Some questions on Hopf algebras

(in relation with Compact Quantum Groups)

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How to describe the compact Lie groups?

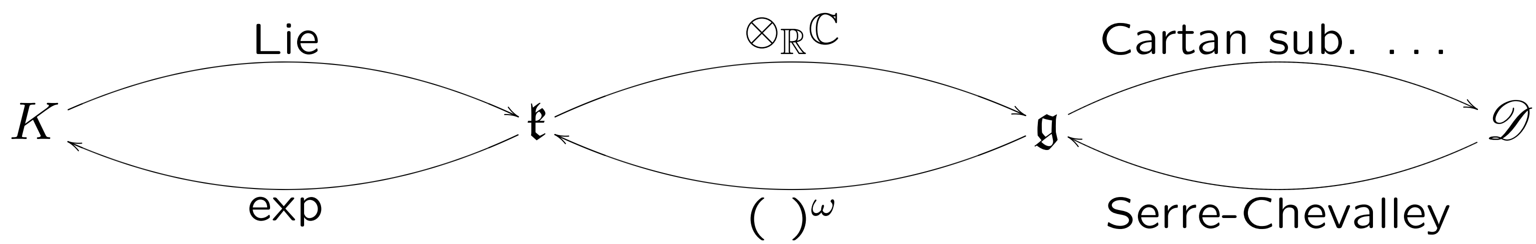
By exhibition: $U(n)$, $SU(n)$, $O(n)$, $SO(n)$, $Spin(n)$, $Sp(n)$,

the exceptional (compact) Lie groups,

the torus T^n,

finite groups.

By structure:



compact (connected, semisimple) Lie group

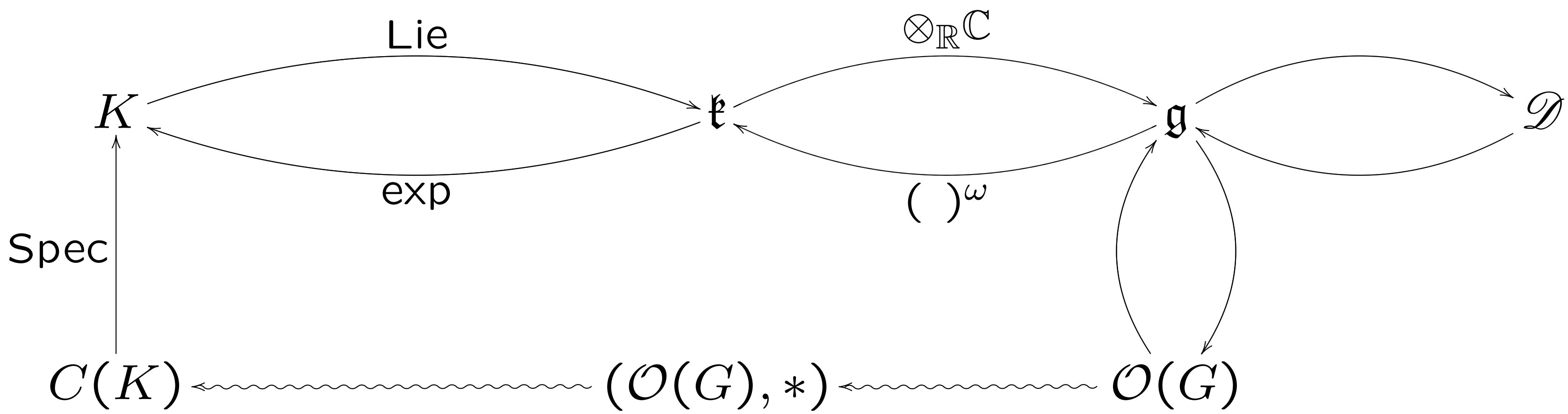
compact Lie algebra (Killing negative)

semisimple Lie algebra

Dynkin diagram

ω compact involution:
existence from Lie algebra structure (Cartan).

Tannakian approach:



$\mathcal{O}(G) = \bigoplus_{V \in \mathrm{Irr}\,\mathfrak{g}} \mathrm{End}(V)^* =$ algebra of polynomial functs. on G

$(\mathcal{O}(G), *)$ is a $*$-Hopf algebra, $*$ induced by the compact inv. ω

$C(K) = C^*$-algebra obtained by completion

Remark. $\mathcal{O}(G)$ is a commutative (complex) Hopf algebra.

H commutative Hopf alg. $\Longrightarrow$ $H = \mathcal{O}(G)$, G pro-algebraic grp.

- $H = \bigcup$ finitely generated Hopf subalgebras.

- H finitely generated commutative Hopf subalgebra $\Longleftrightarrow$ $H = \mathcal{O}(G)$, G an algebraic group.

- In this case, G connected $\Longleftrightarrow$ H domain.

- Also G reductive $\Longleftrightarrow$ H cosemisimple.

- H finitely generated commutative $\Longrightarrow$ H noetherian & Krull dim $H < \infty$.

Woronowicz CQG: $(A, \mathcal{A})$

$(\mathcal{A}, *)$ is a $*$-Hopf algebra **not necessarily commutative**, $*$ positive definite in a suitable sense, generated by a matrix coalgebra.

$A = C^*$-algebra obtained by completion from $\mathcal{A}$.

