On finite-dimensional Hopf algebras

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Abstract. This is a survey on the state-of-the-art of the classification of finite-dimensional complex Hopf algebras. This general question is addressed through the consideration of different classes of such Hopf algebras. Pointed Hopf algebras constitute the class best understood; the classification of those with abelian group is expected to be completed soon and there is substantial progress in the non-abelian case.

1. Introduction

Hopf algebras were introduced in the 1950's from three different perspectives: algebraic groups in positive characteristic, cohomology rings of Lie groups, and group objects in the category of von Neumann algebras. The study of non-commutative non-cocommutative Hopf algebras started in the 1960's. The fundamental breakthrough is Drinfeld's report [25]. Among many contributions and ideas, a systematic construction of solutions of the quantum Yang-Baxter equation (qYBE) was presented. Let V be a vector space. The qYBE is equivalent to the braid equation:

$$(c \otimes \mathrm{id})(\mathrm{id} \otimes c)(c \otimes \mathrm{id}) = (\mathrm{id} \otimes c)(c \otimes \mathrm{id})(\mathrm{id} \otimes c), \qquad c \in GL(V \otimes V). \qquad (1.1)$$

If c satisfies (1.1), then (V, c) is called a braided vector space; this is a down-to-the-earth version of a braided tensor category [54]. Drinfeld introduced the notion of quasi-triangular Hopf algebra, meaning a pair (H, R) where H is a Hopf algebra and $R \in H \otimes H$ is invertible and satisfies the appropriate conditions, so that every H-module V becomes a braided vector space, with c given by the action of R composed with the usual flip. Furthermore, every finite-dimensional Hopf algebra H gives rise to a quasi-triangular Hopf algebra, namely the Drinfeld double $D(H) = H \otimes H^*$ as vector space. If H is not finite-dimensional, some precautions have to be taken to construct $D(H)$, or else one considers Yetter-Drinfeld modules, see §2.2. In conclusion, every Hopf algebra is a source of solutions of the braid equation. Essential examples of quasi-triangular Hopf algebras are the quantum groups $U_q(\mathfrak{g})$ [25, 53] and the finite-dimensional variations $\mathfrak{u}_q(\mathfrak{g})$ [59, 60].

In the approach to the classification of Hopf algebras exposed in this report, braided vector spaces and braided tensor categories play a decisive role; and the finite quantum groups are the main actors in one of the classes that splits off.

By space limitations, there is a selection of the topics and references included. Particularly, we deal with finite-dimensional Hopf algebras over an algebraically closed field of characteristic zero with special emphasis on description of examples and classifications. Interesting results on Hopf algebras either infinite-dimensional, or over other fields, unfortunately can not be reported. There is no account of the many deep results on tensor categories, see [30]. Various basic fundamental results are not explicitly cited, we refer to [1, 62, 66, 75, 79, 83] for them; classifications of Hopf algebras of fixed dimensions are not evoked, see [21, 71, 86].

2. Preliminaries

Let $\theta \in \mathbb{N}$ and $\mathbb{I} = \mathbb{I}_\theta = \{1, 2, \ldots, \theta\}$. The base field is $\mathbb{C}$. If X is a set, then $|X|$ is its cardinal and $\mathbb{C}X$ is the vector space with basis $(x_i)_{i \in X}$. Let G be a group: we denote by $\operatorname{Irr} G$ the set of isomorphism classes of irreducible representations of G and by $\widehat{G}$ the subset of those of dimension 1; by G^x the centralizer of $x \in G$; and by $\mathcal{O}_x^G$ its conjugacy class. More generally we denote by $\operatorname{Irr} \mathfrak{C}$ the set of isomorphism classes of simple objects in an abelian category $\mathfrak{C}$. The group of n-th roots of 1 in $\mathbb{C}$ is denoted $\mathbb{G}_n$; also $\mathbb{G}_\infty = \bigcup_{n \geq 1} \mathbb{G}_n$. The group presented by $(x_i)_{i \in I}$ with relations $(r_j)_{j \in J}$ is denoted $\langle (x_i)_{i \in I} | (r_j)_{j \in J} \rangle$. The notation for Hopf algebras is standard: Δ, ε, $\mathcal{S}$, denote respectively the comultiplication, the counit, the antipode (always assumed bijective, what happens in the finite-dimensional case). We use Sweedler's notation: $\Delta(x) = x_{(1)} \otimes x_{(2)}$. Similarly, if C is a coalgebra and V is a left comodule with structure map $\delta : V \to C \otimes V$, then $\delta(v) = v_{(-1)} \otimes v_{(0)}$. If D, E are subspaces of C, then $D \wedge E = \{c \in C : \Delta(c) \in D \otimes C + C \otimes E\}$; also $\wedge^0 D = D$ and $\wedge^{n+1} D = (\wedge^n D) \wedge D$ for $n > 0$.

2.1. Basic constructions and results. The first examples of finite-dimensional Hopf algebras are the group algebra $\mathbb{C}G$ of a finite group G and its dual, the algebra of functions $\mathbb{C}^G$. Indeed, the dual of a finite-dimensional Hopf algebra is again a Hopf algebra by transposing operations. By analogy with groups, several authors explored the notion of extension of Hopf algebras at various levels of generality; in the finite-dimensional context, every extension $\mathbb{C} \to A \to C \to B \to \mathbb{C}$ can be described as C with underlying vector space $A \otimes B$, via a heavy machinery of actions, coactions and non-abelian cocycles, but actual examples are rarely found in this way (extensions from a different perspective are in [9]). Relevant exceptions are the so-called *abelian extensions* [56] (rediscovered by Takeuchi and Majid): here the input is a matched pair of groups (F, G) with mutual actions $\triangleright, \triangleleft$ (or equivalently, an exact factorization of a finite group). The actions give rise to a Hopf algebra $\mathbb{C}^G \# \mathbb{C}F$. The multiplication and comultiplication can be further modified by compatible cocycles (σ, κ), producing to the abelian extension $\mathbb{C} \to \mathbb{C}^G \to \mathbb{C}^G{}^\kappa \#_\sigma \mathbb{C}F \to \mathbb{C}F \to \mathbb{C}$. Here (σ, κ) turns out to be a 2-cocycle in the total complex associated to a double complex built from the matched pair; the relevant H^2 is computed via the so-called Kac exact sequence.

It is natural to approach Hopf algebras by considering algebra or coalgebra invariants. There is no preference in the finite-dimensional setting but coalgebras and comodules are locally finite, so we privilege the coalgebra ones to lay down general methods. The basic coalgebra invariants of a Hopf algebra H are:

    $\circ$ The group $G(H) = \{g \in H - 0 : \Delta(g) = x \otimes g\}$ of group-like elements of H.

○ The space of skew-primitive elements $\mathcal{P}_{g,h}(H)$, $g, h \in G(H)$; $\mathcal{P}(H) := \mathcal{P}_{1,1}(H)$.

○ The coradical H_0, that is the sum of all simple subcoalgebras.

○ The coradical filtration $H_0 \subset H_1 \subset \ldots$, where $H_n = \wedge^n H_0$; then $H = \bigcup_{n \geq 0} H_n$.

2.2. Modules. The category $_H\mathcal{M}$ of left modules over a Hopf algebra H is monoidal with tensor product defined by the comultiplication; ditto for the category $^H\mathcal{M}$ of left comodules, with tensor product defined by the multiplication. Here are two ways to deform Hopf algebras without altering one of these categories.

• Let $F \in H \otimes H$ be invertible such that $(1 \otimes F)(\mathrm{id} \otimes \Delta)(F) = (F \otimes 1)(\Delta \otimes \mathrm{id})(F)$ and $(\mathrm{id} \otimes \varepsilon)(F) = (\varepsilon \otimes \mathrm{id})(F) = 1$. Then H^F (the same algebra with comultiplication $\Delta^F := F\Delta F^{-1}$) is again a Hopf algebra, named the *twisting* of H by F [26]. The monoidal categories $_H\mathcal{M}$ and $_{H^F}\mathcal{M}$ are equivalent. If H and K are finite-dimensional Hopf algebras with $_H\mathcal{M}$ and $_K\mathcal{M}$ equivalent as monoidal categories, then there exists F with $K \simeq H^F$ as Hopf algebras (Schauenburg, Etingof-Gelaki). Examples of twistings not mentioned elsewhere in this report are in [31, 65].

• Given a linear map $\sigma : H \otimes H \to \mathbb{C}$ with analogous conditions, there is a Hopf algebra H_σ (same coalgebra, multiplication twisted by σ) such that the monoidal categories $^H\mathcal{M}$ and $^{H_\sigma}\mathcal{M}$ are equivalent [24].

A Yetter-Drinfeld module M over H is left H-module and left H-comodule with the compatibility $\delta(h.m) = h_{(1)}m_{(-1)}\mathcal{S}(h_{(3)}) \otimes h_{(2)} \cdot m_{(0)}$, for all $m \in M$ and $h \in H$. The category $_H^H\mathcal{YD}$ of Yetter-Drinfeld modules is braided monoidal. That is, for every $M, N \in {}_H^H\mathcal{YD}$, there is a natural isomorphism $c : M \otimes N \to N \otimes M$ given by $c(m \otimes n) = m_{(-1)} \cdot n \otimes m_{(0)}$, $m \in M$, $n \in N$. When H is finite-dimensional, the category $_H^H\mathcal{YD}$ is equivalent, as a braided monoidal category, to $_{D(H)}\mathcal{M}$.

The definition of Hopf algebra makes sense in any braided monoidal category. Hopf algebras in $_H^H\mathcal{YD}$ are interesting because of the following facts–discovered by Radford and interpreted categorically by Majid, see [62, 75]:

◇ If R is a Hopf algebra in $_H^H\mathcal{YD}$, then $R\#H := R \otimes H$ with semidirect product and coproduct is a Hopf algebra, named the *bosonization* of R by H.

◇ Let π, ι be Hopf algebra maps as in $K \underset{\pi}{\overset{\iota}{\rightleftarrows}} H$ with $\pi\iota = \mathrm{id}_H$. Then $R = H^{\mathrm{co}\pi} := \{x \in K : (\mathrm{id} \otimes \pi)\Delta(x) = 1 \otimes x\}$ is a Hopf algebra in $_H^H\mathcal{YD}$ and $K \simeq R\#H$.

For instance, if $V \in {}_H^H\mathcal{YD}$, then the tensor algebra $T(V)$ is a Hopf algebra in $_H^H\mathcal{YD}$, by requiring $V \hookrightarrow \mathcal{P}(T(V))$. If $c : V \otimes V \to V \otimes V$ satisfies $c = -\tau$, τ the usual flip, then the exterior algebra $\Lambda(V)$ is a Hopf algebra in $_H^H\mathcal{YD}$.

There is a braided adjoint action of a Hopf algebra R in $_H^H\mathcal{YD}$ on itself, see e.g. [12, (1.26)]. If $x \in \mathcal{P}(R)$ and $y \in R$, then $\mathrm{ad}_c(x)(y) = xy - \mathrm{mult}\, c(x \otimes y)$.

2.2.1. Triangular Hopf algebras. A quasitriangular Hopf algebra (H, R) is *triangular* if the braiding induced by R is a symmetry: $c_{V \otimes W}c_{W \otimes V} = \mathrm{id}_{W \otimes V}$ for all $V, W \in {}_H\mathcal{M}$. A finite-dimensional triangular Hopf algebra is a twisting of a bosonization $\Lambda(V)\#\mathbb{C}G$, where G is a finite group and $V \in {}_G^G\mathcal{YD}$ has $c = -\tau$ [6]. This lead eventually to the classification of triangular finite-dimensional Hopf algebras [29]; previous work on the semisimple case culminated in [28].

2.3. Semisimple Hopf algebras. The algebra of functions $\mathbb{C}^G$ on a finite group G admits a Haar measure, i.e., a linear function $\int : \mathbb{C}^G \to \mathbb{C}$ invariant under left and right translations, namely $\int =$ sum of all elements in the standard basis of $\mathbb{C}G$. This is adapted as follows: a right integral *on* a Hopf algebra H is a linear function $\int : H \to \mathbb{C}$ which is invariant under the left regular coaction: analogously there is the notion of left integral. The notion has various applications. Assume that H is finite-dimensional. Then an integral *in* H is an integral on H^*; the subspace of left integrals in H has dimension one, and there is a generalization of Maschke's theorem for finite groups: H is semisimple if and only if $\varepsilon(\Lambda) \neq 0$ for any integral $0 \neq \Lambda \in H$. This characterization of semisimple Hopf algebras, valid in any characteristic, is one of several, some valid only in characteristic 0. See [79]. Semisimple Hopf algebras can be obtained as follows:

◇ A finite-dimensional Hopf algebra H is semisimple if and only if it is cosemisimple (that is, H^* is semisimple).

◇ Given an extension $\mathbb{C} \to K \to H \to L \to \mathbb{C}$, H is semisimple iff K and L are. Notice that there are semisimple extensions that are not abelian [40, 69, 74].

