COMPACT QUANTUM GROUPS ARISING FROM THE FRT-CONSTRUCTION

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SYNOPSIS

*Let S be a solution of the Braid equation satisfing the Hecke condition. We review the construction of the "quantum analog of $GL(N)$" associated to S. We show that it gives rise to a compact quantum group when S is selfadjoint. We classify such matrices S among those found by Gurevich in (*GUREVICH *1991). We discuss when the associated "quantum exterior algebra" is Frobenius.*

SINOPSIS

*Sea S una solución de la ecuación de Trenzas que verifica la condición de Hecke. Se revisa la construcción del "análogo cuántico de $GL(N)$" asociado a S. Se muestra que da lugar a un grupo cuántico compacto cuando S es autoadjunta. Se clasifican dichas matrices S entre las encontradas por Gurevich en (*GUREVICH *1991). Se discute cuándo el "algebra exterior cuántica" asociada es Frobenius.*

1. INTRODUCTION

The purpose of this article is to present new examples of compact quantum groups (CQG, for short). These new examples arise from the celebrated FRT-construction (FADEEV, RESHETIKHIN & TAKHTAJAN 1990).

Given a vector space V of finite dimension n and an automorphism R of $V \otimes V$, we consider the algebra $A(R)$ generated by the entries of a generic $n \times n$ matrix T, with relations $RT_1T_2 = T_2T_1R$, $T_1 = T \otimes \mathrm{id}$,

$T_2 = \mathrm{id} \otimes T$. It is well-known that $A(R)$ bears a bialgebra structure (FADEEV, RESHETIKHIN & TAKHTAJAN 1990), and one would like to pass from $A(R)$ to a Hopf algebra. There is a universal way indicated in (MANIN 1988), Ch. 7, but it seems to be difficult to handle. When R is the Drinfeld-Jimbo solution of the Quantum Yang-Baxter equation, there is a second approach followed in (FADEEV, RESHETIKHIN & TAKHTAJAN 1990). (See (TAKEUCHI 1990) for a related point of view; see also (HAYASHI 1992), (DOI 1993)). The extension of this method to more general automorphisms R was developed by D. Gurevich (GUREVICH 1991) (see also (TSYGAN 1993)).

The strategy of this second approach applies in presence of several restrictions. First, R should be a solution of the Quantum Yang-Baxter equation (QYBE, for short). To state the next conditions, it is more convenient to work with $S = \tau R$, where τ is the usual transposition. It is well-known that S is a solution of the Braid equation if and only if R is a solution of the QYBE. The second restriction is that S should be of Hecke type, that is, it should satisfy the equation $(S-q)(S+1) = 0$, where we assume $q \neq 0$, $q^m \neq 1$ if $m \geq 2$, or $q = 1$. Hence the Hecke algebras act on the various tensor powers of V. Third, the "quantum exterior algebra" $\wedge_q(V) := T(V)/\langle \mathrm{Ker}\,(S-q)\rangle$ should be a "Frobenius algebra" in the sense of (MANIN 1988), Ch. 8. In particular, there is a "quantum determinant", a group-like element $d \in A(R)$. The localization of $A(R)$ at d and the quotient of $A(R)$ by the relation $d = 1$ are Hopf algebras that we denote $H(R)$ and $K(R)$; they can be considered as quantum analogues of $GL(n)$ and $SL(n)$. As is well-known, the Hopf algebra $K(R)$ degenerates if the determinant is not central.

Starting from this point, we look for conditions on S implying the existence of $*$-bialgebra structures in $H(R)$ and $K(R)$. There is a natural condition: for a fixed inner product on V, S is a selfadjoint operator with the respect to the extension of the inner product to $V \otimes V$. As a consequence, the Hopf algebras $H(R)$ and $K(R)$ are Compact Quantum Groups. We obtain finally new examples of CQG from known solutions of the QYBE. Concretely, solutions S as above such that $\wedge_q(V)$ has rank two were classified by Gurevich in (GUREVICH 1991). We completely determine the selfadjoint ones among them. See Theorem 5.0.2.

For the convenience of the reader, we review in this work the above sketched procedure to obtain the Hopf algebras $H(R)$ and $K(R)$ from the solution R of the QYBE. In fact, we do not need the full set of requirements meaning "$\wedge_q(V)$ is a Frobenius algebra", but only some of them. This led us to investigate the relationship between the full definition of Frobenius and the condition we need. In this direction, we prove that, if the homogeneous component $\wedge_q^p(V)$ is not zero and $\wedge_q^{p+1}(V) = 0$, then $\wedge_q(V)$ is a Frobenius algebra. The reason is that $\wedge_q(V)$ is a braided Hopf algebra, and it admits an integral $\int : \wedge_q(V) \to \mathbb{C}$ whenever it is finite dimensional.

The paper is organized as follows. In the second section, we recall the definition of a CQG and a criterion to pass from the algebraic to the C^*-algebraic framework. In the third, we construct the Hopf algebras $H(R)$ and $K(R)$. In the fourth section we present our criterion. The fifth is devoted to explicit examples; we analize the list of solutions of the QYBE from (GUREVICH 1991), characterize the selfadjoint ones and find out new compact quantum groups.

The authors, specially the second one, thank D. Gurevich for many interesting conversations along the years.

Conventions. We shall work over the field $\mathbb{C}$ of complex numbers. Tensor products and homorphisms are over $\mathbb{C}$ unless explicitly stated. The n-fold tensor product of a vector space W is denoted $W^{\otimes n}$ or $T^n W$. All the algebras we consider are associative with unit; all the coalgebras we consider are coassociative with counit. We denote by Δ, ϵ and $\mathcal{S}$ respectively the comultiplication, counit and antipode of a Hopf algebra (or bialgebra) H. We shall mainly consider right comodules, whose coactions we denote by $\delta : V \to V \otimes H$. We use Sweedler's notation for the comultiplication and coaction: $\Delta(x) = \sum x_{(1)} \otimes x_{(2)}$, $\delta(v) = \sum v_{(0)} \otimes v_{(1)}$, but we omit most of the time the summation sign. We denote by $G(H) = \{0 \neq g \in H : \ \Delta(g) = g \otimes g\}$ the semigroup of group-like elements in a bialgebra H. We adopt the usual notation for the categories of modules and comodules: ${}_A\mathcal{M}$, resp. $\mathcal{M}^{\mathcal{C}}$, is the category of left modules over an algebra A , respectively right comodules over a coalgebra C. We denote by Vec the category of complex vector spaces.

2. COMPACT QUANTUM GROUPS

The theory of compact quantum groups was developed by Woronowicz in a series of papers (Woronowicz 1987), (Woronowicz 1988), (Woronowicz 1989). At some points we follow (Guichardet 1995).

We first recall the classical notion of coalgebras with involution:

Definition 2.0.1. A $\circ$-*coalgebra* is a pair $(B, \circ)$, where B is a coalgebra and $\circ : B \to B$ is an anti-comultiplicative conjugate-linear involution.

If B is a $\circ$-coalgebra, the dual algebra B^* is a $*$-algebra with respect to the involution given by

$$\langle \alpha^*, x \rangle = \overline{\langle \alpha, x^\circ \rangle}, \tag{1}$$

$\alpha \in B^*$, $x \in B$. We can then consider two different notions of bialgebras with involution.

Definition 2.0.2. A $*$-*bialgebra* is a pair $(B, *)$, where B is a bialgebra and $* : B \to B$ is a conjugate-linear involution which is anti-multiplicative and comultiplicative. That is,

$$\begin{aligned} (xy)^* &= y^* x^*, \\ \Delta(x^*) = \Delta(x)^{*\otimes *} &= \sum \left(x_{(1)}\right)^* \otimes \left(x_{(2)}\right)^* . \end{aligned} \tag{2}$$

Dually, a $\circ$-*bialgebra* is a pair $(B, \circ)$, where B is a bialgebra and $\circ : B \to B$ is a conjugate-linear involution which is multiplicative and anti-comultiplicative. This means

$$\begin{aligned} (xy)^\circ &= x^\circ y^\circ, \\ \Delta(x^\circ) = \Delta^{\mathrm{op}}(x)^{\circ\otimes\circ} &= \sum \left(x_{(2)}\right)^\circ \otimes \left(x_{(1)}\right)^\circ . \end{aligned} \tag{3}$$

The first definition is probably due to G. I. Kats. The second is attributed to Drinfeld in (Manin 1988).

It follows from the definition that $\epsilon(x^*) = \overline{\epsilon(x)}$, or $\epsilon(x^\circ) = \overline{\epsilon(x)}$, according to the case. If B is a Hopf algebra, then it is a consequence of the definition that $(\mathcal{S}*)^2 = \mathrm{id}$ or $(\mathcal{S}\circ)^2 = \mathrm{id}$.

The bialgebras $A(R)$ from the FRT construction can be endowed with a $\circ$-structure, under certain conditions. To pass to a $*$-structure we shall use the following Lemma.

