The Double Pendulum Fractal

Jeremy S. Heyl
Department of Physics and Astronomy, University of British Columbia
6224 Agricultural Road, Vancouver, British Columbia, Canada, V6T 1Z1; Canada Research Chair
(Dated: August 11, 2008)

The double pendulum is an excellent example of a chaotic system. Small differences in initial conditions yield exponetially diverging outcomes. Although the chaotic nature of the double pendulum is well studied, the fractal nature of the dependence of the outcomes on the initial conditions has not been examined. Numerical simulations of a double pendulum over long timescales reveal the fractal structure of the time evolution of this system. Energetics determine the gross shape of the fractal. It also exhibits quasi-self-similar structure, and other hallmarks of classic fractals such as the Mandelbrot and Julia sets. Additionally, in common with other Hamiltonian systems the double pendulum exhibits a fat fractal structure; specfically for a small fraction of the initial conditions, it is impossible to determine with certainty whether either pendulum will flip in the future even with arbitrarily accurate initial conditions.

I. INTRODUCTION

The double pendulum is one of the simplest systems to exhibit chaotic motion[1–3]. A double pendulum consists of one pendulum hanging from another. The system is governed by a set of coupled non-linear ordinary differnential equations. Although the structure of the system is relatively simple, above a certain initial energy the motion of the system is chaotic.

Many researchers have examined the dynamics of the double pendulum from various points of view from how to make a better baseball bat[4] or golf swing[5], the dynamics of church bells[6] and the adequacy of Newton's axioms to describe its dynamics[7–9], to the quantum mechanics of the system[10,11]. However, the long-term evolution of the system as a function of initial conditions has not been studied in great detail.

This paper presents the results of numerical simulations of a double pendulum over many thousands of oscillations and examines the fractal nature of the relationship between the initial conditions and the final outcome (specifically whether either part of the pendulum flips). § II introduces the physical system and sets a system of coordinates, and § III derives the Lagrangian and the equations of motion. § IV presents the results of the simulations.

II. COORDINATES

Specifically, let us model the double pendulum by two identical thin rods ($I = \frac{1}{12}Ml^2$) connected by a pivot and the end of the upper rod suspended from a pivot. For comparison Shinbrot *et al.*[2] studied two nearly identical pendulums and Levien and Tan[3] looked at a set of pendulums with widely different masses.

It is natural to define the coordinates to be the angles between each rod and the vertical. These coordinates are denoted by θ_1 and θ_2 (see Fig. 1). This are the same coordinates used in by Shinbrot *et al.*[2]. Although one could neglect the masses of the rods as Shinbrot *et al.*[2] did in their Appendix B, including the moment of inertia of the rods better reflects the physical system that they describe, so the equations of motion derived here will differ slightly.

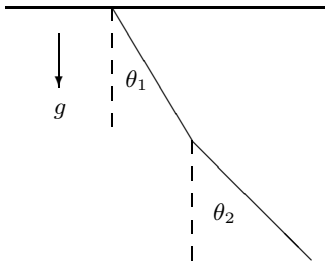

FIG. 1: Coordinates for the double pendulum

The position of the centre of mass of the two rods in terms of these coordinates is

$$x_1 = \frac{l}{2}\sin\theta_1, x_2 = l\left(\sin\theta_1 + \frac{1}{2}\sin\theta_2\right), \quad (1)$$

and

$$y_1 = -\frac{l}{2}\cos\theta_1 y_2 = -l\left(\cos\theta_1 + \frac{1}{2}\cos\theta_2\right). \quad (2)$$

This is enough information to write out the Lagrangian.

III. LAGRANGIAN

The Lagrangian is given by

$$L = \frac{1}{2}m\left(v_1^2 + v_2^2\right) + \frac{1}{2}I\left(\dot\theta_1^2 + \dot\theta_1^2\right) - mg\left(y_1 + y_2\right) \quad (3)$$

where the kinetic energy is the sum of the kinetic energy of the centre of mass of each rod and the kinetic energy about the centres of mass of the rods[2]. The potential energy of a body in a uniform gravitational field is given by the potential energy at the centre of mass.