◇ If H is semisimple, then so are H^F and H_σ, for any twist F and cocycle σ. If G is a finite simple group, then any twisting of $\mathbb{C}G$ is a simple Hopf algebra (i.e., not a non-trivial extension) [73], but the converse is not true [37].

◇ A bosonization $R\#H$ is semisimple iff R and H are.

To my knowledge, all examples of semisimple Hopf algebras arise from group algebras by the preceding constructions; this was proved in [68, 70] in low dimensions and in [32] for dimensions $p^a q^b$, pqr, where p, q and r are primes. See [1, Question 2.6]. An analogous question in terms of fusion categories: is any semisimple Hopf algebra weakly group-theoretical? See [32, Question 2].

There are only finitely many isomorphism classes of *semisimple* Hopf algebras in each dimension [81], but this fails in general [13, 20].

Conjecture 2.1 (Kaplansky). *Let H be a semisimple Hopf algebra. The dimension of every $V \in \mathrm{Irr}\,_H\mathcal{M}$ divides the dimension of H.*

The answer is affirmative for iterated extensions of group algebras and duals of group algebras [67] and notably for semisimple quasitriangular Hopf algebras [27].

3. Lifting methods

3.1. Nichols algebras. Nichols algebras are a special kind of Hopf algebras in braided tensor categories. We are mainly interested in Nichols algebras in the braided category $^H_H\mathcal{YD}$, where H is a Hopf algebra, see page 3. In fact, there is a functor $V \mapsto \mathfrak{B}(V)$ from $^H_H\mathcal{YD}$ to the category of Hopf algebras in $^H_H\mathcal{YD}$. Their first appearence is in the precursor [72]; they were rediscovered in [85] as part of a "quantum differential calculus", and in [61] to present the positive part of $U_q(\mathfrak{g})$. See also [76, 78].

There are several, unrelated at the first glance, alternative definitions. Let $V \in {}^H_H\mathcal{YD}$. The first definition uses the representation of the braid group $\mathbb{B}_n$ in n strands on $V^{\otimes n}$, given by $\varsigma_i \mapsto \mathrm{id} \otimes c \otimes \mathrm{id}$, c in $(i, i+1)$ tensorands; here recall that $\mathbb{B}_n = \langle \varsigma_1, \ldots, \varsigma_{n-1} | \varsigma_i \varsigma_j = \varsigma_j \varsigma_i, |i-j| > 1, \varsigma_i \varsigma_j \varsigma_i = \varsigma_j \varsigma_i \varsigma_j, |i-j| = 1 \rangle$. Let $M : \mathbb{S}_n \to \mathbb{B}_n$ be the Matsumoto section

and let $\mathcal{Q}_n : V^{\otimes n} \to V^{\otimes n}$ be the quantum symmetrizer, $\mathcal{Q}_n = \sum_{s \in \mathbb{S}_n} M(s) : V^{\otimes n} \to V^{\otimes n}$. Then define

$$\mathfrak{J}^n(V) = \ker \mathcal{Q}_n, \qquad \mathfrak{J}(V) = \oplus_{n \geq 2} \mathfrak{J}^n(V), \qquad \mathfrak{B}(V) = T(V)/\mathfrak{J}(V). \tag{3.1}$$

Hence $\mathfrak{B}(V) = \oplus_{n \geq 0} \mathfrak{B}^n(V)$ is a graded Hopf algebra in $^H_H \mathcal{YD}$ with $\mathfrak{B}^0(V) = \mathbb{C}$, $\mathfrak{B}^1(V) \simeq V$; by (3.1) the algebra structure depends only on c. To explain the second definition, let us observe that the tensor algebra $T(V)$ is a Hopf algebra in $^H_H \mathcal{YD}$ with comultiplication determined by $\Delta(v) = v \otimes 1 + 1 \otimes v$ for $v \in V$. Then $\mathfrak{J}(V)$ coincides with the largest homogeneous ideal of $T(V)$ generated by elements of degree ≥ 2 that is also a coideal. Let now $T = \oplus_{n \geq 0} T^n$ be a graded Hopf algebra in $^H_H \mathcal{YD}$ with $T^0 = \mathbb{C}$. Consider the conditions

$$T^1 \text{ generates } T \text{ as an algebra}, \tag{3.2}$$

$$T^1 = \mathcal{P}(T). \tag{3.3}$$

These requirements are dual to each other: if T has finite-dimensional homogeneous components and $R = \oplus_{n \geq 0} R^n$ is the graded dual of T, i.e., $R^n = (T^n)^*$, then T satisfies (3.2) if and only if R satisfies (3.3). These conditions determine $\mathfrak{B}(V)$ up to isomorphisms, as the unique graded connected Hopf algebra T in $^H_H \mathcal{YD}$ that satisfies $T^1 \simeq V$, (3.2) and (3.3). There are still other characterizations of $\mathfrak{J}(V)$, e.g. as the radical of a suitable homogeneous bilinear form on $T(V)$, or as the common kernel of some suitable skew-derivations. See [15] for more details.

Despite all these different definitions, Nichols algebras are extremely difficult to deal with, e.g. to present by generators and relations, or to determine when a Nichols algebra has finite dimension or finite Gelfand-Kirillov dimension. It is not even known a priori whether the ideal $\mathfrak{J}(V)$ is finitely generated, except in a few specific cases. For instance, if c is a symmetry, that is $c^2 = \mathrm{id}$, or satisfies a Hecke condition with generic parameter, then $\mathfrak{B}(V)$ is quadratic. By the efforts of various authors, we have some understanding of finite-dimensional Nichols algebras of braided vector spaces either of diagonal or of rack type, see §3.5, 3.6.

3.2. Hopf algebras with the (dual) Chevalley property. We now explain how Nichols algebras enter into our approach to the classification of Hopf algebras. Recall that a Hopf algebra has the *dual Chevalley property* if the tensor product of two simple comodules is semisimple, or equivalently if its coradical is a (cosemisimple) Hopf subalgebra. For instance, a *pointed* Hopf algebra, one whose simple comodules have all dimension one, has the dual Chevalley property and its coradical is a group algebra. Also, a *copointed* Hopf algebra (one whose coradical is the algebra of functions on a finite group) has the dual Chevalley property. The Lifting Method is formulated in this context [13]. Let H be a Hopf algebra with the dual Chevalley property and set $K := H_0$. Under this assumption, the graded coalgebra $\mathrm{gr}\, H = \oplus_{n \in \mathbb{N}_0} \mathrm{gr}^n H$ associated to the coradical filtration becomes a Hopf algebra and considering the homogeneous projection π as in $R = H^{\mathrm{co}\,\pi} \hookrightarrow \mathrm{gr}\, H \xleftarrow[\pi]{\longleftarrow} K$ we

see that $\mathrm{gr}\, H \simeq R \# K$. The subalgebra of coinvariants R is a graded Hopf algebra in $^K_K \mathcal{YD}$ that inherits the grading with $R^0 = \mathbb{C}$; it satisfies (3.3) since the grading comes from the coradical filtration. Let R' be the subalgebra of R generated by $V := R^1$; then $R' \simeq \mathfrak{B}(V)$. The braided vector space V is a basic invariant of H called its *infinitesimal braiding*. Let us fix then a semisimple Hopf algebra K. To classify all finite-dimensional Hopf algebras H with $H_0 \simeq K$ as Hopf algebras, we have to address the following questions.

(a) Determine those $V \in {}^K_K\mathcal{YD}$ such that $\mathfrak{B}(V)$ is finite-dimensional, and give an efficient defining set of relations of these.

(b) Investigate whether any *finite-dimensional* graded Hopf algebra R in ${}^K_K\mathcal{YD}$ satisfying $R^0 = \mathbb{C}$ and $P(R) = R^1$, is a Nichols algebra.

(c) Compute all Hopf algebras H such that $\operatorname{gr} H \simeq \mathfrak{B}(V) \# K$, V as in (a).

Since the Nichols algebra $\mathfrak{B}(V)$ depends as an algebra (and as a coalgebra) only on the braiding c, it is convenient to restate Question (a) as follows:

(a$_1$) Determine those braided vector spaces (V, c) in a suitable class such that $\dim \mathfrak{B}(V) < \infty$, and give an efficient defining set of relations of these.

(a$_2$) For those V as in (a$_1$), find in how many ways, if any, they can be realized as Yetter-Drinfeld modules over K.

For instance, if $K = \mathbb{C}\Gamma$, Γ a finite abelian group, then the suitable class is that of braided vector spaces of diagonal type. In this context, Question (a$_2$) amounts to solve systems of equations in Γ. The answer to (a) is instrumental to attack (b) and (c). Question (b) can be rephrased in two equivalent statements:

(b$_1$) Investigate whether any *finite-dimensional* graded Hopf algebra T in ${}^K_K\mathcal{YD}$ with $T^0 = \mathbb{C}$ and generated as algebra by T^1, is a Nichols algebra.

(b$_2$) Investigate whether any *finite-dimensional* Hopf algebra H with $H_0 = K$ is generated as algebra by H_1.

We believe that the answer to (b) is affirmative at least when K is a group algebra. In other words, by the reformulation (b$_2$):

Conjecture 3.1 ([14]). *Every finite-dimensional pointed Hopf algebra is generated by group-like and skew-primitive elements.*

As we shall see in §3.7, the complete answer to (a) is needed in the approach proposed in [14] to attack Conjecture 3.1. It is plausible that the answer of (b$_2$) is affirmative for every semisimple Hopf algebra K. Question (c), known as *lifting* of the relations, also requires the knowledge of the generators of $\mathfrak{J}(V)$, see §3.8.

3.3. Generalized lifting method. Before starting with the analysis of the various questions in §3.2, we discuss a possible approach to more general Hopf algebras [5]. Let H be a Hopf algebra; we consider the following invariants of H:

○ The *Hopf coradical* $H_{[0]}$ is the subalgebra generated by H_0.

○ The *standard filtration* $H_{[0]} \subset H_{[1]} \subset \ldots$, $H_{[n]} = \wedge^{n+1} H_{[0]}$; then $H = \bigcup_{n \geq 0} H_{[n]}$.

If H has the dual Chevalley property, then $H_{[n]} = H_n$ for all $n \in \mathbb{N}_0$. In general, $H_{[0]}$ is a Hopf subalgebra of H with coradical H_0 and we may consider the graded Hopf algebra $\operatorname{gr} H = \oplus_{n \geq 0} H_{[n]}/H_{[n-1]}$. As before, if $\pi : \operatorname{gr} H \to H_{[0]}$ is the homogeneous projection, then $R = (\operatorname{gr} H)^{\operatorname{co} \pi}$ is a Hopf algebra in ${}^{H_{[0]}}_{H_{[0]}}\mathcal{YD}$ and $\operatorname{gr} H \cong R \# H_{[0]}$. Furthermore, $R = \oplus_{n \geq 0} R^n$ with grading inherited from $\operatorname{gr} H$. This discussion raises the following questions.

(A) Let C be a finite-dimensional cosemisimple coalgebra and $S : C \to C$ a bijective anti-coalgebra map. Classify all finite-dimensional Hopf algebras L generated by C, such that $\mathcal{S}|_C = S$.

219   (B) Given L as in the previous item, classify all finite-dimensional connected graded Hopf
220   algebras R in ${}^L_L\mathcal{YD}$.

221   (C) Given L and R as in previous items, classify all deformations or liftings, that is, clas-
222   sify all Hopf algebras H such that gr $H \cong R\#L$.

223   Question (A) is largely open, except for the remarkable [82, Theorem 1.5]: if H is a
224   Hopf algebra generated by an $\mathcal{S}$-invariant 4-dimensional simple subcoalgebra C, such that
225   $1 < \mathrm{ord}(\mathcal{S}^2|_C) < \infty$, then H is a Hopf algebra quotient of the quantized algebra of functions
226   on SL_2 at a root of unity ω. Nichols algebras enter into the picture in Question (B); if
227   $V = R^1$, then $\mathfrak{B}(V)$ is a subquotient of R. Question (C) is completely open, as it depends
228   on the previous Questions.

229   **3.4. Generalized root systems and Weyl groupoids.** Here we expose two important no-
230   tions introduced in [51].

      Let $\theta \in \mathbb{N}$ and $\mathbb{I} = \mathbb{I}_\theta$. A *basic datum* of type $\mathbb{I}$ is a pair $(\mathcal{X}, \rho)$, where $\mathcal{X} \neq \emptyset$ is
a set and $\rho : \mathbb{I} \to \mathbb{S}_\mathcal{X}$ is a map such that $\rho_i^2 = \mathrm{id}$ for all $i \in \mathbb{I}$. Let $\mathcal{Q}_\rho$ be the quiver
$\{\sigma_i^x := (x, i, \rho_i(x)) : i \in \mathbb{I}, x \in \mathcal{X}\}$ over $\mathcal{X}$, with $t(\sigma_i^x) = x$, $s(\sigma_i^x) = \rho_i(x)$ (here t means
target, s means source). Let $F(\mathcal{Q}_\rho)$ be the free groupoid over $\mathcal{Q}_\rho$; in any quotient of $F(\mathcal{Q}_\rho)$,
we denote

$$\sigma_{i_1}^x \sigma_{i_2} \cdots \sigma_{i_t} = \sigma_{i_1}^x \sigma_{i_2}^{\rho_{i_1}(x)} \cdots \sigma_{i_t}^{\rho_{i_{t-1}}\cdots\rho_{i_1}(x)}; \tag{3.4}$$

231   i.e., the implicit superscripts are those allowing compositions.

