\alpha_r\}$ in Ψ. Let $C\subset i\mathfrak{t}^*$ be a W_0-invariant strict cone. Then $C\cap\sum_j \mathbb{R}\alpha_j=\{0\}$. In fact, if $X\in C\cap\sum_j\mathbb{R}\alpha_j$ the convex hull of $\{wX, w\in W_0\}$ is contained in $C\cap\sum\mathbb{R}\alpha_j$. Hence, unless $X=0$, there is a non zero vector and its opposite in C.

As before, for a reductive Lie algebra $\mathfrak{g}$, let $\mathfrak{g}_{ss}$ denote its semisimple factor and $\mathfrak{z}_\mathfrak{g}$ its center. Hence, a Cartan subalgebra $\mathfrak{b}$ of $\mathfrak{g}$ splits as $\mathfrak{b}=\mathfrak{z}_\mathfrak{g}\oplus\mathfrak{b}_{ss}$ with $\mathfrak{b}_{ss}$ contained in $\mathfrak{g}_{ss}$.

Corollary 6. $\mathbb{R}^+\Psi_n\cap i(\mathfrak{t}_2^*)_{ss}=0$.

Corollary 7. $\mathbb{R}^+\Psi_n\cap i\mathfrak{t}_2^*=0$ *is equivalent to* $\mathbb{R}^+\Psi_n\cap i\mathfrak{z}_\mathfrak{k}^*=0$.

In fact, for $X\in\mathbb{R}^+\Psi_n\cap i\mathfrak{t}_2^*$, we write $X=X_1+X_2, X_1\in i\mathfrak{z}_\mathfrak{k}^*, X_2\in i(\mathfrak{t}_2^*)_{ss}$. We now proceed as in Remark ..... and recall that for $w\in W_{\mathfrak{k}_2}, w\Psi_n=\Psi_n$.

Corollary 8. *Assume* $\mathfrak{z}_\mathfrak{k}\neq 0$.

$$\text{If } 0\neq\mathfrak{k}_1(\Psi)\neq\mathfrak{k}_{ss}, \text{ then } \mathbb{R}^+\Psi_n\cap i\mathfrak{t}_2^*=0.$$

According to the classification of symmetric spaces the hypothesis forces $\mathfrak{g}$ is isomorphic to either $\mathfrak{su}(p,q)$ or $\mathfrak{so}(2,4)\equiv\mathfrak{su}(2,2)$. We dealt with these cases in §2.3 iv)$\mathfrak{su}(p,q)$ and Table 1.

Corollary 9. *Whenever K is a semisimple group, for any positive root system Ψ, we have* $\mathbb{R}^+\Psi_n\cap i\mathfrak{t}_2^*=\{0\}$.

Remark 11. Example 2.13 in Kobayashi, [?] implies that for K a semisimple group or when $\mathbb{R}^+\Psi_n\cap i\mathfrak{t}_2^*=0$, then a discrete series representation whose Harish-Chandra parameter is dominant with respect to Ψ, restricted to $\mathfrak{k}_1(\Psi)$ is admissible.

We now present a simple direct proof of the statement in Remark ... when we restrict to either $\mathfrak{k}_1(\Psi)$ or $\mathfrak{k}_1(\Psi)+\mathfrak{z}_\mathfrak{k}$. For this, we write

$$\mathfrak{k}=\mathfrak{l}_1\oplus\mathfrak{l}_2$$

as a sum of ideals, we denote the corresponding splitting for $\mathfrak{t}=\mathfrak{b}_1\oplus\mathfrak{b}_2$.

Proposition 4. *Let $\pi(\lambda)$ be a discrete series representation of G whose Harish-Chandra parameter Λ is dominant with respect to Ψ so that $\mathbb{R}^+\Psi_n\cap i\mathfrak{b}_2^*=0$. Then the restriction of $\pi(\lambda)$ to L_1 is an admissible representation.*

Proof: Since $\mathbb{R}^+\Psi_n$ is a strict cone in $i\mathfrak{t}^*$ and $\mathbb{R}^+\Psi_n\cap i\mathfrak{b}_2^*=0$ lemma ...., implies that there exists a nonzero vector $v\in i\mathfrak{b}_1^*$ so that $(v,\mathbb{R}^+\Psi_n-\{0\})>0$. We choose v so that its inner product with any root in Ψ_n is bigger or equal to one. Schmid in [?] express the highest weight μ of a $K-$type of $\pi(\lambda)$, computed with respect to Δ, as

$\mu = \Lambda + \rho_n - \rho_c + \beta_1 + \cdots + \beta_S$ with $\beta_1, \cdots, \beta_S$ roots in Ψ_n. The splitting $i\mathfrak{t}^* = i\mathfrak{b}_1^* + i\mathfrak{b}_2^*$ yields $\mu = \mu_1 + \mu_2$ with $\mu_1 \in i\mathfrak{b}_1^*$ and $\mu_2 \in i\mathfrak{b}_2^*$. Hence, $(\mu, v) = (\mu_1 + \mu_2, v) = (\mu_1, v) = (\Lambda + \rho_n - \rho_c, v) + (\beta_1 + \cdots + \beta_S, v) \geq (\Lambda + \rho_n - \rho_c, v) + S$. Therefore, for a fixed μ_1, S is bounded and hence μ_2 has finite many possibilities. Thus, the multiplicity of μ_1 in $\pi(\lambda)$ is finite. □

Corollary 10. *Assume the Harish-Chandra parameter for $\pi(\lambda)$ is dominant with respect to Ψ. Then, $\pi(\lambda)$ has an admissible restriction to $K_1(\Psi)Z_{\mathfrak{k}}$. Besides, if $\mathbb{R}^+\Psi_n \cap i\mathfrak{z}_{\mathfrak{k}}^\star = \{0\}$, then $\pi(\lambda)$ has an admissible restriction to $K_1(\Psi)$.*

Remark 12. When $\mathfrak{z}_{\mathfrak{k}} \neq 0$ admissible restriction to $K_1(\Psi)$ may be false. We now analyze this question.

It is known that $\pi(\lambda)$ has an admissible restriction to $Z_{\mathfrak{k}}$ if and only if Ψ is holomorphic. For a reference cf [**?**].

It follows that $\mathbb{R}^+\Psi_n \cap i\mathfrak{z}_{\mathfrak{k}}^* = 0$ is equivalent to there exist $v \in i\mathfrak{t}_1^*$ so that $(v, \beta) > 0$, for all noncompact roots in Ψ. Gross-Wallach in [**?**] have shown for Ψ is small and non holomorphic, then $\pi(\lambda)$ has an admissible restriction to $K_1(\Psi)$. They construct $v \in i\mathfrak{t}_1^*$ as above

Second, we have shown that whenever $\mathbb{R}^+\Psi_n \cap i\mathfrak{z}_{\mathfrak{k}}^* = 0$, then $\pi(\lambda)$ restricted to $K_1(\Psi)$. We the converse is not true. In Example 4 we show: a holomorphic discrete series has an admissible restriction to the semisimple factor of K if and only if G/K is not a tube domain. For a holomorphic chamber we have $\mathbb{R}^+\Psi_n \cap i\mathfrak{z}_{\mathfrak{k}} = i\mathfrak{z}_{\mathfrak{k}}$. For $G = SU(p, 2)$ in.... we obtain non holomorphic discrete series with admissible restriction to $K_1 = SU(p) \times SU(2)$ and such that $\mathbb{R}^+\Psi_n \cap i\mathfrak{z}_{\mathfrak{k}} \neq \{0\}$.

Fourth, in subsection...... for G locally isomorphic to $SO(2, 2q + 1)$ we show no discrete series admits admissible restriction to $K_1(\Psi)$. Here, $\mathbb{R}^+\Psi_n \cap i\mathfrak{z}_{\mathfrak{k}} \neq \{0\}$ For G locally isomorphic to $SO(2n, 2)$ admissible restriction to$K_1(\Psi) = SO(2n)$ is equivalent to $\mathbb{R}^+\Psi_n \cap i\mathfrak{z}_{\mathfrak{k}}^* = 0$. In this case there are two systems of positive roots $\Psi_{\pm q}$ so that the discrete series has admissible restriction to K_{ss}.

In order to show the strictness of the set $p_{\mathfrak{t}_1}(\Psi_n)$ the following is useful,

Lemma 21. *If $\mathfrak{g}$ is simple, not locally isomorphic to $\mathfrak{so}(2, 2q+1)$ and $\mathfrak{k}_1(\Psi)$ is nontrivial, then every noncompact root has a nonzero restriction to $\mathfrak{t}_1$. For $\mathfrak{so}(2, 2q+1)$ and a short noncompact root the projection to $\mathfrak{t}_1$ is zero.*

Proof: We show that if $\mathfrak{g}$ is a simple real Lie algebra which is not isomorphic to $\mathfrak{so}(2, 2q+1)$, then any root γ orthogonal to $\mathfrak{t}_1(\Psi)$ lies in $\Phi(\mathfrak{k}_2(\Psi))$. In fact, if γ is compact we have nothing to verify. If γ were noncompact, and $\mathfrak{k}$ semisimple, since γ is orthogonal to $\mathfrak{t}_1(\Psi)$ Lemma ... implies that γ is a linear combination of compact simple roots lying in $\Psi(\mathfrak{k}_2)$. Thus, γ lies in $\Phi(\mathfrak{k}_2)$. When $\mathfrak{k}$ has a nontrivial center and $\mathfrak{g}$ is isomorphic to $\mathfrak{su}(p, q)$ the implication follows by inspection on the roots. We are left the case $\mathfrak{k}$ has a nontrivial center, real rank of $\mathfrak{g}$ bigger or equal than two, and $\mathfrak{k}_{ss}$ is simple. Thus, γ lies in the center of $\mathfrak{k}$. Hence, (γ, β) is a positive number for every noncompact root β lying in a fixed holomorphic system. Since, $\mathfrak{g}$ is not isomorphic to $\mathfrak{so}(2, 2q + 1)$, there is at least one noncompact root which is orthogonal to γ. Contradiction, hence γ is compact. For $\mathfrak{so}(2, 2q + 1)$ the noncompact short root is not orthogonal to any other noncompact root. □

Example 1: For G such that $\mathfrak{k}$ is a semisimple algebra the inclusion $\mathfrak{k}_1(\Psi) \subseteq \mathfrak{l}$ is sufficient to assure for Λ $\Psi-$dominant that $\pi(\lambda)$ has an admissible restriction to $\mathfrak{l}$. For G/K is a Hermitian symmetric space, $\mathfrak{k}_1(\Psi) \subseteq \mathfrak{l}$ is not sufficient to assure that $\pi(\lambda)$ has an admissible restriction to $\mathfrak{l}$. In fact, the group $SO(2,4)$ has six families of discrete series representation [?]. Among these six families, two of them restricts to an admissible representation of $SO(4)$ and the other four do not have an admissible restriction to $SO(4)$. However, for any of these six positive root systems Ψ, the subgroup $\mathfrak{k}_1(\Psi)$ is contained in $SO(4)$. More precisely, for $\mathfrak{g} = \mathfrak{so}(2,4) \equiv \mathfrak{su}(2,2)$ in a toroidal Cartan subalgebra we have a basis $\{\epsilon, \delta_1, \delta_2\}$ so that a system of positive compact roots is $\Delta = \{\delta_1 \pm \delta_2\}$ and $\Phi_n = \{\pm\epsilon \pm \delta_j\}$. When we identify, via the Killing form, the toroidal Cartan subalgebra with its dual space, the center of $\mathfrak{k}$ is equal to $\mathbb{C}\epsilon$. For each of six systems of positive roots containing Δ, in the first column we list the positive noncompact roots, the second column gives $\mathfrak{k}_1(\Psi)$, the third column exhibits $\mathbb{R}^+\Psi_n \cap i\mathfrak{t}_2^*(\Psi)$, the fourth column indicates whether or not discrete series with dominant parameters with respect to Ψ restricts discretely to $\mathfrak{k}_1(\Psi)$, it turns out of the computation that for this case a discrete has admissible restriction to K_1 if and only if has admissible restriction to $\mathfrak{so}(4)$. A proof of the last statement can be found in [?]. The fifth colum shows when $p_{\mathrm{u}}(\Psi_n)$ is strict. The sixth column computes the number $m(\Psi)$, c.f. §2.3 iv).

Ψ_n	$\mathfrak{k}_1(\Psi)$	$\mathbb{R}^+\Psi_n \cap i\mathfrak{t}_2^*(\Psi)$	Ad R	Str	m(Ψ)
$\delta_1 \pm \epsilon, \pm\epsilon - \delta_2$	$\mathfrak{su}_2(\delta_1 - \delta_2)$	0	Y	Y	2
$\pm\epsilon + \delta_i$	$\mathfrak{su}_2(\delta_1 + \delta_2)$	0	Y	Y	2
$\pm\epsilon + \delta_1, \epsilon \pm \delta_2$	$\mathfrak{so}(4)$	$i\mathfrak{z}_{\mathfrak{k}}^*$	N	N	3
$\pm\epsilon + \delta_1, -\epsilon \pm \delta_2$	$\mathfrak{so}(4)$	$i\mathfrak{z}_{\mathfrak{k}}^*$	N	N	3
$\epsilon \pm \delta_i$	0	$\mathbb{R}^+\Psi_n$	N	N	1
$-\epsilon \pm \delta_i$	0	$\mathbb{R}^+\Psi_n$	N	N	1

Table 1

The first two root systems are small.

Example 3: Whenever K is a semisimple Lie group and L is a closed subgroup of K so that $\pi(\lambda)$ restricted to L is admissible, we would like to show that $K_1(\Psi)$ is a subgroup of L. We now show that this is not true, although, later on, we will show this fact holds if L is a maximal compact subgroup of H for (G,H) a reductive symmetric pair. The example is $G = Sp(p,1), p \geq 1$ Here, we consider as in ..... $\Psi_0 = \{\delta > \epsilon_1 > \cdots \epsilon_p\}$ Then, $K_1 = Sp(1), K_2 = Sp(p)$. In [?] they show that such a discrete series has an admissible restriction to both, $Sp(p)$ and $Sp(1)$. Berger's classification of reductive symmetric pairs cf. [?] shows that neither $Sp(n)$ nor $Sp(1)$ are maximal compact subgroups for H so that $(Sp(n,1),H)$ is a reductive symmetric pair.

Example 4: We fix (G,K) an Hermitian Symmetric pair and let K_{ss} denote the semisimple factor of K. Let Ψ denote a holomorphic chamber in Φ and λ a discrete series parameter dominant with respect to Ψ. Then,

- $\mathbb{R}^+\Psi_n \cap i\mathfrak{z}_{\mathfrak{k}}^* = i\mathfrak{z}_{\mathfrak{k}}^*$.
- $\pi(\lambda)$ restricted to K_{ss} is admissible if and only if G/K is not of tube type.

The first assertion follows from that $\rho_n(\Psi)$ is orthogonal to all the compact simple roots in Ψ and that for a holomorphic chamber the simple roots in Δ are the compact simple roots for Ψ. Before we justify the second assertion we recall a few facts. Since Ψ is a holomorphic system of positive roots, that is, the sum of two noncompact roots in Ψ never is a root, there exists a set of strongly orthogonal noncompact roots, $\gamma_1, \dots, \gamma_r$ so that

i) $\sum_{j=1}^{r}\gamma_j$ is in $i\mathfrak{z}_{\mathfrak{k}}^*$ if and only if $\sum_j \mathbb{R}^+\gamma_j \cap i\mathfrak{z}_{\mathfrak{k}}^* \neq \{0\}$ if and only if G/K is a domain of tube type
$\sum_j \mathbb{R}^+\gamma_j \cap i\mathfrak{z}_{\mathfrak{k}}^* = 0$ is equivalent to the second statement. In fact, $\sum_j c_j\gamma_j \in i\mathfrak{z}_{\mathfrak{k}}^*$, lead us to $(\sum_j c_j\gamma_j, \gamma_1) = \cdots = (\sum_j c_j\gamma_j, \gamma_r)$, the orthogonality of the γ_j yields $c_1 = \cdots = c_r$, hence $c_1 = ... = c_r = 0$. For a proof of the second equivalence [?]. Let $\mathfrak{p}_- := \mathfrak{p}_{\Psi_n}$ and consider the symmetric algebra $S(\mathfrak{p}_-)$ of $\mathfrak{p}_-$. Then in [?] we find a proof of

ii) The Δ highest weight of an irreducible $K-$submodule is equal to $-n_1\gamma_1 - \cdots - n_r\gamma_r$, with $n_1 \geq \cdots \geq n_r \geq 0$, and it occurs in degree $\sum_j n_j$.

We claim: G/K is of tube type if and only if some irreducible representation of K_{ss} has infinite multiplicity in $S(\mathfrak{p}_-)$.
Indeed, in [?] it is shown $S(\mathfrak{p}_-)^{K_{ss}} = \mathbb{C}$ if and only if the multiplicity of each $K_{ss}-$type is finite. Since the trivial representation has finite multiplicity in $S(\mathfrak{p}_-)$ if and only if $\sum_j \gamma_j$ lies in the center of $\mathfrak{k}$ the claim follows.
Let V_μ be the lowest $K-$type of $\pi(\lambda)$. Thus, the space of $K-$finite vectors of $\pi(\lambda)$ is equivalent to $S(\mathfrak{p}_-) \otimes V_\mu$. Assume that π_Λ restricted to K_{ss} is admissible, if G/K were a tube domain, we would have that $\sum_j \gamma_j \in i\mathfrak{z}_{\mathfrak{k}}^*$ and hence the $K_{ss}-$irreducible representation of highest weight $\mu - n\sum_j \gamma_j$ would had infinite multiplicity, a contradiction. For the converse of the first statement if some $K_{ss}-$submodule had infinite multiplicity in $\pi(\lambda)$ then some K_{ss} highest weight of $S(\mathfrak{p}_-)$ would had infinite multiplicity and hence G/K would be of tube type. A contradiction.

Another way of showing the first statement is by mean of a Theorem of Kobayashi [?], [?] . Because, the asymptotic cone of $\pi(\lambda)$ is $AS_K(\pi(\lambda)) := -\sum_{j=1}^{r}\mathbb{R}^+\gamma_j$ and the cone for $G' := K_{ss}$ is $cone(G') = i\mathfrak{t}^* \cap (Ad(K)\mathfrak{k}_{ss}^{\perp} = i\mathfrak{z}_{\mathfrak{k}}^*$. The Theorem of Kobayashi asserts that $\pi(\lambda)$ restricted to K_{ss} is admissible if and only if $As_K(\pi(\lambda)) \cap cone(G') = 0$, hence, we conclude $\pi(\lambda)$ restricted to K_{ss} is admissible if and only if $\sum_j \mathbb{R}^+\gamma_j \cap i\mathfrak{z}_{\mathfrak{k}}^* = 0$.

We would like to notice that the unique Hermitian symmetric pair (G, K) so that there exists a reductive symmetric pair (G, H) in such a way that $K_{ss} = H \cap K = L$ is $G = SO(n, 2)$ and $H = SO(n, 1)$. In this very particular case we may apply [?] and [?] in order to obtain both equivalences.

We also notice that this example shows that the sufficient condition for admissibility, $\mathbb{R}^+\Psi_n \cap i\mathfrak{t}_-^* = 0$, may not be necessary when (G, H) is not a reductive symmetric pair. Compare with [?].