Example. $\mathcal{O}_q(SL(2))$ is a well-known Hopf algebra. For q real or imaginary it has two involutions: one is positive definite $\rightsquigarrow$ (completion) $C_q(SU(2))$

Remark. (Vaksman-Soibelman). The spectrum of $C_q(SU(2))$ is parametrized by the symplectic leaves of the Poisson structure on $SU(2)$ behind the quantization.

Woronowicz CQG: enough to have: a $*$-Hopf algebra $(\mathcal{A}, *)$, where

$\mathcal{A}$ is a Hopf algebra generated by a matrix coalgebra ($\iff$ finitely generated $\iff$: affine).

$*$ is positive definite in a suitable sense (compact involution).

Problem 1. Classify affine Hopf algebras, having a compact involution $*$.

We split the problem in two:

Problem 2. Classify affine cosemisimple Hopf algebras.

Problem 3. Given an affine cosemisimple Hopf algebra, decide when it has $*$ positive definite. **(not always!)**
If $\mathcal{S}^2 = \text{id}$? **(not even in this case!)**

Remark. Two compact involutions are conjugated by a Hopf algebra automorphism.

Some variations of Problem 2:

Problem 4. Classify affine cosemisimple Hopf algebras with **finite Gelfand-Kirillov dimension**.

Remark. The right quantum analogue of algebraic group seems to be affine with finite GK-dim (instead of noetherian).

Problem 5. Classify affine coss Hopf algs. with finite GK-dim that are **domains**. (i.e. connected quantum groups).

Still on Problem 2:

Problem 6. Classify **finite-dimensional** cosemisimple Hopf algebras.

Problem 7. Can affine cosemisimple Hopf algebras (with finite GK-dim) be described in terms of those that are domains and finite-dimensional ones?

Problem 8. Classify / characterize cosemisimple Hopf algebras with finite GK-dim that are quotients of the CQG in the first talk of the Seminar.

Examples.

• Let G be a finitely generated group. Then the (cosemisimple) group algebra $\mathbb{C}G$ has finite GK-dim $\iff$ G is nilpotent-by-finite (Gromov, Milnor, Wolf, . . .).

• Let G be a semisimple algebraic group, K its compact form. The Hopf algebra $\mathcal{O}_q(G)$ (suitable q) has a compact involution $\rightsquigarrow$ (completion) $C_q(K)$.

Remark. (Soibelman) The spectrum of $C_q(K)$ is parametrized by the symplectic leaves of the Poisson structure on K behind the quantization.

• There are also multiparametric versions: the Hopf algebra $\mathcal{O}_{q,F}(G)$ (suitable q, F) has a compact involution $\rightsquigarrow$ (completion) $C_{q,F}(K)$.

Remark. (Levendorskii-Soibelman) Again there is a relation between the spectrum of $C_{q,F}(K)$ and the symplectic leaves of the Poisson structure on K.

• (Ohn). Classification of cosemisimple Hopf algebras with the same corepresentation theory of $SL(2)$ and $SL(3)$ (compact involutions?).

Remark. H Hopf algebra, $\sigma : H \otimes H \to \mathbb{C}$ a 2-cocycle
$\rightsquigarrow H_\sigma =$ same comultiplication, multiplication conjugated by σ

• H cosemisimple Hopf alg., σ 2-cocycle $\Longrightarrow$ H_σ cosemisimple.

Problem 9. Classify all 2-cocycles σ on $H = \mathcal{O}(G)$ where G is semisimple (Etingof, Gelaki).
When $\mathcal{O}(G)_\sigma$ admits a compact involution?

Woronowicz CQG Tannakian formalism: enough to have a rigid unitary tensor category generated by one object.

Problem 10. When a semisimple tensor category is unitary?

Let U be a complex Hopf algebra, $\rho : U \to \operatorname{End} V$ a fin.-dim. rep., $C_\rho :=$ Image $\rho^t : (\operatorname{End} V)^* \to U^*$, a subcoalgebra of U°.

Let $\mathcal{C}$ be a tensor subcategory of rep U (i.e. an abelian subcategory closed under $\otimes$ and $(\ )^*$) $\rightsquigarrow A(\mathcal{C}) = \sum_{(V,\rho)\in\mathcal{C}} C_\rho \leq U^\circ$.

Problem 11. Classify all $(U, \mathcal{C})$ s.t. $A(\mathcal{C})$ is cosemisimple. When $A(\mathcal{C})$ is a domain, resp. has finite GK?

Problem 12. Given an affine cosemisimple Hopf algebra H (domain, with finite GK-dim) does there exist $(U, \mathcal{C})$ s.t. $H \simeq A(\mathcal{C})$?

A Hopf algebra U is **reductive** if rep U (the category of fin.-dim. reps.) is semisimple.

Problem 13. Classify reductive FDR Hopf algebras.

Remark. All pointed Hopf algebras H with abelian $G(H)$, domains, with finite GK-dim and *reductive* are classified (A-Radford-Schneider) $\rightsquigarrow$ close to multiparameter quantum groups.

Finally, an algebraic group is a nonsingular affine variety.

Problem 14. Does an affine coss. Hopf alg. (domain, finite GK-dim) satisfy cohomological properties indicating regularity? (See surveys by Brown, Zhang, Goodearl . . .).