Lemma 2.0.1. *(Manin 1988) In a Hopf algebra H, $*$-structures and $\circ$-structures are equivalent.*

Proof. Explicitly, the equivalence is as follows: to $* : H \to H$, it corresponds $\circ : H \to H$ given by $x^\circ := (\mathcal{S}(x))^*$, $x \in H$; then $x^* := (\mathcal{S}(x))^\circ$, $x \in H$. As $\mathcal{S}$ is an antialgebra and anticoalgebra map, $*$ satisfies (2) if and only if $\circ$ satisfies (3). □

Definition 2.0.3. Let B be a $\circ$-coalgebra. A $\circ$-*comodule* is a pair $(V, (\,|\,))$, where V is a (right) comodule and $(\,|\,)$ is an hermitian form on V satisfying

$$(w_{(0)}|v) w_{(1)} = (w|v_{(0)}) {v_{(1)}}^\circ,$$

for all $v, w \in V$. If the hermitian form is an inner product (*i.e.* if it is positive definite), then we shall say that V is a *unitary comodule*.

If V is a right comodule over a coalgebra B, then V is a left module over the dual algebra B^*. We fix an hermitian form on V. Then V is a $\circ$-comodule, if and only if the corresponding representation $\rho : B^* \to \operatorname{End} V$ is a $*$-representation. That is, $(\rho(x)v|w) = (v|\rho(x^*)w)$, for all $x \in B^*$, $v, w \in V$. Indeed,

$$\begin{aligned}(\alpha^* v|w) &= (v_{(0)}|w)\langle\alpha^*, v_{(1)}\rangle = (v_{(0)}|w)\overline{\langle\alpha, v_{(1)}^\circ\rangle} \\ &= \overline{\langle\alpha, (w|v_{(0)})v_{(1)}^\circ\rangle} = \overline{\langle\alpha, (w_{(0)}|v)w_{(1)}\rangle} \\ &= (v|\langle\alpha, w_{(1)}\rangle w_{(0)}) = (v|\alpha w).\end{aligned}$$

The space of *matrix coefficients* of a comodule V is

$$C(V) = \{(\alpha \otimes \mathrm{id})\delta(v) : \alpha \in V^*, v \in V\} \subset B.$$

$C(V)$ is a subcoalgebra of B. If $\{v_1, \dots, v_n\}$ is a basis of V and $\delta(v_i) = \sum_j v_j \otimes T_{ji}$, then the T_{ij} span $C(V)$. If V is irreducible, they form a basis of $C(V)$. One has $\Delta(T_{ij}) = \sum_k T_{ik} \otimes T_{kj}$.

To recall the definition of compact quantum linear groups we need one more ingredient. A C^*-norm in a $*$-algebra H is a norm such that

$$\|xy\| \le \|x\|\|y\|, \quad \|1\| = 1, \quad \|xx^*\| = \|x\|^2,$$

for all $x, y \in H$. We shall consider the norm in $H \otimes H$ given by

$$\|\xi\| := \sup \|\pi(\xi)\|, \tag{4}$$

for all π in the set of $*$-representations of $H \otimes H$ whose restrictions to $1 \otimes H$ and $H \otimes 1$ are continuous. This is again a C^*-norm, see *e.g.* (TAKESAKI 1979), Ch. IV, §4.

Definition 2.0.4. A *compact linear quantum group* is a $*$-Hopf algebra H provided with a C^*-norm such that

- **(a):** The matrix coefficients of a finite dimensional unitary comodule V generate H as algebra.
- **(b):** The comultiplication Δ is continuous with respect to (4).

In this case, we shall say that $*$ is a *compact involution* of H. (In this paper, we only consider CQG which are linear. Accordingly, we also write CQG for compact linear quantum group.)

Remark 2.0.1. The completion of H with respect to the norm in (b) is a C^*-algebra; this is the original setting in (WORONOWICZ 1987). The canonical map from H into its completion is in fact injective; see (GUICHARDET 1995), (DIJKHUIZEN & KOORNWINDER 1994).

Remark 2.0.2. Any finite dimensional right comodule over a CQG H admits an hermitian form which is positive definite, i.e. an inner product (WORONOWICZ 1987), Th. 5.2. This implies that H is cosemisimple; that is, any H-comodule is completely reducible. By the structure theory of cosemisimple Hopf algebras (SWEEDLER 1969), H is the direct sum of its simple subcoalgebras:

$$H = \oplus_{\nu \in \widehat{H}} C(\nu). \tag{5}$$

Here $\widehat{H}$ denotes the set of classes of irreducible H-comodules, and $C(\nu)$ is the space of matrix coefficients of any irreducible comodule in the class ν; $C(\nu)$ is a simple subcoalgebra.

We denote by $\int : H \to C(\epsilon) = \mathbb{C}1$ the projection with respect to (5). Then $\int$ is a left and right integral in the sense of augmented algebras (SWEEDLER 1969). It is also called a Haar measure in the literature.

We consider the sesquilinear form $(\,|\,)_\ell$ defined by $(v|w)_\ell = \langle \int, w^* v\rangle$; it is hermitian, invariant for the right coaction given by the comultiplication and satisfies $(uv|w)_\ell = (v|u^*w)_\ell$. It can be shown that $(H, *)$ is a compact quantum group if and only if the hermitian form $(\,|\,)_\ell$ is positive definite. See (ANDRUSKIEWITSCH 1994).

Theorem 2.0.1. *Let H be a $*$-Hopf algebra generated (as algebra) by the matrix coefficients of a unitary finite dimensional comodule. Then H, provided with the sup of C^*-seminorms, is a CQG.*

Proof. This is (DIJKHUIZEN & KOORNWINDER 1994), Th. 4.4. See also (WORONOWICZ 1987), (ANDRUSKIEWITSCH 1992), (GUICHARDET 1995), p. 37. □

Remark 2.0.3. A Hopf algebra H carries usually many diferent $*$-involutions. Two compact involutions of a cosemisimple Hopf algebra are necessarily conjugated by a Hopf algebra automorphism (ANDRUSKIEWITSCH 1994), Th. 2.6. The problem of existence of compact involutions on cosemisimple Hopf algebras remains open. We have however the following restriction:

If a cosemisimple Hopf algebra H has a compact involution, then $\mathcal{S}^2$ has real eigenvalues.

In fact, $\mathcal{S}^2 * = * \mathcal{S}^{-2}$ and hence, as $\mathcal{S}^2$ is a Hopf algebra automorphism,

$$\begin{aligned}(\mathcal{S}^2 v|w)_\ell &= \left\langle \int, w^* \mathcal{S}^2 v\right\rangle = \left\langle \int, \mathcal{S}^{-2}(w^*) v\right\rangle \\ &= \left\langle \int, (\mathcal{S}^2 w)^* v\right\rangle = (v|\mathcal{S}^2 w)_\ell .\end{aligned}$$

It follows that the Hopf algebra $\mathbb{C}_q[G]$ (quantum version of the algebra of rational functions on the complex simple algebraic group G, *cf.* (LUSZTIG 1990)), which is cosemisimple if q is not a root of 1, has no compact form if q is not a real number.

Indeed, let M be a finite dimensional module, $v \in M$, $\alpha \in M^*$. Assume that v, resp. α is a weight vector of weight λ, resp. μ. In the universal enveloping algebra $U_q(g)$, $\mathcal{S}^2$ is conjugation by the group-like element ρ corresponding to the half-sum of the positive roots, see (DRINFELD 1990). Hence

$$\mathcal{S}^2\left(\phi^M_{\alpha,v}\right) = q^{\langle\rho,\lambda-\mu\rangle}\phi^M_{\alpha,v}.$$

Here $\phi^M_{\alpha,v}$ denotes the matrix coefficient corresponding to v, α.

3. THE HOPF ALGEBRAS $H(R)$ AND $K(R)$

We state the results of this section over $\mathbb{C}$ but they are valid over any field.

3.1. The bialgebra A(R).

In this Subsection, we give a coordinate-free presentation of the FRT-construction following (DOI 1993).

Let W and U be vector spaces. We denote by $\tau : W \otimes U \to U \otimes W$ the usual transposition.

We shall identify $(W^*)^{\otimes n}$ with a subspace of $(W^{\otimes n})^*$ by the duality

$$\langle \alpha_1 \otimes \cdots \otimes \alpha_n, v_n \otimes \cdots \otimes v_1\rangle = \langle \alpha_n, v_n\rangle \ldots \langle \alpha_1, v_1\rangle. \tag{6}$$

Let C be a coalgebra and $A = C^*$ its dual algebra; this means that the multiplication of A is the restriction of the comultiplication of C, via (6). Let $T^nC = C^{\otimes n}$, $T^nA = A^{\otimes n}$ be the n-fold tensor product coalgebra, resp. algebra. We identify T^nA with a subspace of $(T^nC)^*$ via (6); T^nA is a subalgebra of the dual algebra of T^nC.

The tensor algebra $T(C) = \oplus_{n\geq 0}T^nC$ is a bialgebra where the coalgebra structure is the direct sum of the subcoalgebras T^nC.