Example ... For $\mathfrak{g}$ isomorphic to $\mathfrak{su}(p, q)$, $\mathfrak{k}_1(\Psi)$ a proper subalgebra of $\mathfrak{k}_{ss}$ and Λ dominant with respect to Ψ. Then, $\pi(\lambda)$ has an admissible restriction to $\mathfrak{k}_1(\Psi)$. Indeed, we construct a non zero vector v in $i\mathfrak{t}_1^*$ so that v has a positive inner product with every non compact roots in Ψ. For unexplained notation c.f. subsection 2.3 iv) paragraph $\mathfrak{su}(p, q)$. For Ψ_a, set $v_0 = p(\sum_{j=1}^{a}\epsilon_j) + a(\sum_{k=1}^{q}\delta_k)$. For Ψ_b', set $v_0 = b(\sum_{j=1}^{p}\epsilon_j) + q(\sum_{k=1}^{b}\delta_k)$ and let $v = p_{\mathfrak{t}_1}(v_0)$. We now apply Corollary 6. It is straightforward to verify that all this systems satisfy condition (C). This example, also follows from [?]. For the group $SU(p, 1), p \geq 1$ any discrete series representation has an admissible restriction to $SU(p)$.

11.3. **Condition C versus K_1.** We now show a relation between condition (C) and the subalgebra $\mathfrak{k}_1(\Psi)$. Henceforth, for this subsection G is simple. Let L denote a compact connected subgroup of K. We choose a maximal torus U for L contained in T. As before, $\Phi_{\mathfrak{z}}$ denotes the roots in $\Phi_{\mathfrak{k}}$ that vanishes on $\mathfrak{u}$. We fix a system of positive roots Ψ in $\Phi_{\mathfrak{g}}$ containing Δ.

Definition 1. *We say that condition (C) holds for Ψ and L if there exists a system of positive roots Δ_1 in $\Phi_{\mathfrak{k}}$ so that $q_{\mathfrak{u}}(\Delta_1 - \Phi_{\mathfrak{z}})$ is strict, and for each w in the compact Weyl group of G the multiset*

$$q_{\mathfrak{u}}(w\Psi_n) \cup q_{\mathfrak{u}}(\Delta_1 - \Phi_{\mathfrak{z}}) - \Phi_{\mathfrak{l}}$$

is strict.

Proposition 5. *Assume $\mathfrak{k}_1(\Psi) + \mathfrak{z}_{\mathfrak{k}}$ is contained in $\mathfrak{l}$, then condition (C) holds for Ψ and L and for any system Δ_1 so that $q_{\mathfrak{u}}(\Delta_1 - \Phi_{\mathfrak{z}})$ is strict.*
For the converse we assume Condition (C) holds. Then, $\mathfrak{k}_1(\Psi)$ is contained in $\mathfrak{l}$.

Proof: First statement. Since $\mathfrak{t}_1 + \mathfrak{z} \subseteq \mathfrak{u}$ we have the equality $\Delta_1 - \Phi_{\mathfrak{z}} = (\Delta_1 \cap \Phi_{\mathfrak{k}_1}) \cup (\Delta_1 \cap \Phi_{\mathfrak{k}_2} - \Phi_{\mathfrak{z}})$. The hypothesis $p_{\mathfrak{u}}(\Delta_1 - \Phi_{\mathfrak{z}})$ is strict implies there exists v_1 in $i\mathfrak{u}$ so that $\alpha(v_1) > 0$ for every $\alpha \in \Delta_1 - \Phi_{\mathfrak{z}}$. We fix an upper bound C for the numbers $|\gamma(v_1)|$, $\gamma \in \Phi_{\mathfrak{g}}$. For $w \in W_{\mathfrak{k}}$, $\mathbb{R}^+ w\Psi_n$ is a strick cone. It follows from Remark .... $\mathbb{R}^+ w\Psi_n \cap i(\mathfrak{t}_2)^\star_{ss} = \{0\}$. Therefore, Lemma 3 let us obtain that $q_{\mathfrak{t}_1+\mathfrak{z}_{\mathfrak{k}}}(\mathbb{R}^+ w\Psi_n)$ is a strict cone in $i\mathfrak{t}_1^* + i\mathfrak{z}_{\mathfrak{k}}^*$. Hence, we may choose $v_0 \in i\mathfrak{t}_1 + i\mathfrak{z}$ so that $\gamma(v_0) \geq 2C$ for every root in $w\Psi_n$. We define $v = v_0 + v_1$. Thus, for either $\gamma \in q_{\mathfrak{u}}(\Delta_1 - \Phi_{\mathfrak{z}}) - \Phi_{\mathfrak{l}}$ or $\gamma \in w\Psi_n$ we have that $\gamma(v)$ is a positive number. Since $\mathfrak{k}_1 + \mathfrak{z}_{\mathfrak{k}}$ is contained in $\mathfrak{u}$, the multiset $q_{\mathfrak{u}}(w\Psi_n) \cup q_{\mathfrak{u}}(\Delta_1 - \Phi_{\mathfrak{z}}) - \Phi_{\mathfrak{l}}$ is strict. This concludes the proof of the first statement.

Next, we show the second statement. We first verify that $q_{\mathfrak{u}}(\gamma)$ is nonzero for every root in $\Psi_n \cup \Delta_1 \cap \Phi_{\mathfrak{k}_1}$. From the hypothesis we have $q_{\mathfrak{u}}(\beta) \neq O$ for $\beta \in \Psi_n$. Choose $\beta \in \Psi_n$, we show that $q_{\mathfrak{u}}(\beta)$ is not equal to $q_{\mathfrak{u}}(\alpha)$ for every $\alpha \in \Delta_{\mathfrak{k}_1}$. Indeed, assume $q_{\mathfrak{u}}(\beta) = q_{\mathfrak{u}}(\alpha)$, since $K_1(\Psi) \neq 0$ the chamber Ψ is non holomorphic, thus, there exists $w \in W_{\mathfrak{k}}$ so that $w\beta = -\beta$. Hence $\pm q_{\mathfrak{u}}(\alpha)$ belong to the strict multiset associated to w, a contradiction. For $\alpha \in \Delta_1 \cap \Phi_{\mathfrak{k}_1}, q_{\mathfrak{u}}(\alpha)$ is nonzero. Indeed, corollary ....implies that there exists non compact roots β_1, β_2 in Ψ and w in the Weyl group of $K_1(\Psi)$ so that $\alpha = w(\beta_1 + \beta_2)$. Thus, $q_{\mathfrak{u}}(\alpha)$ lies in $q_{\mathfrak{u}}(w\mathbb{R}^+\Psi_n)$, the hypothesis that the multiset associated to w is strict implies that $q_{\mathfrak{u}}(\alpha)$ is non zero. We now verify: each $\alpha \in \Delta_1 \cap \Phi_{\mathfrak{k}_1}$, $q_{\mathfrak{u}}(\alpha)$ is root for the pair $(\mathfrak{l}, \mathfrak{u})$. We already have that $q_{\mathfrak{u}}(\alpha)$ belongs to either $\Phi(\mathfrak{l}, \mathfrak{u})$ or $\Phi(\mathfrak{k}/\mathfrak{l}, \mathfrak{u})$. If $q_{\mathfrak{u}}(\alpha)$ does not belong to $\Phi(\mathfrak{l}, \mathfrak{u})$, then lies in $\Phi(\mathfrak{k}/\mathfrak{l}, \mathfrak{u})$. Because of corollary.... there exist w in the Weyl group of $\mathfrak{k}_1$ so that $-\alpha = w(\beta_1 + \beta_2)$. Hence, $\pm q_{\mathfrak{u}}(\alpha)$ belongs to the strict multiset $q_{\mathfrak{u}}(w\Psi_n) \cup q_{\mathfrak{u}}(\Delta_1 - \Phi_{\mathfrak{z}}) - \Phi_{\mathfrak{l}}$, contradiction and we have shown that $q_{\mathfrak{u}}(\alpha)$ is a root for $(\mathfrak{l}, \mathfrak{u})$.
We show: if $\gamma \in \Delta_1$ and $\alpha \in \Delta_1 \cap \Phi_{\mathfrak{k}_1}$ have the same restriction to $\mathfrak{u}$, then they are equal. Indeed, we choose w in the Weyl group of $K_1(\Psi)$ so that $-\alpha \in w\Psi_n$. Then, $-q_{\mathfrak{u}}(\alpha)$ belongs to the cone spanned by the multiset associated to w, besides if γ were different from α, $q_{\mathfrak{u}}(\alpha)$ would had at least multiplicity two in the multiset $q_{\mathfrak{u}}(\Delta_1 - \Phi_{\mathfrak{z}})$. Hence, $\pm q_{\mathfrak{u}}(\alpha)$ belongs to the strict cone spanned by the multiset associated to w, a contradiction. It readily follows that $\mathfrak{k}_1$ is an ideal of $\mathfrak{l}$. □

Corollary 11. *Assume $\mathfrak{z}_{\mathfrak{k}}$ is contained in $\mathfrak{l}$ and K_1 is not the trivial group. Then, if the second statement of the proposition holds for a system Δ_0 the first statement holds for arbitrary Δ_1.*

Corollary 12. *Let $\mathfrak{l}$ be an ideal of $\mathfrak{k}$ which contains $\mathfrak{z}_{\mathfrak{k}}$. Then, $\mathfrak{k}_1(\Psi) \subset \mathfrak{l}$ if and only if $p_{\mathfrak{u}}(\Psi_n)$ is strict.*

Since $\mathfrak{l}$ is an ideal, $p_{\mathfrak{u}}$ is $W_{\mathfrak{k}}-$ map. Hence, the corollary follows from the Proposition.

Remark 13. The hypothesis $\mathfrak{z}_\mathfrak{k}$ contained in $\mathfrak{l}$ is not redundant. In fact, $G = SU(2,2) \equiv SO(4,2)$ and $L = SO(4)$, Table 1 rows 1,2 show example of $\mathfrak{k}_1$ contained in $\mathfrak{l}$ and $p_\mathfrak{u}(w\Psi_n) \cup p_\mathfrak{u}(\Delta - \Psi_\mathfrak{z}) - \Psi_\mathfrak{l} = p_\mathfrak{u}(w\Psi_n)$ is strict for every $w \in W_\mathfrak{k}$.
For $SO(4,2)$, Table 1 rows 3,4 show example of $\mathfrak{k}_1$ contained in $\mathfrak{l}$ so that $p_\mathfrak{u}(\Psi_n)$ is not strict. In example ......we verify for $SU(p,q)$ condition C holds for Ψ and $K_1(\Psi)$ whenever K_1 is a proper subgroup of K_{ss}.

With minor modifications on the proof of the previous proposition it follows,

Proposition 6. *Assume L is normalized by the compact Cartan subgroup T. Then,*

$$\mathfrak{l} \supseteq \mathfrak{k}_1(\Psi) \text{ if and only if } w\Psi_n \cup \Psi(\mathfrak{k}/\mathfrak{l})$$

is strict for each $w \in W_\mathfrak{k}$.

We note that if replace in the Proposition $w\Psi_n$... strict for $p_\mathfrak{u}(w\Psi_n)$... strict, the statement is false as Remark ...shows.

11.4. **Reductive symmetric pairs.** Let H be a connected reductive subgroup of G so that (G,H) is a reductive symmetric pair. That is, H is the fixed points of an involution σ of G. Kobayashi, in [?], [?] has shown different criterions to assure $\pi(\lambda)$ has an admissible restriction to H. The aim of this subsection is to show the equivalence of his criterion and condition (C). To begin with, we show Kobayashi condition implies $\mathfrak{k}_1$ is contained in $\mathfrak{l}$. For this we recall notation to formulate the condition of Kobayashi.

As before we assume $L := H \cap K$ is a maximal compact subgroup of H. Let $\mathfrak{g} = \mathfrak{k} + \mathfrak{s}$ be the Cartan decomposition associated to K. We assume that G has a compact Cartan subgroup $B \subset K \subset G$. We choose B as follows:
$\sigma(\mathfrak{b}) = \mathfrak{b}$
$\mathfrak{b}_- := \{X \in \mathfrak{b} : \sigma X = -X\}$ is a maximal Cartan subspace for the symmetric pair (K,L).
As usual, $\mathfrak{b}_+ := \{X \in \mathfrak{b} : \sigma X = X\}$.

Henceforth, we fix compatible systems of positive roots $\Delta \subset \Phi(\mathfrak{k},\mathfrak{b})$ and $\Sigma \subset \Phi(\mathfrak{k},\mathfrak{b}_-)$.

We fix a system of positive roots Ψ in $\Phi(\mathfrak{g},\mathfrak{b})$ which contains Δ,

Let

$$\mathbb{R}^+\Psi_n := \sum_{\beta\in\Psi_n} \mathbb{R}^+\beta = \{\sum_{\beta\in\Psi_n} n_\beta\beta : n_\beta \geq 0\} \subset i\mathfrak{b}^*.$$

Lemma 22. *Assume Kobayashi condition holds for Ψ and H, that is, $\mathbb{R}^+\Psi_n \cap i\mathfrak{b}^*_- = \{0\}$. Then $K_1(\Psi)$ is a subgroup of L.*

The data $(SO(2n,2), SO(2n,1), Psi =^{\{} \epsilon_1 > \cdots > \epsilon_n > \delta\}$ shows that the condition of Kobayashi for Psi and H does not implies the center of K is contained in L.

Proof: We show $\mathfrak{k}_1$ is a subalgebra of $\mathfrak{l}$. For this we first prove that if α, β are in Ψ_n and $\alpha+\beta$ is a root, then $\alpha+\beta$ vanishes on $\mathfrak{b}_-$. In fact, the hypothesis together with Lemma.... imply that $p_{\mathfrak{b}_+}(\mathbb{R}^+\Psi_n)$ is a strict cone. Hence, there exists $v \in \mathfrak{b}_+$ so that $\beta(v) > 0$ for every $\beta \in \Psi_n$. Thus, if $\beta \in \Psi_n$, then $\sigma(\beta) \in \Psi_n$. Hence, $\sigma(\alpha+\beta) \in \Delta$. Thus, if $p_{\mathfrak{b}_-}(\alpha+\beta)$ were non zero, we would have $p_{\mathfrak{b}_-}\sigma(\alpha+\beta) = -p_{\mathfrak{b}_-}(\alpha+\beta)$ is nonzero negative root, a contradiction. Since $\mathfrak{b}_-$ is a Cartan subspace of the symmetric pair (K,L) we obtain that the root vectors of $\alpha+\beta$ belong to $\mathfrak{l}$. From lemma .... it follows that $\mathfrak{k}_1(\Psi)$ is an ideal in $\mathfrak{l}$. $\square$

Remark 14. *A consequence of the above Lemma is: for (G,H) a reductive symmetric pair, with H a noncompact group, such that there exists a discrete series of G with admissible restriction to H, then K is not a simple Lie group. However, any irreducible representation for $SO(2n,1)$ has an admissible restriction to $U(n)$.*

The next proposition involves two Cartan subgroups for the pair (K,L), to avoid cumbersome notation in its formulation, we state it in words. In the course of the proof we set up the required mathematical language.

Proposition 7. *Assume (G,H) is a reductive symmetric pair. Then, condition (C) holds if and only if Kobayashi's condition holds.*

Proof: We first assume Condition (C) holds and show that the statement of Kobayashi holds. Condition (C) is stated in a compact Cartan subgroup T so that $U = L \cap T$ is a Cartan subgroup for L. Whereas Kobayashi's condition is stated on a $\sigma-$invariant Cartan subgroup B so that $\mathfrak{b}_-$ is a Cartan subspace for the symmetric pair (K,L). By means of Cayley transform $c \in K$ associated to a set of strongly orthogonal roots S in $\Phi_{\mathfrak{k}}$ we may arrange that $c(\mathfrak{t}) = \mathfrak{b}$. Here, we write c for $Ad(c)$. Because of Lemma ...., $\mathfrak{k}_1(\Psi)$ is contained in $\mathfrak{l}$, and $\mathfrak{t}_1 \subset \mathfrak{u}$. Hence, $S \subset \Phi_{\mathfrak{k}_2/\mathfrak{l}}$ and we have the orthogonal decomposition $\mathfrak{u} = \mathfrak{t}_1 + < S > +V$. Since, $\mathfrak{t} = \mathfrak{u} \oplus \mathfrak{z}_{\mathfrak{k}/\mathfrak{l}}(\mathfrak{u})$ we have that $\mathfrak{b}_+ = \mathfrak{t}_1 + V, \mathfrak{b}_- = \mathfrak{z}_{\mathfrak{k}/\mathfrak{l}}(\mathfrak{u}) + c(< S >)$. Also c acts like the identity on the subspaces $V, \mathfrak{t}_1, \mathfrak{z}_{\mathfrak{k}/\mathfrak{l}}(\mathfrak{u})$. Lemma....implies the equality $\mathbb{R}^+\Psi_n \cap i(t_2)^*_{ss} = \{0\}$. Hence, whenever $\mathfrak{z}_{\mathfrak{k}} \subset \mathfrak{u}$ we have that Kobayashi condition $\mathbb{R}^+(\Psi_1)_n \cap i\mathfrak{b}^*_- = \{0\}$, holds for the system $\Psi_1 := (c^t)^{-1}(\Psi)$. In order to arrange that the restriction to $\mathfrak{b}_-$ of the set of compact roots in Ψ_1 is strict we replace Ψ by convenient $w\Psi$ with $w \in W_{\mathfrak{k}_2}$, owing to Lemma ... such a w stabilizes the set Ψ_n. When, $\mathfrak{z}_{\mathfrak{k}}$ is non trivial and is not contained $\mathfrak{u}$, the classification due to Berger of reductive symmetric pairs, cf [?], implies that $(\mathfrak{g}, \mathfrak{h})$ is one of the pairs $(SU(2p,2q), Sp(p,q)); (e_{6(-14)}, f_{4(-20)}), (SO(n,2), SO(n,1))$. The fact that $\mathfrak{k}_1 \subset \mathfrak{l}$ together with Proposition.... implies that the only pair we must consider is $(SO(n,2), SO(n,1)$. For n odd in ...we compute that condition (C) never holds, and for n even in ...we determine those systems Ψ where conditions (C) holds. For this systems we check that $\mathbb{R}^+\Psi_n \cap i\mathfrak{z}^*_{\mathfrak{k}} = \{0\}$, since in this case $\mathfrak{b}_- = \mathfrak{z}_{\mathfrak{k}}$ we conclude the proof.

Conversely, Assume Kobayashi's condition holds. Hence, Lemma .... implies $\mathfrak{k}_1 \subset \mathfrak{l}$. If $\mathfrak{z}_{\mathfrak{k}}$ is contained in $\mathfrak{l}$ then Proposition .... let us obtain that condition (C) holds. Whenever $\mathfrak{z}_{\mathfrak{k}}$ is nontrivial and is not contained in $\mathfrak{l}$, as in the proof of the direct affirmation we are left to consider the case $(SO(2n,2), SO(2n,1))$. In this case, notation as in....., only the systems $\Psi_{\pm q}$ satisfy Kobayashi's condition, and also they satisfies condition (C). Hence we conclude the proof. □

Corollary 13. *For (G,H) reductive symmetric pair, condition (C) for Ψ and L implies condition (C) for $K_1(\Psi)$. Thus, we have admissible restriction to K_1.*

Remark 15. *Proper do not imply condition c for non symmetric, so(4,1), restriction to k-2.*

Sp(p,1),delta $>$ epsilon $- 1 >$ ....,here pi restricted to k-2 is admisible (gross...) k-2T is the set of fixed points of an involution of K, we claim: this involution cannot extend to an involution of G, otherwise we would have sp-1(delta) contained in k-2T . Also G.lambda.....$>$ k-2 proper and condition C is not satisfied.

for holomorphic chamber, condition c never holds on k-ss, however sometimes there is admissible restriction, sometimes not.

11.5. **Sufficient condition for proper projection.** As before, let G be a connected simple Lie group and T a Cartan subgroup of K. We fix a system of positive roots Ψ in $\Phi(\mathfrak{g},\mathfrak{t})$ Let $\mathfrak{k}_1 := \mathfrak{k}_1(\Psi), \mathfrak{k}_2$ be as usual. Let Λ be the differential of a character of T. We now show,

Lemma 23. *Assume Λ is regular and dominant for Ψ. Let Ω denote the coadjoint orbit of $\lambda := (-i)\Lambda \in \mathfrak{g}^*$. Then, the map*

$$p_{\mathfrak{k}_1+\mathfrak{z}_\mathfrak{k}} : \Omega \to \mathfrak{k}_1^* + \mathfrak{z}_\mathfrak{k}^*$$

is proper. Moreover, if condition (C) holds for $\mathfrak{k}_1(\Psi_n)$, then

$$p_{\mathfrak{k}_1} : \Omega \to \mathfrak{k}_1^*$$

is proper.