Plugging in the coordinates above and doing a bit of algebra gives

$$L = \frac{1}{6}ml^2\left[\dot{\theta}_2^2 + 4\dot{\theta}_1^2 + 3\dot{\theta}_1\dot{\theta}_2\cos(\theta_1-\theta_2)\right] + \frac{1}{2}mgl\left(3\cos\theta_1 + \cos\theta_2\right) \quad (4)$$

Before looking at the equations of motion, there is only one conserved quantity (the energy), and no conserved momenta. The two momenta are

$$p_{\theta_1} = \frac{\partial L}{\partial\dot{\theta}_1} = \frac{1}{6}ml^2\left[8\dot{\theta}_1 + 3\dot{\theta}_2\cos(\theta_1-\theta_2)\right] \quad (5)$$

and

$$p_{\theta_2} = \frac{\partial L}{\partial\dot{\theta}_2} = \frac{1}{6}ml^2\left[2\dot{\theta}_2 + 3\dot{\theta}_1\cos(\theta_1-\theta_2)\right] \quad (6)$$

Let's invert these expressions to get

$$\dot{\theta}_1 = \frac{6}{ml^2}\frac{2p_{\theta_1} - 3\cos(\theta_1-\theta_2)p_{\theta_2}}{16 - 9\cos^2(\theta_1-\theta_2)} \quad (7)$$

and

$$\dot{\theta}_2 = \frac{6}{ml^2}\frac{8p_{\theta_2} - 3\cos(\theta_1-\theta_2)p_{\theta_1}}{16 - 9\cos^2(\theta_1-\theta_2)}. \quad (8)$$

The remaining equations of motion are

$$\dot{p}_{\theta_1} = \frac{\partial L}{\partial\theta_1} = -\frac{1}{2}ml^2\left[3\frac{g}{l}\sin\theta_1 + \dot{\theta}_1\dot{\theta}_2\sin(\theta_1-\theta_2)\right], \quad (9)$$

and

$$\dot{p}_{\theta_2} = \frac{\partial L}{\partial\theta_2} = -\frac{1}{2}ml^2\left[\frac{g}{l}\sin\theta_2 - \dot{\theta}_1\dot{\theta}_2\sin(\theta_1-\theta_2)\right]. \quad (10)$$

These equations of motion cannot be integrated analytically but they are straightforward to integrate numerically[12]. Here they are integrated using a fourth-order Runge-Kutta technique with adaptive stepsizes[13]. A straightforward question is whether a particular set of initial conditions with both pendulums stationary will result with either pendulum flipping and if one does how long is it before it flips.

IV. FRACTALS

Fig. 2 shows when and whether either pendulum flips. The boundary of the central white region is defined in part by energy conservation. Our initial conditions have the rods stationary so our initial Hamiltonian is

$$H_0 = -\frac{1}{2}mgl\left(3\cos\theta_1 + \cos\theta_2\right). \quad (11)$$

Because the Lagrangian is not a function of time explicitly the Hamiltonian is conserved. For either pendulum to flip barely, $\theta_1 = \pi$ or $\theta_2 = \pi$. Looking at the Hamiltonian the latter requires less energy so

$$H_0 = -\frac{1}{2}mgl\left(3\cos\theta_1 + \cos\theta_2\right) > -\frac{1}{2}mgl(3-1) = -mgl \quad (12)$$

Rearranging yields

$$3\cos\theta_1 + \cos\theta_2 > 2 \quad (13)$$

so within the following curve $3\cos\theta_1 + \cos\theta_2 = 2$ it is energetically impossible for either pendulum to flip. Outside this region, the pendulums can flip but this is different from determining when they will flip.

Energetics cannot determine whether either pendulum will flip within a set period of time. The question of the future evolution of the system is much more complicated. Chaotic motion is intimately connected with fractals. Fig. 2 exhibits a fractal structure. First there is a large island of stability where neither pendulum can flip. Two large self-similar buds (and many smaller ones) grow off of the main island of instability. Furthermore, there are small islands of relative stability within the regions where the pendulums flip rather quickly (see Fig. 3). These small islands are distorted copies of the main central stable region.