3.4.1. Coxeter groupoids. A *Coxeter datum* is a triple $(\mathcal{X}, \rho, \mathbf{M})$, where $(\mathcal{X}, \rho)$ is a basic
datum of type $\mathbb{I}$ and $\mathbf{M} = (\mathbf{m}^x)_{x \in \mathcal{X}}$ is a family of Coxeter matrices $\mathbf{m}^x = (m_{ij}^x)_{i,j \in \mathbb{I}}$ with

$$s((\sigma_i^x \sigma_j)^{m_{ij}^x}) = x, \qquad\qquad i, j \in \mathbb{I}, \qquad\qquad x \in \mathcal{X}. \tag{3.5}$$

The *Coxeter groupoid* $\mathcal{W}(\mathcal{X}, \rho, \mathbf{M})$ associated to $(\mathcal{X}, \rho, \mathbf{M})$ [51, Definition 1] is the groupoid
presented by generators $\mathcal{Q}_\rho$ with relations

$$(\sigma_i^x \sigma_j)^{m_{ij}^x} = \mathrm{id}_x, \qquad\qquad i, j \in \mathbb{I}, x \in \mathcal{X}. \tag{3.6}$$

3.4.2. Generalized root system. A generalized root system (GRS for short) is a collection
$\mathcal{R} := (\mathcal{X}, \rho, \mathcal{C}, \Delta)$, where $\mathcal{C} = (C^x)_{x \in \mathcal{X}}$ is a family of generalized Cartan matrices $C^x = (c_{ij}^x)_{i,j \in \mathbb{I}}$, cf. [57], and $\Delta = (\Delta^x)_{x \in \mathcal{X}}$ is a family of subsets $\Delta^x \subset \mathbb{Z}^\mathbb{I}$. We need the
following notation: Let $\{\alpha_i\}_{i \in \mathbb{I}}$ be the canonical basis of $\mathbb{Z}^\mathbb{I}$ and define $s_i^x \in GL(\mathbb{Z}^\mathbb{I})$ by
$s_i^x(\alpha_j) = \alpha_j - c_{ij}^x \alpha_i$, $i, j \in \mathbb{I}$, $x \in \mathcal{X}$. The collection should satisfy the following axioms:

$$c_{ij}^x = c_{ij}^{\rho_i(x)} \qquad\qquad \text{for all } x \in \mathcal{X}, i, j \in \mathbb{I}. \tag{3.7}$$

$$\Delta^x = \Delta_+^x \cup \Delta_-^x, \qquad\qquad \Delta_\pm^x := \pm(\Delta^x \cap \mathbb{N}_0^\mathbb{I}) \subset \pm\mathbb{N}_0^\mathbb{I}; \tag{3.8}$$

$$\Delta^x \cap \mathbb{Z}\alpha_i = \{\pm\alpha_i\}; \tag{3.9}$$

$$s_i^x(\Delta^x) = \Delta^{\rho_i(x)}; \tag{3.10}$$

$$(\rho_i\rho_j)^{m_{ij}^x}(x) = (x), \qquad\qquad m_{ij}^x := |\Delta^x \cap (\mathbb{N}_0\alpha_i + \mathbb{N}_0\alpha_j)|, \tag{3.11}$$

232   for all $x \in \mathcal{X}$, $i \neq j \in \mathbb{I}$. We call Δ_+^x, respectively Δ_-^x, the set of *positive*, respec-
233   tively *negative*, roots. Let $\mathcal{G} = \mathcal{X} \times GL_\theta(\mathbb{Z}) \times \mathcal{X}$, $\varsigma_i^x = (x, s_i^x, \rho_i(x))$, $i \in \mathbb{I}$, $x \in$

$\mathcal{X}$, and $\mathcal{W} = \mathcal{W}(\mathcal{X}, \rho, \mathcal{C})$ the subgroupoid of $\mathcal{G}$ generated by all the ς_i^x, i.e., by the image of the morphism of quivers $\mathcal{Q}_\rho \to \mathcal{G}$, $\sigma_i^x \mapsto \varsigma_i^x$. There is a Coxeter matrix $\mathbf{m}^x = (m_{ij}^x)_{i,j \in \mathbb{I}}$, where m_{ij}^x is the smallest natural number such that $(\varsigma_i^x \varsigma_j^x)^{m_{ij}^x} = \mathrm{id}\, x$. Then $\mathbf{M} = (\mathbf{m}^x)_{x \in \mathcal{X}}$ fits into a Coxeter datum $(\mathcal{X}, \rho, \mathbf{M})$, and there is an isomorphism of groupoids $\mathcal{W}(\mathcal{X}, \rho, \mathbf{M}) \longrightarrow \mathcal{W} = \mathcal{W}(\mathcal{X}, \rho, \mathcal{C})$ [51]; this is called the *Weyl groupoid* of $\mathcal{R}$. If $w \in \mathcal{W}(x, y)$, then $w(\Delta^x) = \Delta^y$, by (3.10). The sets of *real* roots at $x \in \mathcal{X}$ are $(\Delta^{\mathrm{re}})^x = \bigcup_{y \in \mathcal{X}} \{w(\alpha_i) : i \in \mathbb{I}, w \in \mathcal{W}(y, x)\}$; correspondingly the *imaginary* roots are $(\Delta^{\mathrm{im}})^x = \Delta^x - (\Delta^{\mathrm{re}})^x$. Assume that $\mathcal{W}$ is connected. Then the following conditions are equivalent [22, Lemma 2.11]:

- Δ^x is finite for some $x \in \mathcal{X}$,
- Δ^x is finite for all $x \in \mathcal{X}$,
- $(\Delta^{\mathrm{re}})^x$ is finite for all $x \in \mathcal{X}$,
- $\mathcal{W}$ is finite.

If these hold, then all roots are real [22]; we say that $\mathcal{R}$ is *finite*. We now discuss two examples of GRS, central for the subsequent discussion.

Example 3.2 ([2]). Let $\mathbf{k}$ be a field of characteristic $\ell \geq 0$, $\theta \in \mathbb{N}$, $\mathbf{p} \in \mathbb{G}_2^\theta$ and $A = (a_{ij}) \in \mathbf{k}^{\theta \times \theta}$. We assume $\ell \neq 2$ for simplicity. Let $\mathfrak{h} = \mathbf{k}^{2\theta - \mathrm{rk}\, A}$. Let $\mathfrak{g}(A, \mathbf{p})$ be the Kac-Moody Lie superalgebra over $\mathbf{k}$ defined as in [57]; it is generated by $\mathfrak{h}$, e_i and f_i, $i \in \mathbb{I}$, and the parity is given by $|e_i| = |f_i| = p_i$, $i \in \mathbb{I}$, $|h| = 0$, $h \in \mathfrak{h}$. Let $\Delta^{A, \mathbf{p}}$ be the root system of $\mathfrak{g}(A, \mathbf{p})$. We make the following technical assumptions:

$$a_{jk} = 0 \implies a_{kj} = 0, \qquad\qquad j \neq k; \qquad\qquad (3.12)$$

$$\mathrm{ad}\, f_i \text{ is locally nilpotent in } \mathfrak{g}(A, \mathbf{p}), \qquad i \in \mathbb{I}. \qquad\qquad (3.13)$$

The matrix A is *admissible* if (3.13) holds [80]. Let $C^{A, \mathbf{p}} = (c_{ij}^{A, \mathbf{p}})_{i, j \in \mathbb{I}}$ be given by

$$c_{ij}^{A, \mathbf{p}} := -\min\{m \in \mathbb{N}_0 : (\mathrm{ad}\, f_i)^{m+1} f_j = 0\}, i \neq j \in \mathbb{I}, \qquad c_{ii}^{A, \mathbf{p}} := 2. \qquad (3.14)$$

We need the following elements of $\mathbf{k}$:

$$\text{if } p_i = 0, \qquad d_m = m\, a_{ij} + \binom{m}{2} a_{ii}; \qquad\qquad\qquad\qquad (3.15)$$

$$\text{if } p_i = 1; \qquad d_m = \begin{cases} k\, a_{ii}, & m = 2k, \\ k\, a_{ii} + a_{ij}, & m = 2k+1; \end{cases} \qquad\qquad\qquad (3.16)$$

$$\nu_{j,0} = 1, \qquad \nu_{j,n} = \prod_{t=1}^n (-1)^{p_i((t-1)p_i + p_j)} d_t; \qquad\qquad\qquad (3.17)$$

$$\mu_{j,0} = 0, \qquad \mu_{j,n} = (-1)^{p_i p_j} n \left(\prod_{t=2}^n (-1)^{p_i((t-1)p_i + p_j)} d_t \right) a_{ji}. \qquad (3.18)$$

With the help of these scalars, we define a reflection $r_i(A, \mathbf{p}) = (r_i A, r_i \mathbf{p})$, where

$r_i\mathbf{p} = (\overline{p}_j)_{j\in\mathbb{I}}$, with $\overline{p}_j = p_j - c_{ij}^{A,\mathbf{P}}p_i$, and $r_iA = (\overline{a}_{jk})_{j,k\in\mathbb{I}}$, with

$$
\overline{a}_{jk} = \begin{cases}
-c_{ik}^{A,\mathbf{P}}\mu_{j,-c_{ij}^{A,\mathbf{P}}}a_{ii} + \mu_{j,-c_{ij}^{A,\mathbf{P}}}a_{ik} \\
\quad -c_{ik}^{A,\mathbf{P}}\nu_{j,-c_{ij}^{A,\mathbf{P}}}a_{ji} + \nu_{j,-c_{ij}^{A,\mathbf{P}}}a_{jk}, & j,k \neq i; \\
c_{ik}^{A,\mathbf{P}}a_{ii} - a_{ik}, & j = i \neq k; \\
-\mu_{j,-c_{ij}^{A,\mathbf{P}}}a_{ii} - \nu_{j,-c_{ij}^{A,\mathbf{P}}}a_{ji}, & j \neq k = i; \\
a_{ii}, & j = k = i.
\end{cases}
\tag{3.19}
$$

Theorem 3.3. *There is an isomorphism* $T_i^{A,\mathbf{P}} : \mathfrak{g}(r_i(A,\mathbf{p})) \to \mathfrak{g}(A,\mathbf{p})$ *of Lie superalgebras given (for an approppriate basis* (h_i) *of* $\mathfrak{h}$*) by*

$$
\begin{aligned}
T_i^{A,\mathbf{P}}(e_j) &= \begin{cases} (\operatorname{ad} e_i)^{-c_{ij}^{A,\mathbf{P}}}(e_j), & i \neq j \in \mathbb{I}, \\ f_i, & j = i \end{cases} \\[2mm]
T_i^{A,\mathbf{P}}(f_j) &= \begin{cases} (\operatorname{ad} f_i)^{-c_{ij}^{A,\mathbf{P}}}f_j, & j \in \mathbb{I}, j \neq i, \\ (-1)^{p_i}e_i, & j = i, \end{cases} \\[2mm]
T_i^{A,\mathbf{P}}(h_j) &= \begin{cases} \mu_{j,-c_{ij}^{A,\mathbf{P}}}h_i + \nu_{j,-c_{ij}^{A,\mathbf{P}}}h_j, & i \neq j \in \mathbb{I} \\ -h_i, & j = i, \\ h_j, & \theta + 1 \leq j \leq 2\theta - \operatorname{rk} A. \end{cases}
\end{aligned}
\tag{3.20}
$$

Assume that $\dim \mathfrak{g}(A,\mathbf{p}) < \infty$; then (3.12) and (3.13) hold. Let

$$\mathcal{X} = \{r_{i_1}\cdots r_{i_n}(A,\mathbf{p}) \,|\, n \in \mathbb{N}_0, i_1,\ldots,i_n \in \mathbb{I}\}.$$

Then $(\mathcal{X}, r, \mathcal{C}, \Delta)$, where $\mathcal{C} = (C^{(B,\mathbf{q})})_{(B,\mathbf{q})\in\mathcal{X}}$ and $\Delta = (\Delta^{(B,\mathbf{q})})_{(B,\mathbf{q})\in\mathcal{X}}$, is a finite GRS, an invariant of $\mathfrak{g}(A,\mathbf{p})$.

Example 3.4. Let H be a Hopf algebra, assumed semisimple for easiness. Let $M \in {}^H_H\mathcal{YD}$ be finite-dimensional, with a fixed decomposition $M = M_1 \oplus \cdots \oplus M_\theta$, where $M_1,\ldots,M_\theta \in \operatorname{Irr} {}^H_H\mathcal{YD}$. Then $T(M)$ and $\mathfrak{B}(M)$ are $\mathbb{Z}^\theta$-graded, by $\deg x = \alpha_i$ for all $x \in M_i$, $i \in \mathbb{I}_\theta$. Recall that $\mathbb{Z}^\theta_{\geq 0} = \sum_{i\in\mathbb{I}_\theta} \mathbb{Z}_{\geq 0}\alpha_i$.