Proof: We now recall two theorems due to [?] which show,

$p_\mathfrak{k}$ is a proper map.

$p_\mathfrak{k}(\Omega) \subseteq Ad^*(K)(\lambda + (-i)\mathbb{R}^+\Psi_n)$

Set $\mathfrak{t}_-$ equal to the orthogonal complement of $\mathfrak{t}_1 + \mathfrak{z}_\mathfrak{k}$. Hence $\mathfrak{t}_-$ is a Cartan subalgebra of the semisimple factor of $\mathfrak{k}_2$. Because of lemma.... $\mathbb{R}^+\Psi_n \cap \mathfrak{t}_- = 0$. Thus, $p_{\mathfrak{k},\mathfrak{l}} : \mathbb{R}^+\Psi_n \to \mathfrak{l}^*$ is a proper map. Hence, $p_{\mathfrak{k},\mathfrak{l}} : Ad(K)(\lambda + \mathbb{R}^+\Psi_n) \longrightarrow \mathfrak{l}^*$ is also a proper map. If condition C holds, then we have that $\mathbb{R}^+\Psi_n \cap \mathfrak{t}_2 = 0$, now the proof follows as before. □

Proposition 8. *Assume (G,H) is a reductive symmetric pair. Then Kobayashi's condition holds for Ψ if and only if $p_\mathfrak{l} : \Omega \to \mathfrak{l}^*$ is proper.*

Since (G,H) is a reductive symmetric pair , the hypothesis forces that condition (C) holds for $K_1(\Psi_n)$, hence the map is proper.

for the other implication.....

Corollary 14. *Assume (G,H) is a reductive symmetric pair and Kobayashi's condition holds for Ψ. Then a discrete series whose Harish-Chandra parameter is dominant with respect to Ψ has an admissible restriction to H.*

11.6. **Computing $\mathfrak{k}_1(\Psi)$.** In this subsection we determine the ideals $\mathfrak{k}_1(\Psi)$ for each simple real Lie algebra $\mathfrak{g}$ who has a compact Cartan subalgebra. The final result is,

Proposition 9. *The zero ideal is equal to a $\mathfrak{k}_1(\Psi)$ if and only if G/K is Hermitian. Any non zero, semisimple ideal in $\mathfrak{k}$ is equal to an ideal $\mathfrak{k}_1(\Psi)$ except for: four ideals in $\mathfrak{so}(4) \times \mathfrak{so}(4) \subset \mathfrak{so}(4,4)$; $\mathfrak{so}(2q+1)$ in $\mathfrak{so}(2p, 2q+1), p \geq 2$; $\mathfrak{so}(4)$ in $\mathfrak{so}(4,n), (n \neq 2)$.*

i) We have that $\mathfrak{k}_1(\Psi) = 0$ if and only if Ψ is either a holomorphic or a nonholomorphic system. Hence, for any other system of positive roots $\mathfrak{k}_1(\Psi)$ is not equal to to zero. Thus, the first claim is proved. It follows from Cartan classification of Symmetric Spaces, that for G/K Hermitian, $\mathfrak{k}_{ss}$ has more than one simple factor only when G is locally isomorphic to $SU(p,q)$ or $SO(4,2) \equiv SU(2,2)$. We dealt with this case in iv).

ii) We now show for an exceptional simple Lie group G whose symmetric space is not Hermitian, that any nonzero ideal in $\mathfrak{k}$ is equal to an ideal $\mathfrak{k}_1(\Psi)$ for a convenient choice of Ψ. It follows from the table in [?] page 518 that either $\mathfrak{k}$ is simple or $\mathfrak{k} = \mathfrak{su}_2 + \mathfrak{k}'$ with $\mathfrak{k}'$ simple, and $\mathfrak{su}_2$ corresponds to a long compact root. Thus, when $\mathfrak{k}$ is not simple, then G/K is a quaternionic symmetric space, [?]. Therefore, our claim follows from

Proposition 10. *Let G be so that $\mathfrak{k} = \mathfrak{su}_2(\alpha) + \mathfrak{k}'$, with $\mathfrak{k}'$ a simple ideal and G/K is a quaternionic symmetric space. Then the ideals $\mathfrak{su}_2(\alpha), \mathfrak{k}', \mathfrak{k}$ are $\mathfrak{k}_1(\Psi)$.*

In the course of the proof we explicitly write down the ideal that corresponds to each Ψ.
Proof: Fix a Borel de Siebenthal system of positive roots Ψ in $\Phi(\mathfrak{g},\mathfrak{t})$ so that the maximal root α is compact, simple for Ψ_c, and corresponds to the $\mathfrak{su}_2$ factor. That is, in Ψ all simple roots but one are compact and the noncompact simple has multiplicity two in the highest root. For a proof of the existence of such a system c.f. [?]. For this system, all the simple roots for Δ different from α are simple for Ψ. Hence, $[\mathfrak{u}_\Psi,\mathfrak{u}_\Psi]\subseteq\mathfrak{su}_2$. Thus, $\mathfrak{k}_1(\Psi)=\mathfrak{su}_2$. Any other system of positive roots is equal to $w\Psi$ with w in the complex Weyl group. We now show that if $\mathfrak{k}_1(w\Psi)=\mathfrak{su}_2$, then $w\Psi$ is a Borel de Siebenthal system of positive roots. In fact, $\mathfrak{k}_2(w\Psi)=\mathfrak{k}'$. Hence, Lemma 2 let us conclude that all the simple roots for $w\Psi\cap\Phi(\mathfrak{k}')$ are simple for $w\Psi$. Thus, $w\Psi$ has $rank(\mathfrak{k})-1$ compact simple roots and one noncompact simple root β. Owing to the hypothesis that $\mathfrak{k}'$ is an ideal in $\mathfrak{k}$ we obtain that the root associated to the $\mathfrak{su}_2$ factor is orthogonal to every compact simple root for $w\Psi$, hence, this root is, up to a scalar, equal to the fundamental weight associated to β. It follows from the extended Dynkin diagrams as in [?] page 77 that $w\Psi$ is a Borel de Siebenthal system.

We claim that $\mathfrak{k}_1(w\Psi)=\mathfrak{k}'$ if and only if $w^{-1}\alpha$ is a simple root for Ψ. Indeed, let w be so that $w^{-1}\alpha$ is a simple root for Ψ. Hence, $\alpha=ww^{-1}\alpha$ is simple for $w\Psi$. Therefore, the sum of two noncompact roots in $w\Psi$ can never be α and hence $[\mathfrak{u}_{w\Psi},\mathfrak{u}_{w\Psi}]\subseteq\mathfrak{k}'$. The fact that $\mathfrak{k}'$ is a simple ideal let us conclude that $\mathfrak{k}_1(w\Psi)=\mathfrak{k}'$. Conversely, if $\mathfrak{k}_1(w\Psi)=\mathfrak{k}'$, then Lemma 2 implies that α is a simple root for $w\Psi$. We are left to show that there is a system of positive roots whose $\mathfrak{k}_1$ is equal to $\mathfrak{k}$. For this, we count the number of systems whose $\mathfrak{k}_1(\Psi)$ is either $\mathfrak{su}_2$ or $\mathfrak{k}'$. We note that the following holds:

A) If Ψ_1,Ψ_2 are systems of positive roots so that $\Delta\subset\Psi_j$, α is a simple root for both of them and the bond for α in the respective Dynkin diagram is the same. Then $\Psi_1=\Psi_2$. This follows from: a) $\mathfrak{k}'$ is simple, b) the simple roots adjacent to α are noncompact, c) Any simple root not adjacent to α is compact.

B) Two Borel de Siebenthal systems containing Δ so that its $\mathfrak{k}_1$ ideal is $\mathfrak{su}_2$ are equal.

Therefore, the number of positive roots systems containing Δ and whose $\mathfrak{k}_1$ is equal to $\mathfrak{k}$ is equal to $card[W(\Phi)/W(\Phi_c)]-(1+$ number of long simple roots in $\Psi)$. One can check that this number is positive and we have concluded the proof of Proposition 6 $\square$
Note 2: When G/K is quaternionic and $\mathfrak{k}'$ is semisimple but not simple then there are systems $w\Psi$ containing Δ so that α is simple for $w\Psi$ and $\mathfrak{k}_1(w\Psi)\neq\mathfrak{k}'$. From the tables in [?] we need to consider $\mathfrak{so}(4,n)$, $n\geq 3$. For $\mathfrak{so}(4,2q)=\mathfrak{su}(e_1+e_2)\oplus\mathfrak{k}'$, $q\geq 3$. For the systems $w\Psi=\Psi_{\pm q}$ (c.f. ....) $e_1\pm e_2$ are simple for $w\Psi$ and $\mathfrak{k}_1(w\Psi)=\mathfrak{so}(2q)$. For $\mathfrak{so}(4,2q+1)$ there is not system Ψ so that $K_1(\Psi)=SO(2q+1)$.

Next for each quaternionic exceptional Symmetric Space we compute $m(w\Psi)$:=sum of the coefficients of the noncompact simple roots in the highest root for $w\Psi$ for each system of positive roots $w\Psi$ such that α is simple for $w\Psi$. For this, we write the simple roots as in [?] page 478, for $E_6\subset E_7\subset E_8$ as $\alpha_1,\alpha_3,\alpha_4,...,\alpha_8,\alpha_2$, where $\alpha_1,\alpha_3,\alpha_4,...,\alpha_8$ is the A_7 sub diagram and α_2 has a non zero inner product only with α_4. For F_4, the simple roots are denoted by α_j, $\alpha_j, j=3,4$ are the long roots and α_2 is adjacent to α_3. For G_2, α_1 is the long root.

α	$E_{6(-2)}$	$E_{7(-5)}$	$E_{8(-24)}$	$F_{4(4)}$	$G_{2(2)}$
α_1	2	3	4		3
α_2	3	4	6		
α_3	4	6	8	3	
α_4	6	8	12	6	
α_5	4	6	10		
α_6	2	4	8		
α_7		2	6		
α_8			3		

iii) $\mathfrak{g} = \mathfrak{g}_2$. Here, $\Phi = \{\pm(r\alpha + s\beta), 1 \leq r \leq 3, 1 \leq s \leq 2\}$. $\Delta = \{\alpha, 2\beta + 3\alpha\}$. There are three systems of positive roots containing Δ. We list each of them and the corresponding $\mathfrak{k}_1$. $\Psi = \{(r\alpha + s\beta), 1 \leq r \leq 3, 1 \leq s \leq 2\}$, $\mathfrak{k}_1(\Psi) = \mathfrak{su}(2\beta + 3\alpha)$; $\mathfrak{k}_1(S_\beta\Psi) = \mathfrak{su}(2\beta + 3\alpha) + \mathfrak{su}(\alpha)$; $\mathfrak{k}_1(S_{\beta+\alpha}S_\beta\Psi) = \mathfrak{su}(\alpha)$. We not that Ψ is a small and that $S_{\beta+\alpha}S_\beta\Psi$ is not small, $m(S_{\beta+\alpha}S_\beta\Psi) = 3$.
Thus, every non zero ideal of $\mathfrak{k}$ is equal to an ideal $\mathfrak{k}_1(\Psi)$.

iv) We are left to determine the ideals $\mathfrak{k}_1(\Psi)$ for $\mathfrak{g}$ a classical algebra and $\mathfrak{k}_{ss}$ is semisimple but not simple. For all these particular cases we may choose a basis $\{\epsilon_1, \dots, \epsilon_p, \delta_1, \dots, \delta_q\}$ for $i\mathfrak{t}^*$ and describe the roots system by means of this basis. For each case, we fix a system of positive roots Δ for $\Phi(\mathfrak{k}, \mathfrak{t})$ and describe the roots in Φ_n. The number $m(\Psi) :=$ sum of multiplicity of each noncompact simple root in the highest root, is computed.

$\mathfrak{g} = \mathfrak{su}(p, q)$, $p \geq 1, q \geq 1$ we fix

$$\Delta = \{\epsilon_i - \epsilon_j, \delta_r - \delta_s, 1 \leq i < j \leq p,\ 1 \leq r < s \leq q, \}.$$

Here, $\Phi_n = \{\pm\epsilon_i \pm \delta_j,\ 1 \leq i \leq p, 1 \leq j \leq q\}$. There are $\frac{(p+q)!}{p!q!}$ positive root systems containing Δ. They are given by shuffling the delta's and the epsilon's. For example,

$$\epsilon_1 > \dots \epsilon_a > \delta_1 > \dots > \delta_b > \epsilon_{a+1} > \dots > \epsilon_{a+c} > \delta_{b+1} > \dots > \delta_{b+d} \dots$$

The positive root systems $\epsilon_1 > \cdots > \epsilon_p > \delta_1 > \cdots > \delta_q$, or $\delta_1 > \cdots > \delta_q > \epsilon_1 > \cdots > \epsilon_p$ gives $\mathfrak{k}_1(\Psi) = 0$.
A situation $\epsilon_i > \delta_j > \epsilon_k$ forces $\epsilon_i - \epsilon_j \in \mathfrak{k}_1(\Psi)$, hence $\mathfrak{su}(p) \subseteq \mathfrak{k}_1(\Psi)$, and $\delta_j > \epsilon_i > \delta_k$ implies that $\mathfrak{su}(q) \subseteq \mathfrak{k}_1(\Psi)$.
Thus, the unique way of obtaining $\mathfrak{su}(p) = \mathfrak{k}_1(\Psi)$ is by a system

$$\Psi'_a := \{\epsilon_1 > \cdots > \epsilon_a > \delta_1 > \cdots > \delta_q > \epsilon_{a+1} > \cdots > \epsilon_p\}, 1 \leq a < p,$$

and the unique way of obtaining $\mathfrak{su}(q) = \mathfrak{k}_1(\Psi)$ is by means of

$$\Psi_b := \{\delta_1 > \cdots > \delta_b > \epsilon_1 > \cdots > \epsilon_p > \delta_{b+1} > \cdots \delta_q\}, 1 \leq b < q.$$

Any of this systems, is small. For any other nonholomorphic system $\mathfrak{k}_1(\Psi)$ is equal the semisimple factor of $\mathfrak{k}$.
Thus, for $\mathfrak{su}(p, q)$ every ideal of $\mathfrak{k}_{ss}$ is equal to an ideal $\mathfrak{k}_1(\Psi)$.

$\mathfrak{g} = \mathfrak{so}(2p, 2q + 1)$, $p \geq 3, q \geq 0$ we fix

$$\Delta = \{\epsilon_i \pm \epsilon_j, \delta_r \pm \delta_s, \delta_t, 1 \leq i < j \leq p,\ 1 \leq r < s \leq q, 1 \leq t \leq q\}.$$

Here, $\Phi_n = \{\pm\epsilon_i \pm \delta_j, \pm\epsilon_i,\ 1 \leq i \leq p, 1 \leq j \leq q\}$. Hence for any system of positive roots Ψ one of $\pm\epsilon_1$ and one of $\pm\epsilon_2$ are in Ψ_n. Thus, one of the four $\pm\epsilon_1 \pm \epsilon_2$ is a root for $\mathfrak{k}_1(\Psi)$. Hence, $\mathfrak{so}(2p), (p \geq 3)$ is contained in $\mathfrak{k}_1(\Psi)$ and $\mathfrak{so}(2q+1)$ cannot be equal to any $\mathfrak{k}_1(\Psi)$.

In total, there are $2\frac{(p+q)!}{p!q!}$ positive root systems in $\Phi(\mathfrak{g},\mathfrak{t})$ containing Δ. They are given by shuffling the delta's and the epsilon's or applying the Weyl group element S_{ϵ_p} to a shuffling . For example,

$$\epsilon_1 > \dots \epsilon_a > \delta_1 > \dots > \delta_b > \epsilon_{a+1} > \dots \epsilon_{a+c} > \delta_{b+1} > \dots > \delta_{b+d} \dots$$

We observe

$$\epsilon_i > \delta_j > \epsilon_k \,\text{implies}\, \epsilon_i - \delta_j + \delta_j - \epsilon_k = \epsilon_i - \epsilon_k \in \mathfrak{k}_1(\Psi)$$

$$\delta_i > \epsilon_j > \delta_k \,\text{implies}\, \delta_i - \epsilon_j + \epsilon_j - \delta_k = \delta_i - \delta_k \in \mathfrak{k}_1(\Psi)$$

$$\delta_i > \epsilon_i \,\text{implies}\, \delta_i - \epsilon_i + \epsilon_i = \delta_i \in \mathfrak{k}_1(\Psi)$$

Hence, the unique possibility of obtaining for Ψ that $\mathfrak{k}_1(\Psi) = \mathfrak{so}(2p)$ is when do not hold the last two situations. That is, when $\Psi_n = \{\epsilon_j \pm \delta_k\}$ or $\Psi_n = \{\epsilon_j \pm \delta_k, 1 \leq j < p, -\epsilon_p \pm \delta_k\}$. Both of these systems are small.
Here, every ideal of $\mathfrak{k}$ is equal to $\mathfrak{k}_1(\Psi)$ but zero or $\mathfrak{so}(2q+1)$.
$\mathfrak{g} = \mathfrak{so}(4, 2q+1)$, $q \geq 0$ we fix

$$\Delta = \{\epsilon_1 \pm \epsilon_2, \delta_r \pm \delta_s, \delta_t,\, 1 \leq r < s \leq q, 1 \leq t \leq q\}.$$

Here, $\Phi_n = \{\pm\epsilon_i \pm \delta_j, \pm\epsilon_i,\, 1 \leq i \leq 2, 1 \leq j \leq q\}$. One of the four roots $\pm\epsilon_1 \pm \epsilon_2$ is a root for $\mathfrak{k}_1(\Psi)$. Thus, either $\mathfrak{su}_2(\epsilon_1+\epsilon_2)$ or $\mathfrak{su}_2(\epsilon_1-\epsilon_2)$ is contained in $\mathfrak{k}_1(\Psi)$. For $\Psi_0 = \{\epsilon_1 > \epsilon_2 > \delta_1 > \cdots > \delta_q\}$ we obtain that $\epsilon_1+\epsilon_2 \in \mathfrak{k}_1(\Psi_0)$ and

$$\mathfrak{k}_1(\Psi_0) = \mathfrak{su}_2(\epsilon_1+\epsilon_2), \quad m(\Psi_0) = 2.$$

For $S_{\epsilon_2}\Psi_0$ we obtain that $\epsilon_1 - \epsilon_2 \in \mathfrak{k}_1(S_{\epsilon_2}\Psi_0)$ and

$$\mathfrak{k}_1(S_{\epsilon_2}\Psi_0) = \mathfrak{su}_2(\epsilon_1-\epsilon_2), \quad m(S_{\epsilon_2}\Psi_0) = 2.$$

There are two cases left, first there is no δ on the right of the epsilon's, that is, $\Psi_{\pm q} = \{\delta_1 > \cdots > \delta_q > \epsilon_1 > \sigma\epsilon_2\}$, $\sigma \in \{\pm 1\}$, i.e $\Psi_{-q} = S_{\epsilon_2}\Psi_q$, we have

$$\mathfrak{k}_1(\Psi_{\pm q}) = \mathfrak{su}_2(\epsilon_1 + (\pm 1)\epsilon_2) \oplus \mathfrak{so}(2q+1), \quad m(\Psi_{\pm q}) = 4, q \geq 2.$$

$$\mathfrak{k}_1(\Psi_{\pm 1}) = \mathfrak{su}_2(\epsilon_1 + (\pm 1)\epsilon_2) \oplus \mathfrak{so}(3), \quad m(\Psi_{\pm 1}) = 3, q = 1.$$

