On the representation theory of a quantum group attached to the Fomin-Kirillov algebra $\mathcal{FK}_3$.

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Introduction

Pogorelsky, V.

- Verma and simple modules for quantum groups at non-abelian groups. Adv. Math. 301 (2016).
- On the representation theory of a quantum group attached to the Fomin-Kirillov algebra $\mathcal{FK}_3$. arXiv:1707.02091.

Introduction

We are interested on the representations of the Drinfeld double

$$\mathcal{D} = \mathcal{D}(\mathfrak{B}(V)\#H)$$

of the bozonization of a Nichols algebra and a Hopf algebra.

Why?

- These are natural generalization of (small) quantum groups.
- The category of graded $\mathcal{D}$-modules is highest-weight [Bellamy-Thiel].
- Categorification of $\mathbb{Z}$-fusion datum associated with cyclic complex reflection groups [Bonnafé-Rouquier].
- These could give information about the Nichols algebra.
- To construct new examples of fusion categories.

Simple $\mathcal{D}$-modules

- $\mathcal{D}$ admits a triangular decomposition:
$$\mathfrak{B}(V) \otimes \mathcal{D}(H) \otimes \mathfrak{B}(\overline{V}) \xrightarrow{\sim} \mathcal{D}.$$
- $\mathcal{D}$ is a graded Hopf algebra
$$\mathcal{D}^n = \oplus_{n=j-i} \mathfrak{B}^i(V) \otimes \mathcal{D}(H) \otimes \mathfrak{B}^j(\overline{V}).$$
- Λ = set of weights = simple $\mathcal{D}(H)$-modules.
- $\mathsf{M}(\lambda) = \mathcal{D} \otimes_{\mathfrak{B}(\overline{V})\#\mathcal{D}(H)} \lambda$ = Verma module of $\lambda \in \Lambda$.
- $\mathsf{L}(\lambda)$ = the head of $\mathsf{M}(\lambda)$.

Theorem

1. $\mathsf{L}(\lambda)$ is simple and graded for all $\lambda \in \Lambda$.
2. Every graded simple module is isomorphic to a shift of $\mathsf{L}(\lambda)$ for some $\lambda \in \Lambda$.

Example

Let T_q be the Taft algebra with q a primitive n-root of unity.
$\Rightarrow$ $\mathcal{D}(T_q) = \langle x, g, h, y \rangle$ with relations

$$x^n = 0 = y^n, \quad g^n = 1 = h^n, \quad gh = hg$$
$$gx = qxg,\ gy = qyg, \quad hx = q^{-1}xh,\ hy = q^{-1}yh,$$
$$xy - qyx = (1 - gh)$$

The representation theory of $\mathcal{D}(T_q)$ was studied by [Chen, Erdmann-Green-Snashall-Taillefer].

Example

The Drinfeld double of the bozonization of Fomin-Kirillov $\mathcal{FK}_3$ over $\Bbbk\mathbb{S}_3$ is generated by

Warning!

- $\dim \lambda \geq 1$
- $\lambda \otimes \mu = \oplus_{\nu \in \Lambda} N^{\nu}_{\lambda,\mu}\, \nu$

with relations

$$x_{(ij)}^2 = x_{(ij)}x_{(ik)} + x_{(jk)}x_{(ij)} + x_{(ik)}x_{(jk)} = 0,$$

Objective

Explain how to handle this problems.

$$y_{(ij)}x_{(ij)} + x_{(ij)}y_{(ij)} = 1 + (ij)(\delta_{(ij)} - \delta_e),$$

$$y_{(ik)}x_{(ij)} + x_{(ij)}y_{(jk)} = (ij)(\delta_{(ik)} - \delta_{(ik)(ij)}),$$

Let $\mathsf{N} = \oplus_{i\in\mathbb{Z}}\mathsf{N}(i)$ be a graded $\mathcal{D}$-module and

$$\mathrm{ch}^{\bullet}\,\mathsf{N} = \sum_{i\in\mathbb{Z}} \mathrm{ch}\,\mathsf{N}(i)\,t^i \in \Lambda[t^{\pm 1}]$$

the graded character of N.

