RESTRICTION OF DISCRETE SERIES

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ABSTRACT. Let $G = KAN$ be a connected linear semisimple Lie group. Let π be a Discrete Series representation for G. Let $H = LA_1N_1$ be a connected semisimple subgroup of G. 1- We provide a continuous immersion into π of a Discrete Series representation for H, whose lowest $L-$type occurs in the lowest $K-$type of π. 2- Tensor product has admissible diagonal decomposition then holomorphic 3- in continuous factor integral along whole a_P 4- Irreducible $H-$subrepresentation cuts smooth vectors, then π is admissible.

Introduction

For any Lie group we will denote its Lie algebra by the corresponding German lower case letter. Let G be a connected matrix semisimple Lie group. We fix an Iwasawa decomposition for G, $G = KAN$. Let H be a connected semisimple subgroup of G. Henceforth, we choose an Iwasawa decomposition $H = LA_1N_1$ such that $L \subset K, A_1 \subset A$ and $N_1 \subset N$.

1. EXPLICIT IMMERSION

We also assume that the Discrete Series for G is nonempty. Let (π, V) be a Discrete Series representation for G. Let (τ, W) be the lowest $K-$type for (π, V). Let (σ, Z) be an irreducible $L-$component of the restriction of τ to L. Finally, we assume that there exists a Discrete Series representation (ρ, V_1) for H such that its lowest $L-$type is σ. In [**?**] we have shown

Proposition 1. *(ρ, V_1) is contained in the restriction of (π, V) to H.*

We now construct an explicit intertwining linear map from ρ into π.

The main steps of the proof of the above proposition were:

a) For $K-$finite vectors v, w of (π, V), the restriction to H of a matrix coefficient $(\pi(?)v, w)$ is in $L^p(H)$, for p in an interval $(p_0, 2]$ with $p_0 < 2$.

b) An explicit formula for the spherical trace functions attached to the lowest $K-$type of irreducible square integrable representation. The formula is proved in Proposition 7.4 of [F-J] and it is:

As before (π, V) (resp. (ρ, V_1)) is a Discrete Series representation for G, (H) and (τ, W) ((σ, Z)) its lowest $K-$type ($L-$type) respectively. Let P_τ

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(resp. P_σ) be the orthogonal projection of V (resp. V_1) onto W (Z). Then the respective spherical trace functions are:

$$\phi_G(x) = trace(P_\tau \pi(x) P_\tau)\ (x \in G) \text{ and } \phi_H(x) = trace(P_\sigma \rho(x) P_\sigma)\ (x \in H).$$

Let P (resp. Q) be the orthogonal projection from W (Z) onto the line that contains the highest weight vector for W $(Z\ )$ respectively. The formula is:

$$\phi_G(k_1 a k_2) = \int_K trace(\tau(k^{-1}k_2k_1k)P)\, C(a,k) dk$$
$$\text{for } a \in exp(A_i^+), k_1, k_2 \in K.$$

where $C(a,k)$ is a continuous, nowhere vanishing and nonnegative real valued function, and dk is Haar measure on K. Certainly, a similar formula holds for ϕ_H.

Let w (z) be norm one highest weight vector for W (Z) respectively. Then,

$$(\tau(k^{-1}k_2k_1k)w, w) = trace(\tau(k^{-1}k_2k_1k)P)$$
$$(\sigma(r^{-1}k_2k_1r)z, z) = trace(\sigma(r^{-1}k_2k_1r)Q)$$

Here $(,)$ denotes the inner product on W.

c) $\int_H \overline{\phi_G(x)}\phi_H(x)dx$ is a positive number.

d) There exists $w_1\ \in W, z_1\ \in Z$ so that

$$\int_H \overline{(\pi(h)w_1, w_1)}(\rho(h)z_1, z_1)dh$$

is nonzero.

We now show

Proposition 2. *For a $K-$finite vector $w \in V$ and $v \in V$ the function $(\pi(?)v, w)$ belongs to $L^2(H)$. Besides, the restriction map*

$$r : V \longrightarrow L^2(H)$$
$$V \ni v \longrightarrow (H \ni h \rightarrow (\pi(h)v, w))$$

is Hilbert continuous.

Note that for each choice of w there is a map r. In order to avoid a cumbersome notation we will write just r rather than a notation that also involves w.