We consider T^nC as T^nA-bimodule by transposition of the left and right regular actions. In terms of the coalgebra structure, these actions are given by

$$R\eta = \eta_{(1)}\langle \eta_{(2)}, R\rangle, \quad \eta R = \langle \eta_{(1)}, R\rangle \eta_{(2)},$$

$\eta \in T^nC, \quad R \in T^nA.$
We fix an element $R \in T^2A$ and set

$$\begin{aligned} J_R &= \{\eta R - \tau(R\eta) : \eta \in T^2C\} \\ &= \operatorname{span}\big\{\langle \alpha_{(1)} \otimes \beta_{(1)}, R\rangle\, \alpha_{(2)} \otimes \beta_{(2)} - \\ &\quad \beta_{(1)} \otimes \alpha_{(1)}\, \langle \alpha_{(2)} \otimes \beta_{(2)}, R\rangle : \alpha, \beta \in C\big\}. \end{aligned}$$

The two-sided ideal $\langle J_R\rangle$ generated by J_R is a bi-ideal; hence the quotient algebra

$$A(R) := T(C)/\langle J_R\rangle$$

is a quotient bialgebra. It is a graded algebra, but not a graded coalgebra. We shall identify C with the image of the coalgebra homomorphism $C \to T(C) \to A(R)$.

Let now V be a right C-comodule. The coaction $\delta : V \to V \otimes C$ extends to a unique coaction $\delta : T(V) \to T(V) \otimes T(C)$; hence $T(V)$ is a $T(C)$-comodule algebra. The coaction is graded in the following sense: $\delta(T^nV) \subset T^nV \otimes T^nC$, for all n. By corestriction, $T(V)$ is a graded $A(R)$-comodule algebra.

For any pair V, W, of right C-comodules there is a linear transformation $S_{V,W} : V \otimes W \to W \otimes V$ induced by R:

$$S_{V,W}(v \otimes w) = w_{(0)} \otimes v_{(0)}\langle R, v_{(1)} \otimes w_{(1)}\rangle.$$

We shall denote S_V for $S_{V,V}$. Sometimes we shall simply write S instead of $S_{V,W}$.

The following theorem, a generalization of (Larson & Towber 1991) Th. 3.1, characterizes $A(R)$. A related characterization is given in (Doi 1993). Compare also with (Majid 1995), pp. 438 and 476.

Theorem 3.1.1. *The bialgebra $A(R)$ is universal among the bialgebras B satisfying the following properties:*

- **(a):** *There exist a coalgebra map $C \to B$.*
- **(b):** *Given two C-comodules M and N, $S_{M,N}$ is a B-comodule map when M and N are regarded as B-comodules via the map considered in (a).*

If a bialgebra B satisfies (a) and (b), then there exists a bialgebra map $\phi : A(R) \to B$ which verifies the following: for any C-comodule M, if δ_B and $\delta_{A(R)}$ are the structure maps given as in (b), then

$$\delta_B = (\mathrm{id} \otimes \phi) \circ \delta_{A(R)}.$$

Proof. We first show that $A(R)$ satisfies the desired properties. For (a) consider the injective coalgebra map defined by $C \to T(C) \to A(R)$. As for (b), let M, N be two C-comodules. We compute the maps

$(S\otimes \mathrm{id})\delta, \delta S : M\otimes N \to N\otimes M\otimes C\otimes C$:

$$\begin{aligned}(S\otimes \mathrm{id})\delta(m\otimes n) &= n_{(0)}\otimes m_{(0)}\otimes \\ &\qquad \langle m_{(1)}\otimes n_{(1)}, R\rangle m_{(2)}\otimes n_{(2)}, \\ \delta S(m\otimes n) &= n_{(0)}\otimes m_{(0)}\otimes n_{(1)}\otimes \\ &\qquad m_{(1)}\langle m_{(2)}\otimes n_{(2)}, R\rangle.\end{aligned}$$

This implies that property (b) holds for $A(R)$.

Let B be a bialgebra satisfying (a) and (b), and $\phi : C\to B$ the corresponding map. We extend ϕ to a morhism of (bi)algebras, still denoted as ϕ, from $T(C)$ to B. As C is a right comodule over itself by the comultiplication, we get that S_{CC} is a morphism in $\mathcal{M}^B$, i.e.

$$\begin{aligned}&n_{(1)}\otimes m_{(1)}\otimes \langle m_{(2)}\otimes n_{(2)}, R\rangle\, \phi\left(m_{(3)}\right)\phi\left(n_{(3)}\right) = \\ &\quad n_{(1)}\otimes m_{(1)}\otimes \phi\left(n_{(2)}\right)\phi\left(m_{(2)}\right)\langle m_{(3)}\otimes n_{(3)}, R\rangle .\end{aligned}$$

Applying ϵ to the first two factors, we conclude that ϕ factorizes trough $A(R)$ by a morphism still named ϕ. Let now $M\in\mathcal{M}^C$, and call δ_B and $\delta_{A(R)}$ the structure maps obtained when we regard $M\in\mathcal{M}^B$ and $M\in\mathcal{M}^{A(R)}$ respectively. Then

$$\begin{aligned}\left((\mathrm{id}\otimes\phi)\circ\delta_{A(R)}\right)(m) &= (\mathrm{id}\otimes\phi)\left(m_{(0)}\otimes m_{(1)}\right) = \\ &= m_{(0)}\otimes\phi\left(m_{(1)}\right) = \delta_B(m).\end{aligned}$$

□

In this paper, we are concerned with the following important particular case of the construction above. Let $A = \mathrm{End}\,(V)$ and $C = (\mathrm{End}\,(V))^*$, where V is a finite dimensional vector space. The natural action of A on V gives rise to a right comodule structure on V. Explicitly, $\delta : V\to V\otimes C$ is given by

$$(\mathrm{id}\otimes T)\delta(v) = T(v), \quad v\in V,\ T\in A.$$

We present $A(R)$ in terms of coordinates. We fix a basis $\{v_1,\cdots,v_m\}$ of V, and let $\{e_{ij}\}$ be the basis of A given by $e_{ij}(v_k) = \delta_{jk}v_i$. Let $\{e^{ij}\}\subset C$ the basis dual to $\{e_{ij}\}$. Then $\delta(v_i) = \sum_j v_j\otimes e^{ji}$. Let $t_{ij}\in A(R)$ be the image of e^{ij} under the canonical projection. Then

$$\delta(v_i) = \sum_j v_j\otimes t_{ji}. \tag{7}$$

Now, if $R(v_i\otimes v_j) = \sum_{kl} R_{ij}^{kl}\, v_k\otimes v_l$, then $A(R)$ can be presented as the algebra generated by t_{ij}, $1\le i,j\le n$, with relations:

$$\sum_{mn} R_{ij}^{nm} t_{km}\, t_{ln} = \sum_{mn} R_{mn}^{lk}\, t_{mi}\, t_{nj},\ \forall i,j,k,l. \tag{8}$$

This means that $A(R)$ coincides with the bialgebra defined in (Fadeev, Reshetikhin & Takhtajan 1990). Moreover,

$$S_V = S = \tau R.$$

A version of the following Proposition is proved in (FADEEV, RESHETIKHIN & TAKHTAJAN 1990).

Proposition 3.1.1. *Let C be a coalgebra, A its dual algebra, $R \in A \otimes A$, V a finite dimensional C-comodule. Let q be an eigenvalue of S_V. Let $I_q = \ker(S_V - q) \subseteq T^2V$. Let $\wedge_q(V) = \oplus_{n\geq 0} \wedge_q^n(V)$ be the quotient of $T(V)$ by the homogeneous two-sided ideal $\langle I_q \rangle$. Then $\wedge_q(V)$ is an $A(R)$-comodule algebra. Moreover, $\delta(\wedge_q^n(V)) \subseteq \wedge_q^n(V) \otimes A(R)^n$.*

Proof. The map $S_V : T^2V \to T^2V$ is a $A(R)$-comodule map; hence I_q is a $A(R)$-subcomodule. As the coaction is an algebra map, the two-sided ideal $\langle I_q \rangle$ is a $A(R)$-subcomodule of $T(V)$. The Proposition follows from this. □

The homogeneous components $\wedge_q^n(V) = T^nV/I_q^n$ can be described as follows: $I_q^2 = I_q$ and

$$I_q^n := T^nV \cap \langle I_q \rangle = \sum_{i=1}^{n-1} I_q^{n,i},$$
$$I_q^{n,i} = T^{i-1}V \otimes I_q \otimes T^{n-i-1}V,$$

for $i = 1, \cdots, n-1$, $n = 2, 3, \cdots$. We identify V with $\wedge_q^1(V)$ and write $\wedge$ for the product in $\wedge_q(V)$. We shall denote $\wedge_{S,q}(V)$ instead of $\wedge_q(V)$, when emphasis on S is needed.

Let $\Lambda = \oplus_{p\geq 0}\Lambda^p$ be a graded algebra. We consider the following conditions on Λ, for $p \in \mathbb{N}$, $0 \leq j \leq p$:

$F_0(p)$: $\dim \Lambda^p = 1$.

$F_j(p)$: $\dim \Lambda^p = 1$ and the multiplication $\Lambda^{p-j} \times \Lambda^j \xrightarrow{\wedge} \Lambda^p$ is non-degenerate.

A graded algebra Λ is *Frobenius* of rank p if, for a fixed p, $F_j(p)$ holds for any j, $0 \leq j \leq p$ and $\Lambda^j = 0$, for $j > p$. We shall not need, for the construction of the antipode, all the conditions $F_j(p)$; see Theorem 3.3.2, and the discussion in Subsection 3.7. The following Proposition appears in (MANIN 1988).