1. The action maps

$$\mathfrak{B}(V)\otimes\mathsf{N}\longrightarrow\mathsf{N} \quad \text{and} \quad \mathfrak{B}(\overline{V})\otimes\mathsf{N}\longrightarrow\mathsf{N}$$

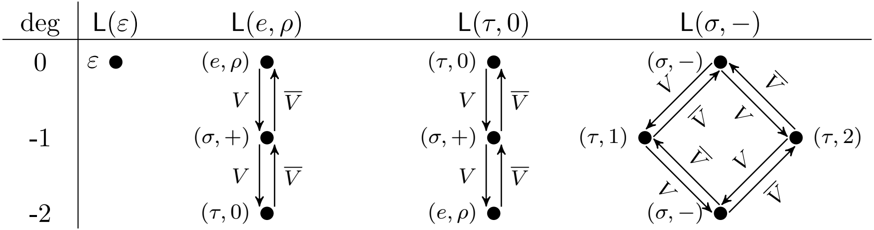
   are morphism in the category of graded $\mathcal{D}(H)$-modules.
2. The graded characters of simple modules form a $\mathbb{Z}[t^{\pm 1}]$-basis of the Grothendieck ring of the category of graded $\mathcal{D}$-modules.

Notation

- $\mathcal{D} = \mathcal{D}(\mathfrak{B}(V)\#\Bbbk\mathbb{S}_3) \simeq \mathfrak{B}(V) \otimes \mathcal{D}(\mathbb{S}_3) \otimes \mathfrak{B}(\overline{V})$.
- The family Λ of simple $\mathcal{D}(\mathbb{S}_3)$-modules is

weight	ε	$(e,-)$	(e,ρ)	$(\tau,0)$	$(\tau,1)$	$(\tau,2)$	$(\sigma,\pm)$
dimension	1	1	2	2	2	2	3

- The Verma modules $\mathsf{M}(e,-)$, $\mathsf{M}(\tau,1)$, $\mathsf{M}(\tau,2)$ and $\mathsf{M}(\sigma,-)$ are simple projective.
- The reminder simple modules can be depicted by



$\mathsf{L}(\tau, 0)$ explicitly

$(12)a_1 = a_2$	$(13)a_1 = \zeta^2 a_2$	$(23)a_1 = \zeta a_2$
$(12)a_2 = a_1$	$(13)a_2 = \zeta a_1$	$(23)a_2 = \zeta^2 a_1$
$(12)a_3 = a_3$	$(13)a_3 = a_4$	$(23)a_3 = a_3$
$(12)a_4 = a_5$	$(13)a_4 = a_3$	$(23)a_4 = a_5$
$(12)a_5 = a_4$	$(13)a_5 = a_5$	$(23)a_5 = a_4$
$(12)a_6 = a_7$	$(13)a_6 = a_7$	$(23)a_6 = a_7$
$(12)a_7 = a_6$	$(13)a_7 = a_6$	$(23)a_7 = a_6$
$x_{(12)}a_1 = 0$	$x_{(13)}a_1 = 0$	$x_{(23)}a_1 = 0$
$x_{(12)}a_2 = 0$	$x_{(13)}a_2 = 0$	$x_{(23)}a_2 = 0$
$x_{(12)}a_3 = a_1 - a_2$	$x_{(13)}a_3 = \zeta^2 a_1 - \zeta a_2$	$x_{(23)}a_3 = 0$
$x_{(12)}a_4 = 0$	$x_{(13)}a_4 = 0$	$x_{(23)}a_4 = \zeta a_1 - \zeta^2 a_2$
$x_{(12)}a_5 = 0$	$x_{(13)}a_5 = 0$	$x_{(23)}a_5 = 0$
$x_{(12)}a_6 = a_5$	$x_{(13)}a_6 = a_5$	$x_{(23)}a_6 = a_3$
$x_{(12)}a_7 = -a_4$	$x_{(13)}a_7 = -a_4$	$x_{(23)}a_7 = -a_5$
$y_{(12)}a_1 = a_3$	$y_{(13)}a_1 = \zeta a_3$	$y_{(23)}a_1 = \zeta^2 a_4$
$y_{(12)}a_2 = -a_3$	$y_{(13)}a_2 = -\zeta^2 a_3$	$y_{(23)}a_2 = -\zeta a_4$
$y_{(12)}a_3 = 0$	$y_{(13)}a_3 = 0$	$y_{(23)}a_3 = a_6$
$y_{(12)}a_4 = -a_7$	$y_{(13)}a_4 = -a_7$	$y_{(23)}a_4 = 0$
$y_{(12)}a_5 = a_6$	$y_{(13)}a_5 = a_6$	$y_{(23)}a_5 = -a_7$
$y_{(12)}a_6 = 0$	$y_{(13)}a_6 = 0$	$y_{(23)}a_6 = 0$
$y_{(12)}a_7 = 0$	$y_{(13)}a_7 = 0$	$y_{(23)}a_7 = 0$

Extensions of simple modules

Assume that E is a graded $\mathcal{D}$-module such that

$$0 \longrightarrow \mathsf{L}(\lambda) \xrightarrow{i} \mathsf{E} \xrightarrow{\pi} \mathsf{L}(\mu) \longrightarrow 0$$

is a short exact sequence with $\lambda, \mu \in \{\varepsilon, (e, \rho), (\tau, 0), (\sigma, -)\}$.