Proof: We consider the isometric immersion $(\|w\| = 1)$

$$V \longrightarrow L^2(G)$$
$$x \rightarrow (\pi(?)x, w)$$

We have that if $(\pi(?)x_n, w)$ converges in L^2-norm, then the sequence x_n converges in V. Therefore, the sequence $(\pi(?)x_n, w)$ converges pointwise. In order to avoid a cumbersome notation, the image of the immersion will again be denoted by V. Note that the elements of V are continuous functions. Let V_F be the subspace of $K-$finite vectors for V. Then, owing to a) the restriction map $r : V_F \longrightarrow L^2(H)$ is a well defined linear map. Let D be the subspace of the elements f in V such that the function $r(f)$ is in $L^2(H)$. We claim that $r : D \longrightarrow L^2(H)$ is a closed linear map. In fact, let $f_n \in D$ which converges to $f \in V$ so that $r(f_n)$ converges to $g \in L^2(H)$, want to show that $f \in D$ and $r(f) = g$. We already know that f_n converges pointwise to f and f is continuous. The Riesz-Fisher Theorem implies that $r(f_{n_j})$ converges almost everywhere to g in H. Thus, $r(f)$ is equal to g almost everywhere, and hence $f \in D$. For f in the domain of $rr^\star$ we have,

$$rr^\star(f) = d_\pi(\pi(?)w, w) \star_H f = d_\pi \int_H (\pi(xh^{-1}w, w)f(h)dh.$$

Indeed, firs of all we recall the identity [?] Cor. 4.5.9.4

$$\int_G (gw, w)(g^{-1}hv, w)dg = \frac{1}{d_\pi}(\pi(h)v, w)(w, w)$$

and let $\tilde{f} = (\pi(?)v, w) \in D, f \in L^2(H)$ then

$$\begin{aligned}(r(\tilde{f}), f)_{L^2(H)} &= \int_H (\pi(h)v, w)\overline{f(h)}dh \\ &= d_\pi \int_H \int_G (gw, w)(g^{-1}hv, w)dg\overline{f(h)}dh \\ &= d_\pi \int_G \int_H \overline{f(h)}(ghv, w)(g^{-1}w, w)dhdg \\ &= d_\pi \int_G \int_H \overline{f(h)(gh^{-1}w, w)}dh\tilde{f}(g)dg\end{aligned}$$

Hence, $rr^\star(f) = d_\pi(\pi(?)w, w) \star_H f$

We now apply the Kunze-Stein phenomenon and obtain that $rr^\star$ extends to a bounded linear operator in $L^2(H)$. Thus, $r^\star = V(rr^\star)^{\frac{1}{2}}$ is continuous and hence r is continuous and $D = V$. □

Corollary 1. *For each $w \in W$, the adjoint map $r^\star : L^2(H) \longrightarrow V$ is given by*

$$r^\star(f)(x) = d_\pi \int_H (\pi(xh^{-1})w, w)f(h)dh, \ x \in G, f \in L^2(H).$$

Corollary 2. *Let w_1, z_1 as in c), fix the immersion of V_1 associated to z_1 and consider the map r corresponding to w_1. Then $r^\star$ restricted to V_1 is an injective map.*

In fact,

$$(r(?)w_1, z_1) = (w_1, r^\star(z_1))$$

and because of c) the right hand side is nonzero. Hence, Schur's lemma concludes the corollary.

Next, we construct an explicit non zero linear intertwining map T from V_1 into V when we consider the realization of V given by Hota. To start with we fix w_1, z_1 as in c) and complete w_1 to an orthonormal basis w_j of W.

We realize V as the kernel in $L^2(G \times_K W)$ of the homogeneous differential operator $\Omega - [(\lambda,\lambda) - (\rho,\rho)]$. That is, V is an eigenspace of the Casimir operator. We fix the realization of V_1 in $L^2(H)$ provided by z_1 and we consider the linear map T defined by convolution in H by the spherical function attached to the lowest $K-$type of π evaluated at w_1. Thus, for $f \in V_1, x \in G$,

$$T(f)(x) = \sum_j [(\pi(?)w_1, w_j) \star_H f](x) w_j$$

$$= \sum_j [\int_H (\pi(xh^{-1}w_1, w_j) f(h) dh] w_j$$

Hence

$$T(f)(kx) = \sum_j [\int_H (\pi(kxh^{-1}w_1, k^{-1}w_j) f(h) dh] w_j$$

$$= \sum_j [\int_H ((xh^{-1}w_1, k^{-1}w_j) f(h) dh] w_j$$

Since $k^{-1}w_j = \sum_r (k^{-1}w_j, w_r)$ we obtain

$$T(f)(kx) = \sum_j [\int_H (\pi(xh^{-1}w_1, w_r) f(h) dh] \overline{(k^{-1}w_j, w_r)} w_j = \tau(k) T(f)(x)$$

By hypothesis, π has infinitesimal character λ and $w_j \in V$ hence we have that the function $T(f)$ belongs to the eigenspace of the Casimir operator for the eigenvalue $(\lambda,\lambda) - (\rho,\rho)$. When we apply T to the function $(\rho(?)z_1, z_1)$ the first component of $T(f)(x)$ is nonzero, hence T is a nontrivial map. It is clear that T commutes with the action of H. The Lemma of Schur implies the statement.