One can estimate the fractal dimension of the boundary between the stable and unstable initial conditions. Because this boundary is indeed a fractal, its area depends on the scale of measurement. The area of the boundary is estimated by counting the number of initial conditions on a rectilinear grid of a given scale that do not result in either pendula flipping after a long time. Only those initial conditions that have at least a single neighbor (again at a particular scale) that does flip are counted as members of the boundary; an estimate of the area of the boundary is the product of the number of points in the boundary and the square of the distance between the grid points.

The entire domain of initial conditions $-\pi < \theta_1, \theta_2 < \pi$ is divided into a Cartesian grid of dimension $2^{15}\times 2^{15}$ for a total of $2^{30} \approx 10^9$ unique sets of initial conditions. The number of points in the boundary are determined on the full grid and for a grid consisting of every second grid point in each direction, every fourth grid point and so on. To minimize counting errors, the sparse grids are shifted over the domain and averaged. For example, at the second to finest grid spacing, the number of boundary points is determined for four different grids each shifted relative to each other. The spacing of the nth finest grid is $\pi 2^{n-15}\times\pi 2^{n-15}$ and there are 2^{2n-2} such grids. In this way the number of initial conditions sampled in each grid is 2^{30}, independent of n. The area of the boundary $\mu(\epsilon)$ is defined as the product of the number of grid points in the boundary and the square of the grid spacing ϵ^2.

The left panel of Fig. 4 shows the area of the boundary as defined above for the entire set of initial conditions (Fig. 2) and those initial conditions with $\theta_1\theta_2 > 0$ and $\theta_1\theta_2 < 0$. Different parts of the boundary have approximately the same fractal dimensions but the region with