The second case is when at least one δ is on the left of the $\epsilon' s$. These are the systems

$$\Psi_{\pm a} := \{\delta_1 > \cdots > \delta_a > \epsilon_1 > \pm\epsilon_2 > \delta_{a+1} > \cdots > \delta_q\},\ 1 \leq a \leq q.$$

Then,

$$\mathfrak{k}_1(\Psi_{\pm a}) = \mathfrak{su}_2(\epsilon_1 \pm \epsilon_2) \oplus \mathfrak{so}(2q+1), \quad m(\Psi_{\pm a}) = 4,\ 1 < |a| \leq q$$
$$\mathfrak{k}_1(\Psi_{\pm 1}) = \mathfrak{su}_2(\epsilon_1 \pm \epsilon_2) \oplus \mathfrak{so}(2q+1), \quad m(\Psi_{\pm 1}) = 3,$$

Whenever, the system is of the type, $\epsilon_1 > \delta_j > \pm\epsilon_2$ then $\mathfrak{k}_1(\Psi)$ is equal to $\mathfrak{k}$.
The ideals $\{0\}, \mathfrak{so}(4), \mathfrak{so}(2q+1)$ are not equal to a $\mathfrak{k}_1(\Psi)$. We point out that no discrete series of G has an admissible restriction to $\mathfrak{so}(2q+1)$. Otherwise, the discrete series of $\mathfrak{so}(1, 2q+1)$ would be non empty.
$\mathfrak{g} = \mathfrak{s}o(2, 2q+1)$ Here

$$\Delta = \{\epsilon_1 \pm \delta_k, \delta_k, k = 1, \cdots, q\}$$
$$\Phi_n = \{\pm\epsilon_1, \pm\epsilon_1 \pm \delta_j, 1 \leq j \leq q\}.$$

Obviously, for Ψ a non holomorphic system of positive roots we have $\mathfrak{k}_1\Psi = \mathfrak{so}(2q+1)$. The center of $\mathfrak{k}$ is $\mathbb{C}\epsilon_1$. We now show that for every system of positive roots Ψ, then $\mathbb{R}^+\Psi_n \cap i\mathfrak{z}^* \neq \{0\}$. In fact, for any system of positive roots system Ψ in $\Phi(\mathfrak{g},\mathfrak{t})$, we have that $\pm\epsilon_1$ belongs to Ψ. Thus, $\pm\epsilon_1 + \sum 0(\epsilon_1 \pm \delta_i) \in \mathbb{R}^+\Psi_n$ and $\mathbb{R}^+\Psi_n \cap i\mathfrak{z}_\mathfrak{k}^* \neq 0$. We would like to point out that no discrete series for $SO(2, 2q+1)$ restricts discretely either to $SO(1, 2q+1)$ or to $SO(2q+1)$. Indeed, $SO(1, 2q+1)$ has no discrete series representation, so if discrete series had an admissible restriction to any of the subgroups in question, we would have that $SO(1, 2q+1)$ had a non empty discrete series. This shows of $K_1(\Psi)$ so that the discrete series attached to Ψ has no admissible restriction to $K_1(\Psi)$.

$\mathfrak{g} = \mathfrak{sp}(p, q)$, $p \geq 1, q \geq 1$, we fix

$$\Delta = \{\epsilon_i \pm \epsilon_j, 2\epsilon_k, \delta_r \pm \delta_s, 2\delta_t, 1 \leq i < j \leq p,\ 1 \leq r < s \leq q, 1 \leq t \leq q\}.$$

Here, $\Phi_n = \{\pm\epsilon_i \pm \delta_j,\ 1 \leq i \leq p, 1 \leq j \leq q\}$. In total there are $\frac{(p+q)!}{p!q!}$ systems of positive root in $\Phi(\mathfrak{g}, \mathfrak{t})$ containing Δ. They are given by shuffling the delta's and the epsilon's. We observe

$$\epsilon_i > \delta_j > \epsilon_k \text{ implies } \epsilon_i - \delta_j + \delta_j \pm \epsilon_k = \epsilon_i \pm \epsilon_k \in \mathfrak{k}_1(\Psi)$$
$$\delta_j > \epsilon_k \text{ implies } \delta_j - \epsilon_k + \delta_j + \epsilon_k = 2\delta_j \in \mathfrak{k}_1(\Psi)$$

$$\epsilon_i > \delta_j \text{ implies } \epsilon_i - \delta_j + \delta_j \pm \epsilon_i = 2\epsilon_i \in \mathfrak{k}_1(\Psi)$$
$$\delta_i > \epsilon_j > \delta_k \text{ implies } \delta_i - \epsilon_j + \epsilon_j \pm \delta_k = \delta_i \pm \delta_k \in \mathfrak{k}_1(\Psi)$$

Hence, the unique possibility of obtaining for Ψ that $\mathfrak{k}_1(\Psi) = \mathfrak{sp}(p)$ is for $\Psi_0 := \{\epsilon_1 > \cdots > \epsilon_p > \delta_1 > \cdots \delta_q\}$, and the unique way of obtaining $\mathfrak{sp}(q) = \mathfrak{k}_1(\Psi)$ is by means of $\Psi_q := \{\delta_1 > \cdots > \delta_q > \epsilon_1 > \cdots > \epsilon_p\}$. For any other system $\mathfrak{k}_1(\Psi)$ is equal to $\mathfrak{k}$. Ψ_q, Ψ_0 are small.

The trivial ideal is the unique ideal not equal to a $\mathfrak{k}_1(\Psi)$.

$\mathfrak{g} = \mathfrak{so}(2, 2q), q \geq 2$, we fix

$$\Delta = \{\delta_r \pm \delta_s,\ 1 \leq r < s \leq q\}.$$

Here, $\Phi_n = \{\pm\epsilon_1 \pm \delta_j,\ 1 \leq j \leq q\}$. The center of $\mathfrak{k}$ is equal to $\mathbb{C}\epsilon_1$. There are $2(q+1)$ systems of positive roots that contain Δ. They are:

$$\Psi_a := \{\delta_1 > \cdots > \delta_a > \epsilon_1 > \delta_{a+1} > \cdots > \delta_q > 0\},$$
$$\Psi_{-a} := S_{\epsilon_1 - \delta_q} S_{\epsilon_1 + \delta_q} \Psi_a,\ 0 \leq a \leq q.$$

Ψ_0, Ψ_{-0} are the holomorphic and antiholomorphic systems. For $q \geq 3, 1 \leq a \leq q$, we obtain that $\mathfrak{k}_1(\Psi_a) = \mathfrak{k}_1(\Psi_{-a}) = \mathfrak{so}(2q)$.

For $q = 2$, $\mathfrak{so}(4)$ has three non zero ideals, they are $\mathfrak{so}(\delta_1 \pm \delta_2)$ and $\mathfrak{so}(4)$, all of them are equal to a $\mathfrak{k}_1(\Psi)$ as Table 1 shows. Hence, all the ideals of $\mathfrak{k}_{ss}$ are equal to a $\mathfrak{k}_1\Psi$.

For any $q \geq 2$, we have that

$$\mathbb{R}^+(\Psi_q)_n \cap i\mathfrak{z}_\mathfrak{k}^* = \mathbb{R}^+(\Psi_{-q})_n \cap i\mathfrak{z}_\mathfrak{k}^* = 0.$$

In fact, $(\sum_j a_j(\delta_j - \epsilon_1 + b_j(\delta_j + \epsilon_1)) \in \mathbb{C}\epsilon_1, a_j \geq 0, b_j \geq 0$ forces $a_j = b_j = 0$, for all j.

For any other systems Ψ,

$$\mathbb{R}^+\Psi_n \cap i\mathfrak{z}_\mathfrak{k}^* \neq 0.$$

Because, $(\epsilon_1 - \delta_q) + (\epsilon_1 + \delta_q) \in \mathbb{R}^+\Psi_n \cap i\mathfrak{z}_\mathfrak{k}^*$.

We have that

$$m(\Psi_1) = m(S_{\epsilon_1+\delta_q} S_{\epsilon_1-\delta_q} \Psi_1) = 3$$
$$m(\Psi_a) = m(S_{\epsilon_1+\delta_q} S_{\epsilon_1-\delta_q} \Psi_a) = 4,\ 1 < a < q.$$
$$m(\Psi_q) = m(\Psi_{-q}) = 2.$$

Corollary ... implies that a discrete series with dominant parameter with respect to either

Ψ_q or Ψ_{-q} have an admissible restriction to $SO(2q)$. The other discrete series have no admissible restriction to $K_1(\Psi) = SO(2q)$. One way of showing this last statement is to observe that $SO(2q) = SO(1,2q) \cap SO(2q)$. Thus, $SO(2q)$ is the maximal compact subgroup of the fixed point subgroup for the reductive symmetric pair $(SO(2,2q), SO(1,2q))$ and $\mathfrak{z}_{\mathfrak{k}}$ is a maximal Cartan subspace for the symmetric pair $(SO(2)\times SO(2q), SO(1)\times SO(2q))$ so that $\mathbb{R}^+\Psi_n \cap i\mathfrak{z}_{\mathfrak{k}}^* \neq \{0\}$. Then, we apply Theorem 4.2 in [?] . We would like to notice that it follows from Cartan's classification of bounded symmetric domains G/K and from Berger's classification of the reductive symmetric pairs (G,H) that the unique G/K such that K_{ss} is a maximal compact subgroup of H, for a suitable H, is $G = SO(p,2)$ and $H = SO(p,1)$.

$\mathfrak{g} = \mathfrak{so}(4,4)$, we fix

$$\Delta = \{\epsilon_1 \pm \epsilon_2, \delta_1 \pm \delta_2, \}.$$

Here, $\Phi_n = \{\pm\epsilon_i \pm \delta_j,\ 1 \leq i \leq 2, 1 \leq j \leq 2\}$. For this algebra there are twelve systems of positive roots containing Δ. For each system the first column shows the positive noncompact roots, the second column $\mathfrak{k}_1(\Psi)$ and the third computes the number $m(\Psi) :=$ sum of multiplicity of each noncompact simple roots in the highest root.

Ψ_n	$\mathfrak{k}_1(\Psi)$	m
$\epsilon_i \pm \delta_j$	$\mathfrak{su}(\epsilon_1 + \epsilon_2)$	2
$\epsilon_1 \pm \delta_j, -\epsilon_2 \pm \delta_j$	$\mathfrak{su}(\epsilon_1 - \epsilon_2)$	2
$\pm\epsilon_i + \delta_j$	$\mathfrak{su}(\delta_1 + \delta_2)$	2
$\pm\epsilon_j + \delta_1, \pm\epsilon_j - \delta_2$	$\mathfrak{su}(\delta_1 - \delta_2)$	2
$\pm\epsilon_j + \delta_1, (-1)^{j+1}\epsilon_j \pm \delta_2$	$\mathfrak{su}(\epsilon_1 - \epsilon_2) + \mathfrak{su}(\delta_1 - \delta_2) + \mathfrak{su}(\delta_1 + \delta_2)$	3
$\pm\epsilon_j + \delta_1, \epsilon_j \pm \delta_2$	$\mathfrak{su}(\epsilon_1 + \epsilon_2) + \mathfrak{su}(\delta_1 - \delta_2) + \mathfrak{su}(\delta_1 + \delta_2)$	3
$\epsilon_1 \pm \delta_j, \pm\epsilon_2 + (-1)^{j+1}\delta_j$	$\mathfrak{su}(\epsilon_1 - \epsilon_2) + \mathfrak{su}(\epsilon_1 + \epsilon_2) + \mathfrak{su}(\delta_1 - \delta_2)$	3
$\epsilon_1 + \delta_j, \pm\epsilon_2 + \delta_j$	$\mathfrak{su}(\epsilon_1 - \epsilon_2) + \mathfrak{su}(\epsilon_1 + \epsilon_2) + \mathfrak{su}(\delta_1 + \delta_2)$	3

Table 2

The other systems of positive roots gives $\mathfrak{k}_1(\Psi) = \mathfrak{k}$.
The ideals $\{0\}, \mathfrak{su}_2(\epsilon_1 + \sigma\epsilon_2) + \mathfrak{su}_2(\delta_1 + \tau\delta_2)$, $\sigma, \tau \in \{\pm 1\}, \mathfrak{so}(4) \times \{0\} = \mathfrak{su}_2(\epsilon_1 + \epsilon_2) + \mathfrak{su}_2(\epsilon_1 - \epsilon_2), \{0\} \times \mathfrak{so}(4) = \mathfrak{su}_2(\delta_1 + \delta_2) + \mathfrak{su}_2(\delta_1 - \delta_2)$ are not a $\mathfrak{k}_1(\Psi)$.

$\mathfrak{g} = \mathfrak{so}(4,2q)$, $q \geq 3$, we fix

$$\Delta = \{\epsilon_1 \pm \epsilon_2, \delta_r \pm \delta_s,\ 1 \leq r < s \leq q\}.$$

Here, $\Phi_n = \{\pm\epsilon_i \pm \delta_j,\ 1 \leq i \leq 2, 1 \leq j \leq q\}$. For $\Psi_0 = \{\epsilon_1 > \epsilon_2 > \delta_1 > \cdots > \delta_q > 0\}$ we obtain that $\epsilon_1 + \epsilon_2 \in \mathfrak{k}_1(\Psi)$ and then $\mathfrak{su}_2(\epsilon_1 + \epsilon_2) = \mathfrak{k}_1(\Psi)$. For $S_{\epsilon_2+\delta_q}S_{\epsilon_2-\delta_q}\Psi_0$ we obtain that $\epsilon_1 - \epsilon_2 \in \mathfrak{k}_1(\Psi)$ and then $\mathfrak{su}_2(\epsilon_1 - \epsilon_2) = \mathfrak{k}_1(\Psi)$. Both systems are small.
For $\Psi_a = \{\delta_1 > \cdots > \delta_a > \epsilon_1 > \epsilon_2 > \delta_{a+1} > \cdots > \delta_q\}, 1 \leq a \leq q$, we have for $1 \leq a < q$

$$\mathfrak{k}_1(\Psi_a) = \mathfrak{su}_2(\epsilon_1 + \epsilon_2) \oplus \mathfrak{so}(2q)$$

and

$$\mathfrak{k}_1(S_{\epsilon_2+\delta_q}S_{\epsilon_2-\delta_q}\Psi_a) = \mathfrak{su}_2(\epsilon_1 - \epsilon_2) \oplus \mathfrak{so}(2q).$$

It follows that

$$m(\Psi_1) = m(S_{\epsilon_2+\delta_q}S_{\epsilon_2-\delta_q}\Psi_1) = 3$$

$$m(\Psi_a) = m(S_{\epsilon_2+\delta_q}S_{\epsilon_2-\delta_q}\Psi_a) = 4 \ for\ 1 < a < q$$

For $q \geq 3$ we have $\mathfrak{k}_1(\Psi_q) = \mathfrak{so}(2q)$ as well as $\mathfrak{k}_1(\Psi_{-q} := S_{\epsilon_2+\delta_q}S_{\epsilon_2-\delta_q}\Psi_q) = \mathfrak{so}(2q)$. The systems Ψ_q, $S_{\epsilon_2+\delta_q}S_{\epsilon_2-\delta_q}\Psi_q$ are small. For the system $\epsilon_1 > \delta_1 > \cdots > \delta_q > \epsilon_2$, since $q \geq 3$ we have that $\mathfrak{k}_1(\Psi) = \mathfrak{k}$. Any other system of positive roots must contain both

$\delta_j > \epsilon_i > \delta_k$ and $\epsilon_i > \delta_j > \epsilon_k$ and for them $\mathfrak{k}_1(\Psi)$ is equal to $\mathfrak{k}$.
Every ideal is equal to a $\mathfrak{k}_1(\Psi)$ but $\{0\}, \mathfrak{so}(4)$.
$\mathfrak{g} = \mathfrak{so}(2p, 2q)$, $p \geq 3, q \geq 3$, we fix

$$\Delta = \{\epsilon_i \pm \epsilon_j, \delta_r \pm \delta_s, 1 \leq i < j \leq p,\, 1 \leq r < s \leq q, \}.$$

Here, $\Phi_n = \{\pm\epsilon_i \pm \delta_j,\, 1 \leq i \leq p, 1 \leq j \leq q\}$. As before, let Ψ that contains Δ, a situation $\epsilon_i > \delta_j > \epsilon_k$ forces $\epsilon_i - \epsilon_j \in \mathfrak{k}_1(\Psi)$, hence $\mathfrak{so}(2p) \subseteq \mathfrak{k}_1(\Psi)$, and $\delta_j > \epsilon_i > \delta_k$ implies that $\mathfrak{so}(2q) \subseteq \mathfrak{k}_1(\Psi)$. Thus, the unique way of obtaining $\mathfrak{so}(2p) = \mathfrak{k}_1(\Psi)$ is for the system

$$\Psi_0 := \{\epsilon_1 > \cdots > \epsilon_p > \delta_1 > \cdots > \delta_q\},$$

and $\Psi_{-0} := S_{\epsilon_p+\delta_q} S_{\epsilon_p-\delta_q} \Psi_0$. The unique way of obtaining $\mathfrak{so}(2q) = \mathfrak{k}_1(\Psi)$ is by means of

$$\Psi_q := \{\delta_1 > \cdots > \delta_q > \epsilon_1 > \cdots > \epsilon_p\}$$

and $\Psi_{-q} := S_{\epsilon_p+\delta_q} S_{\epsilon_p-\delta_q} \Psi_q$. For any other system $\mathfrak{k}_1(\Psi) = \mathfrak{k}$. We have that

$$m(\Psi_0) = m(\Psi_{-0}) = m(\Psi_q) = m(\Psi_{-q}) = 2,$$

Every ideal but zero is equal to a $\mathfrak{k}_1(\Psi)$.

11.7. **Strongly elliptic elements.** Let us recall that an element of a semisimple Lie algebra $\mathfrak{h}$ is called strongly elliptic (resp. elliptic) in $\mathfrak{h}$ if its centralizer in the Adjoint group of $\mathfrak{h}$ is a compact subgroup (resp. is a compact Cartan subgroup). In [?] we find a characterization of the strongly elliptic elements in $\mathfrak{g}$ by means of the action of G on the symmetric space G/K. We also find a proof that strongly elliptic elements in $\mathfrak{g}$ do exists if and only if $rank\, \mathfrak{k} = rank \mathfrak{g}$. We have:

Proposition 11. *Let $\Omega = G \cdot \lambda$ be a coadjoint orbit in $\mathfrak{g}^*$. If λ is strongly elliptic and $p_{\mathfrak{h}} : \Omega \to \mathfrak{h}$ is a proper map. Then, $p_{\mathfrak{h}}(\Omega)$ is contained in the set of strongly elliptic elements of $\mathfrak{h}^*$.*

Proof: Let $\gamma \in \Omega$ and $\mu := p_{\mathfrak{h}}(\gamma)$, then the centralizer, $C_H(\mu)$, of μ in H acts on the fiber $p_{\mathfrak{h}}^{-1}(\mu) \cap \Omega$. Since $C_H(\mu)$ is an algebraic group one of its orbits in $p_{\mathfrak{h}}^{-1}(\mu) \cap \Omega$ is closed. Thus, the hypothesis that $p_{\mathfrak{h}}$ is a proper map gives us that $Ad^*(C_H(\mu))\gamma_0$ is compact for some $\gamma_0 \in p_{\mathfrak{h}}^{-1}(\mu)$. Now, since γ_0 is $Ad^*(G)-$conjugate to λ we have that $C_G(\gamma_0)$ is a compact subgroup of G. Thus, $C_H(\mu)/(C_H(\mu) \cap C_G(\gamma_0))$ is a compact homogeneous space. Hence, $C_H(\mu)$ is compact. This proves the proposition. □

Corollary 15. *Assume H has at least one noncompact factor, then, $p_{\mathfrak{h}}(\lambda) \neq 0$*

This is due to the fact that under the above hypothesis, $p_{\mathfrak{h}}(\lambda)$ is a strongly elliptic element of $\mathfrak{h}^*$.