Lemma

- If $\lambda \neq (\sigma, -) \neq \mu$, then $\mathsf{E} \simeq \mathsf{L}(\lambda)[s] \oplus \mathsf{L}(\mu)[t]$.
- If $\lambda = (\sigma, -) \neq \mu$, then E is isomorphic to either

$$\mathsf{M}(\mu)/soc(\mathsf{M}(\mu)), \quad \mathsf{W}(\mu)/soc(\mathsf{W}(\mu)) \quad \text{or} \quad \mathsf{E}_\varepsilon \quad \text{for } \mu = \varepsilon.$$

Proposition

$\mathcal{D}$ is of wild representation type.

Tensor products of simple modules

Proposition

$\mathsf{L}(\tau, 0) \otimes \mathsf{L}(e, \rho) \simeq \mathsf{L}(\tau, 1) \oplus \mathsf{L}(\tau, 2) \oplus \mathsf{L}(\varepsilon)[-2].$

Proof:

$$\begin{aligned} \operatorname{ch}^{\bullet}\left(\mathsf{L}(\tau,0) \otimes \mathsf{L}(e,\rho)\right) &= \operatorname{ch}^{\bullet} \mathsf{L}(\tau,0)\ \operatorname{ch}^{\bullet} \mathsf{L}(e,\rho) \\ &= \operatorname{ch}^{\bullet} \mathsf{L}(\tau,1) + \operatorname{ch}^{\bullet} \mathsf{L}(\tau,2) + t^{-2} \operatorname{ch}^{\bullet} \mathsf{L}(\varepsilon). \end{aligned}$$

Proposition

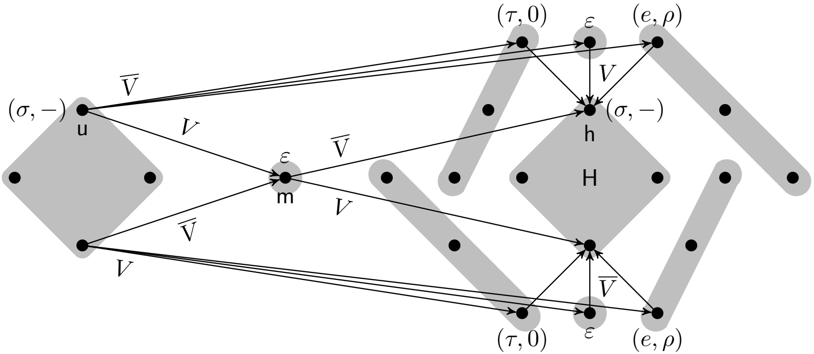
There exists an indecomposable graded $\mathcal{D}$-module A such that

$$\mathsf{L}(\sigma,-)\otimes\mathsf{L}(\sigma,-)\simeq\mathsf{L}(\tau,1)\oplus\mathsf{L}(\tau,2)\oplus\mathsf{L}(\varepsilon)\oplus\mathsf{A}.$$

and

$$\begin{aligned} soc\mathsf{A} &= t^{-1}\mathsf{L}(\sigma,-),\\ soc^2\mathsf{A}/soc\mathsf{A} &\simeq (1+t^{-2}+t^{-4})\mathsf{L}(\varepsilon)\\ &\qquad \oplus(1+t^{-2})\mathsf{L}(e,\rho)\oplus(1+t^{-2})\mathsf{L}(\tau,0),\\ soc^3\mathsf{A}/soc^2\mathsf{A} &\simeq t^{-1}\mathsf{L}(\sigma,-),\\ soc^3\mathsf{A} &= \mathsf{A}. \end{aligned}$$

Picture of A


$(\tau, 0)$
ε
(e, ρ)
$\overline{V}$
V
$(\sigma, -)$
u
V
$(\sigma, -)$
h
ε
$\overline{V}$
m
H
V
$\overline{V}$
V
$\overline{V}$
$(\tau, 0)$
ε
(e, ρ)