Proposition 3.1.2. *Let V be a finite dimensional vector space, $C = (\operatorname{End} V)^*$, $R : V \otimes V \to V \otimes V$ a linear map, $S = \tau R$, q an eigenvalue of S.*

(a): *If $\wedge_q(V)$ verifies $F_0(p)$, then there exists $d \in A(R)^p$, $d \neq 0$, such that $\Delta d = d \otimes d$.*

(b): *If $\wedge_q(V)$ verifies $F_1(p)$, then there exists $A_{ij} \in A(R)^{p-1}$, $i, j = 1, \cdots, m$ such that*

$$\sum_{k=1}^{m} A_{ik}t_{kj} = \delta_{ij}d. \tag{9}$$

(c): *If $\wedge_q(V)$ verifies $F_{p-1}(p)$, then there exists $\tilde{A}_{ij} \in A(R)^{p-1}$, $i, j = 1, \cdots, m$ such that*

$$\sum_{k=1}^{m} t_{ki}\tilde{A}_{kj} = \delta_{ij}d. \tag{10}$$

Proof. (a). This follows from Proposition 3.1.1: if $0 \neq \mu \in \wedge_q^pV$, then $\delta(\mu) = \mu \otimes d$, $d \in A(R)^p$; d does not depend on the choice of μ.

(b). We fix $0 \neq \mu \in \wedge_q^pV$ and consider the basis $\{\eta_1, \cdots, \eta_m\}$ of $\wedge_q^{p-1}(V)$ determined by $\eta_i \wedge v_j = \delta_{ij}\mu$; we define $A_{ij} \in A(R)^{p-1}$ by $\delta(\eta_i) = \sum_{j=1}^m \eta_j \otimes A_{ij}$, $i = 1, \cdots, m$. As δ is an algebra map, (9) follows.

(c). Analogous to the preceding, considering this time $V \times \wedge_q^{p-1}V \xrightarrow{\wedge} \wedge_q^pV$. □

By abuse of notation, we shall say that S satisfies $F_j(p)$ if $\wedge_q(V)$ does.

Definition 3.1.1. Let V be a finite dimensional vector space, $\mathbf{S} \in \mathrm{End}\,(V \otimes V)$, $q \in \mathbb{C}$, $q \neq 0, -1$. We shall say that $\mathbf{S}$ satisfies the *Hecke condition with label* q if

$$(\mathbf{S} - q)(\mathbf{S} + 1) = 0.$$

We have then $\mathrm{Ker}\,(\mathbf{S} - q) = \mathrm{Im}\,(\mathbf{S} + 1)$,
$\mathrm{Ker}\,(\mathbf{S} + 1) = \mathrm{Im}\,(\mathbf{S} - q)$. We denote $\mathrm{Sym}_{\mathbf{S}} V = \wedge_{-1} V = T(V)/\langle \mathrm{Im}\,(\mathbf{S} - q)\rangle$, $\wedge_{\mathbf{S}} V$ for $\wedge_q V$ and $I_{\mathbf{S}}$ for I_q.

If C is a coalgebra, A its dual algebra, $S \in A \otimes A$, V a finite dimensional C-comodule and $\mathbf{S} = S_V$, then we denote $\mathrm{Sym}_S V = \mathrm{Sym}_{\mathbf{S}} V$, $\wedge_S V = \wedge_{\mathbf{S}} V$ and $I_S = I_{\mathbf{S}}$; Proposition 3.1.2 applies to both of $\mathrm{Sym}_S V$ and $\wedge_S V$.

If S is the usual transposition, then $\mathrm{Sym}_S V$ and $\wedge_S V$ are respectively the usual symmetric and exterior algebra on V.

Let $\overline{S} \in \mathrm{End}\,(T^2V)$ be given by $\overline{S}(v_i \otimes v_j) = \sum_{kl} \overline{S}_{ij}^{kl} v_k \otimes v_l$; this means we consider the real form $\sum_i \mathbb{R} v_i$ of V, and the corresponding involutions of V, T^2V, $\mathrm{End}\, T^2V$. The following result will be needed in Section 4.

Lemma 3.1.1. *If $S \in \mathrm{End}\,(T^2V)$ satisfies the Hecke condition with label q, then*

- **(a):** *$\overline{S} \in \mathrm{End}\,(T^2V)$ satisfies the Hecke condition with label $\overline{q}$; and*
- **(b):** *S verifies $F_j(p)$ if and only if $\overline{S}$ verifies $F_j(p)$.* □

3.2. **The Hecke algebra.** We recall in this Subsection some facts about the Hecke algebra (see for instance (Harpe, Kervaire & Weber 1986)).

Let $q \in \mathbb{C}$, $q \neq 0$, $n \in \mathbb{N}$, $n \geq 2$. The *Hecke algebra* $H_q(n)$ is the $\mathbb{C}$-algebra presented by generators $\sigma_1, \cdots, \sigma_{n-1}$ with relations

- **(a):** $(\sigma_i - q)(\sigma_i + 1) = 0$, $i = 1, \cdots, n-1$
- **(b):** $\sigma_i \sigma_j = \sigma_j \sigma_i$, $|i - j| \geq 2$.
- **(c):** $\sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1}$, $i = 1, \cdots, n-2$.

We shall assume that $q \neq 0$, $q^n \neq 1$ if $n \geq 2$ or $q = 1$. Then the Hecke algebra $H_q(n)$ is semisimple. It has two one-dimensional representations:

- **(a):** the trivial, which sends any σ_i to q, and
- **(b):** the sign, which sends any σ_i to -1.

For $q = 1$, $H_q(n)$ is the group algebra of the symmetric group $\mathbb{S}_n$; this motivates the preceding notation for the characters. In fact, it is well known that $H_q(n)$ is isomorphic to the group algebra of the symmetric group $\mathbb{S}_n$, provided that the above restrictions on q hold.

Let $B(n)$ be the set of all monomials of the form

$$(\sigma_{i_1} \sigma_{i_1 - 1} \cdots \sigma_{h_1}) \cdots (\sigma_{i_p} \sigma_{i_p - 1} \cdots \sigma_{h_p})$$

where $1 \leq h_1 \leq i_1, \cdots, 1 \leq h_p \leq i_p$ and $1 \leq i_1 < i_2 < \cdots < i_p \leq n-1$. It is known that $B(n)$ is a basis of $H_q(n)$. The set $B(n)$ is identified with $\mathbb{S}_n$, via

$$(\sigma_{i_1} \cdots \sigma_{h_1}) \cdots (\sigma_{i_p} \cdots \sigma_{h_p}) \leftrightarrow (t_{i_1} \cdots t_{h_1}) \cdots (t_{i_p} \cdots t_{h_p}),$$

where t_i is the transposition $(i, i+1)$.

We recall the definition of the quantum factorial number

$$[n]_q! = [n]_q[n-1]_q \cdots [2]_q[1]_q,$$
$$[n]_q = 1 + q + \cdots + q^{n-1}, \ n \geq 1.$$

We observe that

$$[n]_q! = q^{\frac{n(n-1)}{2}} [n]_{q^{-1}}!, \qquad \sum_{t \in B(n)} q^{|t|} = [n]_q!,$$

where $|t| = h_1 + \cdots + h_p$ if

$$t = (\sigma_{i_1}\sigma_{i_1-1} \cdots \sigma_{h_1}) \cdots (\sigma_{i_p}\sigma_{i_p-1} \cdots \sigma_{h_p}).$$

Since $q^m \neq 1$, if $m \geq 2$ or $q = 1$, it is $[n]_q! \neq 0$. We consider the following elements of the Hecke algebra:

$$M_- := \frac{1}{[n]_{q^{-1}}!} \sum_{t \in B(n)} (-q)^{-|t|} t,$$
$$M_\epsilon := \frac{1}{[n]_q!} \sum_{t \in B(n)} t.$$

We are interested in these elements because of the following reason. If W is a representation of $H_q(n)$, then the action of M_- on W (resp., M_ϵ) is the $H_q(n)$-invariant projection onto its isotypical component of sign (resp., trivial) type.

Let $\# : H_q(n) \to H_q(n)$ be the unique algebra anti-isomorphism such that $\sigma_i^\# = \sigma_i, \ i = 1, \cdots, n-1$. Let V, W be $H_q(n)$-modules. One says that a bilinear map $\langle\langle \ , \ \rangle\rangle : V \times W \to \mathbb{C}$ is $H_q(n)$-*invariant* if

$$\langle\langle x \cdot v, w \rangle\rangle = \left\langle\left\langle v, x^\# \cdot w \right\rangle\right\rangle,$$

for all $v \in V$, $w \in W$, $x \in H_q(n)$.

For instance, if V is a $H_q(n)$-module, then we regard $V^* = \operatorname{Hom}(V, \mathbb{C})$ as $H_q(n)$-module by:

$$\langle x \cdot \phi, v \rangle := \left\langle \phi, x^\# \cdot v \right\rangle, \tag{11}$$

for all $v \in V$, $\phi \in V^*$, $x \in H_q(n)$. In this way, the canonical duality between V and V^* is $H_q(n)$-invariant.