We now show an example so that properness does not imply $p_{\mathfrak{h}}(\lambda)$ is $\mathfrak{h}-$regular. In fact, we consider $\mathfrak{g} = \mathfrak{so}(4, 3)$ and $\mathfrak{h}$ equal to the normal real form of $\mathfrak{g}_2$ immersed in the usual way. A reference for this is [?]. Hence, $i\mathfrak{t}^*$ admits an orthogonal basis and a system of positive roots Ψ so that $\Psi_n = \{\epsilon_1 \pm \delta, -\epsilon_2 \pm \delta, \epsilon_1, -\epsilon_2\}$, $\Psi_{\mathfrak{k}} = \{\epsilon_1 \pm \epsilon_2, \delta\}$. Here, $\mathfrak{k}_1(\Psi) = \mathfrak{su}(\epsilon_1 - \epsilon_2)$. The torus $\mathfrak{u}$ is the orthogonal to $\epsilon_1 + \epsilon_2 + \delta$. The roots of $\mathfrak{u}$ in $\mathfrak{h}$ are: $\pm(\epsilon_1 - \delta), \pm(\epsilon_2 - \delta), \pm(\epsilon_1 - \epsilon_2)$ and $(1, 1, -2)$ together with its change of signs and permutations. It follows that a vector in $p_{\mathfrak{u}}(\mathcal{C}(\Psi) \cap \{\lambda_1 + \lambda_2 = 2\lambda_3\})$ is orthogonal to the compact root $(1, 1, -2)$.

Corollary 16. *Let $\Omega = G \cdot \lambda$ be a strongly elliptic coadjoint orbit and assume $p_{\mathfrak{h}} : \Omega \to \mathfrak{h}^*$ is a proper map. Then, $p_{\mathfrak{l}} : \Omega \to \mathfrak{l}^*$ is a proper map.*

Proof: The decompositions $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{s} = \mathfrak{h} \oplus \mathfrak{q}$ and the fact that the Cartan involution associated to $\mathfrak{k}$ leaves invariant $\mathfrak{h}$ and $\mathfrak{q}$ imply that we may write in a unique manner each $x \in \mathfrak{g}$, $x = x_l + x_{hs} + x_{qk} + x_{qs}$ with $x_l \in \mathfrak{l}, x_{hs} \in \mathfrak{h} \cap \mathfrak{s}, x_{qk} \in \mathfrak{q} \cap \mathfrak{k}, x_{qs} \in \mathfrak{q} \cap \mathfrak{s}$. This decomposition is orthogonal with respect to the Killing form. For the purpose of this proof, we denote the Killing form evaluated in (v,v) by $\pm\|v\|^2$ according to that the Killing form in (v,v) is positive or negative. Thus, we have for $x \in G \cdot \lambda$ that

$$-\|\lambda\|^2 = -\|x_l\|^2 - \|x_{qk}\|^2 + \|x_{hs}\|^2 + \|x_{qs}\|^2.$$

The hypothesis implies that whenever x_l, x_{hs} runs in a bounded set then x_{qk}, x_{qs} varies on a suitable bounded set. On the other hand, Proposition 7 implies that $-\|x_l\|^2 + \|x_{hs}\|^2 \leq 0$ for every point $x_l + x_{hs}$ in $p_{\mathfrak{h}}(\Omega)$. Therefore, whenever x_l runs in a compact set, x_{hs} runs on a bounded set. Since Ω is closed in $\mathfrak{g}^*$, we have that $p_{\mathfrak{l}}$ is a proper map. $\square$

In order to state the next Proposition we set up some notation, we fix $U \subset T$ maximal tori of L and K respectively, and consider $\lambda \in \mathfrak{t}^*$ $\mathfrak{g}-$regular so that the map $p_{\mathfrak{h}} : \Omega \to \mathfrak{h}^*$ is proper. Next, we choose, $\mathcal{C}_{\mathfrak{l},\lambda,+}$, a closed Weyl chamber for $\Phi(\mathfrak{l},\mathfrak{u})$ in $\mathfrak{u}^*$ so that $p_{\mathfrak{h}}(i\lambda)$ is dominant. Whenever $p_{\mathfrak{h}}(\lambda)$ is $\mathfrak{h}-$regular the chamber is unique.

Proposition 12. *There exists a unique open Weyl chamber $C_{\mathfrak{h},\lambda,+}$ for $\Phi(\mathfrak{h},\mathfrak{u})$ contained in $\mathcal{C}_{\mathfrak{l},\lambda,+}$ such that $p_{\mathfrak{h}}(i\Omega) \cap \mathcal{C}_{\mathfrak{l},\lambda,+}$ is contained in the closure of $C_{\mathfrak{h},\lambda,+}$ relative to $\mathcal{C}_{\mathfrak{l},\lambda,+}$.*

Proof: We write

$$\mathcal{C}_{\mathfrak{l},\lambda,+} - \cup_{\beta\in\Phi_n(\mathfrak{h},\mathfrak{u})} Ker(\beta)$$

as a union of disjoint, convex sets $C_1, \cdots, C_r$. Each of this convex sets is the relative closure in $\mathcal{C}_{\mathfrak{l},\lambda,+}$ of an open Weyl chambers for $\Phi(\mathfrak{h},\mathfrak{u})$ contained in $\mathcal{C}_{\mathfrak{l},\lambda,+}$. Since, the centralizer of any strongly elliptic element is a compact Lie algebra, we have that the set of strongly elliptic elements of $\mathfrak{h}^*$ is equal to the set $\mathcal{D} := \cup_{j=1}^r Ad^*(H)C_j$. Next, we show that $Ad(H)C_j$ is both a closed and an open relative set in $\mathcal{D}$. In fact, let $\hat{C}_j$ denote the interior of C_j. Thus, $\hat{C}_j$ is an open Weyl chamber for $\Phi(\mathfrak{h},\mathfrak{u})$. We will show:

$$Ad(H)C_j = \text{ } relative\, closure\, in\, \mathcal{D}\, of \text{ } Ad(H)\hat{C}_j.$$

For this, we verify that if $j \neq k$ then the relative closure of $Ad(H)\hat{C}_j$ does not intersects the relative closure of $Ad(H)\hat{C}_k$. In fact, let $X_n \in \hat{C}_j, Y_n \in \hat{C}_k, x_n, y_n \in H, h \in H, z \in \cup_s C_s$ so that $Ad(x_n)X_n$ and $Ad(y_n)Y_n$ converge to $Ad(h^{-1})z$. Replacing x_n by hx_n and y_n by hy_n we may and will assume that $h = 1$. Since the eigenvalues of X_n (resp. Y_n) are the same as of $Ad(x_n)X_n$ (resp. $Ad(y_n)Y_n$) and X_n, Y_n are semisimple matrices, there exist a subsequence of X_n (resp. Y_n) which converges, for a proof cf [**?**], for simplicity, we denote such subsequences by X_n, Y_n. Let β be a noncompact root such that $\beta(\hat{C}_j) > 0$ and $\beta(\hat{C}_k) < 0$. Thus, we have that $\beta(z)$ is non zero, it is the limit of a sequence of positive numbers $(\beta(X_n))$ and it is the limit of a sequence of negative numbers $(\beta(Y_n))$, a contradiction. Since,

$$Ad(H)C_j \subsetneqq relative\, closure\, of\, Ad(H)\hat{C}_j\, in\, \mathcal{D},$$

and the set of strongly elliptic elements equal to the disjoint unions $\cup_j Ad(H)C_j$, we have the equality that were looking for. Now the hypothesis, H is connected, leads us that $Ad(H)C_j, j = 1, \cdots, r$ are the connected components of $\mathcal{D}$.

Finally, $p_{\mathfrak{h}}(i\Omega)$ is a closed connected subset of the set of strongly elliptic elements, and each $Ad(H)C_j$ is closed and open relative set in $\mathcal{D}$. Thus, $p_{\mathfrak{h}}(i\Omega)$ is contained in one of the $Ad(H)C_j$. $\square$

The convexity Theorem showed by Weinstein in [?] together with Proposition 7 and Proposition 8, applied to (Weinstein notation) $\mathcal{U} = Ad(H)C_j$ lead us to:

Corollary 17. *$p_{\mathfrak{h}}(i\Omega) \cap \mathcal{C}_{\mathfrak{l},\lambda,+}$ is a closed convex, locally polyhedral subset of $\mathcal{C}_{\mathfrak{l},\lambda,+} - \cup_\beta Ker\beta$, and $p_{\mathfrak{h}}^{-1}(\mu)$ is connected for every $\mu \in p_{\mathfrak{h}}(\Omega)$.*

In order to apply the Theorem of Weinstein, we need to verify that $Ad(H)C_j \cap \mathcal{C}_{\mathfrak{l},\lambda,+}$ is a convex set. Actually, $Ad(H)C_j \cap \mathcal{C}_{\mathfrak{l},\lambda,+} = C_j$. This follows from i) U is a compact Cartan subgroup, ii) Since H is connected, the normalizer of U in H is equal to the normalizer of U in L, iii) Two elements of a closed Weyl chamber in compact Lie algebra which are conjugated by an inner automorphism are equal.

Remark 16. Theorem 1 together with Proposition 8,9 and their Corollaries reflects the Theorem which shows (cf. Kobayashi, [?]) that whenever a discrete series representation of G restricts discretely to a subgroup H, then the irreducible factors are discrete series for H, and the Theorem that if a discrete series representation has an admissible restriction to H, then it has an admissible restriction to L (c.f. 3.1). Corollaries 9, 10 represent the fact that whenever $\pi(\lambda)$ restricts discretely to H, the Harish-Chandra parameters of the $H-$irreducible factors belong to a unique Weyl chamber of $\mathfrak{h}^*$ in $\mathfrak{u}^*$. This generalizes the result that when a holomorphic discrete series restricts discretely, then the irreducible factors are again holomorphic discrete series and similar results [?] and Loke on quaternionic representations. In [?], we find examples of discrete series for $SO(4,1)$ whose restriction to $SO(2,1)$ contains both holomorphic and antiholomorphic discrete series as discrete factors.

Proposition 13. *Assume that $\pi(\lambda)$ has an admissible restriction to H. Then, the set of Harish-Chandra parameters for each irreducible factor of the restriction of $\pi(\lambda)$ to H is contained in a unique Weyl chamber for $\Phi(\mathfrak{h},\mathfrak{u})$.*

Proof:

To talk about 1. *For large rank groups, for example, $Sp(2n,\mathbb{R}), n > 11$, there are chambers and μ so that $\mu + \rho_n(\mu)$ does not lie in the chamber of μ. Pick up any system of positive roots and fix β a noncompact simple root for Ψ and define $\Psi' := S_\beta\Psi$. Then, $\rho_n(\Psi') = \rho_n(\Psi) - \beta$. Now $-\beta$ is a simple root for Ψ' and $2(\rho_n(\Psi'),-\beta)/(\beta,\beta) = -2(\rho_n(\Psi),\beta)/(\beta,\beta)+2$. For the usual chamber $\rho_n = \frac{n}{2}(e_1+\cdots+e_n)$, and $\beta = 2e_n$ hence, $2(\rho_n(\Psi'),-\beta)/(\beta,\beta) = -\frac{n}{2}+2$ and thus $\rho(\Psi')$ and $\rho(\Psi')+\rho_n(\Psi')$ do not lie in the same Weyl chamber for n large.*

For the Borel de Siebenthal chamber Ψ, ρ_n is dominant wrt Ψ . For a proof Hecht-Schmid,

For rank 2 groups as well as for $so(2n,1)$ ρ_n is dominant.

For a chamber Ψ such that $m(\Psi)$ is smaller or equal than 2, ρ_n is not always dominant wrt Ψ . For example, in $su(p,1)$ for the chamber

$$\Psi_a := \{e_1 > \cdots > e_a > \delta > e_{a+1} > \cdots > e_p\},\ 0 \leq a \leq p,$$

$\rho_n(\Psi_a) = \frac{1}{2}(e_1+\cdots+e_a-e_{a+1}-\cdots-e_p+(p-2a)\delta)$ Hence, $2(\rho_n,\delta-e_a)) = (p-2a)-1$ and $2(\rho_n, e_{a+1}-\delta) = (2a-p)-1$ Thus, whenever $p-2a$ has absolute value not equal to 0 or 1, $\rho_n(\Psi_a)$ is not dominant for Ψ_a. The table bellow shows the systems of positive roots Ψ_b such that $\rho_n(\Psi_a)$ is dominant with respect to Ψ_b.

$p-2a>1$	Ψ_0
$p-2a=1$	$\Psi_b,\ 0\leq b\leq a$
$p-2a=0$	Ψ_a
$p-2a=-1$	$\Psi_b,\ a\leq b\leq p$
$p-2a<-1$	Ψ_p

MI-JO so we must look for examples in su(n,2) restricted to su(n,1)

Kobyashi has examples of $A_q(\lambda)$ lambda singular so that the restriction to H has $A_q(\lambda)$ components for different parabolic His examples are nonunitary principal series coming from a maximal parabolic q,

Wallach proves the proposition when he restricts to $K_1(\Psi)$ BY MEANS of the $A_q(\lambda)$.

11.8. **Comparing K_1 of Gross-Wallach and $K_1(\Psi)$.** We note that if Ψ is a holomorphic system of positive roots. That is, the sum of two roots in Ψ_n never is a root, then $\mathfrak{k}_1(\Psi)=\{0\}$. Whereas for this case Gross-Wallach in [?], Proposition 1 have defined $\mathfrak{k}_1$ equal to the center of $\mathfrak{k}$. We claim that when Ψ is small, and is a non holomorphic system, then the Lie algebra $\mathfrak{k}_1$ defined by Gross-Wallach agree with ours. In fact, Ψ small and nonholomorphic is equivalent to either Ψ has a unique non compact simple root β with multiplicity two in the highest root α or Ψ has exactly two noncompact simple roots, β_1,β_2 with multiplicity one in the highest root. We recall Gross-Wallach defined K_1 to be the simple ideal of $\mathfrak{k}$ that contains the $\mathfrak{su}_2(\alpha)$ spanned by the highest root. In the first case the simple roots for Ψ_c are the compact simple roots for Ψ, $\{\alpha_1,\cdots,\alpha_s\}$, and $\gamma=2\beta+\sum n_i\alpha_i$. Let C_γ be the roots in $\{\alpha_1,\cdots,\alpha_s,\gamma\}$ which belong to the connected component of γ in the Dynkin diagram for Ψ_c. Hence, if β, σ are in Ψ_n whose sum is a root, then $\beta+\sigma$ lies in the subroot system spanned by C_γ. Therefore, $\mathfrak{k}_1(\Psi)$ is contained in the simple ideal of $\mathfrak{k}$ whose root system is $<C_\gamma>$. Hence, $\mathfrak{k}_1=\mathfrak{k}_1(\Psi)$. For the other case Gross-Wallach in [?] Proposition 1, have shown that $\mathfrak{k}$ has a one dimensional center and that the simple roots for Ψ_c are $\{\alpha_1,\cdots,\alpha_s,\gamma\}$ with $\gamma=\beta_1+\beta_2+\sum n_i\alpha_i$. Now the proof follows as before.

Besides, [?] show that when (G,H) is a reductive symmetric pair and $K_1(\Psi)Z_K$ is contained in L, then the Harish-Chandra parameter of the irreducible factors of the restriction of $\pi(\lambda)$ to H lie in a unique Weyl chamber $\mathcal{C}(\Psi_\mathfrak{h})$ constructed as follows. Let Ψ be the system of positive roots determinate by Λ. For a compact semisimple Lie group L let S_L denote the element of the Weyl group that transform each system of positive roots into its opposite. Under the hypothesis of Gross-Wallach we further assume that $T\subset L=H\cap K\subset K, K_1:=K_1(\Psi)\subset L$ and recall their definition of $P_G:=-s_K\Psi, P_H:=\Phi(\mathfrak{h})\cap P_G, \Psi_H=-s_LP_H$. We would show $\Psi_H=\Psi\cap\Phi(\mathfrak{h})$. In fact, $K=K_1\times K_2, L=K_1\times L_2$ Hence, $s_K=(s_{K_1}\times 1)(1\times s_{K_2})$. Since $T\subset L, s_L=(s_{K_1}\times 1)(1\times s_{L_2})$. It follows from Lemma 2 that s_{K_2} (resp. s_{L_2}) is a product of reflections about compact simple roots in Ψ (resp. $\Psi_c\cap\Phi(\mathfrak{h})$). Thus, $(1\times s_{K_2})\Psi_n=\Psi_n,\ (1\times s_{L_2})(\Psi\cap\Phi(\mathfrak{h}))_n=(\Psi\cap\Phi(\mathfrak{h}))_n$. Since $\Psi=\Psi(K_1)\cup\Psi(K_2)\cup\Psi_n$, together with the observations of above lead us to

$$s_Ks_L\Psi=(1\times s_{K_2})(1\times s_{L_2})\Psi$$

Thus, $\Psi_H=s_Ks_L\Psi\cap s_L\Phi(\mathfrak{h})=\Psi(K_1)\cup\Psi(L_2)\cup\Psi_n\cap\Phi(\mathfrak{h})$.

12. Algorithms to compute multiplicities

12.1. **Discrete methods.** As before G denotes a connected matrix, semisimple Lie group and K a maximal compact subgroup for G. Let H be a connected reductive subgroup of G. We choose K so that $H \cap K = L$ is a maximal compact subgroup of H. From now on, we assume that G, as well as H, has compact Cartan subgroups $U \subset T$. Let P_T denote the lattice of differentials at the identity of the characters of T. That is, P_T denotes the set of analytically integral forms on T. We choose $\Lambda \in P_T$ and for the purpose of this section we assume that Λ is regular. That is, the inner product of Λ with any root in $\Phi(\mathfrak{g}, \mathfrak{t})$ is nonzero. Let

$$\Psi := \{\alpha \in \Phi(\mathfrak{g}, \mathfrak{t}) : (\Lambda, \alpha) > 0\}.$$

and let $\Delta\,(\mathrm{resp.}\Psi_{\mathrm{n}})$ be the compact (resp. non compact) roots in Ψ. The set of analytically integral forms and dominant with respect to Δ is denoted by C_K. We also fix a system of positive roots Δ_L in $\Phi(\mathfrak{l}, \mathfrak{u})$ which is compatible with Δ and denote by C_L the set of analytically integral characters of U dominant with respect to Δ_L. In order to choose Δ_L we might need to conjugate by an element of W to Δ and Λ. C_L^r denotes the subset of $\mathfrak{h}-$regular elements in C_L. From now on, the highest weight or the infinitesimal character of an irreducible representation of K (resp.L) is computed in C_K (resp. C_L). Let Θ_Λ be the Character of the discrete series representation of Harish-Chandra parameter Λ. Thus

$$\Theta_\Lambda(X) = (-1)^{\frac{1}{2} dim G/K} \frac{\sum_{w \in W} \epsilon(w) e^{w\Lambda(X)}}{\prod_{\alpha \in \Psi} (e^{\alpha/2} - e^{-\alpha/2})(X)}$$

We further assume that the irreducible unitary representation $(\pi(\lambda), V_\Lambda)$ corresponding to Θ_Λ has a discrete and admissible restriction to H. We claim that the restriction of $\pi(\lambda)$ to L is of finite multiplicity. In fact,since, each irreducible discrete factor is a discrete series for H and that each irreducible $L-$type is contained in at most finitely many irreducible square integrable representations (Harish-Chandra) we have that the restriction of $\pi(\lambda)$ to L has finite multiplicity. Moreover, T. Kobayashi has shown that the subspace of $K-$finite vector agrees with the subspace of $L-$finite vectors. In our setting, T. Kobayashi [?] has proved that the wave front set for Θ_Λ is transversal to H, hence we may restrict Θ_Λ to H and apply remark on page 22 in Duflo-Vergne [?], which shows that the Character of the restriction of $\pi(\lambda)$ to H is equal to the restriction of Θ_Λ to H. For each $\mathfrak{h}-$regular μ in C_L we denote by σ_μ the character of the discrete series representation associated to the Harish-Chandra parameter μ. If $m(\Lambda, \mu)$ denotes the multiplicity of σ_μ in $(V_\Lambda)_{|H}$ we may write

$$(\Theta_\Lambda)_{|H} = \sum_{\mu \in C_L^r} m(\Lambda, \mu) \sigma_\mu. \tag{3}$$

In [?] it is shown that $m(\Lambda, \mu)$ is at most of polynomial growth in μ. We define $m(\Lambda, \mu) = 0$ for $\mu \in C_L - C_L^r$.