Proposition

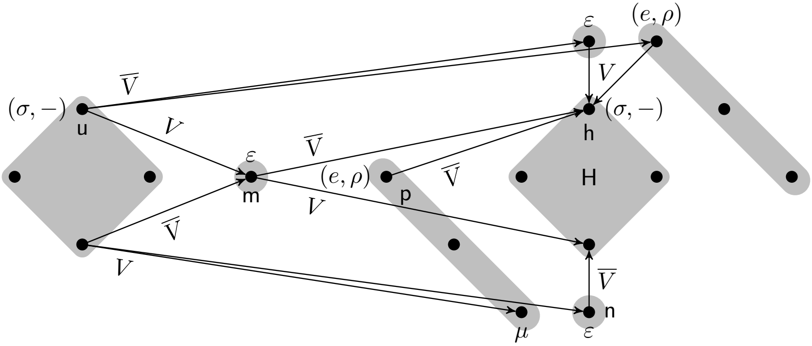
There exists an indecomposable graded $\mathcal{D}$-module B such that

$$\mathsf{L}(e,\rho) \otimes \mathsf{L}(e,\rho) \simeq \mathsf{L}(e,-) \oplus \mathsf{B}$$

and

$$\begin{aligned} soc\mathsf{B} &\simeq t^{-1}\mathsf{L}(\sigma,-), \\ soc^2\mathsf{B}/soc\mathsf{B} &\simeq (1+t^{-2}+t^{-4})\mathsf{L}(\varepsilon) \oplus (1+t^{-2})\mathsf{L}(e,\rho), \\ soc^3\mathsf{B}/soc^2\mathsf{B} &\simeq t^{-1}\mathsf{L}(\sigma,-), \\ soc^3\mathsf{B} &= \mathsf{B}. \end{aligned}$$

Picture of B


ε
(e, ρ)
$\overline{V}$
V
$(\sigma, -)$
u
V
$(\sigma, -)$
h
ε
$\overline{V}$
(e, ρ)
$\overline{V}$
H
m
p
V
$\overline{V}$
V
$\overline{V}$
n
μ
ε

Proposition

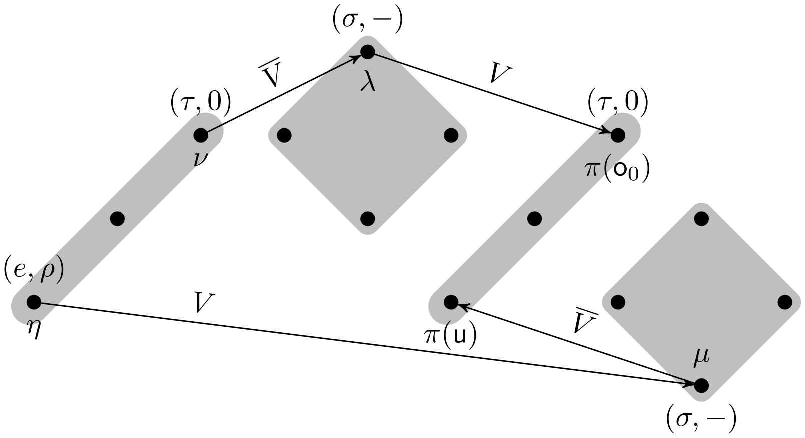
There exists an indecomposable graded $\mathcal{D}$-module C such that

$$\mathsf{L}(\sigma,-)\otimes\mathsf{L}(e,\rho)\simeq\mathsf{L}(\sigma,+)\oplus\mathsf{C}$$

and

$$\begin{aligned} soc\mathsf{C} &\simeq t^{-1}\mathsf{L}(\tau,0),\\ soc^2\mathsf{C}/soc\mathsf{C} &\simeq (1+t^{-2})\mathsf{L}(\sigma,-),\\ soc^3\mathsf{C}/soc^2\mathsf{C} &\simeq t^{-1}\mathsf{L}(\tau,0),\\ soc^3\mathsf{C} &= \mathsf{C}. \end{aligned}$$

Picture of C


$(\sigma,-)$
$\overline{V}$
λ
V
$(\tau,0)$
$(\tau,0)$
ν
$\pi(\mathsf{o}_0)$
(e,ρ)
V
η
$\pi(\mathsf{u})$
$\overline{V}$
μ
$(\sigma,-)$

On projective modules

- Let $\mathsf{P}(\lambda)$ be the projective cover of $\mathsf{L}(\lambda)$.