Conversely, if V, W are $H_q(n)$-modules and

$$\langle\langle \ , \ \rangle\rangle : V \times W \to \mathbb{C}$$

is a bilinear $H_q(n)$-invariant map, then $\Upsilon : V \to W^*$, $\langle \Upsilon(v), w \rangle := \langle\langle v, w \rangle\rangle$, $v \in V, w \in W$, is a map of $H_q(n)$-modules.

There is a unique algebra isomorphism (of order 2) $\Psi : H_q(n) \to H_q(n)$ such that

$$\Psi(\sigma_i) = \sigma_{n-i}, \ 1 \leq i \leq n-1.$$

Clearly,

$$\Psi(M_-) = (M_-)^\# = M_-,$$
$$\Psi(M_\epsilon) = (M_\epsilon)^\# = M_\epsilon.$$

Let us consider $H_q(n)$ as a subalgebra of $H_q(n+1)$ by identifying the generator σ_i of $H_q(n)$ with the generator σ_i of $H_q(n+1)$. Let

$$\left(M_-^n\right)_2 = \frac{1}{[n]_{q^{-1}}!} \sum_{t \in B(n)} (-q)^{-|t|} \hat{t} \in H_q(n+1),$$

where $\hat{t} = \sigma_{i_1+1} \cdots \sigma_{i_l+1}$ if $t = \sigma_{i_1} \cdots \sigma_{i_l}$.

Proposition 3.2.1. *(*GUREVICH *1991) We have*

1.

$$M_-^n \, \sigma_n \, M_-^n = - \, [n+1]_q \, [n]_q^{-1} \, M_-^{n+1} + \\ q^n \, [n]_q^{-1} M_-^n .$$

2. *If* $M_-^{n+1} = 0$*, then*

$$M_-^n \, \left(M_-^n\right)_2 \, M_-^n = q^{n-1} \, [n]_q^{-2} M_-^n .$$

3.3. **Duality between** $\wedge_S^n(V)$ **and** $\wedge_{tS}^n(V^*)$**.** Let C be a coalgebra, A its dual algebra, $R \in A \otimes A$, V a C-comodule.

Definition 3.3.1. We say that $S = S_V$ is a solution of the Braid equation if

$$S^{12} S^{23} S^{12} = S^{23} S^{12} S^{23}. \tag{12}$$

Here we adopt the usual notation: for $1 \le i < j \le n$, $S^{ij} \in \mathrm{End}\, T^n V$ acts as S on the factors i, j , and as the identity on the others.

We denote by $R = R_V$ the endomorphism of $V \otimes V$ given by the action of R. Then $R_V = \tau S_V$. It is well-known that (12) is equivalent to the Quantum Yang-Baxter equation:

$$R^{12} R^{13} R^{23} = R^{23} R^{13} R^{12}. \tag{13}$$

Note that (13) makes sense if $R \in A \otimes A$ where A is any algebra.

We assume from now on that S_V verifies both the Hecke condition with label q and the Braid equation; we also suppose that $q^m \neq 1$, if $m \ge 2$ or $q = 1$. Hence the Hecke algebra $H_q(n)$ acts linearly on $T^n V$ via $\sigma_i \mapsto S^{i,i+1} = S^i \in \mathrm{End}\,(T^n V)$. Note that $I_S^{n,i} = \mathrm{Im}\,(S^i + 1)$ for all $i = 1, \cdots, n-1$.

Proposition 3.3.1. *The action of* $H_q(n)$ *and the coaction of* $A(R)$ *in* $T^n V$ *commute,* i.e.

$$(x \cdot w)_{(0)} \otimes (x \cdot w)_{(1)} = (x \cdot w_{(0)}) \otimes w_{(1)},$$

for all $x \in H_q(n)$ *and* $w \in T^n V$.

Proof. This follows because S_V is a morphism of $A(R)$-comodules. □

It can be shown that one of these actions is the commutant of the other (it cf.) [(HAI 1997), Thm. 2.1].

Proposition 3.3.2. *(*GUREVICH *1991)* $\wedge_S^n V$ *is a* $H_q(n)$*-module quotient of* $T^n V$*, isomorphic to* $(T^n V)_-$*.*

Proof. From

$$S^i (S^j + 1) = (S^i + 1)(S^j + 1) - (S^j + 1), \quad \forall i, j,$$

we deduce $\sigma_i \cdot I_S^n \subset I_S^n$; hence $\wedge_S^n V$ is a $H_q(n)$-module quotient of $T^n V$. As $M_- \sigma_i = -M_-$ for all $i = 1, \cdots, n-1$, we have

$$\begin{aligned} &M_-(\sigma_i + 1) = 0, \ \forall i \\ \Rightarrow &M_- \circ (S^i + 1) = 0, \ \forall i \\ \Rightarrow &M_-|_{I_S^{n,i}} = 0, \ \forall i \\ \Rightarrow &M_-|_{I_S^n} = 0 \\ \Rightarrow &(T^n V)_- \cap I_S^n = \{0\}; \end{aligned}$$

that is, the restriction of the canonical projection $\pi_\wedge|_{(T^n V)_-} : (T^n V)_- \to \wedge_S^n V$ is injective. On the other hand, $\mathrm{Im}\,(S^i + 1) = I_S^{n,i} \subset I_S^n$, hence $H_q(n)$ acts on $\wedge_S^n V$ by the sign representation. Thus $\pi_\wedge(M_- \cdot w) = \pi_\wedge(w)$ for any $w \in T^n V$, and $\pi_\wedge|_{(T^n V)_-} : (T^n V)_- \to \wedge_S^n V$ is surjective. □

Remark 3.3.1. Compare with (ANDRUSKIEWITSCH & GRAÑA 1999).

Observe that $I_S^n = \ker(M_-)$, because $\pi_\wedge \circ M_- = \pi_\wedge$. Combining Propositions 3.3.1 and 3.3.2, we conclude:

Proposition 3.3.3. *The action of $H_q(n)$ and the coaction of $A(R)$ on $\wedge_S^n V$ commute.* □

We assume now that $C = (\mathrm{End}\, V)^*$, $\dim V < \infty$. Let $C^\diamond = (\mathrm{End}\,(V^*))^*$. Let ${}^t(\,) : \mathrm{End}\,(V^*) \to \mathrm{End}\,(V)^{\mathrm{op}}$ denote the algebra isomorphism given by the transposition.

Definition 3.3.2. We denote by $\theta : C \to C^\diamond$ the anti-coalgebra isomorphism induced by the transposition ${}^t(\,)$. That is,

$$\langle \theta(\alpha), x \rangle = \langle \alpha, {}^t x \rangle.$$

We still denote by θ the induced algebra isomorphism from $T(C)$ onto $T(C^\diamond)$; θ is in fact a bialgebra isomorphism from $T(C)^{\mathrm{cop}}$ onto $T(C^\diamond)$.

Let now $R : V \otimes V \to V \otimes V$ and $S = \tau R$. We set $\widehat{R} = \tau({}^t R) = \tau {}^t R \tau \in \mathrm{End}\,(V^*) \otimes \mathrm{End}\,(V^*)$. Then $\tau \widehat{R} = {}^t S$.

It is not difficult to prove:

Lemma 3.3.1. **(a):** *S satisfies the Braid equation (resp., the Hecke condition with label q), if and only if ${}^t S$ does.*

(b): *If $J_{\widehat{R}} := \{\widehat{R}\zeta - \tau(\zeta \widehat{R}) : \zeta \in T^2(C^\diamond)\}$, then $\theta(J_R) = J_{\widehat{R}}$. Hence θ induces a bialgebra isomorphism, again denoted by θ, from $A(R)^{\mathrm{cop}}$ onto $A(\widehat{R}) := T(C^\diamond)/\langle J_{\widehat{R}} \rangle$.* □

For a fixed basis $\{v_i\}$ of V, we denote: $\{\alpha_i\}$, its dual basis in V^*; $\{f_{ij}\}$ the basis of $\mathrm{End}\,(V^*)$ given by $f_{ij}(\alpha_k) = \delta_{jk}\alpha_i$, and $\{f^{ij}\}$ the basis of $C^\diamond = (\mathrm{End}\,(V^*))^*$ dual of the basis $\{f_{ij}\}$. We further denote by u_{ij} the class of f^{ij} in $A(\widehat{R})$. Hence

$$\theta(t_{ij}) = u_{ji}.$$

We recall that S satisfies the Braid equation and the Hecke condition with label q.