From now on, we parameterize an irreducible representation of L by its infinitesimal character, $\nu \in C_L$. Thus, τ_ν denotes the character of the irreducible representation of L of highest weight $\nu - \rho(\Delta_L)$. We write $\tau_\nu = 0$ for those $\nu \in C_L$ which are $\mathfrak{l}-$singular. Since the restriction of V_Λ to L has finite multiplicity we also may write

$$(\Theta_\Lambda)_{|L} = \sum_{\nu \in C_L} n(\Lambda, \nu) \tau_\nu. \tag{4}$$

In [?] it is shown that $n(\Lambda, \nu)$ is of polynomial growth in ν. Because of a result of Harish-Chandra, Hecht-Schmid (Blattner's conjecture) we have that there exists nonnegative integers numbers $b(\mu,\nu)$, $\nu \in C_L$ with at most polynomial growth in ν or μ, so that

$$(\sigma_\mu)_{|L} = \sum_{\nu \in C_L} b(\mu,\nu)\tau_\nu.$$

Owing to the fact that we are considering discrete Hilbert direct sum of unitary irreducible representations, the uniqueness theorem implies the identity

$$\sum_{\mu \in C_L^r} m(\Lambda,\mu) b(\mu,\nu) = n(\Lambda,\nu) \tag{6}$$

For a linear functional $\gamma \in i\mathfrak{u}^\star$, let δ_γ denote the Dirac distribution concentrated at γ. For each linear functional α on $i\mathfrak{u}$ we define a distribution y_α on $i\mathfrak{u}^\star$ by the equality

$$y_\alpha = \sum_{n\geq 0} \delta_{\alpha/2+n\alpha}.$$

We fix $\mu \in C_L^r$ and let $\Psi_n(\mu) := \{\beta_1, \cdots, \beta_s\}$ denote the noncompact roots in $\Phi(\mathfrak{h},\mathfrak{u})$ who has a positive inner product with μ. We define, $Q_{\Psi_n(\mu)}$, the twisted partition function associated to $\Psi_n(\mu)$ by the formula

$$\sum_{\nu \in P_L} Q_{\Psi_n(\mu)}(\nu)\delta_\nu = y_{\beta_1} \star \cdots \star y_{\beta_s}$$

Hence, if $\rho_n = \frac{1}{2}\sum_{1\leq j\leq s}\beta_j$, then

$$Q_{\Psi_n(\mu)}(\nu) = card\{(n_j)_{1\leq j\leq s} : n_j \geq 0,\ \rho_n + \sum_{1\leq j\leq s} n_j\beta_j = \nu\}.$$

Since we parameterize the irreducible representations of L by their infinitesimal character, $\nu \in C_L$, rather than by its highest weight $\nu - \rho(\Delta_L)$. Blattner's formula for σ_μ now reads:

$$b(\mu,\nu) = \sum_{w \in W_L} \epsilon(w) Q_{\Psi_n(\mu)}(w\nu - \mu).$$

We extend the functions $b(\mu,...)$, $m(\Lambda,...)$ to the weight lattice P_L to be an antisymmetric function for W_L. Thus,

$$b(\mu, w\nu) = \epsilon(w) b(\mu,\nu), \text{ for } w \in W_L, \nu \in P_L$$

$$m(\Lambda, w\mu) = \epsilon(w) m(\Lambda,\mu), \text{ for } w \in W_L, \mu \in P_L.$$

For each linear functional α on $i\mathfrak{u}$ we define a linear operator d_α on the linear space of distributions, $\mathcal{D}'(i\mathfrak{u}^\star)$, by means of the convolution

$$d_\alpha(f) = (\delta_{-\alpha/2} - \delta_{\alpha/2}) \star f,\ f \in \mathcal{D}'(i\mathfrak{u}^\star).$$

We also define

$$d^{n,\mathfrak{h}} = \prod_{\alpha \in \Psi_n(\mu)} d_\alpha$$

The following equalities hold in the space of distributions on $i\mathfrak{u}^\star$.

Proposition 14.
1. $\sum_{\nu \in P_L} b(\mu,\nu)\delta_\nu = \sum_{w \in W_L} \epsilon(w)\delta_{w\mu} \star w(\bigstar_{\alpha \in \Psi_n(\mu)}\ y_\alpha)$
2. $d^{n,\mathfrak{h}}\ [\sum_{\nu \in P_L} b(\mu,\nu)\delta_\nu] = \sum_{w \in W_L} \epsilon(w)\delta_{w\mu}$
3. $d^{n,\mathfrak{h}}\ [\sum_{\nu \in P_L} n(\Lambda,\nu)\,\delta_\nu] = \sum_{\mu \in P_L} m(\Lambda,\mu)\delta_\mu$

Proposition 10 together with the formulae of above allow us to compute the $H-$multiplicities from the $L-$multiplicities and conversely.

Proof : Since all the series involved in the statement of the Proposition converge absolutely in the topology of the space of distributions on $i\mathfrak{u}^\star$, we may proceed in a formal manner. The left hand side of the equality in (1) is computed by means of Blattner's formulae. For the right hand side, since $\Psi_n(\mu) = \{\beta_1, \cdots, \beta_s\}$.

$$\sum_{w \in W_L} \epsilon(w)\delta_{w\mu} \star w(\, y_{\beta_1} \star \cdots \star y_{\beta_s})$$

$$= \sum_w \epsilon(w)\delta_{w\mu} \star \sum_{n_1, \cdots, n_s} \delta_{w(\beta_1 + \cdots + \beta_s)/2 + w\sum_j n_j\beta_j}$$

$$= \sum_w \epsilon(w)\delta_{w\mu} \star (\sum_{\theta \in P_L} Q_{\Psi_n(\mu)}(w^{-1}\theta)\ \delta_\theta)$$

$$= \sum_{w \in W_L, \nu \in P_L} \epsilon(w) Q_{\Psi_n(\mu)}(w(\nu) - \mu)\delta_\nu$$

$$= \sum_{\nu \in P_L} b(\mu, \nu)\delta_\nu$$

In order to show (2) we notice that

$$d_\alpha(y_\alpha) = \delta_0, \ \ d_\alpha y_{-\alpha} = -\delta_0$$

and recall that for each $w \in W_L$ the number of elements in $w(\Psi_n(\mu)) \cap -\Psi_n(\mu)$ is even. Thus,

$$d^{n,\mathfrak{h}}\,[\sum_{\nu \in P_L} b(\mu, \nu)\delta_\nu] = \sum_{w \in W_L} \epsilon(w)\delta_{w\mu} \bigstar_{1 \le j \le s}((\delta_{-\beta_j/2} - \delta_{\beta_j/2}) \star y_{w\beta_j})$$

Now, $d^{n,\mathfrak{h}} w(y_{\beta_1} \star \cdots \star y_{\beta_s}) = (-1)^{card(w(\Psi_n \cap \Phi(\mathfrak{h},\mathfrak{t})) \cap (-\Psi_n))}\delta_0$ and we have verified (2). We finally show (3). The equality

$$\sum_{\mu \in C_L^r} m(\Lambda, \mu) b(\mu, \nu) = n(\Lambda, \nu)$$

implies that

$$\sum_{\nu \in P_L} \sum_{\mu \in C_L^r} m(\Lambda, \mu) b(\mu, \nu)\delta_\nu = \sum_{\nu \in P_L} n(\Lambda, \nu)\delta_\nu$$

Thus

$$d^{n,\mathfrak{h}}[\sum_\nu n(\Lambda, \nu)\delta_\nu] = \sum_\mu m(\Lambda, \mu)(\sum_{w \in W_L} \epsilon(w)\delta_{w\mu})$$

$$= \sum_{\mu \in P_L,\, w \in W_L} \epsilon(w) m(\Lambda, w^{-1}\mu)\delta_\mu = \sum_\mu m(\Lambda, \mu)\delta_\mu.$$

$\square$

THE FINAL RESULT ???

The hypothesis in the next theorem are sufficient conditions to assure that $\pi(\lambda)$ restricted to H is admissible and to be able to compute multiplicities by means of partition

functions. Let $G, H, L, T, U, \Lambda, \Psi, K_1(\Psi)$,
$Z_K, \pi(\lambda), n(\Lambda, \nu)$ as usual. For next theorem we assume $K_1(\Psi)$ is contained in L. Hence if Z_K, is contained in L, then Corollary 6 implies $\pi(\lambda)$ has an admissible restriction to L. The proof shows the theorem is also true whenever $\pi(\lambda)$ has an admissible restriction to $K_1(\Psi)$.

Theorem 1. *Suppose that condition (C) holds and either $\pi(\lambda)$ has admissible restriction to K_1 or $K_1(\Psi)Z_K \subset L$. Then,*

$$\sum_{\nu \in P_L} n(\Lambda, \nu)\delta_\nu$$
$$= \sum_{w \in W} \epsilon(w)\, \varpi(w\lambda)\, \delta_{p_\mathfrak{u}(w\Lambda)} \star \bigstar_{\beta \in p_\mathfrak{u}(w\Psi_n) \cup (-1)p_\mathfrak{u}(\Delta - \Psi_\mathfrak{z}) - \Delta_L} y_\beta$$

Proof: The proof of the Theorem is done in several steps. We first work under the assumption $\Phi_\mathfrak{z}$ is empty, then, we consider the case L is normalized by the maximal torus T and finally we reduce the general case to these particular cases. To begin with we show the right hand side of the equality in the theorem is a well defined distribution on $i\mathfrak{u}^\star$. This follows from the hypothesis and Propositon ???. Also, the following equality holds

$$(p_\mathfrak{u})_\star(\delta_\Lambda \star \bigstar_{\beta \in \Psi_n} y_\beta) = \delta_{p_\mathfrak{u}(\Lambda)} \star \bigstar_{\beta \in \Psi_n} y_{p_\mathfrak{u}(\beta)} \tag{11}$$

Here, $(p_\mathfrak{u})_\star$ denotes push-forward of distributions. In order to show (11) we writeF $\mathfrak{t} = \mathfrak{u} \oplus \mathfrak{u}^\perp$. The hypothesis $K_1(\Psi) \subset L$ implies $\mathfrak{u}^\perp \subset \mathfrak{t}_2$ (resp. $K_1(\Psi)Z \subset L$ implies $\mathfrak{u}^\perp \subset \mathfrak{t}_{2ss}$) Therefore, our hypothesis, Corollary 3 and Corollary 5 imply that in any case

$$\mathbb{R}^+\Psi_n \cap i(\mathfrak{u}^\perp)^\star = 0 \tag{12}$$

Hence, (12) and the proof of Theorem 1 let us conclude that for any compact subset R of $i\mathfrak{t}^\star$ the map

$$p_\mathfrak{u} : R + (\cup_{w \in W} w(\Lambda + \mathbb{R}^+\Psi_n)) \to i\mathfrak{u}^\star \tag{13}$$

is proper. Thus, Lemma 3 let us obtain that $p_\mathfrak{u}(\Lambda + \mathbb{R}^+\Psi_n)$ is a proper cone in $i\mathfrak{u}^\star$ and therefore the convolution product

$$\delta_{p_\mathfrak{u}(\Lambda)} \star \bigstar_{\beta \in \Psi_n} y_{p_\mathfrak{u}(\beta)}$$

is well defined. On the other hand, (13) forces that $p_\mathfrak{u}(\delta_\Lambda \star \bigstar_{\beta \in \Psi_n} y_\beta)$ is well defined. Now, general properties of the push-forward of distributions implies equality (11).
Next, we write a proof of the Theorem under the assumption $\Phi_\mathfrak{z} = \emptyset$. According to Heckman we have

$$\sum_{\nu \in P_L} R(\xi, \nu)\delta_\nu = \sum_{w \in W} \epsilon(w)\delta_{p_\mathfrak{u}(w\xi)} \star (\bigstar_{\alpha \in p_\mathfrak{u}(\Delta) - \Delta_L} y_{-\alpha})$$

This equality justifies the last step in,

$$\begin{aligned}\sum_{\nu\in P_L} n(\Lambda,\nu)\delta_\nu \\ &= \sum_{\xi\in P_T,\nu\in P_L} b(\Lambda,\xi)R(\xi,\nu)\delta_\nu \\ &= \sum_\xi b(\Lambda,\xi)\sum_\nu R(\xi,\nu)\delta_\nu \\ &= \sum_\xi b(\Lambda,\xi)\sum_{w\in W}\epsilon(w)\delta_{p_{\mathfrak{u}}(w\xi)}\star(\bigstar_{\alpha\in p_{\mathfrak{u}}(\Delta)-\Delta_L} y_{-\alpha})\end{aligned}$$

Now, since $b(\Lambda, w\xi) = \epsilon(w)b(\Lambda,\xi)$, P_T is $W-$invariant and we are assuming $\Psi_{\mathfrak{z}} = \emptyset$ we have that

$$\begin{aligned}\sum_{\nu\in P_L} n(\Lambda,\nu)\delta_\nu \\ &= \sum_\xi b(\Lambda,\xi)\delta_{p_{\mathfrak{u}}(\xi)}\star(\bigstar_{\alpha\in p_{\mathfrak{u}}(\Delta)-\Delta_L} y_{-\alpha}) \\ &= p_{\mathfrak{u}}(\sum_\xi b(\Lambda,\xi)\delta_\xi)\star(\bigstar_{\alpha\in p_{\mathfrak{u}}(\Delta)-\Delta_L}\ y_{-\alpha}) \\ &= p_{\mathfrak{u}}(\sum_{w\in W}\epsilon(w)\ \delta_{w\Lambda}\bigstar_{\beta\in\Psi_n} y_{w\beta})\star(\bigstar_{\alpha\in p_{\mathfrak{u}}(\Delta)-\Delta_L} y_{-\alpha}) \\ &= \sum_{w\in W}\epsilon(w)\ \delta_{p_{\mathfrak{u}}(w\Lambda)}\star\bigstar_{\beta\in\Psi_n} y_{p_{\mathfrak{u}}(w\beta)}\star(\bigstar_{\alpha\in p_{\mathfrak{u}}(\Delta)-\Delta_L} y_{-\alpha})\end{aligned}$$

For the last three equalities we have applied: that the support of the distribution $\sum b(\Lambda,\xi)$ is contained $\cup_{w\in W} w(\Lambda+\mathbb{R}^+\Psi_n)$, that push-forward of distributions is a continuous map when we consider a proper map as (13) , Duflo-Heckman-Vergne version of Blattner's formula, and (11). In this manner we obtain Theorem 3 when $\Phi_{\mathfrak{z}} = \emptyset$. □

In particular, we already know that the theorem is true when $U = T$.

Next we prove the theorem for the case L is normalized by T. Hence LT is a subgroup of K of equal rank. Let $n_1(\lambda,\mu)$ denote the multiplicity of the irreducible representation of LT of infinitesimal character μ in $\pi(\lambda)$. Thus, we may write

$$\sum_{\mu\in P_T} n_1(\lambda,\mu)\delta_\mu = \sum_{w\in W}\epsilon(w)\ \delta_{w\Lambda}\star\bigstar_{\beta\in\Psi_n} y_{w\beta}\star(\bigstar_{\alpha\in(\Delta)-\Delta_L} y_{-\alpha})$$

We recall the equality

$$(p_{\mathfrak{u}})_\star(\sum_{\mu\in P_T} n_1(\lambda,\mu)\delta_\mu) = \sum_{\nu\in P_L} n(\lambda,\nu)\delta_\nu$$

and formula ??? to obtain

$$\sum_{\mu\in P_L} n(\lambda,\nu)\delta_\nu) = \sum_{w\in W}\epsilon(w)\ \delta_{p_{\mathfrak{u}}(w\Lambda)}\,\varpi(w\lambda)\star\bigstar_{\beta\in\Psi_n} y_{p_{\mathfrak{u}}(w\beta)}\star(\bigstar_{\alpha\in p_{\mathfrak{u}}(\Delta-\Psi_{\mathfrak{z}})-\Delta_L} y_{-\alpha})$$

In order to obtain the equality of the theorem for any L subject to our hypothesis we need the following Proposition.

Consider $\pi(\lambda)$, G, H, L as before. Let L_1 be a compact subgroup of L We fix maximal torus $T_1 \subset L_1, U \subset L$ so that $T_1 \subset U \subset T$ and assume that $\pi(\lambda)$ restricted to L_1 is admissible. By a result of Kobayashi, $\pi(\lambda)$ restricted to L is also admissible. We now show how to compute $L-$multiplicities from L_1-multiplicities. For this we make the following assumption

U is maximal torus of both L and L_1.

Hence both groups share the same system of roots $\Phi_{\mathfrak{z}}$ and have the same ϖ function.

We fix a system of positive roots Δ_1 in $\Phi(\mathfrak{l}_1, \mathfrak{u})$ compatible with Δ_L. For $\nu \in P_L$ (resp. $\nu_1 \in C_{P_1}$) let $n(\Lambda, \nu)$ (resp. $n_1(\Lambda, \nu_1)$) denote the multiplicity of the representation of L (resp L_1) of infinitesimal character ν (resp. ν_1) in $\pi(\lambda)$. As before, we extend both multiplicity functions to the weight lattice to be skew-symmetric functions.

We have,

Proposition 15.

$$\bigstar_{\alpha\in\Delta_L-\Delta_1} d_\alpha(\sum_{\nu_1\in P_{L_1}} n_1(\Lambda,\nu_1)\delta_{\nu_1}) = \sum_{\nu\in P_L} n(\Lambda,\nu)\delta_\nu$$

Proof: In [?] we find a proof of the following multiplicity formula. Let τ be a representation of K of infinitesimal character ξ dominant with respect Δ. For $\nu \in P_L$ (resp. $\nu_1 \in P_{L_1}$) let $R(\xi, \nu)$ (resp. $R_1(\xi, \nu_1)$) denote the multiplicity of the representation of L (resp. L_1) of infinitesimal character ν (resp. ν_1) in τ. For a finite multiset $\Sigma = \{\sigma_1, \cdots, \sigma_r\}$ of $i\mathfrak{t}_1^\star$, let Q_Σ denote the twisted partition function defined by

$$\sum_{\nu\in P_{T_1}} Q_\Sigma(\nu) = y_{\sigma_1} \star \cdots \star y_{\sigma_r}.$$

Thus, $Q_\Sigma(\nu) = card\{(n_j)_{j=1}^r : \nu = \sum_j (\frac{1}{2} + n_j)\sigma_j\}$.