Theorem 3.5 ([46, 48]). *If* $\dim \mathfrak{B}(M) < \infty$*, then M has a finite GRS.*

We discuss the main ideas of the proof. Let $i \in \mathbb{I} = \mathbb{I}_\theta$. We define $M_i' = V_i^*$,

$$
\begin{aligned}
c_{ij}^M &= -\sup\{h \in \mathbb{N}_0 : \operatorname{ad}_c^h(M_i)(M_j) \neq 0 \text{ in } \mathfrak{B}(M)\}, & i \neq j, \quad c_{ii}^M = 2; \\
M_j' &= \operatorname{ad}_c^{-c_{ij}}(M_i)(M_j), & \rho_i(M) = M_1' \oplus \cdots \oplus M_\theta'.
\end{aligned}
$$

Then $\dim \mathfrak{B}(M) = \dim \mathfrak{B}(\rho_i(M))$ and $C^M = (c_{ij}^M)_{i,j\in\mathbb{I}}$ is a generalized Cartan matrix [12]. Also, M_j' is irreducible [12, 3.8], [46, 7.2]. Let $\mathcal{X}$ be the set of objects in ${}^H_H\mathcal{YD}$ with fixed decomposition (up to isomorphism) of the form

$$\{\rho_{i_1}\cdots\rho_{i_n}(M) \,|\, n \in \mathbb{N}_0, i_1,\ldots,i_n \in \mathbb{I}\}.$$

Then $(\mathcal{X}, \rho, \mathcal{C})$, where $\mathcal{C} = (C^N)_{N\in\mathcal{X}}$, satisfies (3.7). Next we need:

- [46, Theorem 4.5]; [42] There exists a totally ordered index set $(L, \leq)$ and families $(W_l)_{l \in L}$ in Irr $_H^H \mathcal{YD}$, $(\beta_l)_{l \in L}$ such that $\mathfrak{B}(M) \simeq \otimes_{l \in L} \mathfrak{B}(W_l)$ as $\mathbb{Z}^\theta$-graded objects in $_H^H \mathcal{YD}$, where $\deg x = \beta_l$ for all $x \in W_l, l \in L$.

Let $\Delta_\pm^M = \{\pm\beta_l : l \in L\}$, $\Delta^M = \Delta_+^M \cup \Delta_-^M$, $\Delta = (\Delta^N)_{N \in \mathcal{X}(M)}$. Then $\mathcal{R} = (\mathcal{X}, \rho, \mathcal{C}, \Delta)$ is a finite GRS.

Theorem 3.6 ([23]). *The classification of all finite GRS is known.*

The proof is a combinatorial tour-de-force and requires computer calculations. It is possible to recover from this result the classification of the finite-dimensional contragredient Lie superalgebras in arbitrary characteristic [2]. However, the list of [23] is substantially larger than the classifications of the alluded Lie superalgebras or the braidings of diagonal type with finite-dimensional Nichols algebra.

3.5. Nichols algebras of diagonal type. Let G be a finite group. We denote $_G^G \mathcal{YD} = _H^H \mathcal{YD}$ for $H = \mathbb{C}G$. So $M \in _G^G \mathcal{YD}$ is a left G-module with a G-grading $M = \oplus_{g \in G} M_g$ such that $t \cdot M_g = M_{tgt^{-1}}$, for all $g, t \in G$. If $M, N \in _G^G \mathcal{YD}$, then the braiding $c : M \otimes N \to N \otimes M$ is given by $c(m \otimes n) = g \cdot n \otimes m$, $m \in M_g$, $n \in N$, $g \in G$. Now assume that $G = \Gamma$ is a finite abelian group. Then every $M \in _\Gamma^\Gamma \mathcal{YD}$ is a Γ-graded Γ-module, hence of the form $M = \oplus_{g \in \Gamma, \chi \in \widehat{\Gamma}} M_g^\chi$, where M_g^χ is the χ-isotypic component of M_g. So $_\Gamma^\Gamma \mathcal{YD}$ is just the category of $\Gamma \times \widehat{\Gamma}$-graded modules, with the braiding $c : M \otimes N \to N \otimes M$ given by $c(m \otimes n) = \chi(g)n \otimes m$, $m \in M_g^\eta$, $n \in N_t^\chi$, $g, t \in G$, $\chi, \eta \in \widehat{\Gamma}$. Let $\theta \in \mathbb{N}$, $\mathbb{I} = \mathbb{I}_\theta$.

Definition 3.7. Let $\mathbf{q} = (q_{ij})_{i,j \in \mathbb{I}}$ be a matrix with entries in $\mathbb{C}^\times$. A braided vector space (V, c) is of *diagonal type* with matrix $\mathbf{q}$ if V has a basis $(x_i)_{i \in \mathbb{I}}$ with

$$c(x_i \otimes x_j) = q_{ij}x_j \otimes x_i, \qquad\qquad i, j \in \mathbb{I}. \qquad (3.21)$$

Thus, every finite-dimensional $V \in _\Gamma^\Gamma \mathcal{YD}$ is a braided vector space of diagonal type. Question (a), more precisely (a$_1$), has a complete answer in this setting. First we can assume that $q_{ii} \neq 1$ for $i \in \mathbb{I}$, as otherwise $\dim \mathfrak{B}(V) = \infty$. Also, let $\mathbf{q}' = (q'_{ij})_{i,j \in \mathbb{I}} \in (\mathbb{C}^\times)^{\mathbb{I} \times \mathbb{I}}$ and V' a braided vector space with matrix $\mathbf{q}'$. If $q_{ii} = q'_{ii}$ and $q_{ij}q_{ji} = q'_{ij}q'_{ji}$ for all $j \neq i \in \mathbb{I}_\theta$, then $\mathfrak{B}(V) \simeq \mathfrak{B}(V')$ as braided vector spaces.

Theorem 3.8 ([44]). *The classification of all braided vector spaces of diagonal type with finite-dimensional Nichols algebra is known.*

The proof relies on the Weyl groupoid introduced in [43], a particular case of Theorem 3.5. Another fundamental ingredient is the following result, generalized at various levels in [42, 46, 48].

Theorem 3.9 ([58]). *Let V be a braided vector space of diagonal type. Every Hopf algebra quotient of $T(V)$ has a PBW basis.*

The classification in Theorem 3.8 can be organized as follows:

◇ For most of the matrices $\mathbf{q} = (q_{ij})_{i,j \in \mathbb{I}_\theta}$ in the list of [44] there is a field $\mathbf{k}$ and a pair $(A, \mathbf{p})$ as in Example 3.2 such that $\dim \mathfrak{g}(A, \mathbf{p}) < \infty$, and $\mathfrak{g}(A, \mathbf{p})$ has the same GRS as the Nichols algebra corresponding to $\mathbf{q}$ [2].

◇ Besides these, there are 12 (yet) unidentified examples.

We believe that Theorem 3.8 can be proved from Theorem 3.6, via Example 3.2.

Theorem 3.10 ([18, 19]). *An efficient set of defining relations of each finite-dimensional Nichols algebra of a braided vector space of diagonal type is known.*

The proof uses most technical tools available in the theory of Nichols algebras; of interest in its own is the introduction of the notion of convex order in Weyl groupoids. As for other classifications above, it is not possible to state precisely the list of relations. We just mention different types of relations that appear.

- ○ Quantum Serre relations, i.e., $\mathrm{ad}_c(x_i)^{1-a_{ij}}(x_j)$ for suitable $i \neq j$.
- ○ Powers of root vectors, i.e., $x_\beta^{N_\beta}$, where the x_β's are part of the PBW basis.
- ○ More exotic relations; they involve 2, 3, or at most 4 i's in $\mathbb{I}$.

3.6. Nichols algebras of rack type. We now consider Nichols algebras of objects in $^G_G\mathcal{YD}$, where G is a finite not necessarily abelian group. The category $^G_G\mathcal{YD}$ is semisimple and the simple objects are parametrized by pairs $(\mathcal{O}, \rho)$, where $\mathcal{O}$ is a conjugacy class in G and $\rho \in \mathrm{Irr}\, G^x$, for a fixed $x \in \mathcal{O}$; the corresponding simple Yetter-Drinfeld module $M(\mathcal{O}, \rho)$ is $\mathrm{Ind}_{G^x}^G \rho$ as a module. The braiding c is described in terms of the conjugation in $\mathcal{O}$. To describe the related suitable class, we recall that a rack is a set $X \neq \emptyset$ with a map $\rhd : X \times X \to X$ satisfying

- ○ $\varphi_x := x \rhd \_\_$ is a bijection for every $x \in X$.
- ○ $x \rhd (y \rhd z) = (x \rhd y) \rhd (x \rhd z)$ for all $x, y, z \in X$ (self-distributivity).

For instance, a conjugacy class $\mathcal{O}$ in G with the operation $x \rhd y = xyx^{-1}$, $x, y \in \mathcal{O}$ is a rack; actually we only consider racks realizable as conjugacy classes. Let X be a rack and $\mathfrak{X} = (X_k)_{k \in I}$ a decomposition of X, i.e., a disjoint family of subracks with $X_l \rhd X_k = X_k$ for all $k, l \in I$.

Definition 3.11. [10] A *2-cocycle of degree* $\mathbf{n} = (n_k)_{k \in I}$, associated to $\mathfrak{X}$, is a family $\mathbf{q} = (q_k)_{k \in I}$ of maps $q_k : X \times X_k \to \mathbf{GL}(n_k, \mathbb{C})$ such that

$$q_k(i, j \rhd h)q_k(j, h) = q_k(i \rhd j, i \rhd h)q_k(i, h), \qquad i, j \in X, \ h \in X_k, \ k \in I. \qquad (3.22)$$

Given such $\mathbf{q}$, let $V = \oplus_{k \in I} \mathbb{C}X_k \otimes \mathbb{C}^{n_k}$ and let $c^{\mathbf{q}} \in \mathbf{GL}(V \otimes V)$ be given by

$$c^{\mathbf{q}}(x_i v \otimes x_j w) = x_{i \rhd j} q_k(i, j)(w) \otimes x_i v, \qquad i \in X_l, \ j \in X_k, \ v \in \mathbb{C}^{n_l}, \ w \in \mathbb{C}^{n_k}.$$

Then $(V, c^{\mathbf{q}})$ is a braided vector space called of *rack type*; its Nichols algebra is denoted $\mathfrak{B}(X, \mathbf{q})$. If $\mathfrak{X} = (X)$, then we say that $\mathbf{q}$ is *principal*.

Every finite-dimensional $V \in {}^G_G\mathcal{YD}$ is a braided vector space of rack type [10, Theorem 4.14]. Question (a_1) in this setting has partial answers in three different lines: computation of some finite-dimensional Nichols algebras, Nichols algebras of reducible Yetter-Drinfeld modules and collapsing of racks.

3.6.1. Finite-dimensional Nichols algebras of rack type. The algorithm to compute a Nichols algebra $\mathfrak{B}(V)$ is as follows: compute the space $\mathfrak{J}^i(V) = \ker \mathcal{Q}_i$ of relations of degree i, for $i = 2, 3, \ldots, m$; then compute the m-th partial Nichols algebra $\widehat{\mathfrak{B}_m}(V) =$

$T(V)/\langle\oplus_{2\leq i\leq m}\mathfrak{J}^i(V)\rangle$, say with a computer program. If lucky enough to get $\dim\widehat{\mathfrak{B}_m}(V)<\infty$, then check whether it is a Nichols algebra, e.g. via skew-derivations; otherwise go to $m+1$. The description of $\mathfrak{J}^2(V)=\ker(\mathrm{id}+c)$ is not difficult [38] but for higher degrees it turns out to be very complicated. We list all known examples of finite-dimensional Nichols algebras $\mathfrak{B}(X,\mathbf{q})$ with X indecomposable and $\mathbf{q}$ principal and abelian ($n_1=1$).