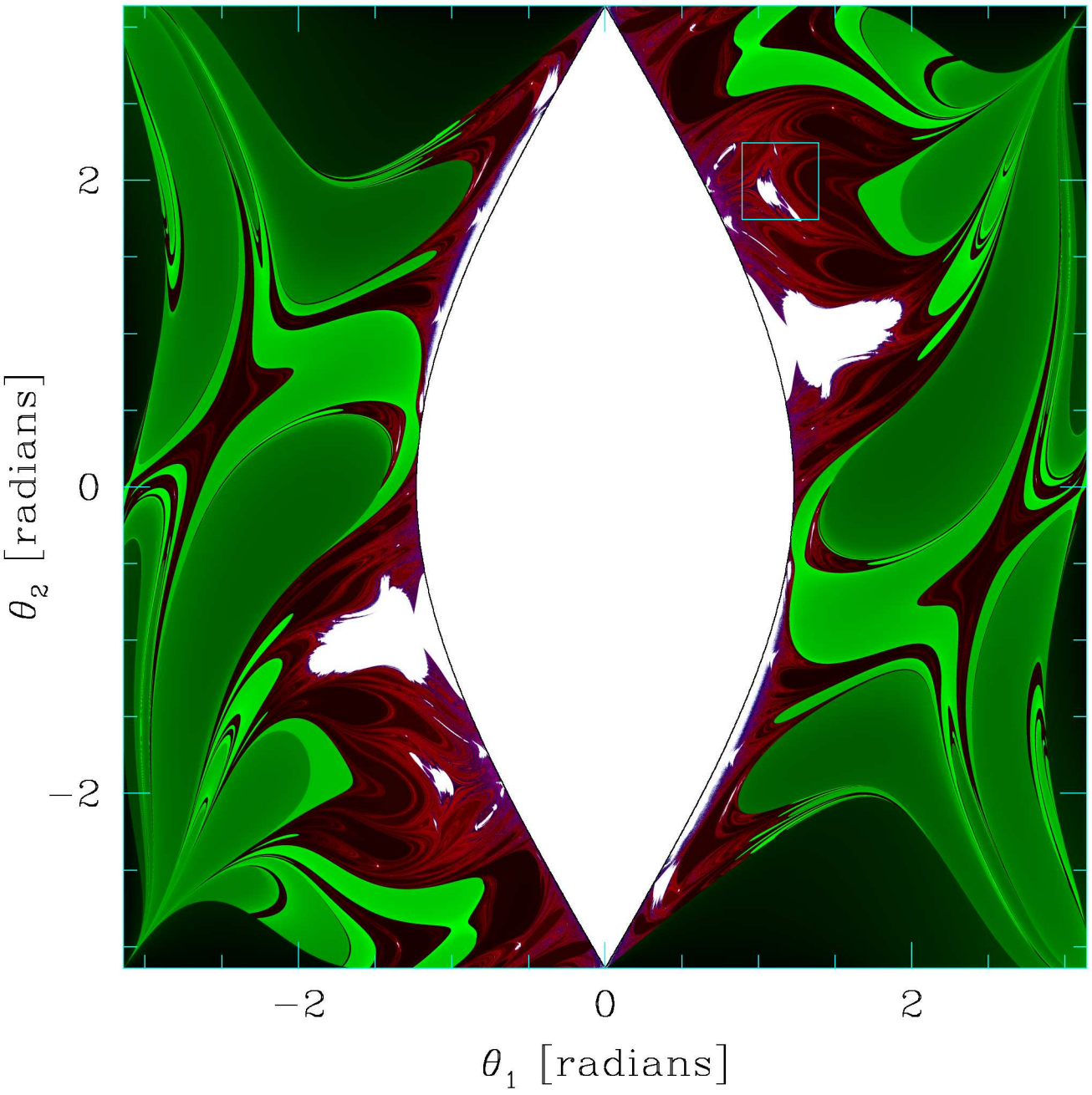

FIG. 2: Outcomes for the double pendulum. The initial angles of the two rods θ_1 and θ_2 range from $-\pi$ to π. The color of each pixel indicates how much time elapses before either pendulum flips in units of $\sqrt{l/g}$. Green indicates the one of pendulums flips within 10 time units, red indicates between 10 and 100 units, purples denotes between 100 and 1000 units and blue shows those initial conditions that lead to flips only after 1000 to 10000 $\sqrt{l/g}$ have elapsed. Those that do not flip within 10000 $\sqrt{l/g}$ are plotted white. For reference the period of oscillation of the lower pendulum is $2\pi\sqrt{3/2}\sqrt{l/g} \approx 7.7\sqrt{l/g}$. Within each colour range, the darker shades flip sooner. Within the solid black curve, it is not energetically possible for either pendulum to flip.

$\theta_1\theta_2 > 0$ contributes more to the boundary area for the finer grids than the region with $\theta_1\theta_2 < 0$. However, at the smallest grid scales the area of the boundary does not go to zero as the grid scale goes to zero. This is evidence of a "fat fractal".

For each of the regions the area of the boundary is fit with the sum of a power-law and a constant, $\mu(\epsilon) = A\epsilon^{\beta} + \mu(0)$[14,15]. If $\mu(0) = 0$, the boundary is a thin fractal and β is equal to the codimension $D - d$, where d is the fractal dimension and D is the dimension of the