The Hecke algebra $H_q(n)$ acts on $T^n(V^*)$ by $\sigma_i \mapsto ({}^t S)^i$. The duality $\langle\ ,\ \rangle : T^n V \times T^n(V^*) \to \mathbb{C}$ is $H_q(n)$-invariant in the following "twisted" sense:

$$\langle x \cdot \phi, v \rangle = \langle \phi, (\Psi(x))^{\#} \cdot v \rangle.$$

We denote, according to our previous conventions,

$$\begin{aligned} I_{{}^tS} &= \operatorname{Im}\left({}^tS+1\right) \subset T^2(V^*), \\ \wedge_{{}^tS}(V^*) &= T(V^*)/\langle I_{{}^tS}\rangle = \oplus_{n=0}^{\infty} \wedge_{{}^tS}^n (V^*). \end{aligned}$$

Theorem 3.3.1. *The bilinear form $\langle\ ,\ \rangle : \wedge_S^n V \times \wedge_{{}^tS}^n(V^*) \to \mathbb{C}$ given by $\langle w + I_S^n, \alpha + I_{{}^tS}^n\rangle := \langle w, M_- \cdot \alpha\rangle$, $w \in T^nV$, $\alpha \in T^n(V^*)$, is non degenerate, and $H_q(n)$-invariant.*

Proof. The canonical duality $\langle\ ,\ \rangle : T^nV \times T^n(V^*) \to \mathbb{C}$ induces a non-degenerate $H_q(n)$-invariant bilinear form $\langle\ ,\ \rangle : T^nV/I_S^n \times (I_S^n)^\perp \to \mathbb{C}$ by $\langle w + I_S^n, \alpha\rangle := \langle w, \alpha\rangle$, $w \in T^nV$, $\alpha \in (I_S^n)^\perp$. As $I_S^n = \ker M_-^{T^nV}$,

$$\begin{aligned} (I_S^n)^\perp &= \left(\ker M_-^{T^nV}\right)^\perp = \operatorname{Im} M_-^{T^n(V^*)} = \\ &= (T^n(V^*))_- \simeq \wedge_{{}^tS}^n(V^*). \end{aligned}$$

So we have a non-degenerate $H_q(n)$-invariant bilinear form $\langle\ ,\ \rangle : \wedge_S^n V \times \wedge_{{}^tS}^n(V^*) \to \mathbb{C}$. But the isomorphism $(T^n(V^*))_- \simeq \wedge_{{}^tS}^n(V^*)$ is given by $M_- \cdot \alpha \leftrightarrow \alpha + (I_{{}^tS})^n$; hence $\langle\ ,\ \rangle : \wedge_S^n V \times \wedge_{{}^tS}^n(V^*) \to \mathbb{C}$ is explicitly defined by $\langle w + I_S^n, \alpha + (I_{{}^tS})^n\rangle := \langle w, M_- \cdot \alpha\rangle$, $w \in T^nV$, $\alpha \in T^n(V^*)$. □

The hypothesis of part (b) of the Theorem below will be discussed in more detail in Subsection 3.7.

Theorem 3.3.2. *Let V be a finite dimensional vector space, $C = (\operatorname{End} V)^*$, $R : V \otimes V \to V \otimes V$ a linear map, $S = \tau R$.*

- **(a):** *Assume that S verifies $F_0(p)$. Then ${}^tS$ also verifies $F_0(p)$; and if $d \in A(R)$, $d' \in A(\widehat{R})$ are those elements given by Proposition 3.1.2, then $\theta(d) = d'$.*
- **(b):** *If S verifies $F_1(p)$ and ${}^tS$ verifies $F_{p-1}(p)$, then there exist $A_{ij},\ B_{ij} \in A(R)^{p-1}$ $i, j = 1, \cdots, m$ such that*

$$\sum_{k=1}^{m} A_{ik} t_{kj} = \sum_{k=1}^{m} t_{ik} B_{kj} = \delta_{ij} d.$$

Proof. **Step 1.** The following equality holds:

$$(\langle\ ,\ \rangle \otimes \mathrm{id})(\mathrm{id} \otimes \delta) = (\langle\ ,\ \rangle \otimes \theta)\tau^{23}(\delta \otimes \mathrm{id}),$$

as maps from $T^n(V) \otimes T^n(V^*)$ to $T(C^\diamond)$.

This follows by induction on n, the case $n = 1$ being clear. Assume it holds for $m < n$, and let $v \in T^mV$, $w \in T^{n-m}V$, $\alpha \in T^m(V^*)$, $\beta \in T^{n-m}(V^*)$. We have to prove

$$\begin{aligned} &\langle v \otimes w, (\alpha \otimes \beta)_{(0)}\rangle(\alpha \otimes \beta)_{(1)} = \\ &\quad \langle (v \otimes w)_{(0)}, \alpha \otimes \beta\rangle \theta\left((v \otimes w)_{(1)}\right). \end{aligned}$$

As $\delta : T(V) \to T(V) \otimes T(C)$, $\delta : T(V^*) \to T(V^*) \otimes T(C^\diamond)$ and $\theta : T(C) \to T(C^\diamond)$ are algebra morphisms,

$$\begin{aligned} &\langle v \otimes w, (\alpha \otimes \beta)_{(0)}\rangle(\alpha \otimes \beta)_{(1)} = \\ &= \langle v \otimes w, \alpha_{(0)} \otimes \beta_{(0)}\rangle \alpha_{(1)} \otimes \beta_{(1)} \\ &= \langle v, \alpha_{(0)}\rangle \alpha_{(1)} \otimes \langle w, \beta_{(0)}\rangle \beta_{(1)} \\ &= \langle v_{(0)}, \alpha\rangle \theta(v_{(1)}) \otimes \langle w_{(0)}, \beta\rangle \theta(w_{(1)}) \\ &= \langle v_{(0)} \otimes w_{(0)}, \alpha \otimes \beta\rangle \theta\left(v_{(1)} \otimes w_{(1)}\right) \\ &= \langle (v \otimes w)_{(0)}, \alpha \otimes \beta\rangle \theta\left((v \otimes w)_{(1)}\right). \end{aligned}$$

Step 2. The following equality holds:

$$(\langle\ ,\ \rangle \otimes \mathrm{id})(\mathrm{id} \otimes \delta) = (\langle\ ,\ \rangle \otimes \theta)\tau^{23}(\delta \otimes \mathrm{id}),$$

as maps $\wedge^n(V) \otimes \wedge^n(V^*) \to A(\widehat{R})$.

To avoid confusions, we shall denote in this Step $\langle\langle\ ,\ \rangle\rangle$ for the duality between $\wedge^n_S V$ and $\wedge^n_{{}^tS}(V^*)$, $\langle\ ,\ \rangle$ for the canonical duality between T^nV and $T^n(V^*)$ and $\overline{\theta}$ for the isomorphism from $A(R)$ onto $A(\widehat{R})$ induced by $\theta : T(C) \to T(C^\diamond)$.

Let $v \in T^nV$, $\alpha \in T^n(V^*)$, $\overline{v} = v + I^n_S \in \wedge^n_S V$, $\overline{\alpha} = \alpha + I^n_{{}^tS} \in \wedge^n_{{}^tS}(V^*)$. We have to prove

$$\langle\langle \overline{v}, \overline{\alpha}_{(0)} \rangle\rangle \overline{\alpha}_{(1)} = \langle\langle \overline{v}_{(0)}, \overline{\alpha} \rangle\rangle \overline{\theta}\left(\overline{v}_{(1)}\right).$$

By Step 1, and using that the action of $H_q(n)$ and the coaction of $A(R)$ in T^nV commute, we have

$$\begin{aligned}
&\langle\langle \overline{v}_{(0)}, \overline{\alpha} \rangle\rangle \overline{\theta}\left(\overline{v}_{(1)}\right) = \\
&= \langle\langle \overline{v_{(0)}}, \overline{\alpha} \rangle\rangle \overline{\theta}\left(\overline{v_{(1)}}\right) = \langle M_- \cdot v_{(0)}, \alpha \rangle \overline{\theta}\left(\overline{v_{(1)}}\right) \\
&= \langle (M_- \cdot v)_{(0)}, \alpha \rangle \overline{\theta}\left(\overline{(M_- \cdot v)_{(1)}}\right) \\
&= \overline{\langle (M_- \cdot v)_{(0)}, \alpha \rangle \theta\left((M_- \cdot v)_{(1)}\right)} \\
&= \overline{\langle M_- \cdot v, \alpha_{(0)} \rangle \alpha_{(1)}} = \langle M_- \cdot v, \alpha_{(0)} \rangle \overline{\alpha_{(1)}} \\
&= \langle\langle \overline{v}, \overline{\alpha_{(0)}} \rangle\rangle \overline{\alpha_{(1)}} = \langle\langle \overline{v}, \overline{\alpha}_{(0)} \rangle\rangle \overline{\alpha}_{(1)}.
\end{aligned}$$

Step 3. Proof of (a).

${}^tS$ verifies $F_0(p)$ because of the duality between $\wedge^p_S V$ and $\wedge^p_{{}^tS}(V^*)$. Let then $0 \neq \mu \in \wedge^p_S V$, $0 \neq \mu' \in \wedge^p_{{}^tS}(V^*)$, $d \in A(R)^p$ and $d' \in A(\widehat{R})^p$ such that

$$\delta(\mu) = \mu \otimes d, \quad \delta(\mu') = \mu' \otimes d'.$$

We have

$$\begin{aligned}
&(\langle\ ,\ \rangle \otimes \mathrm{id})(\mathrm{id} \otimes \delta)(\mu \otimes \mu') = \\
&\quad (\langle\ ,\ \rangle \otimes \mathrm{id})(\mu \otimes \mu' \otimes d') = \langle \mu, \mu' \rangle d', \\
&(\langle\ ,\ \rangle \otimes \theta)\tau^{23}(\delta \otimes \mathrm{id})(\mu \otimes \mu') = \\
&\quad (\langle\ ,\ \rangle \otimes \theta)(\mu \otimes \mu' \otimes d) = \langle \mu, \mu' \rangle \theta(d).
\end{aligned}$$

By Step 2, and because $\langle\ ,\ \rangle : \wedge^p_S V \times \wedge^p_{{}^tS}(V^*) \to \mathbb{C}$ is non-degenerate, $\theta(d) = d'$.