Example 3.12. Let $\mathcal{O}_d^m$ be the conjugacy class of d-cycles in $\mathbb{S}_m$, $m\geq 3$. We start with the rack of transpositions in $\mathbb{S}_m$ and the cocycles -1, χ that arise from the $\rho\in\mathrm{Irr}\,\mathbb{S}_m^{(12)}$ with $\rho(12)=-1$, see [64, (5.5), (5.9)]. Let V be a vector space with basis $(x_{ij})_{(ij)\in\mathcal{O}_2^m}$ and consider the relations

$$x_{ij}^2=0, \qquad (ij)\in\mathcal{O}_2^m; \tag{3.23}$$
$$x_{ij}x_{kl}+x_{kl}x_{ij}=0, \qquad (ij),(kl)\in\mathcal{O}_2^m,\ |\{i,j,k,l\}|=4; \tag{3.24}$$
$$x_{ij}x_{kl}-x_{kl}x_{ij}=0, \qquad (ij),(kl)\in\mathcal{O}_2^m,\ |\{i,j,k,l\}|=4; \tag{3.25}$$
$$x_{ij}x_{ik}+x_{jk}x_{ij}+x_{ik}x_{jk}=0, \qquad (ij),(ik),(jk)\in\mathcal{O}_2^m,\ |\{i,j,k\}|=3; \tag{3.26}$$
$$x_{ij}x_{ik}-x_{jk}x_{ij}-x_{ik}x_{jk}=0, \qquad (ij),(ik),(jk)\in\mathcal{O}_2^m,\ |\{i,j,k\}|=3. \tag{3.27}$$

The quadratic algebras $\mathfrak{B}_m:=\widehat{\mathfrak{B}}_2(\mathcal{O}_2^m,-1)=T(V)/\langle(3.23),(3.24),(3.26)\rangle$ and $\mathcal{E}_m:=\widehat{\mathfrak{B}}_2(\mathcal{O}_2^m,\chi)=T(V)/\langle(3.23),(3.25),(3.27)\rangle$ were considered in [64], [36] respectively; $\mathcal{E}_m$ are named the *Fomin-Kirillov algebras*. It is known that

- The Nichols algebras $\mathfrak{B}(\mathcal{O}_2^m,-1)$ and $\mathfrak{B}(\mathcal{O}_2^m,\chi)$ are twist-equivalent, hence have the same Hilbert series. Ditto for the algebras $\mathfrak{B}_m$ and $\mathcal{E}_m$ [84].

- If $3\leq m\leq 5$, then $\mathfrak{B}_m=\mathfrak{B}(\mathcal{O}_2^m,-1)$ and $\mathcal{E}_m=\mathfrak{B}(\mathcal{O}_2^m,\chi)$ are finite-dimensional [36, 38, 64] (for $m=5$ part of this was done by Graña). In fact

$$\dim\mathfrak{B}_3=12, \qquad \dim\mathfrak{B}_4=576, \qquad \dim\mathfrak{B}_5=8294400.$$

  But for $m\geq 6$, it is not known whether the Nichols algebras $\mathfrak{B}(\mathcal{O}_2^m,-1)$ and $\mathfrak{B}(\mathcal{O}_2^m,\chi)$ have finite dimension or are quadratic.

Example 3.13 ([10]). The Nichols algebra $\mathfrak{B}(\mathcal{O}_4^4,-1)$ is quadratic, has the same Hilbert series as $\mathfrak{B}(\mathcal{O}_2^4,-1)$ and is generated by $(x_\sigma)_{\sigma\in\mathcal{O}_4^4}$ with defining relations

$$x_\sigma^2=0, \tag{3.28}$$
$$x_\sigma x_{\sigma^{-1}}+x_{\sigma^{-1}}x_\sigma=0, \tag{3.29}$$
$$x_\sigma x_\kappa+x_\nu x_\sigma+x_\kappa x_\nu=0, \qquad \sigma\kappa=\nu\sigma,\ \kappa\neq\sigma\neq\nu\in\mathcal{O}_4^4. \tag{3.30}$$

Example 3.14 ([41]). Let A be a finite abelian group and $g\in\mathrm{Aut}\,A$. The *affine* rack (A,g) is the set A with product $a\triangleright b=g(b)+(\mathrm{id}-g)(a)$, $a,b\in A$. Let $p\in\mathbb{N}$ be a prime, $q=p^{v(q)}$ a power of p, $A=\mathbb{F}_q$ and g the multiplication by $N\in\mathbb{F}_q^\times$; let $X_{q,N}=(A,g)$. Assume that $q=3,4,5$, or 7, with $N=2,\omega\in\mathbb{F}_4-\mathbb{F}_2$, 2 or 3, respectively. Then $\dim\mathfrak{B}(X_{q,N},-1)=q\varphi(q)(q-1)^{q-2}$, φ being the Euler function, and $\mathfrak{J}(X_{q,N},-1)=\langle\mathfrak{J}^2+\mathfrak{J}^{v(q)(q-1)}\rangle$, where $\mathfrak{J}^2$ is generated by

$$x_i^2, \qquad\qquad\qquad\qquad \text{always} \tag{3.31}$$
$$x_ix_j+x_{-i+2j}x_i+x_jx_{-i+2j}, \qquad\qquad \text{for } q=3, \tag{3.32}$$

$$x_i x_j + x_{(\omega+1)i+\omega j} x_i + x_j x_{(\omega+1)i+\omega j}, \qquad \text{for } q = 4, \qquad (3.33)$$

$$x_i x_j + x_{-i+2j} x_i + x_{3i-2j} x_{-i+2j} + x_j x_{3i-2j}, \qquad \text{for } q = 5, \qquad (3.34)$$

$$x_i x_j + x_{-2i+3j} x_i + x_j x_{-2i+3j}, \qquad \text{for } q = 7, \qquad (3.35)$$

with $i, j \in \mathbb{F}_q$; and $\mathfrak{J}^{v(q)(q-1)}$ is generated by $\sum_h T^h(V) \mathfrak{J}^2 T^{v(q)(q-1)-h-2}(V)$ and

$$(x_\omega x_1 x_0)^2 + (x_1 x_0 x_\omega)^2 + (x_0 x_\omega x_1)^2, \qquad \text{for } q = 4, \qquad (3.36)$$

$$(x_1 x_0)^2 + (x_0 x_1)^2, \qquad \text{for } q = 5, \qquad (3.37)$$

$$(x_2 x_1 x_0)^2 + (x_1 x_0 x_2)^2 + (x_0 x_2 x_1)^2, \qquad \text{for } q = 7. \qquad (3.38)$$

Of course $X_{3,2} = \mathcal{O}_2^3$; also $\dim \mathfrak{B}(X_{4,\omega}, -1) = 72$. By duality, we get

$$\dim \mathfrak{B}(X_{5,3}, -1) = \dim \mathfrak{B}(X_{5,2}, -1) = 1280,$$
$$\dim \mathfrak{B}(X_{7,5}, -1) = \dim \mathfrak{B}(X_{7,3}, -1) = 326592.$$

Example 3.15 ([45]). There is another finite-dimensional Nichols algebra associated to $X_{4,\omega}$ with a cocycle $\mathbf{q}$ with values $\pm\xi$, where $1 \neq \xi \in \mathbb{G}_3$. Concretely, $\dim \mathfrak{B}(X_{4,\omega}, \mathbf{q}) = 5184$ and $\mathfrak{B}(X_{4,\omega}, \mathbf{q})$ can be presented by generators $(x_i)_{i \in \mathbb{F}_4}$ with defining relations

$$x_0^3 = x_1^3 = x_\omega^3 = x_{\omega^2}^3 = 0,$$
$$\xi^2 x_0 x_1 + \xi x_1 x_\omega - x_\omega x_0 = 0, \quad \xi^2 x_0 x_\omega + \xi x_\omega x_{\omega^2} - x_{\omega^2} x_0 = 0,$$
$$\xi x_0 x_{\omega^2} - \xi^2 x_1 x_0 + x_{\omega^2} x_1 = 0, \quad \xi x_1 x_{\omega^2} + \xi^2 x_\omega x_1 + x_{\omega^2} x_\omega = 0,$$
$$x_0^2 x_1 x_\omega x_1^2 + x_0 x_1 x_\omega x_1^2 x_0 + x_1 x_\omega x_1^2 x_0^2 + x_\omega x_1^2 x_0^2 x_1 + x_1^2 x_0^2 x_1 x_\omega + x_1 x_0^2 x_1 x_\omega x_1$$
$$+ x_1 x_\omega x_1 x_0^2 x_\omega + x_\omega x_1 x_0 x_1 x_0 x_\omega + x_\omega x_1^2 x_0 x_\omega x_0 = 0.$$

3.6.2. Nichols algebras of decomposable Yetter-Drinfeld modules over groups. The ideas of Example 3.4 in the context of decomposable Yetter-Drinfeld modules over groups were pushed further in a series of papers culminating with a remarkable classification result [50]. Consider the groups

$$\Gamma_n = \langle a, b, \nu | ba = \nu ab, \quad \nu a = a\nu^{-1}, \quad \nu b = b\nu, \quad \nu^n = 1 \rangle, \quad n \geq 2; \qquad (3.39)$$

$$T = \langle \zeta, \chi_1, \chi_2 | \zeta \chi_1 = \chi_1 \zeta, \quad \zeta \chi_2 = \chi_2 \zeta, \quad \chi_1 \chi_2 \chi_1 = \chi_2 \chi_1 \chi_2, \quad \chi_1^3 = \chi_2^3 \rangle. \qquad (3.40)$$

○ [47] Let G be a suitable quotient of Γ_2. Then there exist $V_1, W_1 \in \mathrm{Irr}\,{}^G_G\mathcal{YD}$ such that $\dim V_1 = \dim W_1 = 2$ and $\dim \mathfrak{B}(V_1 \oplus W_1) = 64 = 2^6$.

○ [50] Let G be a suitable quotient of Γ_3. Then there exist $V_2, V_3, V_4, W_2, W_3, W_4 \in \mathrm{Irr}\,{}^G_G\mathcal{YD}$ such that $\dim V_2 = 1$, $\dim V_3 = \dim V_4 = 2$, $\dim W_2 = \dim W_3 = \dim W_4 = 3$ and $\dim \mathfrak{B}(V_2 \oplus W_2) = \dim \mathfrak{B}(V_3 \oplus W_3) = 10368 = 2^7 3^4$, $\dim \mathfrak{B}(V_4 \oplus W_4) = 2304 = 2^{18}$.

○ [49] Let G be a suitable quotient of Γ_4. Then there exist $V_5, W_5 \in \mathrm{Irr}\,{}^G_G\mathcal{YD}$ such that $\dim V_5 = 2$, $\dim W_5 = 4$ and $\dim \mathfrak{B}(V_5 \oplus W_5) = 262144 = 2^{18}$.

○ [49] Let G be a suitable quotient of T. Then there exist $V_6, W_6 \in \mathrm{Irr}\,{}^G_G\mathcal{YD}$ such that $\dim V_6 = 1$, $\dim W_6 = 4$ and $\dim \mathfrak{B}(V_6 \oplus W_6) = 80621568 = 2^{12} 3^9$.

Theorem 3.16 ([50]). *Let G be a non-abelian group and $V, W \in \mathrm{Irr}\, {}^G_G \mathcal{YD}$ such that G is generated by the support of $V \oplus W$. Assume that $c^2_{|V \otimes W} \neq \mathrm{id}$ and that $\dim \mathfrak{B}(V \oplus W) < \infty$. Then $V \oplus W$ is one of $V_i \oplus W_i$, $i \in \mathbb{I}_6$, above, and correspondingly G is a quotient of either Γ_n, $2 \leq n \leq 4$, or T.*

3.6.3. Collapsing racks. Implicit in Question (a$_1$) in the setting of racks is the need to compute all non-principal 2-cocycles for a fixed rack X. Notably, there exist criteria that dispense of this computation. To state them and explain their significance, we need some terminology. All racks below are finite.

- ○ A rack X is *abelian* when $x \triangleright y = y$, for all $x, y \in X$.
- ○ A rack is *indecomposable* when it is not a disjoint union of two proper subracks.
- ○ A rack X with $|X| > 1$ is *simple* when for any projection of racks $\pi : X \to Y$, either π is an isomorphism or Y has only one element.

Theorem 3.17 ([10, 3.9, 3.12], [55]). *Every simple rack is isomorphic to one of:*

(1) *Affine racks $(\mathbb{F}^t_p, T)$, where p is a prime, $t \in \mathbb{N}$, and T is the companion matrix of a monic irreducible polynomial $f \in \mathbb{F}_p[\mathtt{X}]$ of degree t, $f \neq \mathtt{X}, \mathtt{X} - 1$.*

(2) *Non-trivial (twisted) conjugacy classes in simple groups.*

(3) *Twisted conjugacy classes of type (G, u), where $G = L^t$, with L a simple non-abelian group and $1 < t \in \mathbb{N}$; and $u \in \mathrm{Aut}(L^t)$ acts by $u(\ell_1, \ell_2, \ldots, \ell_t) = (\theta(\ell_t), \ell_1, \ell_2, \ldots, \ell_{t-1})$, where $\theta \in \mathrm{Aut}(L)$.*

Definition 3.18 ([7, 3.5]). We say that a finite rack X is *of type D* when there are a decomposable subrack $Y = R \coprod S$, $r \in R$ and $s \in S$ such that $r \triangleright (s \triangleright (r \triangleright s)) \neq s$.

Also, X is *of type F* [4] if there are a disjoint family of subracks $(R_a)_{a \in \mathbb{I}_4}$ and a family $(r_a)_{a \in \mathbb{I}_4}$ with $r_a \in R_a$, such that $R_a \triangleright R_b = R_b$, $r_a \triangleright r_b \neq r_b$, for all $a \neq b \in \mathbb{I}_4$.

An indecomposable rack X *collapses* when $\dim \mathfrak{B}(X, \mathbf{q}) = \infty$ for every finite *faithful* 2-cocycle $\mathbf{q}$ (see [7] for the definition of faithful).