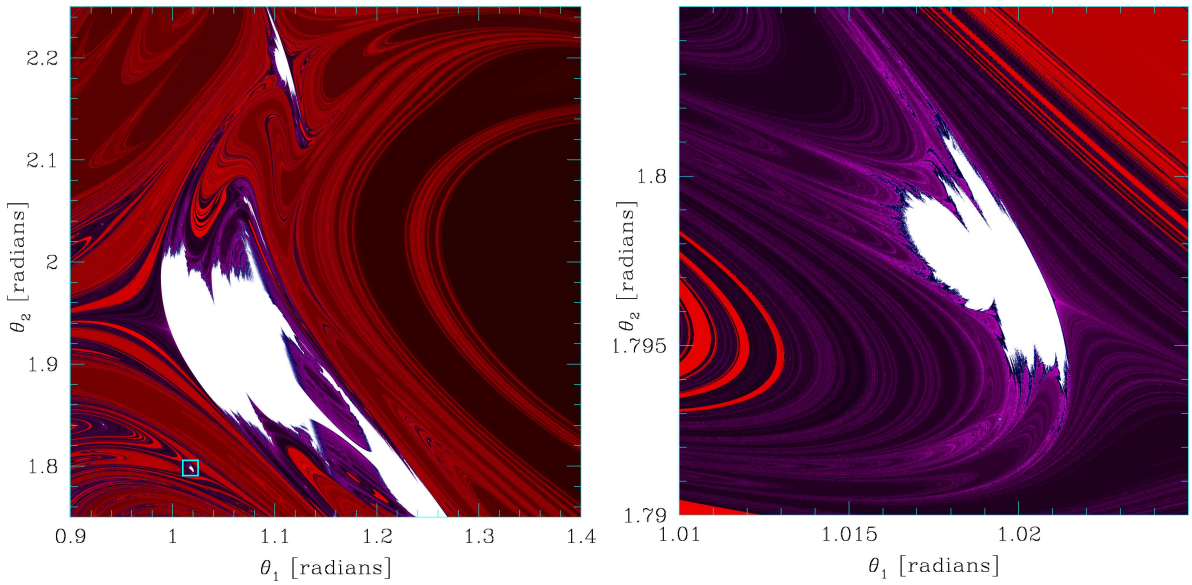

FIG. 3: Outcomes for the double pendulum (detail). The left-hand panel is an enlargement of the region within the cyan box in Fig. 2. The right-hand panel is an enlargement of the region in the region within the cyan box in the left-hand panel. The colors designate the time to flip as in Fig. 2.

embedding space. In general, the value of β is dubbed the *fatness exponent* and can be defined precisely as

$$\beta = \lim_{\epsilon \to 0} \frac{\log\left[\mu(\epsilon) - \mu(0)\right]}{\log \epsilon}. \tag{14}$$

In all three cases, $\mu(0) \neq 0$. For the entire set $\mu(0) \approx 0.057$ or about 0.15% of the entire area of initial conditions $(4\pi^2)$.

Umberger and Farmer[15] studied the Poincaré section of various Hamiltonian mappings and found that the area filled by a particular orbit is greater than zero; more precisely, the closure of the orbit has positive Lebesque measure. This paper examines the totality of possible orbits with static initial conditions for a non-linear Hamiltonian system, the double pendulum, for which no simple analytic mapping exists. Again the dependence of the future evolution of the system on its initial conditions exhibits the structure of a fat fractal. Specifically, the set of initial conditions that eventually results in either pendulum flipping has fat fractal boundary.

Farmer[14] examined the sensitivity of the behaviour of the mapping

$$x_{k+1} = f_r(x_k) = r\left(1 - 2x_k^2\right) \tag{15}$$

on the value of the parameter r, in particular the fraction of chaotic parameter intervals and how the value of this fraction scales (the scaling exponent). In the double pendulum for all initial conditions of sufficient energy the motion is chaotic, *i.e.* the Lyapunov exponent is positive[2,3]. The sensitivity to initial conditions examined here is much more gross. If one defines chaotic as the inability to forecast whether a pendulum will flip with a given level on uncertainty in the initial conditions (ϵ), the analysis here maps nicely onto that of Farmer[14]. Firstly, the results presented in Fig. 4 imply that for a small fraction of the initial conditions of about 0.15% it is impossible to determine with certainty whether either pendulum will flip in the future even with arbitrarily accurate initial conditions. A finite fraction of the initial conditions lie on the boundary between flipping and not flipping. Secondly, the value of $\mu(\epsilon)/(4\pi^2)$ gives the probability that if your initial conditions are specified with an uncertainty of ϵ that the numerical calculation of whether either pendulum flips will be unreliable due to the uncertainty in the initial conditions.