Step 4. Proof of (b).

By Proposition 3.1.2, there exist $A_{ij} \in A(R)^{p-1}$, $C_{ij} \in A(\widehat{R})^{p-1}$ $i, j = 1, \cdots, m$ such that

$$\sum_{k=1}^{m} A_{ik} t_{kj} = \delta_{ij} d, \qquad \sum_{k=1}^{m} u_{ki} C_{kj} = \delta_{ij} d'.$$

Applying θ^{-1} to the second equality, and denoting $B_{ij} = \theta^{-1}(C_{ij})$, we get $\sum_{k=1}^m t_{ik} B_{kj} = \delta_{ij} d$, as desired. □

3.4. **The Koszul complex.** We show that the "quantum Koszul complex" studied by Gurevich, Wambst and Hai – see (Gurevich 1991), (Wambst 1993), (Hai 1997) – fits into the general scheme of Koszul rings. Our main reference in this Subsection is (Beilinson, Ginzburg & Soergel 1996). The results of this subsection are not needed in the rest of the paper.

Let $A = T(V)/\langle\mathcal{R}\rangle$ be a quadratic algebra, *i.e.* a graded algebra generated by the space V of elements of degree 1, with relations "in degree 2". This means that the ideal of relations is generated by $\mathcal{R} \subset V \otimes V$. We assume that V has finite dimension. The quadratic dual of A is

$$A^! := T(V^*)/\langle\mathcal{R}^\perp\rangle,$$

where $\mathcal{R}^\perp \subset V^* \otimes V^*$ is the subspace of linear functionals vanishing on $\mathcal{R}$. The Koszul complex $\cdots \to K^2 \to K^1 \to K^0 = A$ can be explicitly given by

$$K^i = \hom(A^!_i, A) = A \otimes (A^!_i)^*,$$

and the differential by

$$(df)(b) = \sum_j f(b\check{v}_j)v_j,$$

for $f \in \hom(A^!_{i+1}, A)$, $b \in A^!_i$. Here (v_j) is a basis of V and $(\check{v}_j)$ is its dual basis. If $f = x \otimes \omega$, with $x \in A$, $\omega \in \left(A^!_{i+1}\right)^*$, then

$$df = \sum_j xv_j \otimes R(\check{v}_j)(\omega);$$

where R denotes the transpose of the right multiplication.

Let S be a solution of the Braid equation which satisfies the Hecke condition. Let $A = \mathrm{Sym}_S V = T(V)/\langle \mathrm{Im}\,(S-q)\rangle$. Then the quadratic dual of $\mathrm{Sym}_S V$ is $\Lambda_{{}^tS}(V^*)$. Indeed, $\mathcal{R} = \mathrm{Ker}\,(S+1)$ in our case, and $\mathcal{R}^\perp = \mathrm{Ker}\,({}^tS - q)$. Therefore $(\mathrm{Sym}_S V)^! = T(V^*)/\mathrm{Ker}\,({}^tS-q) = \Lambda_{{}^tS}(V^*)$. Now, by Theorem 3.3.1, we have

$$(A^!_i)^* = (\Lambda^i_{{}^tS}(V^*))^* \simeq \Lambda^i_S(V).$$

Hence we can identify the Koszul complex of A with $\mathrm{Sym}_S V \otimes \Lambda_S V$. Let $f = x \otimes \omega \in \mathrm{Sym}_S V \otimes \Lambda^{i+1}_S V$, with $\omega = \omega_1 \wedge \cdots \wedge \omega_{i+1}$. Then $R(\check{v}_j)(\omega) = \langle \check{v}_j, \omega_1\rangle \omega_2 \wedge \cdots \wedge \omega_{i+1}$, because of the identification (6). We further identify $\mathrm{Sym}_S V$ with the subspace $(T^nV)_\epsilon$, thanks to Proposition 3.3.2 applied to $-q^{-1}S$. With all these conventions, we have

$$\begin{aligned} df &= \sum_j M_\epsilon(x \otimes v_j) \otimes \langle \check{v}_j, \omega_1\rangle \omega_2 \wedge \cdots \wedge \omega_{i+1} \\ &= M_\epsilon(x \otimes \omega_1) \otimes \omega_2 \wedge \cdots \wedge \omega_{i+1}. \end{aligned}$$

This is exactly the first complex in (Gurevich 1991), §3. In the same vein, we can show that the Koszul complex of $\Lambda_S(V)$ is the second complex from *loc. cit.* and that $\mathrm{Sym}_S V$ and $\Lambda_S V$ are Koszul, see (Gurevich 1991), (Wambst 1993).

3.5. **The Hopf algebra $H(R)$.**

We first recall a well-known definition from (Hayashi 1992), (Larson & Towber 1991), (Majid 1990).

Definition 3.5.1. A *co-quasitriangular bialgebra* (CQT) is a pair (B, ρ), where B is a bialgebra and ρ is an invertible element of the algebra $(B \otimes B)^*$ that satisfies:

$$\begin{aligned} &\langle\rho, x_{(1)}\otimes y_{(1)}\rangle x_{(2)}y_{(2)} = \\ &\qquad y_{(1)}x_{(1)}\langle\rho, x_{(2)}\otimes y_{(2)}\rangle \qquad (14)\\ &\langle\rho, xy\otimes z\rangle = \\ &\qquad \langle\rho, x\otimes z_{(1)}\rangle\langle\rho, y\otimes z_{(2)}\rangle \qquad (15)\\ &\langle\rho, x\otimes yz\rangle = \\ &\qquad \langle\rho, x_{(1)}\otimes z\rangle\langle\rho, x_{(2)}\otimes y\rangle, \qquad (16)\end{aligned}$$

for any $x, y, z \in B$.

Let (B,ρ), (B',ρ') be CQT-bialgebras and $\phi : B \to B'$ a bialgebra map; we say that ϕ is a *morphism* of CQT-bialgebras if

$$\langle\rho, x\otimes y\rangle = \langle\rho', \phi(x)\otimes\phi(y)\rangle, \quad x, y \in B.$$

We recall that $\mathcal{M}^B$ is a braided category, with braiding $C_{M,N} : M\otimes N \to N\otimes M$ given by

$$C_{M,N}(m\otimes n) = n_{(0)}\otimes m_{(0)}\,\langle\rho, m_{(1)}\otimes n_{(1)}\rangle. \qquad (17)$$

We refer to (Majid 1995) for information about braided categories. In the present case, since the associativity is trivial, the relevant hexagon identities are reduced to the equalities

$$C_{M\otimes N,P} = (C_{M,P}\otimes \mathrm{id}\,_N)(\mathrm{id}\,_M\otimes C_{N,P}) \qquad (18)$$

from $M\otimes N\otimes P$ on $P\otimes M\otimes N$, and

$$C_{M,N\otimes P} = (\mathrm{id}\,_N\otimes C_{M,P})(C_{M,N}\otimes \mathrm{id}\,_P), \qquad (19)$$

from $M\otimes N\otimes P$ on $N\otimes P\otimes M$, for any $M, N, P \in \mathcal{M}^B$. Now (15) implies (18) and (16) implies (19); in turn, (14) means that C is natural, *i.e.* that $C_{M,N}$ is a morphism for any M, N.

Similarly, the category of left comodules ${}^B\mathcal{M}$ is also braided, with braiding $D_{M,N} : M\otimes N \to N\otimes M$ given by

$$D_{M,N}(m\otimes n) = \langle\rho, n_{(-1)}\otimes m_{(-1)}\rangle n_{(0)}\otimes m_{(0)}. \qquad (20)$$

In this case, (15) implies (19), (18) follows from (16), and the naturality of D from (14).

If M is a finite dimensional right B-comodule, then M^* has a natural structure of left B-comodule. If also N is a finite dimensional object of $\mathcal{M}^B$, then the braiding between M^* and N^* is given by the transpose of the braiding in $\mathcal{M}^B$:

$$D_{N^*,M^*} = {}^t(C_{M,N}) \qquad (21)$$

Proof of (21). We shall not work here with the identification (6) but the usual one. Let $\alpha \in N^*$, $\beta \in M^*$, $n \in N$ and $m \in M$, then

$$\begin{aligned}\left\langle {}^tC_{M,N}(\alpha\otimes\beta), m\otimes n\right\rangle &= \langle\alpha\otimes\beta, C_{M,N}(m\otimes n)\rangle\\ &= \left\langle\alpha\otimes\beta, n_{(0)}\otimes m_{(0)}\right\rangle\left\langle\rho, m_{(1)}\otimes n_{(1)}\right\rangle\\ &= \left\langle\rho, \left\langle\beta, m_{(0)}\right\rangle m_{(1)}\otimes\left\langle\alpha, n_{(0)}\right\rangle n_{(1)}\right\rangle\\ &= \left\langle\rho, \beta_{(-1)}\left\langle\beta_{(0)}, m\right\rangle\otimes\alpha_{(-1)}\left\langle\alpha_{(0)}, n\right\rangle\right\rangle\\ &= \left\langle\rho, \beta_{(-1)}\otimes\alpha_{(-1)}\right\rangle\left\langle\beta_{(0)}\otimes\alpha_{(0)}, m\otimes n\right\rangle\\ &= \langle D_{N^*,M^*}(\alpha\otimes\beta), m\otimes n\rangle\,,\end{aligned}$$

as desired $\square$.