Theorem 3.19 ([7, 3.6]; [4, 2.8]). *If a rack is of type D or F, then it collapses.*

The proofs use results on Nichols algebras from [12, 23, 46].

If a rack projects onto a rack of type D (or F), then it is also of type D (or F), hence it collapses by Theorem 3.19. Since every indecomposable rack X, $|X| > 1$, projects onto a simple rack, it is natural to ask for the determination of all *simple* racks of type D or F. A rack is *cthulhu* if it is neither of type D nor F; it is *sober* if every subrack is either abelian or indecomposable [4]. Sober implies cthulhu.

- ○ Let $m \geq 5$. Let $\mathcal{O}$ be either $\mathcal{O}^{\mathbb{S}_m}_\sigma$, if $\sigma \in \mathbb{S}_m - \mathbb{A}_m$, or else $\mathcal{O}^{\mathbb{A}_m}_\sigma$ if $\sigma \in \mathbb{A}_m$. The type of σ is formed by the lengths of the cycles in its decomposition.
  - ◇ [7, 4.2] If the type of σ is (3^2), $(2^2, 3)$, $(1^n, 3)$, (2^4), $(1^2, 2^2)$, $(2, 3)$, (2^3), or $(1^n, 2)$, then $\mathcal{O}$ is cthulhu. If the type of σ is $(1, 2^2)$, then $\mathcal{O}$ is sober.
  - ◇ [33] Let $p \in \mathbb{N}$ be a prime. Assume the type of σ is (p). If $p = 5, 7$ or not of the form $(r^k - 1)/(r - 1)$, r a prime power, then $\mathcal{O}$ is sober; otherwise $\mathcal{O}$ is of type D. Assume the type of σ is $(1, p)$. If $p = 5$ or not of the form $(r^k - 1)/(r - 1)$, r a prime power, then $\mathcal{O}$ is sober; otherwise $\mathcal{O}$ is of type D.

Table 3.1. Classes in sporadic simple groups not of type D

Group	Classes	Group	Classes
T	2A	Co_3	23A, 23B
M_{11}	8A, 8B, 11A, 11B	J_1	15A, 15B, 19A, 19B, 19C
M_{12}	11A, 11B	J_2	2A, 3A
M_{22}	11A, 11B	J_3	5A, 5B, 19A, 19B
M_{23}	23A, 23B	J_4	29A, 43A, 43B, 43C
M_{24}	23A, 23B	Ly	37A, 37B, 67A, 67B, 67C
Ru	29A, 29B	$O'N$	31A, 31B
Suz	3A	Fi_{23}	2A
HS	11A, 11B	Fi_{22}	2A, 22A, 22B
McL	11A, 11B	Fi'_{24}	29A, 29B
Co_1	3A	B	2A, 46A, 46B, 47A, 47B
Co_2	2A, 23A, 23B		

$\diamond$ [7, 4.1] For all other types, $\mathcal{O}$ is of type D, hence it collapses.

$\circ$ [4] Let $n \geq 2$ and q be a prime power. Let $x \in \mathbf{PSL}_n(q)$ not semisimple and $\mathcal{O} = \mathcal{O}_x^{\mathbf{PSL}_n(q)}$. The type of a unipotent element are the sizes of its Jordan blocks.

$\diamond$ Assume x is unipotent. If x is either of type (2) and q is even or not a square, or of type (3) and $q = 2$, then $\mathcal{O}$ is sober. If x is either of type $(2,1)$ and q is even, or of type $(2,1,1)$ and $q = 2$ then $\mathcal{O}$ is cthulhu. If x is of type $(2,1,1)$ and $q > 2$ is even, then $\mathcal{O}$ is not of type D, but it is open if it is of type F.

$\diamond$ Otherwise, $\mathcal{O}$ is either of type D or of type F, hence it collapses.

$\circ$ [8, 35] Let $\mathcal{O}$ be a conjugacy class in a sporadic simple group G. If $\mathcal{O}$ appears in Table 3.1, then $\mathcal{O}$ is not of type D. If $G = M$ is the Monster and $\mathcal{O}$ is one of 32A, 32B, 41A, 46A, 46B, 47A, 47B, 59A, 59B, 69A, 69B, 71A, 71B, 87A, 87B, 92A, 92B, 94A, 94B, then it is open whether $\mathcal{O}$ is of type D. Otherwise, $\mathcal{O}$ is of type D.

3.7. Generation in degree one. Here is the scheme of proof proposed in [14] to attack Conjecture 3.1: Let T be a finite-dimensional graded Hopf algebra in ${}^K_K\mathcal{YD}$ with $T^0 = \mathbb{C}$ and generated as algebra by T^1. We have a commutative diagram of Hopf algebra maps $T \xrightarrow{\ \pi\ } \mathfrak{B}(V)$. To show that π is injective, take a generator r (or a family of

$$T \underset{p \quad T(V)}{\xrightarrow{\ \pi\ } \mathfrak{B}(V)}$$

generators) of $\mathfrak{J}(V)$ such that $r \in \mathcal{P}(T(V))$ and consider the Yetter-Drinfeld submodule $U = \mathbb{C}r \oplus V$ of $T(V)$; if $\dim \mathfrak{B}(U) = \infty$, then $p(r) = 0$. Then p factorizes through $T(V)/\mathfrak{J}_1(V)$, where $\mathfrak{J}_1(V)$ is the ideal generated by primitive generators of $\mathfrak{J}(V)$, and so on.

The Conjecture has been verified in all known examples in characteristic 0 (it is false in positive characteristic or for infinite-dimensional Hopf algebras).

Theorem 3.20. *A finite-dimensional pointed Hopf algebra H is generated by group-like and skew-primitive elements if either of the following holds:*

⁴¹⁴   ◇ [19] *The infinitesimal braiding is of diagonal type, e. g. $G(H)$ is abelian.*

⁴¹⁵   ◇ [11, 38]. *The infinitesimal braiding of H is any of $(\mathcal{O}_2^m, -1)$, $(\mathcal{O}_2^m, \chi)$ ($m = 3, 4, 5$),*
⁴¹⁶   *$(X_{4,\omega}, -1)$, $(X_{5,2}, -1)$, $(X_{5,3}, -1)$, $(X_{7,3}, -1)$, $(X_{7,5}, -1)$.*

⁴¹⁷ **3.8. Liftings.** We address here Question (c) in §3.2. Let X be a finite rack and $q : X \times X \to$
⁴¹⁸ $\mathbb{G}_\infty$ a 2-cocycle. A Hopf algebra H is a *lifting* of (X, q) if H_0 is a Hopf subalgebra, H is
⁴¹⁹ generated by H_1 and its infinitesimal braiding is a realization of $(\mathbb{C}X, c^q)$. See [39] for
⁴²⁰ liftings in the setting of copointed Hopf algebras.

   We start discussing realizations of braided vector spaces as Yetter-Drinfeld modules. Let
$\theta \in \mathbb{N}$ and $\mathbb{I} = \mathbb{I}_\theta$. First, a *YD-datum of diagonal type* is a collection

$$\mathcal{D} = ((q_{ij})_{i,j \in \mathbb{I}}, G, (g_i)_{i \in \mathbb{I}}, (\chi_i)_{i \in \mathbb{I}}), \tag{3.41}$$

where $q_{ij} \in \mathbb{G}_\infty$, $q_{ii} \neq 1$, $i, j \in \mathbb{I}$; G is a finite group; $g_i \in Z(G)$; $\chi_i \in \widehat{G}$, $i \in \mathbb{I}$; such that
$q_{ij} := \chi_j(g_i)$, $i, j \in \mathbb{I}$. Let (V, c) be the braided vector space of diagonal type with matrix
(q_{ij}) in the basis $(x_i)_{i \in \mathbb{I}_\theta}$. Then $V \in {}_G^G\mathcal{YD}$ by declaring $x_i \in V_{g_i}^{\chi_i}$, $i \in \mathbb{I}$. More generally, a
YD-datum of rack type [11, 64] is a collection

$$\mathcal{D} = (X, q, G, \cdot, g, \chi), \tag{3.42}$$

⁴²¹ where X is a finite rack; $q : X \times X \to \mathbb{G}_\infty$ is a 2-cocycle; G is a finite group; $\cdot$ is an action of
⁴²² G on X; $g : X \to G$ is equivariant with respect to the conjugation in G; and $\chi = (\chi_i)_{i \in X}$ is
⁴²³ a family of 1-cocycles $\chi_i : G \to \mathbb{C}^\times$ (that is, $\chi_i(ht) = \chi_i(t)\chi_{t \cdot i}(h)$, for all $i \in X$, $h, t \in G$)
⁴²⁴ such that $g_i \cdot j = i \triangleright j$ and $\chi_i(g_j) = q_{ij}$ for all $i, j \in X$. Let $(V, c) = (\mathbb{C}X, c^q)$ be the
⁴²⁵ associated braided vector space. Then V becomes an object in ${}_G^G\mathcal{YD}$ by $\delta(x_i) = g_i \otimes x_i$ and
⁴²⁶ $t \cdot x_i = \chi_i(t)x_{t \cdot i}$, $t \in G$, $i \in X$.

⁴²⁷   Second, let $\mathcal{D}$ be a YD-datum of either diagonal or rack type and $V \in {}_G^G\mathcal{YD}$ as above; let
⁴²⁸ $\mathcal{T}(V) := T(V)\#\mathbb{C}G$. The desired liftings are quotients of $\mathcal{T}(V)$; write a_i in these quotients
⁴²⁹ instead of x_i to distinguish them from the elements in $\mathfrak{B}(V)\#\mathbb{C}G$. Let $\mathcal{G}$ be a minimal set
⁴³⁰ of generators of $\mathfrak{J}(V)$, assumed homogeneous both for the $\mathbb{N}$- and the G-grading. Roughly
⁴³¹ speaking, the deformations will be defined by replacing the relations $r = 0$ by $r = \phi_r$,
⁴³² $r \in \mathcal{G}$, where $\phi_r \in \mathcal{T}(V)$ belongs to a lower term of the coradical filtration, and the ideal
⁴³³ $\mathfrak{J}_\phi(V)$ generated by ϕ_r, $r \in \mathcal{G}$, is a Hopf ideal. The problem is to describe the ϕ_r's and
⁴³⁴ to check that $\mathcal{T}(V)/\mathfrak{J}_\phi(V)$ has the right dimension. If $r \in \mathcal{P}(T(V))$ has G-degree g, then
⁴³⁵ $\phi_r = \lambda(1 - g)$ for some $\lambda \in \mathbb{C}$; depending on the action of G on r, it may happen that λ
⁴³⁶ should be 0. In some cases, all $r \in \mathcal{G}$ are primitive, so all deformations can be described; see
⁴³⁷ [13] for quantum linear spaces (their liftings can also be presented as Ore extensions [20])
⁴³⁸ and the Examples 3.21 and 3.22. But in most cases, not all $r \in \mathcal{G}$ are primitive and some
⁴³⁹ recursive construction of the deformations is needed. This was achieved in [15] for diagonal
⁴⁴⁰ braidings of Cartan type A_n, with explicit formulae, and in [16] for diagonal braidings of
⁴⁴¹ finite Cartan type, with recursive formulae. Later it was observed that the so obtained liftings
⁴⁴² are cocycle deformations of $\mathfrak{B}(V)\#\mathbb{C}G$, see e.g. [63]. This led to the strategy in [3]: pick
⁴⁴³ an adapted stratification $\mathcal{G} = \mathcal{G}_0 \cup \mathcal{G}_1 \cup \cdots \cup \mathcal{G}_N$ [3, 5.1]; then construct recursively the
⁴⁴⁴ deformations of $T(V)/\langle \mathcal{G}_0 \cup \mathcal{G}_1 \cup \cdots \cup \mathcal{G}_{k-1} \rangle$ by determining the cleft extensions of the
⁴⁴⁵ deformations in the previous step and applying the theory of Hopf bi-Galois extensions [77].
⁴⁴⁶ In the Examples below, $\chi_i = \chi \in \widehat{G}$ for all $i \in X$ by [11, 3.3 (d)].