Note: Since w_j are $K-$finite vectors it is possible to give a direct proof of the fact that the integral which defines $T(f)$ converges and that $T(f)$ belongs to $L^2(G \times_K W)$.

2. Admissible Tensor products

Let G a connected semisimple Lie group having a compact Cartan subgroup T. Fix a maximal compact subgroup K containing T. Let Φ_c, Φ_n denote the set of compact (noncompact) roots in $\Phi_{\mathfrak{g}}$ the root system of the pair $(\mathfrak{g}, \mathfrak{t})$. Once and for all we fix

Δ system of positive roots in Φ_c.

We consider $\Psi, \tilde{\Psi}$ system of positive roots in $\Phi_{\mathfrak{g}}$ both of them contain Δ.

Let λ a $\Psi-$ dominant a Harish-Chandra parameter of a discrete series representation π_λ of G.

Similarly, let μ, a $\tilde{\Psi}-$ dominant ...

w_k denotes the involution in Φ_c which carries Δ onto $-\Delta$.

Proposition 3. *If* $\pi_\lambda \boxtimes \pi_\mu$ *is admissible under the diagonal action of* G, *then* $\Psi = \tilde{\Psi}$ *and* Ψ *is a holomorphic system. The converse statement is also true.*

Proof: In Kobayashi Inv. Math 1994 page 188 it is proven that the hypothesis implies

$$\mathbb{R}^+\Psi_n \cap \mathbb{R}^- w_k \tilde{\Psi}_n = \emptyset$$

Hence

$$\Psi_n \cap -w_k \tilde{\Psi}_n = \emptyset$$

Since $-w_k\tilde{\Psi}$ is another system of positive roots containing Δ we have that

$$\Psi_n = w_k \tilde{\Psi}_n$$

Thus

$$-w_k\tilde{\Psi} = \Delta \cup -\Psi_n$$

Hence,

$$\Delta \cup -\Psi_n \;\; and \;\; \Delta \cup \Psi_n$$

are systems of positive roots. Therefore if the sum of two roots in Ψ_n were a root we would have that a root and its negative belonged to Δ a contradiction. □

3. Structure of the continuous spectrum

Let (π, V) an square integrable irreducible representation of G. As usual (τ, W) denotes the lowest $K-$type of π. Assume that the restriction of π to H is not discretely decomposable. Since π is tempered, the continuous spectrum is Hilbert sum of direct integrals of unitary principal series induced by discrete or limit of discrete series. Hence, a typical piece of the restricction looks like

$$\int_S Ind_{MAN}^H(\sigma \times exp(i\nu) \times 1)m(\sigma,\nu)d\nu.$$

Here MAN is a cuspidal parabolic subgroup of H, σ a discrete series of M, $S \subset \mathfrak{a}$ and $m(\sigma,\nu)$ is nonzero and nonnegative on S. We claim

$$S = \mathfrak{a}.$$

We now write down in detail the statement for the case H is a real rank one group. For this we need to recall a result of [**?**]

Let $H = LAN_1$ with $dim A = 1$, and MAN_1 denote the minimal parabolic of H which contains AN_1. Fix a finite dimensional representation (γ, Z) of L. Thus, $(\gamma_{|_M}, Z) = \sum_j (\sigma_j, Z_j)$ as a sum of irreducible representations. Let P_j denote the orthogonal projection onto Z_j.

Then for $f \in Ind_L^H(\gamma) = L^2(H \times_L Z)$ who also belongs to the Schwartz space, the Helgason-Fourier transform

$$P_{\sigma_j,\nu}(f) \in Ind_{MAN_1}^H(\sigma_j \otimes e^{i\nu} \otimes 1)$$

in the direction $\sigma_j, \nu \in \mathfrak{a}^\star$ for f is given by the formula

$$P_{\sigma_j,\nu}(f)(s) = \int_H P_j(\gamma(k(h^{-1}s)^{-1})f(h))e^{i\nu-\rho_H(H(h^{-1}s))}dx\,(s \in L).$$

Here, $y = k(y)exp(H(y))n(y)$ is the Iwasawa decomposition of y.