V. CONCLUSIONS

Numerical simulations of the double pendulum reveal that the set of initial conditions that do not result in either pendulum flipping within a set period of time has a fractal structure. Specfically, the boundary of this set is a fat fractal similar to that found by Farmer[14] for a simple Hamiltonian logistic map, indicating that fat fractal boundaries may be a hallmark of chaos in non-linear Hamiltonian systems.

Acknowledgments

The Natural Sciences and Engineering Research Council of Canada, Canadian Foundation for Innovation and the British Columbia Knowledge Development Fund supported this work. Correspondence and requests for ma-

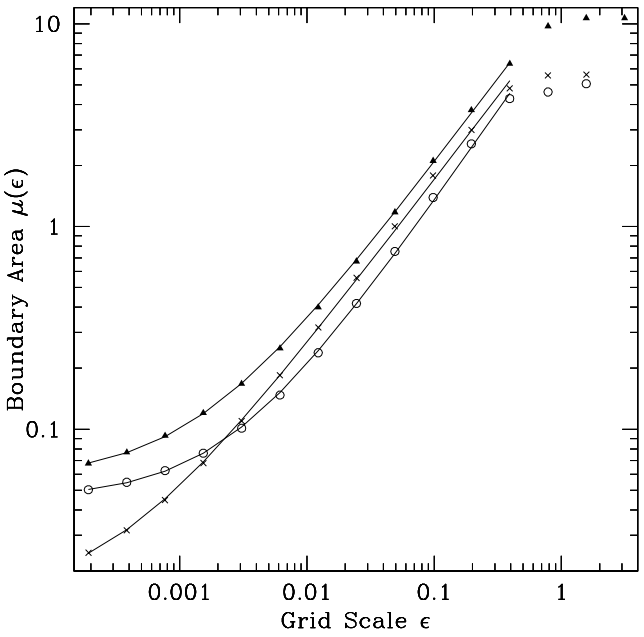

FIG. 4: The panel depicts area of the boundary of the stable set as a function of the measuring scale. At the smallest grid scale, the upper points (solid triangles) are for entire set (Fig. 2). The middle points (open circles) are for $\theta_1\theta_2 < 0$ and the lower points (crosses) are for $\theta_1\theta_2 > 0$. In all cases the asymptotic behaviour of each boundary is fitted by the sum of a power law and a constant.

terials should be addressed to heyl@phas.ubc.ca. This research has made use of NASA's Astrophysics Data System Bibliographic Services

[1] P. H. Richter and H.-J. Scholz, in *Stochastic phenomena and chaotic behaviour in complex systems (A86-16194 05-70)*, edited by P. Schuster (Springer-Verlag, Berlin and New York, 1984), pp. 86–97.

[2] T. Shinbrot, C. Grebogi, J. Wisdom, and J. A. Yorke, American Journal of Physics **60**, 491 (1992).

[3] R. B. Levien and S. M. Tan, American Journal of Physics **61**, 1038 (1993).

[4] R. Cross, American Journal of Physics **73**, 330 (2005).

[5] A. R. Penner, Reports of Progress in Physics **66**, 131 (2003).

[6] S. Ivorra, M. J. Palomo, G. Verdú, and A. Zasso, Meccnica **41**, 46 (2006).

[7] W. Stadler, American Journal of Physics **53**, 233 (1985).

[8] S. F. Felszeghy, American Journal of Physics **53**, 230 (1985).

[9] W. Stadler, American Journal of Physics **50**, 595 (1982).

[10] J. Ford and G. Mantica, American Journal of Physics **60**, 1086 (1992).

[11] L. Perotti, Phys. Rev. E **70**, 066218 (2004).

[12] D. R. Stump, American Journal of Physics **54**, 1096 (1986).

[13] W. H. Press, S. A. Teukolsky, W. T. Vettering, and B. P. Flannery, *Numerical Recipes in C*, 2nd ed. (Cambridge Univ. Press, Cambridge, 1992).

[14] J. D. Farmer, Physical Review Letters **55**, 351 (1985).

[15] D. K. Umberger and J. D. Farmer, Physical Review Letters **55**, 661 (1985).