Example 3.21 ([11, 39]). Let $\mathcal{D} = (\mathcal{O}_2^3, -1, G, \cdot, g, \chi)$ be a YD-datum. Let $\lambda \in \mathbb{C}^2$ be such that

$$\lambda_1 = \lambda_2 = 0, \qquad \text{if } \chi^2 \neq \varepsilon; \tag{3.43}$$

$$\lambda_1 = 0, \qquad \text{if } g_{12}^2 = 1; \qquad \lambda_2 = 0, \qquad \text{if } g_{12}g_{13} = 1. \tag{3.44}$$

Let $\mathfrak{u} = \mathfrak{u}(\mathcal{D}, \lambda)$ be the quotient of $\mathcal{T}(V)$ by the relations

$$a_{12}^2 = \lambda_1(1 - g_{12}^2), \tag{3.45}$$

$$a_{12}a_{13} + a_{23}a_{12} + a_{13}a_{23} = \lambda_2(1 - g_{12}g_{13}). \tag{3.46}$$

Then $\mathfrak{u}$ is a pointed Hopf algebra, a cocycle deformation of $\operatorname{gr}\mathfrak{u} \simeq \mathfrak{B}(V)\#\mathbb{C}G$ and $\dim\mathfrak{u} = 12|G|$; $\mathfrak{u}(\mathcal{D}, \lambda) \simeq \mathfrak{u}(\mathcal{D}, \lambda')$ iff $\lambda = c\lambda'$ for some $c \in \mathbb{C}^\times$. Conversely, any lifting of $(\mathcal{O}_2^3, -1)$ is isomorphic to $\mathfrak{u}(\mathcal{D}, \lambda)$ for some YD-datum $\mathcal{D} = (\mathcal{O}_2^3, -1, G, \cdot, g, \chi)$ and $\lambda \in \mathbb{C}^2$ satisfying (3.43), (3.44).

Example 3.22 ([11]). Let $\mathcal{D} = (\mathcal{O}_2^4, -1, G, \cdot, g, \chi)$ be a YD-datum. Let $\lambda \in \mathbb{C}^3$ be such that

$$\lambda_i = 0, \quad i \in \mathbb{I}_3, \qquad \text{if } \chi^2 \neq \varepsilon; \tag{3.47}$$

$$\lambda_1 = 0, \quad \text{if } g_{12}^2 = 1; \qquad \lambda_2 = 0, \quad \text{if } g_{12}g_{34} = 1; \quad \lambda_3 = 0, \quad \text{if } g_{12}g_{13} = 1. \tag{3.48}$$

Let $\mathfrak{u} = \mathfrak{u}(\mathcal{D}, \lambda)$ be the quotient of $\mathcal{T}(V)$ by the relations

$$a_{12}^2 = \lambda_1(1 - g_{12}^2), \tag{3.49}$$

$$a_{12}a_{34} + a_{34}a_{12} = \lambda_2(1 - g_{12}g_{34}), \tag{3.50}$$

$$a_{12}a_{13} + a_{23}a_{12} + a_{13}a_{23} = \lambda_3(1 - g_{12}g_{13}). \tag{3.51}$$

Then $\mathfrak{u}$ is a pointed Hopf algebra, a cocycle deformation of $\mathfrak{B}(V)\#\mathbb{C}G$ and $\dim\mathfrak{u} = 576|G|$; $\mathfrak{u}(\mathcal{D}, \lambda) \simeq \mathfrak{u}(\mathcal{D}, \lambda')$ iff $\lambda = c\lambda'$ for some $c \in \mathbb{C}^\times$. Conversely, any lifting of $(\mathcal{O}_2^4, -1)$ is isomorphic to $\mathfrak{u}(\mathcal{D}, \lambda)$ for some YD-datum $\mathcal{D} = (\mathcal{O}_2^4, -1, G, \cdot, g, \chi)$ and $\lambda \in \mathbb{C}^3$ satisfying (3.47), (3.48).

Example 3.23 ([39]). Let $\mathcal{D} = (X_{4,\omega}, -1, G, \cdot, g, \chi)$ be a YD-datum. Let $\lambda \in \mathbb{C}^3$ be such that

$$\lambda_1 = \lambda_2 = 0, \qquad \text{if } \chi^2 \neq \varepsilon; \qquad\qquad \lambda_3 = 0, \text{ if } \chi^6 \neq \varepsilon; \tag{3.52}$$

$$\lambda_1 = 0, \text{ if } g_0^2 = 1, \qquad \lambda_2 = 0, \text{ if } g_0 g_1 = 1, \qquad \lambda_3 = 0, \text{ if } g_0^3 g_1^3 = 1. \tag{3.53}$$

Let $\mathfrak{u} = \mathfrak{u}(\mathcal{D}, \lambda)$ be the quotient of $\mathcal{T}(V)$ by the relations

$$x_0^2 = \lambda_1(1 - g_0^2), \tag{3.54}$$

$$x_0x_1 + x_\omega x_0 + x_1 x_\omega = \lambda_2(1 - g_0 g_1) \tag{3.55}$$

$$(x_\omega x_1 x_0)^2 + (x_1 x_0 x_\omega)^2 + (x_0 x_\omega x_1)^2 = \zeta_6 - \lambda_3(1 - g_0^3 g_1^3), \qquad \text{where} \tag{3.56}$$

$$\zeta_6 = \lambda_2(x_\omega x_1 x_0 x_\omega + x_1 x_0 x_\omega x_1 + x_0 x_\omega x_1 x_0) - \lambda_2^3(g_0 g_1 - g_0^3 g_1^3)$$

$$+\lambda_1^2 g_0^2 \big(g_{1+\omega}^2(x_\omega x_3 + x_0 x_\omega) + g_1 g_{1+\omega}(x_\omega x_1 + x_1 x_3) + g_1^2(x_1 x_0 + x_0 x_3)\big)$$
$$-2\lambda_1^2 g_0^2(x_0 x_3 - x_\omega x_3 - x_1 x_\omega + x_1 x_0) - 2\lambda_1^2 g_\omega^2(x_\omega x_3 - x_1 x_3 + x_0 x_\omega - x_0 x_1)$$
$$-2\lambda_1^2 g_1^2(x_\omega x_1 + x_1 x_3 + x_1 x_\omega - x_0 x_3 + x_0 x_1)$$
$$+\lambda_2 \lambda_1(g_\omega^2 x_0 x_3 + g_1^2 x_\omega x_3 + g_0^2 x_1 x_3) + \lambda_2^2 g_0 g_1(x_\omega x_1 + x_1 x_0 + x_0 x_\omega - \lambda_1)$$
$$-\lambda_2 \lambda_1^2(3 g_0^3 g_{1+\omega} - 2 g_0 g_1^3 - g_0^2 g_\omega^2 - 2 g_0^3 g_1 + g_\omega^2 - g_1^2 + g_0^2)$$
$$-\lambda_2(\lambda_1 - \lambda_2)\big(\lambda_1 g_0^2(g_{1+\omega}^2 + g_1 g_{1+\omega} + g_1^2 + 2 g_0 g_1^3) + x_\omega x_1 + x_1 x_0 + x_0 x_\omega\big).$$

Then $\mathfrak{u}$ is a pointed Hopf algebra, a cocycle deformation of $\operatorname{gr}\mathfrak{u} \simeq \mathfrak{B}(V)\#\mathbb{C}G$ and $\dim\mathfrak{u} = 72|G|$; $\mathfrak{u}(\mathcal{D},\lambda) \simeq \mathfrak{u}(\mathcal{D},\lambda')$ iff $\lambda = c\lambda'$ for some $c \in \mathbb{C}^\times$. Conversely, any lifting of $(X_{4,\omega}, -1)$ is isomorphic to $\mathfrak{u}(\mathcal{D},\lambda)$ for some YD-datum $\mathcal{D} = (X_{4,\omega}, -1, G, \cdot, g, \chi)$ and $\lambda \in \mathbb{C}^3$ satisfying (3.52), (3.53).

Example 3.24 ([39]). Let $\mathcal{D} = (X_{5,2}, -1, G, \cdot, g, \chi)$ be a YD-datum. Let $\lambda \in \mathbb{C}^3$ be such that

$$\lambda_1 = \lambda_2 = 0, \qquad \text{if } \chi^2 \neq \varepsilon; \qquad\qquad \lambda_3 = 0, \text{ if } \chi^4 \neq \varepsilon; \qquad\qquad (3.57)$$
$$\lambda_1 = 0, \text{ if } g_0^2 = 1, \qquad \lambda_2 = 0, \text{ if } g_0 g_1 = 1, \qquad \lambda_3 = 0, \text{ if } g_0^2 g_1 g_2 = 1. \qquad (3.58)$$

Let $\mathfrak{u} = \mathfrak{u}(\mathcal{D},\lambda)$ be the quotient of $\mathcal{T}(V)$ by the relations

$$x_0^2 = \lambda_1(1 - g_0^2), \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad (3.59)$$
$$x_0 x_1 + x_2 x_0 + x_3 x_2 + x_1 x_3 = \lambda_2(1 - g_0 g_1), \qquad\qquad\qquad\qquad (3.60)$$
$$(x_1 x_0)^2 + (x_0 x_1)^2 = \zeta_4 - \lambda_3(1 - g_0^2 g_1 g_2), \qquad\qquad \text{where} \qquad (3.61)$$

$\zeta_4 = \lambda_2\,(x_1 x_0 + x_0 x_1) + \lambda_1 g_1^2(x_3 x_0 + x_2 x_3) - \lambda_1 g_0^2(x_2 x_4 + x_1 x_2) + \lambda_2 \lambda_1 g_0^2(1 - g_1 g_2)$.
Then $\mathfrak{u}$ is a pointed Hopf algebra, a cocycle deformation of $\operatorname{gr}\mathfrak{u} \simeq \mathfrak{B}(V)\#\mathbb{C}G$ and $\dim\mathfrak{u} = 1280|G|$; $\mathfrak{u}(\mathcal{D},\lambda) \simeq \mathfrak{u}(\mathcal{D},\lambda')$ iff $\lambda = c\lambda'$ for some $c \in \mathbb{C}^\times$. Conversely, any lifting of $(X_{5,2}, -1)$ is isomorphic to $\mathfrak{u}(\mathcal{D},\lambda)$ for some YD-datum $\mathcal{D} = (X_{5,2}, -1, G, \cdot, g, \chi)$ and $\lambda \in \mathbb{C}^3$ satisfying (3.57), (3.58).

Example 3.25 ([39]). Let $\mathcal{D} = (X_{5,3}, -1, G, \cdot, g, \chi)$ be a YD-datum. Let $\lambda \in \mathbb{C}^3$ be such that

$$\lambda_1 = \lambda_2 = 0, \qquad \text{if } \chi^2 \neq \varepsilon; \qquad\qquad \lambda_3 = 0, \text{ if } \chi^4 \neq \varepsilon; \qquad\qquad (3.62)$$
$$\lambda_1 = 0, \text{ if } g_0^2 = 1, \qquad \lambda_2 = 0, \text{ if } g_1 g_0 = 1, \qquad \lambda_3 = 0, \text{ if } g_0^2 g_1 g_3 = 1. \qquad (3.63)$$

Let $\mathfrak{u} = \mathfrak{u}(\mathcal{D},\lambda)$ be the quotient of $\mathcal{T}(V)$ by the relations

$$x_0^2 = \lambda_1(1 - g_0^2), \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad (3.64)$$
$$x_1 x_0 + x_0 x_2 + x_2 x_3 + x_3 x_1 = \lambda_2(1 - g_1 g_0) \qquad\qquad\qquad\qquad (3.65)$$
$$x_0 x_2 x_3 x_1 + x_1 x_4 x_3 x_0 = \zeta_4' - \lambda_3(1 - g_0^2 g_1 g_3), \qquad\qquad\qquad (3.66)$$

where $\zeta_4' = \lambda_2\,(x_0 x_1 + x_1 x_0) - \lambda_1 g_1^2(x_3 x_2 + x_0 x_3) - \lambda_1 g_0^2(x_3 x_4 + x_1 x_3) + \lambda_1 \lambda_2(g_1^2 + g_0^2 - 2 g_0^2 g_1 g_3)$. Then $\mathfrak{u}$ is a pointed Hopf algebra, a cocycle deformation of $\operatorname{gr}\mathfrak{u} \simeq \mathfrak{B}(V)\#\mathbb{C}G$ and $\dim\mathfrak{u} = 1280|G|$; $\mathfrak{u}(\mathcal{D},\lambda) \simeq \mathfrak{u}(\mathcal{D},\lambda')$ iff $\lambda = c\lambda'$ for some $c \in \mathbb{C}^\times$. Conversely, any lifting of $(X_{5,3}, -1)$ is isomorphic to $\mathfrak{u}(\mathcal{D},\lambda)$ for some YD-datum $\mathcal{D} = (X_{5,3}, -1, G, \cdot, g, \chi)$ and $\lambda \in \mathbb{C}^3$ satisfying (3.62), (3.63).

⁴⁷⁰ **4. Pointed Hopf algebras**

⁴⁷¹ **4.1. Pointed Hopf algebras with abelian group.** Here is a classification from [16]. Let
⁴⁷² $\mathcal{D} = ((q_{ij})_{i,j \in \mathbb{I}_\theta}, \Gamma, (g_i)_{i \in \mathbb{I}_\theta}, (\chi_i)_{i \in \mathbb{I}_\theta})$ be a YD-datum of diagonal type as in (3.41) with
⁴⁷³ Γ a finite abelian group and let $V \in {}_\Gamma^\Gamma \mathcal{YD}$ be the corresponding realization. We say that
⁴⁷⁴ $\mathcal{D}$ is a *Cartan datum* if there is a Cartan matrix (of finite type) $\mathbf{a} = (a_{ij})_{i,j \in \mathbb{I}_\theta}$ such that
⁴⁷⁵ $q_{ij}q_{ji} = q_{ii}^{a_{ij}}, i \neq j \in \mathbb{I}_\theta$.