Following Hota, we realize (π, V) as an eigenspace of the Casimir operator. Let $v \in V - \{0\}$, then in [?] it is shown that some normal derivate of v restricted to H is nonzero. Because of the L^2-continuity of the normal derivate, some $K-$component of v enjoys the same property. Thus there exists some $K-$finite element of v so that has a nontrivial component on the continuous spectrum. Now by means of some normal derivate, we may assume f lies in $L^2(H \times_L Z)$ for a finite dimensional representation (γ, Z) of L. We claim that if π is an integrable representation, then $P_{\sigma_j,\nu}(f)$ is a real analytic function of ν. Indeed, for $\nu \in \mathfrak{a}_{\mathbb{C}}$ we write $\nu = \Re(\nu) + i\Im(\nu)$ Hence,

$$\|P_{\sigma_j,\nu}(f)(s)\|$$

$$\leq \int_{\mathfrak{h}^+} \Delta(Y) \int_L \|(f(k_2 expY)\| \int_L e^{(\Re\nu-\rho_H)(H(exp(-Y)k_1)}dk_1 dk_2 dY.$$

Since π is an integrable representation in [?] it is shown that

$$\|(f(k_2 expY)\| << e^{-(2+\epsilon)\rho_H(Y)}(1 + \|Y\|)^q$$

Therefore,

$$\|P_{\sigma_j,\nu}(f)(s)\|$$

$$\leq \int_{\mathfrak{h}^+} (1 + \|Y\|)^q e^{-\epsilon\rho_H(Y)} \int_L e^{(\Re\nu-\rho_H)(H(exp(-Y)k_1)}dk_1 dY.$$

In [?] we find a proof of a theorem of Helgason-Osborne which shows that the spherical function $\int_L e^{(\Re\nu-\rho_H)(H(exp(-Y)k_1)}dk_1$ is a bounded function of $\Re\nu$ in an open interval containing zero. Thus, the integral defining $P_{\sigma_j,\nu}(f)(s)$ converges absolutely in a band near $\mathfrak{a}$. Hence, it defines a holomorphic function near $\mathfrak{a}$. Therefore $P_{\sigma_j,\nu}(f)(s)$ is real analytic function in $\nu \in \mathfrak{a}$.

Therefore, if π is an integrable representation, $\nu \to P_{\sigma_j,\nu}(f)$ is nonzero in the complement of a numerable set and hence the direct integral must be supported in the whole $\mathfrak{a}$. When π is not an integrable representation we choose a finite dimensional representation F and an integrable representation $\tilde{\pi}$ of G so that π is the result of applying the Zuckerman functor to $\tilde{\pi}$. Since, to apply the Zuckerman functor amounts to perform tensor product for a finite dimensional representation F we get the support of the continuous spectrum of π is the whole $\mathfrak{a}$. For arbitrary π i can prove $\nu \to P_{\sigma_j,\nu}(f)$ is real analytic by means of helgason-johnson, to be typed later on.

4. Discrete factors of restriction of unitary representations

Proposition 4. *Let (π, V) a unitary irreducible representation of G. Assume that there exists an irreducible $H-$subrepresentation V_1 of π so that*

$H-$smooth vectors of V_1 are smooth vectors of V. Then π is Hilbert discrete decomposable as a representation of H.

In order to justify the statement we recall several important facts.

i) Let $\mathscr{S}(H)$ denotes the space of rapidly decreasing functions on H defined by Wallach in Vol 1. page 230. Then Wallach shows in Vol II that the space of smooth vectors of a unitary representation is an $\mathscr{S}(H)-$module, and that the representation is irreducible if and only if the $\mathscr{S}(H)-$module of smooth vectors is algebraically irreducible. Thus, the subspace of $H-$smooth vectors of V_1 is contained in the subspace of smooth vectors of V.

ii) If V_1 is a finite length representation and F is a finite dimensional representation of H. Then $V_1 \boxtimes F$ is a representation of finite length.

We write $\mathfrak{g} = \mathfrak{h} + \mathfrak{q}$ so that $\mathfrak{q}$ is an $Ad(H)-$invariant complement.

We denote the smooth vectors of a representation by adding a subscript ∞ to the vector space. Hence, for nonnegative integer n, $(V_1)_\infty \boxtimes S^n(\mathfrak{q})$ is a representation of finite length of $\mathscr{U}(H)$. Thus, V_∞ has the $\mathscr{U}(H)-$invariant filtration $\sum_{1\leq n\leq N} \pi(S^n(\mathfrak{q}))(V_1)_\infty$, $N = 1, \cdots, \infty$. Since

$$\bigcup_{N=1}^{\infty} \sum_{0\leq n\leq N} \pi(S^n(\mathfrak{q}))(V_1)_\infty$$

is a $\mathscr{U}(G)-$invariant subspace and π is irreducible this union is a dense subspace of V_∞. The fact that π is unitary forces $\pi_{|_H}$ to be discretely decomposable. Compare with Kobayashi Inv. Math.

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