Then

$$\begin{aligned}
\mathrm{ch}^\bullet \mathsf{P}(\varepsilon) &= (1+t^4)\,\mathrm{ch}^\bullet \mathsf{M}(\varepsilon) + (t+t^3)\,\mathrm{ch}^\bullet \mathsf{M}(\sigma,-),\\
\mathrm{ch}^\bullet \mathsf{P}(e,\rho) &= \mathrm{ch}^\bullet \mathsf{M}(e,\rho) + t\,\mathrm{ch}^\bullet \mathsf{L}(\sigma,-) + t^2\,\mathrm{ch}^\bullet \mathsf{M}(\tau,0),\\
\mathrm{ch}^\bullet \mathsf{P}(\sigma,-) &= (1+t^2)\,\mathrm{ch}^\bullet \mathsf{L}(\sigma,-)\\
&\qquad + t\,\mathrm{ch}^\bullet \mathsf{M}(\varepsilon) + t\,\mathrm{ch}^\bullet \mathsf{M}(e,\rho) + t\,\mathrm{ch}^\bullet \mathsf{M}(\tau,0),\\
\mathrm{ch}^\bullet \mathsf{P}(\tau,0) &= \mathrm{ch}^\bullet \mathsf{M}(\tau,0) + t\,\mathrm{ch}^\bullet \mathsf{M}(\sigma,-) + t^2\,\mathrm{ch}^\bullet \mathsf{M}(e,\rho),
\end{aligned}$$

by the BGG Reciprocity [V].

On projective modules

- Let $\mathsf{P}(\lambda)$ be the projective cover of $\mathsf{L}(\lambda)$.

Then, as graded $\mathfrak{B}(V)\#\mathcal{D}(\mathbb{S}_3)$-modules,

$$\begin{aligned}
\mathsf{P}(\varepsilon) &= \mathsf{M}(\varepsilon)[4] \oplus \mathsf{M}(\sigma,-)[3] \oplus \mathsf{M}(\sigma,-)[1] \oplus \mathsf{M}(\varepsilon) \\
\mathsf{P}(e,\rho) &= \mathsf{M}(\tau,0)[2] \oplus \mathsf{M}(\sigma,-)[1] \oplus \mathsf{M}(e,\rho) \\
\mathsf{P}(\tau,0) &= \mathsf{M}(e,\rho)[2] \oplus \mathsf{M}(\sigma,-)[1] \oplus \mathsf{M}(\tau,0) \\
\mathsf{P}(\sigma,-) &= \mathsf{M}(\sigma,-)[2] \oplus \mathsf{M}(\varepsilon)[1] \oplus \mathsf{M}(e,\rho)[1] \oplus \mathsf{M}(\tau,0)[1] \oplus \mathsf{M}(\sigma,-)
\end{aligned}$$

Tensoring by projective modules

Proposition [V]

$$\mathsf{P}(\lambda_1) \otimes \mathsf{P}(\lambda_2) \simeq \oplus_{\lambda,\mu \in \Lambda} \quad p_{\mathsf{P}(\lambda_1),\mathsf{W}(\lambda)} \quad p_{\mathsf{P}(\lambda_2),\mathsf{M}(\mu)} \quad \mathsf{Ind}(\lambda \cdot \mu),$$

for all $\lambda_1, \lambda_2 \in \Lambda$.

Fact: The sets $\{\mathrm{ch}^\bullet \, \mathsf{M}(\lambda) \mid \lambda \in \Lambda\}$ and $\{\mathrm{ch}^\bullet \, \mathsf{W}(\lambda) \mid \lambda \in \Lambda\}$ are $\mathbb{Z}[t, t^{-1}]$-bases of the Grothendieck ring of the category of projective $\mathcal{D}$-modules.

Example

$\mathsf{P}(\varepsilon) \otimes \mathsf{P}(\varepsilon)$ is isomorphic to

$$
\begin{aligned}
& t^{-4}(t^8 + t^6 + 4t^4 + t^2 + 1)\mathsf{P}(\varepsilon) \oplus \\
& 2t^{-1}(1+t^2)^2\, \mathsf{P}(e,-) \oplus t^{-2}(1+t^2)^3\, \mathsf{P}(e,\rho) \oplus \\
& 2t^{-3}(1+t^2+t^4)(1+t^2)^2\, \mathsf{P}(\sigma,-) \oplus \\
& 8t^{-1}(1+t^2+t^4)(1+t^2)\, \mathsf{P}(\sigma,+) \oplus \\
& t^{-2}(1+t^2)^3\, \mathsf{P}(\tau,0) \oplus \\
& t^{-2}\left((1+t^4)^2 + 2(1+t^2)^4\right)(\mathsf{P}(\tau,1) \oplus \mathsf{P}(\tau,2)).
\end{aligned}
$$

GRACIAS!