⁴⁷⁶   Let Φ be the root system associated to $\mathbf{a}$, $\alpha_1, \ldots, \alpha_\theta$ a choice of simple roots, $\mathcal{X}$ the set
⁴⁷⁷ of connected components of the Dynkin diagram of Φ and set $i \sim j$ whenever α_i, α_j belong
⁴⁷⁸ to the same $J \in \mathcal{X}$. We consider two classes of parameters:

⁴⁷⁹   ○ $\lambda = (\lambda_{ij})_{\substack{i < j \in \mathbb{I}_\theta \\ i \nsim j}}$, is a family in $\{0, 1\}$ with $\lambda_{ij} = 0$ when $g_i g_j = 1$ or $\chi_i \chi_j \neq \varepsilon$.

⁴⁸⁰   ○ $\mu = (\mu_\alpha)_{\alpha \in \Phi^+}$ is a family in $\mathbb{C}$ with $\mu_\alpha = 0$ when $\operatorname{supp} \alpha \subset J$, $J \in \mathcal{X}$, and $g_\alpha^{N_J} = 1$
⁴⁸¹   or $\chi_\alpha^{N_J} \neq \varepsilon$. Here $N_J = \operatorname{ord} q_{ii}$ for an arbitrary $i \in J$.

We attach a family $(u_\alpha(\mu))_{\alpha \in \Phi^+}$ in $\mathbb{C}\Gamma$ to the parameter μ, defined recursively on the
length of α, starting by $u_{\alpha_i}(\mu) = \mu_{\alpha_i}(1 - g_i^{N_i})$. From all these data we define a Hopf
algebra $\mathfrak{u}(\mathcal{D}, \lambda, \mu)$ as the quotient of $\mathcal{T}(V) = T(V)\#\mathbb{C}\Gamma$ by the relations

$$ga_i g^{-1} = \chi_i(g)a_i, \tag{4.1}$$

$$\operatorname{ad}_c(a_i)^{1-a_{ij}}(a_j) = 0, \qquad\qquad i \neq j, i \sim j, \tag{4.2}$$

$$\operatorname{ad}_c(a_i)(a_j) = \lambda_{ij}(1 - g_i g_j), \qquad\qquad i < j, i \nsim j, \tag{4.3}$$

$$a_\alpha^{N_J} = u_\alpha(\mu). \tag{4.4}$$

⁴⁸² **Theorem 4.1.** *The Hopf algebra* $\mathfrak{u}(\mathcal{D}, \lambda, \mu)$ *is pointed,* $G(\mathfrak{u}(\mathcal{D}, \lambda, \mu)) \simeq \Gamma$ *and* $\dim \mathfrak{u}(\mathcal{D}, \lambda, \mu)$
⁴⁸³ $= \prod_{J \in \mathcal{X}} N_J^{|\Phi_J^+|}|\Gamma|$. *Let H be a pointed finite-dimensional Hopf algebra and set $\Gamma = G(H)$.*
⁴⁸⁴ *Assume that the prime divisors of $|\Gamma|$ are > 7. Then there exists a Cartan datum $\mathcal{D}$ and pa-*
⁴⁸⁵ *rameters λ and μ such that $H \simeq \mathfrak{u}(\mathcal{D}, \lambda, \mu)$. It is known when two Hopf algebras $\mathfrak{u}(\mathcal{D}, \lambda, \mu)$*
⁴⁸⁶ *and $\mathfrak{u}(\mathcal{D}', \lambda', \mu')$ are isomorphic.*

⁴⁸⁷   The proof offered in [16] relies on [14, 43, 59, 60]. Some comments: the hypothesis on
⁴⁸⁸ $|\Gamma|$ forces the infinitesimal braiding V of H to be of Cartan type, and the relations of $\mathfrak{B}(V)$
⁴⁸⁹ to be just quantum Serre and powers of root vectors. The quantum Serre relations are not
⁴⁹⁰ deformed in the liftings, except those linking different components of the Dynkin diagram;
⁴⁹¹ the powers of the root vectors are deformed to the $u_\alpha(\mu)$ that belong to the coradical. All
⁴⁹² this can fail without the hypothesis, see [52] for examples in rank 2.

⁴⁹³ **4.2. Pointed Hopf algebras with non-abelian group.** We present some classification re-
⁴⁹⁴ sults of pointed Hopf algebras with non-abelian group. We say that a finite group G *collapses*
⁴⁹⁵ whenever any finite-dimensional pointed Hopf algebra H with $G(H) \simeq G$ is isomorphic to
⁴⁹⁶ $\mathbb{C}G$.

⁴⁹⁷   • [7, 8] Let G be either $\mathbb{A}_m$, $m \geq 5$, or a sporadic simple group, different from Fi_{22},
⁴⁹⁸   the Baby Monster B or the Monster M. Then G collapses.

⁴⁹⁹   The proof uses §3.6.3; the remaining Yetter-Drinfeld modules are discarded consider-
⁵⁰⁰ ing abelian subracks of the supporting conjugacy class and the list in [44].

- [12] Let $V = M(\mathcal{O}_2^3, \text{sgn})$ and let $\mathcal{D}$ be the corresponding YD-datum. Let H be a finite-dimensional pointed Hopf algebra with $G(H) \simeq \mathbb{S}_3$. Then H is isomorphic either to $\mathbb{C}\mathbb{S}_3$, or to $\mathfrak{u}(\mathcal{D}, 0) = \mathfrak{B}(V)\#\mathbb{C}\mathbb{S}_3$, or to $\mathfrak{u}(\mathcal{D}, (0, 1))$, cf. Example 3.21.

- [38] Let $H \not\simeq \mathbb{C}\mathbb{S}_4$ be a finite-dimensional pointed Hopf algebra with $G(H) \simeq \mathbb{S}_4$. Let $V_1 = M(\mathcal{O}_2^4, \text{sgn} \otimes \text{id})$, $V_2 = M(\mathcal{O}_2^4, \text{sgn} \otimes \text{sgn})$, $W = M(\mathcal{O}_4^4, \text{sgn} \otimes \text{id})$, with corresponding data $\mathcal{D}_1$, $\mathcal{D}_2$ and $\mathcal{D}_3$. Then H is isomorphic to one of

$$\mathfrak{u}(\mathcal{D}_1, (0, \mu)), \quad \mu \in \mathbb{C}^2; \qquad \mathfrak{u}(\mathcal{D}_2, t), \quad t \in \{0, 1\}; \qquad \mathfrak{u}(\mathcal{D}_3, \lambda), \quad \lambda \in \mathbb{C}^2.$$

  Here $\mathfrak{u}(\mathcal{D}_1, (0, \mu))$ is as in Example 3.22; $\mathfrak{u}(\mathcal{D}_2, t)$ is the quotient of $\mathcal{T}(V_2)$ by the relations $a_{12}^2 = 0$, $a_{12}a_{34} - a_{34}a_{12} = 0$, $a_{12}a_{23} - a_{13}a_{12} - a_{23}a_{13} = t(1 - g_{(12)}g_{(23)})$ and $\mathfrak{u}(\mathcal{D}_3, \lambda)$ is the quotient of $\mathcal{T}(W)$ by the relations

$$a_{(1234)}^2 = \lambda_1(1 - g_{(13)}g_{(24)}); \qquad a_{(1234)}a_{(1432)} + a_{(1432)}a_{(1234)} = 0;$$
$$a_{(1234)}a_{(1243)} + a_{(1243)}a_{(1423)} + a_{(1423)}a_{(1234)} = \lambda_2(1 - g_{(12)}g_{(13)}).$$

  Clearly $\mathfrak{u}(\mathcal{D}_1, 0) = \mathfrak{B}(V_1)\#\mathbb{C}\mathbb{S}_4$, $\mathfrak{u}(\mathcal{D}_2, 0) = \mathfrak{B}(V_2)\#\mathbb{C}\mathbb{S}_4$, $\mathfrak{u}(\mathcal{D}_3, 0) = \mathfrak{B}(W)\#\mathbb{C}\mathbb{S}_4$. Also $\mathfrak{u}(\mathcal{D}_1, (0, \mu)) \simeq \mathfrak{u}(\mathcal{D}_1, (0, \nu))$ iff $\mu = c\nu$ for some $c \in \mathbb{C}^\times$, and $\mathfrak{u}(\mathcal{D}_3, \lambda) \simeq \mathfrak{u}(\mathcal{D}_3, \kappa)$ iff $\lambda = c\kappa$ for some $c \in \mathbb{C}^\times$.

- [7, 38] Let H be a finite-dimensional pointed Hopf algebra with $G(H) \simeq \mathbb{S}_5$, but $H \not\simeq \mathbb{C}\mathbb{S}_5$. It is not known whether $\dim \mathfrak{B}(\mathcal{O}_{2,3}^5, \text{sgn} \otimes \varepsilon) < \infty$. Let $\mathcal{D}_1$, $\mathcal{D}_2$ be the data corresponding to $V_1 = M(\mathcal{O}_2^5, \text{sgn} \otimes \text{id})$, $V_2 = M(\mathcal{O}_2^5, \text{sgn} \otimes \text{sgn})$. If the infinitesimal braiding of H is not $M(\mathcal{O}_{2,3}^5, \text{sgn} \otimes \varepsilon)$, then H is isomorphic to one of $\mathfrak{u}(\mathcal{D}_1, (0, \mu))$, $\mu \in \mathbb{C}^2$ (defined as in Example 3.22), or $\mathfrak{B}(V_2)\#\mathbb{C}\mathbb{S}_5$, or $\mathfrak{u}(\mathcal{D}_2, 1)$ (defined as above).

- [7] Let $m > 6$. Let $H \not\simeq \mathbb{C}\mathbb{S}_m$ be a finite-dimensional pointed Hopf algebra with $G(H) \simeq \mathbb{S}_m$. Then the infinitesimal braiding of H is $V = M(\mathcal{O}, \rho)$, where the type of σ is $(1^{m-2}, 2)$ and $\rho = \rho_1 \otimes \text{sgn}$, $\rho_1 = \text{sgn}$ or ε; it is an open question whether $\dim \mathfrak{B}(V) < \infty$, see Example 3.12. If $m = 6$, there are two more Nichols algebras with unknown dimension corresponding to the class of type (2^3), but they are conjugated to those of type $(1^4, 2)$ by the outer automorphism of $\mathbb{S}_6$.

- [34] Let $m \geq 12$, $m = 4h$ with $h \in \mathbb{N}$. Let $G = \mathbb{D}_m$ be the dihedral group of order $2m$. Then there are infinitely many finite-dimensional Nichols algebras in ${}^G_G\mathcal{YD}$; all of them are exterior algebras as braided Hopf algebras. Let H be a finite-dimensional pointed Hopf algebra with $G(H) \simeq \mathbb{D}_m$, but $H \not\simeq \mathbb{C}\mathbb{D}_m$. Then H is a lifting of an exterior algebra, and there infinitely many such liftings.

4.2.1. Copointed Hopf algebras. We say that a semisimple Hopf algebra K *collapses* if any finite-dimensional Hopf algebra H with $H_0 \simeq K$ is isomorphic to K. Thus, if G collapses, then $\mathbb{C}^G$ and $(\mathbb{C}G)^F$ collapse, for any twist F. Next we state the classification of the finite-dimensional copointed Hopf algebras over $\mathbb{S}_3$ [17]. Let $V = M(\mathcal{O}_2^3, \text{sgn})$ as a Yetter-Drinfeld module over $\mathbb{C}^{\mathbb{S}_3}$. Let $\lambda \in \mathbb{C}^{\mathcal{O}_2^3}$ be such that $\sum_{(ij) \in \mathcal{O}_2^3} \lambda_{ij} = 0$. Let $\mathfrak{v} = \mathfrak{v}(V, \lambda)$ be the quotient of $T(V)\#\mathbb{C}^{\mathbb{S}_3}$ by the relations $a_{(13)}a_{(23)} + a_{(12)}a_{(13)} + a_{(23)}a_{(12)} = 0$, $a_{(23)}a_{(13)} + a_{(13)}a_{(12)} + a_{(12)}a_{(23)} = 0$, $a_{(ij)}^2 = \sum_{g \in \mathbb{S}_3}(\lambda_{ij} - \lambda_{g^{-1}(ij)g})\delta_g$, for $(ij) \in \mathcal{O}_2^3$. Then $\mathfrak{v}$ is a Hopf algebra of dimension 72

and gr $\mathfrak{v} \simeq \mathfrak{B}(V)\#\mathbb{C}^{\mathbb{S}_3}$. Any finite-dimensional copointed Hopf algebra H with $H_0 \simeq \mathbb{C}^{\mathbb{S}_3}$ is isomorphic to $\mathfrak{v}(V, \lambda)$ for some λ as above; $\mathfrak{v}(V, \lambda) \simeq \mathfrak{v}(V, \lambda')$ iff λ and λ' are conjugated under $\mathbb{C}^{\times} \times \mathrm{Aut}\, \mathbb{S}_3$.

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