Then, Heckman shows, in [?] Lemma 3.1, that

$$R(\xi,\nu) = \sum_{w\in W} \epsilon(w)\,\varpi(w\xi)\;Q_{p_{\mathfrak{u}}(\Delta)-\Delta_L}(p_{\mathfrak{u}}(w\xi)-\nu),$$

and

$$R_1(\xi,\nu_1) = \sum_{w\in W} \epsilon(w)\,\varpi(w\xi)\;Q_{p_{\mathfrak{u}}(\Delta)-\Delta_1}(p_{\mathfrak{u}}(w\xi)-\nu_1).$$

It follows from Lemma 3.1 in [?] that if we allow $\xi \in P_T, \nu \in P_{L_1}$ in the above formula, then

$$R(w\xi, v\nu) = \epsilon(w)\epsilon(v)R(\xi,\nu), w \in W, v \in W_L$$

Then, the following equalities hold in the space of distributions on $i\mathfrak{u}^\star$.

$$\sum_{\nu\in P_L} R(\xi,\nu)\delta_\nu = \sum_{w\in W} \epsilon(w)\varpi(w\xi)\delta_{p_{mathfraku}(w\xi)} \star (\bigstar_{\alpha\in p_{\mathfrak{u}}(\Delta)-\Delta_L} y_{-\alpha})$$

and

$$\sum_{\nu\in P_L} R_1(\xi,\nu)\delta_\nu = \sum_{w\in W} \epsilon(w)\varpi(w\xi)\delta_{p_{\mathfrak{u}}(w\xi)} \star (\bigstar_{\alpha\in p_{\mathfrak{u}}(\Delta)-\Delta_1} y_{-\alpha}).$$

Thus, we have

$$\bigstar_{\alpha\in\Delta_L-\Delta_1} d_\alpha(\sum_{\nu_1\in C_{L_1}} R_1(\xi,\nu_1)\delta_{\nu_1}) = \sum_{\nu\in C_L} R(\xi,\nu)\delta_\nu.$$

Now if $b(\Lambda,\xi)$ denotes the multiplicity of the representation of infinitesimal character ξ in $\pi(\lambda)$ we have that

$$n_1(\Lambda,\nu_1)=\sum_{\xi} b(\Lambda,\xi)R_1(\xi,\nu_1)$$

Therefore, we obtain,

$$\begin{aligned}\bigstar_{\alpha\in\Delta_L-\Delta_1} d_\alpha(\sum_{\nu_1\in P_{L_1}} n_1(\Lambda,\nu_1)\delta_{\nu_1})\\ &=\bigstar_{\alpha\in\Delta_L-\Delta_1} d_\alpha(\sum_{\xi,\nu_1} b(\Lambda,\xi)R_1(\xi,\nu_1)\delta_{\nu_1})\\ &=\sum_{\xi} b(\Lambda,\xi)\sum_{\nu} R(\xi,\nu)\delta_\nu\\ &=\sum_{\nu\in P_L} n(\Lambda,\nu)\delta_\nu\end{aligned}$$

Hence, we conclude the proof of Proposition ??? □

This proposition could be stated in more generality?????, for example for the case $T_1 \subsetneq U$, and no compact root in $\Phi(\mathfrak{g})$ has null restriction to $\mathfrak{t}_1$. However it will require more notation. What really matters is the proof and the technique to deduce $L-$multiplicities from L_1-multiplicities.

We are ready to conclude the proof of Theorem ???. Since condition (C) is satisfied we have that $K_1(\Psi)$ is contained in L. The torus T splits as a product $T=UU_1$. Let $L_1=K_1(\Psi)U$. Then, L_1 is normalized by T and U is maximal torus of both L and L_1. Hence, we may apply the previous case and Proposition ??? and conclude the proof of Theorem ???.

More details???

Corollary 18. *We get H-multiplicities applying Proposition 12 item 3.*

Note 4: The same proof will work for the dream hypothesis if we could show $p_{\mathfrak{u}}:\cup_{w\in W} w(\Lambda+\mathbb{R}^+\Psi_n)\longrightarrow i\mathfrak{u}^\star$ is a proper map. In turn, this is implied by (12).

12.2. **Continuous methods.** We now study multiplicity functions by means of the continuous methods introduced by Heckman and DHV. We also derive differential equations which are satisfied by the continuous multiplicity function. Let $G,H,K,Z,L,T,U,\lambda,\Psi,K_1(\Psi),\Omega,p_{\mathfrak{u}}$ have the same meaning as in the previous setting. For a coadjoint orbit $G\cdot\lambda$ let $\beta_{G\cdot\lambda}$ denote the Liouville measure on the coadjoint orbit computed with respect to the symplectic form on Ω arising from the Killing form on $\mathfrak{g}$. The Liouville measure of the coadjoint orbit $G\cdot\lambda$ evaluated on a test function φ is given by

$$\beta_{G\cdot\lambda}(\varphi):=A_G(\varphi)(\lambda)=c_G\prod_{\alpha\in\Psi}(\alpha,\lambda)\int_G\varphi(Ad(u)\cdot\nu)du.$$

Here, du is convenient chosen Haar measure on G. For the constants involved and proofs of the equalities we refer to [?], Chapter VII. From now on, for this subsection we assume $p_{\mathfrak{h}}:\Omega\to\mathfrak{h}^\star$ is proper. Because of Proposition 4, the map $p_{\mathfrak{l}}:\Omega\to\mathfrak{l}^\star$ is also proper. Moreover, in Heckman [?], Lemma 6.2 we find a proof that each of $p_{\mathfrak{l}},p_{\mathfrak{t}}$ is a submersion on an open dense whose complement has Lebesguian measure zero. For a more precise statement cf. [?] Chapter 5. Therefore, there exists a function

$$M_L : \mathfrak{u}^\star \to \mathbb{R}$$

so that the push-forward, $(p_\mathfrak{l})_\star(\beta_\Omega)$ of the measure β_Ω by $p_\mathfrak{l}$ is given by

$$(p_\mathfrak{l})_\star(\beta_\Omega)(\varphi) := (p_\mathfrak{l})_\star(\beta_{G\cdot\lambda})(\varphi) = \int_{\mathfrak{u}^\star} M_L(\mu)A_L(\varphi)(\mu)d\mu.$$

Here, $d\mu$ is a convenient Lebesgue measure on $\mathfrak{u}^\star$ and φ a test function on $\mathfrak{u}^\star$. Besides, M_L is a smooth function on the set of regular values of $p_\mathfrak{l}$ and is a W_L-skew-invariant function. For each root $\beta \in \mathfrak{u}^\star$ let Y_β the Heaviside generalized function on $\mathfrak{u}^\star$ associated to β. Thus,

$$Y_\beta(\varphi) = \int_0^\infty \varphi(t\beta)dt.$$

For the next propositon we assume that

$$K_1(\Psi)Z \subset L \text{ or } K_1(\Psi) \subset L \text{ and } \mathfrak{z}_\mathfrak{k}^\star \cap \mathfrak{u}^\perp = \{0\}.$$

Any of this two hipothesis, oweing to Proposition 3, imply that $p_\mathfrak{l}$ is a proper map and that for each $w \in W$, the convolution of the Heaviside functions

$$w(Y_n^+) := \bigstar_{\beta\in\Psi_n} Y_{p_\mathfrak{u}(w\beta)}, \qquad Y_{\mathfrak{k},\mathfrak{l}}^+ = \bigstar_{\alpha\in p_\mathfrak{u}(\Psi_\mathfrak{k}-\Psi_\mathfrak{z})-\Psi_\mathfrak{l}}\, Y_{-\alpha}.$$

are defined, as well as the product $w(Y_n^+) \star Y_{\mathfrak{k},\mathfrak{l}}$. For a proof see [?]. We now show,

Proposition 16.

$$M_L = \sum_{w\in W} \epsilon(w)\ \varpi(w\lambda)\ \delta_{p_\mathfrak{u}(w\lambda)} \star w(Y_n^+)\ \star\ Y_{\mathfrak{k},\mathfrak{l}}^+.$$

Proof: We have two proofs of this Proposition. the first proof is recalling formuola ???? which shows that $r \star m_L = M_L$. For the second proof we follow the same frame work as in the proof of Theorem ??. We first consider the case $\varpi = 1$, then the case L is normalized by T and then the general case. The maps $p_\mathfrak{k} : \mathfrak{g}^\star \to \mathfrak{k}^\star$, $p_{\mathfrak{k},\mathfrak{l}} : \mathfrak{k}^\star \to \mathfrak{l}^\star$. Thus, $p_\mathfrak{l} = p_{\mathfrak{k},\mathfrak{l}} \circ p_\mathfrak{k}$. In [?] we find a proof that $p_\mathfrak{k}$ restricted to Ω is a proper map. Since $p_\mathfrak{l}$ is a proper map on Ω we have that $p_{\mathfrak{k},\mathfrak{l}}$ is a proper map on $p_\mathfrak{k}(\Omega)$. Hence,

$$(p_\mathfrak{l})_\star(\beta_{G\cdot\lambda}) = (p_{\mathfrak{k},\mathfrak{l}})_\star(p_\mathfrak{k})_\star(\beta_{G\cdot\lambda}).$$

A Theorem of Duflo-Heckman-Vergne [?] affirms that for a test function φ on $\mathfrak{k}^\star$ we have,

$$(p_\mathfrak{k})_\star(\beta_\Omega)(\varphi) =< \sum_{w\in W} \epsilon(w)\delta_{w\lambda} \star w(Y_n^+),\ A_K(\varphi) > .$$

A Theorem of Heckman [?] states that for a test function on $\mathfrak{l}^\star$ we have

$$(p_{\mathfrak{k},\mathfrak{l}})_\star(\beta_{K\cdot\nu})(\varphi) =< \sum_{s\in W_Z\backslash W_K} \epsilon(s)\ \delta_{p_\mathfrak{u}(s\nu)} \star Y_{\mathfrak{k},\mathfrak{l}}^+,\ A_L(\varphi) > .$$

For a test function φ on $\mathfrak{l}^\star$ we have,

$$\begin{aligned}
&(p_\mathfrak{l})_\star(\beta_{G\cdot\lambda})(\varphi) \\
&\qquad = (p_{\mathfrak{k},\mathfrak{l}})_\star((p_\mathfrak{k})_\star(\beta_{G\cdot\lambda}))(\varphi) = (p_\mathfrak{k})_\star(\beta_{G\cdot\lambda})(\varphi \circ p_{\mathfrak{k},\mathfrak{l}}) \\
&\qquad\qquad =< \sum_{w\in W} \epsilon(w)\delta_{w\lambda} \star w(Y_n^+),\ A_K(\varphi \circ p_{\mathfrak{k},\mathfrak{l}}) > .
\end{aligned}$$

Now

$$A_K(\varphi \circ p_{\mathfrak{k},\mathfrak{l}})(\nu) = \int_K (\varphi \circ p_{\mathfrak{k},\mathfrak{l}})\beta_{K\cdot\nu}$$

$$= (p_{\mathfrak{k},\mathfrak{l}})_\star(\beta_{K\cdot\nu})(\varphi)$$

Thus, Heckman's Theorem implies the desired identity, when $\varpi = 1$. For the case L normalized by T as in the discrete case we need lemma.....

□

More details???

Corollary 19. *We further assume $T \subset L$, then*

$$M_L = \sum_{w\in W} \epsilon(w) \star w(Y_n^+) \star Y_{\mathfrak{k},\mathfrak{l}}^+.$$

The next Propositions are the continuous analogue to Proposition 10, item 3. In order to state them, we just assume that $p_{\mathfrak{h}} : \Omega \to \mathfrak{h}^\star$ is a proper map. Thus, because of Proposition ?? we have that $p_{\mathfrak{l}} : \Omega \to \mathfrak{l}^\star$ is a proper map. Hence, we may compute the push-forward into $\mathfrak{h}^\star$ of the Liouville measure on Ω. Once again, we have that $p_{\mathfrak{h}}$ is a submersion on an open dense subset of Ω whose complement has Lebesguian measure zero, besides Proposition 7 shows that $p_{\mathfrak{h}}(\Omega)$ is contained in the set of strongly elliptic elements in $\mathfrak{h}^\star$. Therefore there exists a function

$$M_H : \mathfrak{u}^\star \to \mathbb{R}$$

which is smooth on the regular values of $p_{\mathfrak{h}}$ restricted to Ω so that

$$(p_{\mathfrak{h}})_\star(\beta_{G\cdot\lambda})(\varphi) = \int_{\mathfrak{u}^\star} M_H(\mu)\ A_H(\varphi)(\mu)\ d\mu.$$

Here, as before, $d\mu$ is a Lebesgue measure on $\mathfrak{u}^\star$. We now show,

Proposition 17. *Assume $p_{\mathfrak{h}} : \Omega \to i\mathfrak{h}^\star$ is proper, then*

$$M_H = (\prod_{\alpha\in\Psi_n\cap\Phi(\mathfrak{h},\mathfrak{u})} \frac{\partial}{\partial\alpha})(M_L).$$

Proof: For every test function g on $i\mathfrak{l}^\star$ we have the equalities

$$\int_{i\mathfrak{u}^\star} M_L(\mu)A_L(g)(\mu)d\mu$$
$$= (p_{\mathfrak{l}})_\star(\beta_\Omega)(g) = (p_{\mathfrak{h},\mathfrak{l}})_\star(p_{\mathfrak{h}})_\star(\beta_\Omega)(g) = (p_{\mathfrak{h}})_\star(\beta_\Omega)(g \circ p_{\mathfrak{h},\mathfrak{l}})$$
$$= \int_{i\mathfrak{u}^\star} M_H(\mu)A_H(g \circ p_{\mathfrak{h},\mathfrak{l}})(\mu)d\mu$$

We compute and apply the Theorem of Duflo-Heckman-Vergne to obtain

$$A_H(g \circ p_{\mathfrak{h},\mathfrak{l}})(\mu) = \beta_{H\cdot\mu}(g \circ p_{\mathfrak{h},\mathfrak{l}}) = (p_{\mathfrak{h},\mathfrak{l}})_\star(\beta_{H\cdot\mu})(g)$$
$$=< \sum_{w\in W_L} \epsilon(w)\delta_{w\mu} \star w(Y_{n,\mathfrak{h}}^+),\ A_L(g) >$$

Therefore we have shown for every test function g on $\mathfrak{l}^\star$ that

$$\int_{i\mathfrak{u}^\star} M_L(\mu)A_L(g)(\mu)d\mu$$
$$= \int_{i\mathfrak{u}^\star} M_H(\mu) < \sum_{w\in W_L} \epsilon(w)\delta_{w\mu}\star w(Y^+_{n,\mathfrak{h}}),\ A_L(g) > d\mu$$

The differential operator, $\prod_{\alpha\in\Psi_n\cap\Phi(\mathfrak{h},\mathfrak{u})}(\frac{\partial}{\partial\alpha})$, on $i\mathfrak{u}^\star$ is W_L-invariant. Hence, Chevalley's restriction Theorem implies that there exists a constant coefficient $L-$invariant differential operator, D, on $i\mathfrak{l}^\star$ so that
$\prod_{\alpha\in\Psi_n\cap\Phi(\mathfrak{h},\mathfrak{u})}(\frac{\partial}{\partial\alpha})$ is the radial part of D. A formula of Harish-Chandra says that for every test function f in $\mathfrak{l}^\star$ we have

$$A_L(Df) = \prod_{\alpha\in\Psi_n\cap\Phi(\mathfrak{h},\mathfrak{u})}\frac{\partial}{\partial\alpha}A_L(f).$$

When we apply equality (7) to $g = Df$, Harish-Chandra equality yields,

$$\int_{i\mathfrak{u}^\star}\prod_{\alpha\in\Psi_n(\mathfrak{h})}\frac{\partial}{\partial\alpha}M_L(\mu)A_L(f)(\mu)d\mu$$
$$= (-1)^q\int_{i\mathfrak{u}^\star} M_H(\mu) < \sum_{w\in W_L}\epsilon(w)\delta_{w\mu}\star w(Y^+_{n,\mathfrak{h}}),\ \prod_{\alpha\in\Psi_n(\mathfrak{h})}\frac{\partial}{\partial\alpha}A_L(f) > d\mu$$

For each $w\in W_L$ we have that $card(w\Psi_n(\mathfrak{h})\cap(-\Psi_n(\mathfrak{h}))$ is even, hence

$$< \sum_{w\in W_L}\epsilon(w)\delta_{w\mu}\star w(Y^+_{n,\mathfrak{h}}),\ \prod_{\alpha\in\Psi_n(\mathfrak{h})}\frac{\partial}{\partial\alpha}A_L(f) >$$
$$= (-1)^q < \sum_{w\in W_L}\epsilon(w)\delta_{w\mu},\ A_L(f) >$$

The fact that $A_L(f)(\mu)$ is W_L-skew invariant in μ gives us that

$$\int_{i\mathfrak{u}^\star} M_H(\mu) < \sum_{w\in W_L}\epsilon(w)\delta_{w\mu}\star w(Y^+_{n,\mathfrak{h}}),\ \prod_{\alpha\in\Psi_n(\mathfrak{h})}\frac{\partial}{\partial\alpha}A_L(f) > d\mu$$
$$= \int_{i\mathfrak{u}^\star} M_H(\mu)A_L(f)(\mu)d\mu.$$

Therefore, (7) and (8) conclud the proof of Proposition 13. □

Let $d(w) = (-1)^{card[w\Psi_n\cap\Phi(\mathfrak{g}/\mathfrak{h})]}$.

Corollary 20. *We further assume $T = U \subset L$. Then,*

$$M_H = \sum_{w\in W}\epsilon(w)d(w)\delta_{w\lambda}\star[\bigstar_{\beta\in w\Psi_n\cap\Phi(\mathfrak{g}/\mathfrak{h})}Y_\beta]\star Y^+_{\mathfrak{k}/\mathfrak{l}}$$

Next, we show an analogue to Lemma 6.1 [?]. That is, we show that M_L, M_H are solution to certain differential equation.

Corollary 21. *Assume, $T \subset L$, then*
i) M_L is a solution to the differential equation

$$\prod_{\alpha \in \Psi(\mathfrak{g}/\mathfrak{l})} \frac{\partial}{\partial \alpha} X = \sum_{w \in W} \epsilon(w) \delta_{w\lambda}$$

ii) M_H is a solution to the differential equation

$$\prod_{\alpha \in \Psi(\mathfrak{g}/\mathfrak{h})} \frac{\partial}{\partial \alpha} X = \sum_{w \in W} \epsilon(w) d(w) \delta_{w\lambda}$$

Both equations are in $X \in \mathcal{D}'(i\mathfrak{u}^\star)$.

To talk about 2. *Since the support of $\sum_{\nu \in C_L} b(\mu, \nu)\delta_\nu$ is contained in the cone $\mu + \sum_{j=1}^{s} \mathbb{R}^+ \beta_j$ the uniqueness statement of Holmgren's theorem*
For a proof of Holgrem's Theorem we refer to the book [**?**] *page 70.*

12.3. **Multiplicity formulae.**

To talk about 3. *mi-jo, kobayashi obtained this and better formula for sopq sppq. some of them same type*

In this subsection we apply the results obtained in the previous subsections to obtain explicit multiplicity formulae in the spirit of Kostant or Blattner multiplicity formulae. For this, we keep the notation of previous sections. We first compute multiplicities when we restrict to the subgroup

$$L_1 := K_1(\Psi)T$$

and then we derive multiplicity formulae when we restrict to any reductive subgroup containing $K_1(\Psi)T$. We fix Λ dominant with respect to Ψ. Let $\Delta_1 := \Psi \cap \Phi(\mathfrak{l}_1, \mathfrak{t})$. Then, Corollary 6 assure us that $\pi(\lambda)$ restricted to $K_1(\Psi)T$ is admissible. Therefore, there exists non negative integers $n(\Lambda, \nu), \nu \in C_{L_1}$, so that

$$\Theta_{\Lambda_{|L_1}} = \sum_{\nu_1 \in C_{L_1}} n_1(\Lambda, \nu) \tau_{\nu_1}.$$

As before, τ_ν is the character of the irreducible representation of L_1 of infinitesimal character ν. Let W_j denote the Weyl group of $K_j(\Psi), j = 1, 2$. Thus, $W = W_1 \times W_2$. As before, for a finite subset $\Sigma = \{\sigma_1, \cdots, \sigma_r\}$ of $i\mathfrak{t}^\star$, let Q_Σ denote the twisted partition function defined by

$$\sum_{\nu \in P_T} Q_\Sigma(\nu) = y_{\sigma_1} \star \cdots \star y_{\sigma_r}.$$

Thus, $Q_\Sigma(\nu) = card\{(n_j)_{j=1}^r : \nu = \sum_j (\frac{1}{2} + n_j)\sigma_j\}$. In particular, we consider $Q_{\Psi - \Delta_1}$ the twisted partition function associated to the set

$$\Psi - \Delta_1 := \{\alpha_1, \cdots, \alpha_r, \beta_1, \cdots, \beta_q\}$$

Here the α_j are compact roots and β_k are noncompact roots.
We have,

Lemma 24. *Let Λ be dominant with respect to Ψ, then the representation of L_1 whose infinitesimal character is $\nu \in C_{L_1}$ has multiplicity in the restriction of $\pi(\lambda)$ to $K_1(\Psi)T$ given by*

$$n_1(\Lambda,\nu) = \sum_{w\in W_1, s\in W_2} \epsilon(ws) Q_{\Psi-\Delta_1}(w(\nu) - s\Lambda)$$

If we parameterize representations of L_1 by their highest weight rather than by its infinitesimal character, and we use plain partition function $\tilde{Q}$ associated to $\Psi - \Delta_1$ then the multiplicity formula becomes

$$\sum_{w\in W_1, s\in W_2} \epsilon(ws)\tilde{Q}(w(\tilde{\nu} + \rho_{L_1}) - s(\Lambda + \rho_{G/L_1}))$$

Proof: From Proposition 12 we obtain for a test function g on $\mathfrak{l}_1{}^\star$ the equality

$$(p_{\mathfrak{l}_1})_\star(\beta_{G\cdot\lambda})(g) \\ =< \sum_{w\in W} \epsilon(w)\delta_{w\lambda} \star w(Y_n^+) \star Y_{\mathfrak{k},\mathfrak{l}_1}^+,\ A_{L_1}(g) > \qquad (7)$$

Now, W_2 is spanned by reflection about compact simple roots for Ψ, thus, for $w \in W_2$, we have $w\Psi_n = \Psi_n$ and hence

$$w(Y_{-i\beta_1} \star \cdots \star Y_{-i\beta_q}) = Y_{-i\beta_1} \star \cdots \star Y_{-i\beta_q}$$

Moreover, since $\mathfrak{l}_1 = \mathfrak{k}_1(\Psi) + \mathfrak{t}$ we have that $\Psi(\mathfrak{k}_2) = \{\alpha_1, \cdots, \alpha_r\}$ Therefore, for all $\gamma \in \Psi(\mathfrak{k}_2), w \in W_1$ we have $w\gamma = \gamma$. Thus, from the previous formula we obtain,

$$(8) \sum_{w\in W} \epsilon(w)\delta_{w\lambda} \star w(Y_n^+) \star Y_{\mathfrak{k},\mathfrak{l}_1}^+ \\ = \sum_{w\in W_1, s\in W_2} \epsilon(ws)\delta_{ws\lambda} \star w(Y_{-i\beta_1} \star \cdots \star Y_{-i\beta_q} \star Y_{-\alpha_1} \star \cdots \star Y_{-\alpha_r})$$

Now, β_Ω as well as $p_{\mathfrak{l}_1}(\beta_\Omega)$ are tempered distributions, so equality (7) holds for any tempered test function. In particular, we may apply it for g equal to the Fourier transform of a test function. In order to avoid a cumbersome notation we identify, via Killing form, both $\mathfrak{g}, \mathfrak{l}_1, \mathfrak{t}$ with their dual vector spaces. For a Lie algebra $\mathfrak{v}$ Fourier transform of a test function is denoted by $g^{\wedge_V} = \widehat{g}^V$. In [?], we find a proof of the equality

$$A_{L_1}(\widehat{g}^{L_1}) = \widehat{A_{L_1}(g)}^T.$$

Combining this equality with (7) and (8), lead us to

$$(9) < ((p_{\mathfrak{l}_1})_\star(\beta_{G\cdot\lambda}))^{\wedge_{L_1}}, g > \\ =< \sum_{w\in W_1, s\in W_2} \epsilon(ws)\delta_{ws\lambda} \star w(Y_{-i\beta_1} \star \cdots \star Y_{-i\beta_q} \star Y_{-\alpha_1} \star \cdots \star Y_{-\alpha_r}), \widehat{A_{L_1}(g)}^T > .$$

In [?] we find a proof of the following equalities in the space of distributions,

$$\widehat{\delta_\mu}^T(X) = e^{i\mu X},\ \ \widehat{Y_{-i\beta}}^T = \sum_{n\geq 0} e^{n\beta}\ \frac{e^\beta - 1}{\beta}.$$

Therefore, (9) implies

$$
\begin{aligned}
&(10) < ((p_{\mathfrak{l}_1})_\star(\beta_{G\cdot\lambda}))^{\wedge_{L_1}}, g > \\
&\quad =< \sum_{w\in W_1, s\in W_2, n_j\geq 0, m_k\geq 0} \epsilon(ws) e^{ws\Lambda + n_j w\beta_j + m_k w\alpha_k} \prod_{\gamma\in\Psi-\Delta_1} \frac{e^{w\gamma}-1}{w\gamma}, A_{L_1}(g) > \\
&\qquad =< \sum_{w\in W_1, s\in W_2, n_j\geq 0, m_k\geq 0} \epsilon(ws) e^{ws\Lambda + n_j w\beta_j + m_k w\alpha_k + w\rho_{G/L_1}} \\
&\qquad\qquad \times \prod_{\gamma\in\Psi-\Delta_1} \frac{e^{\gamma/2}-e^{-\gamma/2}}{\gamma}, A_{L_1}(g) > .
\end{aligned}
$$

The last equality follows from the fact that $\prod_{\gamma\in\Psi-\Delta_1} \frac{e^{\gamma/2}-e^{-\gamma/2}}{\gamma}$ is invariant under the group W_1.

We now operate on $< ((p_{\mathfrak{l}_1})_\star(\beta_{G\cdot\lambda}))^{\wedge_{L_1}}, g >$. First of all we recall Proposition 5 in [**?**] which shows,

$$(p_{\mathfrak{l}_1\star}\beta_{G\cdot\lambda})^{\wedge_{L_1}} = p_{\mathfrak{l}_1\star}(\widehat{\beta_{G\cdot\lambda}}^{G}).$$

Let

$$J_G(X) := det(\frac{1-exp(-ad(X)}{ad(X)}) = \prod_{\gamma\in\Psi} \frac{(e^{\gamma/2}-e^{-\gamma/2})}{\gamma}(X),\ for\, X\in\mathfrak{t}.$$

We also recall Rossman-Kirillov character formula which affirms

$$J_G^{\frac{1}{2}}(X)\Theta_\Lambda(\exp X) = \widehat{\beta_{G\cdot\lambda}}^{G}(X)$$

and Harish-Chandra integration formula

$$\int_{\mathfrak{l}_1} \phi(\mu)d\mu = c_{L_1}\int_{\mathfrak{t}} \prod_{\alpha\in\Delta_1} < \alpha, \nu > A_{L_1}(\phi)(\nu)d\nu.$$

Combining these three equalities lead us to,

$$
\begin{aligned}
&< (p_{\mathfrak{l}_1,\star}(\beta_{G\cdot\lambda}))^{\wedge_{L_1}}, g > \\
&\qquad =< p_{\mathfrak{l}_1\star}(\widehat{\beta_{G\cdot\lambda}}^{G}), g > \\
&\qquad =< p_{\mathfrak{l}_1\star}(J_G^{\frac{1}{2}}(X)\Theta_\Lambda(\exp X)), g > \\
&\qquad =< \prod_{\gamma\in\Psi} \frac{(e^{\gamma/2}-e^{-\gamma/2})}{\gamma}(X)\Theta_\Lambda(\exp X), \prod_{\alpha\in\Delta_1}\alpha(X)A_{L_1}(g)(X) > \\
&\qquad =< \frac{\prod_{\gamma\in\Psi}(e^{\gamma/2}-e^{-\gamma/2})}{\prod_{\gamma\in\Psi-\Delta_1}\gamma}(X)\Theta_\Lambda(\exp X), A_{L_1}(g)(X) >
\end{aligned}
$$

Thus, (10) implies for $X \in \mathfrak{t}$ which is $G-$regular

$$\Theta_\Lambda(\exp X)$$
$$= \frac{\sum_{w\in W_1, s\in W_2, n_j\geq 0, m_k\geq 0} \epsilon(ws) e^{ws\Lambda + n_j w\beta_j + m_k w\alpha_k + w\rho_{G/L_1}}}{\prod_{\gamma\in\Delta_1} e^{\gamma/2} - e^{-\gamma/2}}$$
$$= \sum_{w_1\in W_1 s\in W_2,\, \nu\in C_{L_1}} \epsilon(w_1 s) Q_{\Psi-\Delta_1}(w_1(\nu) - s\Lambda)$$
$$\times \frac{\sum_{w\in W_1} \epsilon(w) e^{w\nu}}{\prod_{\gamma\in\Delta_1} e^{\gamma/2} - e^{-\gamma/2}}.$$

Since, on the set of regular elements of G which lie in L_1, the character of $\pi(\lambda)$ restricted to L_1 agrees with the restriction of the character of $\pi(\lambda)$ we have shown Lemma 4 $\square$

Next, we derive a multiplicity formulae for groups G, H so that $K_1(\Psi)T$ is contained in H. As before, we fix Λ dominant with respect to Ψ. Then, the hypothesis $K_1(\Psi)T$ is contained in H let us conclude that $\pi(\lambda)$ restricted to H is admissible. Let L be a maximal compact subgroup of H which contains $K_1(\Psi)T$. Let $C, C_L, C_{L_1}, \sigma_\mu$ have the same meaning as before. Therefore, there exists non negative integers $m(\Lambda,\mu)$ such that

$$(\Theta_\Lambda)_{|H} = \sum_{\mu\in C_L^r} m(\Lambda,\mu)\sigma_\mu$$

Let $Q_{\Psi-\Phi(\mathfrak{h})}$ be the twisted partition function associated to the set $\Psi - \Phi(\mathfrak{h})$.

Theorem 2. *Under the above hypothesis, for $\mu \in C_L$, we have*

$$m(\Lambda,\mu) = \sum_{w\in W_1,\, s\in W_2} \epsilon(ws) Q_{\Psi-\Phi(\mathfrak{h})}(w(\mu) - s\Lambda).$$

For the proof we apply Proposition 10 item 3 and Lemma 4. As before, let $n(\Lambda,\nu)$ denote the multiplicity in $\pi(\lambda)$ of the representation of L whose infinitesimal character is $\nu \in C_L$.

Lemma 25.

$$\sum_{\nu\in P_T} n(\Lambda,\nu)\delta_\nu = \sum_{w\in W_1,\, s\in W_2} \epsilon(ws)\delta_{ws\Lambda} \star w(\bigstar_{\gamma\in\Psi-\Delta_L} y_\gamma)$$

Proof: Because of Lemma 4 we have

$$\sum_{\nu_1\in P_T} n_1(\Lambda,\nu_1)\delta_{\nu_1}$$
$$= \sum_{w\in W_1,\, s\in W_2, \nu\in P_T} \epsilon(ws) Q_{\Psi-\Delta_1}(w\nu - s\Lambda)\delta_\nu$$
$$= \sum_{w\in W_1,\, s\in W_2,\, \nu\in P_T} \epsilon(ws)\delta_{ws\Lambda} \star w(\bigstar_{\gamma\in\Psi-\Delta_1} y_\gamma)$$

Proposition 11 gives us the equality

$$\sum n(\Lambda,\nu)\delta_\nu$$
$$= \bigstar_{\alpha\in\Delta_L-\Delta_1} d_\alpha(\sum n_1(\Lambda,\nu_1)\delta_{\nu_1})$$
$$= \sum_{w\in W_1,\, s\in W_2} \epsilon(ws)\delta_{ws\Lambda}\star w(\bigstar_{\gamma\in\Psi-\Delta_L} y_\gamma).$$

and hence we have shown Lemma 5 □

We continue with the proof of Theorem 2.
From Proposition 11 we have

$$\sum_{\mu\in P_T} m(\Lambda,\mu)\delta_\mu = d^{n,h}(\sum_{\nu\in P_T} n(\Lambda,\nu)\delta_\nu)$$

so we must compute

$$\bigstar_{\alpha\in\Psi_n\cap\Phi(\mathfrak{h})}(\delta_{-\alpha/2}-\delta_{\alpha/2})\star w(\bigstar_{\gamma\in\Psi-\Delta_L} y_\gamma)$$

Since $K_1(\Psi)T$ is contained in L we have that $W_1\subset W_L$, hence for $w\in W_1$ we have that $\prod_{\gamma\in\Psi-\Delta_L} w\gamma = \prod_{\gamma\in\Psi-\Delta_L}\gamma$, and we conclude that $card\{\gamma\in\Psi-\Delta_L : w\gamma<0\}$ is an even number. For $\alpha,\gamma\in\Psi_n\cap\Phi(\mathfrak{h})$ so that $w\gamma=\pm\alpha$ we have that $(\delta_{-\alpha/2}-\delta_{\alpha/2})\star y_{w\gamma}=\pm\delta_0$. Therefore,

$$\bigstar_{\alpha\in\Psi_n\cap\Phi(\mathfrak{h})}(\delta_{-\alpha/2}-\delta_{\alpha/2})\star w(\bigstar_{\gamma\in\Psi-\Delta_L} y_\gamma)$$
$$= \bigstar_{\gamma\in\Psi-\Phi(\mathfrak{h})} w\cdot y_\gamma$$

Combining the last formulaes, we obtain

$$d^{n,h}(\sum_{\nu\in P_T} n(\Lambda,\nu)\delta_\nu)$$
$$= \sum_{w\in W_1,\, s\in W_2}\epsilon(ws)\delta_{ws\Lambda}\star w(\bigstar_{\gamma\in\Psi-\Phi(\mathfrak{h})} y_\gamma)$$
$$= \sum_{\mu\in P_T}\sum_{w\in W_1, s\in W_2}\epsilon(ws)Q_{\Psi-\Phi(\mathfrak{h})}(w\mu-s\Lambda)\delta_\mu.$$

This concludes the proof of Theorem 2 □

We now show a formula for the multiplicities for restriction to the subgroup $K_1(\Psi)Z$. As before, Z is the identity connected component of the center of K. For other unexplained notation we refer to previous subsections. We define $\mathfrak{t}_{1,z} := \mathfrak{t}_1(\Psi)\oplus\mathfrak{z}$. Thus, $\mathfrak{t}_{1,z}$ is a Cartan subalgebra of $K_1(\Psi)Z$. We write $\nu=\nu_1+\nu_2$ with $\nu_1\in\mathfrak{t}_{1,z}, \nu_2\in\mathfrak{t}_{2ss}$. For the irreducible representation of $K_2(\Psi)$ of highest weight Λ_2 let $m(\Lambda_2,\theta)$ denote the multiplicity of the weight $\theta\in i\mathfrak{t}^\star_{2ss}$ and d_2 its dimension.

Proposition 18. *Suppose that Λ is dominant with respect to Ψ. Then the multiplicity, $n_{K_1Z}(\Lambda,\nu)$, of an irreducible representation of $K_1(\Psi)Z$ of highest weight ν in $\pi(\lambda)$ is given by the equality*

$$n_{K_1Z}(\Lambda,\nu) = d_2\sum_{w\in W_1,\nu_2\in P_{T_{2\,ss}}}\epsilon(w)Q_{\Psi_n}(w(\nu+\nu_2)-\Lambda_1)$$

Proof: From Lemma 4 we have the equalities

$$\sum_{\nu\in P_T} n(\Lambda,\nu)\delta_\nu$$

$$= \sum_{w\in W_1,\, s\in W_2,\, \nu\in P_T} \epsilon(ws)Q_{\Psi-\Delta_1}(w\nu - s\Lambda)\delta_\nu$$

$$= \sum_{w\in W_1,\, s\in W_2,\, \nu\in P_T} \epsilon(ws)\delta_{ws\Lambda} \star w(\bigstar_{\gamma\in\Psi-\Delta_1} y_\gamma)$$

Since $\Psi - \Delta_1 = (\Phi(\mathfrak{k}_2)\cap\Psi)\cup\Psi_n$ and $w \in W_1$ we have that

$$w(\bigstar_{\gamma\in\Psi-\Delta_L} y_\gamma) = \bigstar_{\alpha\in\Psi(\mathfrak{k}_2)} y_\alpha \star w(\bigstar_{\beta\in\Psi_n} y_\beta).$$

Let s_0 denote the longest element in the Weyl group of $\Phi(\mathfrak{k}_2)$ Then, we have the identity

$$\sum_{s\in W_2} \epsilon(s)\delta_{s\Lambda_2}\bigstar_{\alpha\in\Psi(\mathfrak{k}_2)} y_\alpha$$

$$= \epsilon(s_0)\sum_{s\in W_2} \epsilon(s)\delta_{ss_0\Lambda_2}\bigstar_{\alpha\in -s_0\Psi(\mathfrak{k}_2)} y_\alpha$$

$$= \epsilon(s_0)\sum_{\theta\in P_{T_2}} m(s_0\Lambda_2,\theta)\delta_\theta$$

We also have that

$$\sum_{\nu\in P_T}\sum_{w\in W_1} \epsilon(w)Q_{\Psi_n}(w\nu - \Lambda_1)\delta_\nu = \sum_{w\in W_1} \epsilon(w)\delta_{w\Lambda_1}\bigstar_{\beta\in\Psi_n} y_\beta$$

Therefore,

$$\sum_{\nu\in P_T} n_1(\Lambda,\nu)\delta_\nu$$

$$= \sum_{\nu_1\in P_{T_{1,z}}} [\sum_{\nu_2,\theta\in P_{T_2}} m(s_0\Lambda_2,\theta)Q_{\Psi_n}(w(\nu_1,\nu_2) - \Lambda_1)\delta_{\nu_2+\theta}] \star \delta_{\nu_1}$$

$$= \sum_{\nu_1\in P_{T_{1,z}}} (\sum_{\nu_2\in P_{T_2}} n_1(\Lambda,\nu_1,\nu_2)\delta_{\nu_2}) \star \delta_{\nu_1}$$

Let $\mathcal{A}$ be the subalgebra of finite linear combinations of $\{\delta_\nu,\ \nu\in P_{T_2}\}$. It readily follows that the set $\{\delta_\nu,\ \nu\in P_{T_{1,z}}\}$ is linearly independent over $\mathcal{A}$. This, leads to the equality

$$\sum_{\nu_2\in P_{T_2}} n_1(\Lambda,\nu_1,\nu_2)\delta_{\nu_2}$$

$$= \sum_{\nu_2}[\sum_{\theta,\,\nu_3\,:\,\theta+\nu_3=\nu_2,\,w\in W_1} \epsilon(w)m(s_0\Lambda_2,\theta)$$

$$\times Q_{\Psi_n}(w(\nu_1,\nu_3) - \Lambda_1)] \star \delta_{\nu_2}$$

Thus, we obtain

$$n_1(\Lambda,\nu_1,\nu_2) = \sum_{\theta,\nu_3\,:\,\theta+\nu_3=\nu_2,w\in W_1} \epsilon(w)m(s_0\Lambda_2,\theta)Q_{\Psi_n}(w(\nu_1,\nu_3) - \Lambda_1)$$

Finally, since $n_{K_1Z}(\Lambda, \nu_1) = \sum_{\nu_2 \in P_{T_2}} n_1(\Lambda, \nu_1, \nu_2)$ we have shown that

$$n_{K_1Z}(\Lambda, \nu_1) = d_2 \sum_{\nu_2, w} \epsilon(w) Q_{\Psi_n}(w(\nu_1, \nu_2) - \Lambda_1).$$

□

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