Theorem 3.5.1. *(*HAYASHI *1992) Let (B,ρ) be a CQT bialgebra. Let $G \subseteq G(B)$ be a subsemigroup. Then G is a (left and right) Ore set. We denote by B_G the localization of B at G. B_G admits a CQT-bialgebra structure such that the natural map $B \to B_G$ is a CQT-bialgebra homomorphism. Explicitly, the comultiplication is determined by $\Delta(g^{-1}) = g^{-1} \otimes g^{-1}$, for $g \in G$, and $\rho_G \in (B_G \otimes B_G)^*$ is given by*

$$\left\langle \rho_G, hg^{-1} \otimes kt^{-1} \right\rangle = \langle \rho, h_{(2)} \otimes k_{(1)} \rangle \left\langle \rho^{-1}, h_{(1)} \otimes t \right\rangle \left\langle \rho^{-1}, g \otimes k_{(2)} \right\rangle \langle \rho, g \otimes t \rangle. \quad \square$$

If $G = \{g^n : n = 0, 1, \cdots\}$ for some $g \in G(B)$, then we shall denote $B_g := B_G$, $\rho_g = \rho_G$. If B_G is Hopf algebra then, by the universal property of the quotients, it is isomorphic to $B_{G(B)}$ ((HAYASHI 1992), Prop. 3.3). Therefore, the question is to find the most economical G, if any, such that B_G is a Hopf algebra. We shall also need the following remark.

Lemma 3.5.1. *Let (B,ρ) be a CQT bialgebra, $G \subseteq G(B)$ a subsemigroup, B_G the localization of B at G. Let $\varphi : B \to H$ be a morphism of bialgebras, where H is a Hopf algebra. Then φ factorizes through B_G. Hence, if B_G is a Hopf algebra, the morphism $B \to B_G$ is universal among the morphisms from B into a Hopf algebra.*

Proof. We need the coquasitriangularity of B only to allow localization at G. If $g \in G(B)$ then $\epsilon(g) = 1$; hence $\varphi(g) \neq 0$, since $\epsilon(\varphi(g)) = \epsilon(g) = 1$. Therefore $\varphi(g)$ is invertible in H, with $\varphi(g)^{-1} = \mathcal{S}\varphi(g)$. The Lemma follows. $\square$

We can apply the preceding Theorem to the bialgebra $A(R)$ because of a well-known result ((HAYASHI 1992), (LARSON & TOWBER 1991), (MAJID 1990)). We state a generalization of this result given in (DOI 1993).

Theorem 3.5.2. *(*DOI *1993) Let C be a coalgebra, $A = C^*$ its dual algebra and $R \in A \otimes A$ be a solution of the Quantum Yang-Baxter equation (13). Then there exists a unique $\rho \in (A(R) \otimes A(R))^*$ such that $(A(R),\rho)$ is a CQT-bialgebra and*

$$\langle \rho, a \otimes b \rangle = \langle R, a \otimes b \rangle, \quad \forall a, b \in C. \quad \square \tag{22}$$

Combining this Theorem with Theorem 3.1.1, we obtain the following Corollary. For $C = (\operatorname{End} V)^*$ compare with (LYUBASHENKO 1986), (SCHAUENBURG 1992), (MAJID 1995), pp. 438 and 476.

Corollary 3.5.1. *The CQT-bialgebra $(A(R),\rho)$ is universal among the CQT-bialgebras (B,ζ) satisfying the following properties:*

(a): *There exist a coalgebra map $\phi : C \to B$ satisfying*

$$\langle R, x \otimes y \rangle = \langle \zeta, \phi(x) \otimes \phi(y) \rangle, \quad \forall x, y \in C.$$

(b): *Given two C-comodules M and N, $S_{M,N}$ coincides with the braiding in $\mathcal{M}^B$.*

If a CQT-bialgebra (B,ζ) satisfies (a) and (b), then we can extend the map of (a) to a CQT-bialgebra map $\phi : A(R) \to B$ which verifies the following: for any $M \in \mathcal{M}^C$, if δ_B and $\delta_{A(R)}$ are the structure maps given by (b), then

$$\delta_B = (\mathrm{id} \otimes \phi) \circ \delta_{A(R)}.$$

Proof. We already know that $A(R)$ satisfies **(a)** from Theorem 3.1.1, and it satisfies **(b)** because of Theorem 3.5.2. Now let (B, ζ) be a CQT-bialgebra fulfilling the above properties. We know from Theorem 3.1.1 the existence of a bialgebra map $\phi : A(R) \to B$. We have to show that

$$\langle \rho, a \otimes b \rangle = \langle \zeta, \phi(a) \otimes \phi(b) \rangle$$

Now let $\tilde{\rho} \in (A(R) \otimes A(R))^*$ defined by $\langle \tilde{\rho}, a \otimes b \rangle = \langle \zeta, \phi(a) \otimes \phi(b) \rangle$. By property **(b)** for B $\langle \tilde{\rho}, a \otimes b \rangle = \langle R, a \otimes b \rangle$ if $a, b \in C$. By Theorem 3.5.2 is $\tilde{\rho} = \rho$. □

We are ready now to localize; to have an antipode in the localization we shall use the following general statement.

Theorem 3.5.3. *(*DOI *1993) Let C be a coalgebra, $A = C^*$ its dual algebra and $R \in A \otimes A$ be a solution of the Quantum Yang-Baxter equation. Let g be a group-like element in $A(R)$. If there exist linear maps $\phi, \psi : C \to A(R)$ such that*

$$\phi(c_{(1)})c_{(2)} = \epsilon(c)g = c_{(1)}\psi(c_{(2)})$$

for all $c \in C$, then $A(R)_g$ is a Hopf algebra. □

We can finally build up the quantum analog of $GL(N)$.

Theorem 3.5.4. *Let V be a finite dimensional vector space, $C = (\operatorname{End} V)^*$, $R : V \otimes V \to V \otimes V$, $S = \tau R$. Assume that S verifies the Braid equation, the Hecke condition, $F_1(p)$, and that ${}^tS$ verifies $F_{p-1}(p)$. Let d be the group-like element given by Proposition 3.1.2 (the "quantum determinant"). Then $H(R) := A(R)_d$ is a CQT-Hopf algebra.*

Proof. By Theorem 3.3.2, there exist A_{ij}, $B_{ij} \in A(R)^{p-1}$ $i, j = 1, \cdots, m$ such that

$$\sum_{k=1}^{m} A_{ik} t_{kj} = \sum_{k=1}^{m} t_{ik} B_{kj} = \delta_{ij} d \text{ in } A(R).$$

The maps $\phi, \psi : C \to A(R)$ given by $\phi(t_{ij}) = A_{ij}$, $\psi(t_{ij}) = B_{ij}$ fulfill the hypothesis of Theorem 3.5.3. The Theorem follows. □

Remark 3.5.1. The preceding Theorem was first proved in (FADEEV, RESHETIKHIN & TAKHTAJAN 1990) for the Drinfeld-Jimbo solution of the QYBE of type A_n; see also (TAKEUCHI 1990). For general solutions of Hecke type, the Theorem appears in (GUREVICH 1991). However, it is proved there only that the proposed antipode satisfies one of the axioms; the other is left to the reader. We were unable to recover the proof and were forced to introduce the hypothesis "${}^tS$ verifies $F_{p-1}(p)$". In Subsection 3.7, we discuss the relation between this hypothesis and Frobenius algebras. We observe, by the way, that one of the axioms of the antipode does not imply the other: see the paper (GREEN, NICHOLS & TAFT 1980). Another proof of Gurevich's Theorem is offered in (TSYGAN 1993); unhappily, it was again not clear to us.

Corollary 3.5.2. *$H(R)$ is generated as algebra by the matrix coefficients of a finite dimensional comodule.*

Proof. We know that $H(R)$ is generated as algebra by the t_{ij}'s and d^{-1}. We can then take the comodule $V \oplus (\wedge_S^p V)^*$. Indeed, d^{-1} generates the space of matrix coefficients of $(\wedge_S^p V)^*$. □

An invertible solution of the QYBE generates a braided rigid category inside the category of vector spaces. See (LYUBASHENKO 1986), (SCHAUENBURG 1992). In our case, we have the following alternative description.

Corollary 3.5.3. *In the hypothesis of Theorem 3.5.4. The CQT-Hopf algebra $(H(R), \rho)$ is universal among the CQT-Hopf algebras (H, ζ) satisfying the following properties:*

(a): *There exist a coalgebra map $\phi : C \to H$ satisfying*

$$\langle R, x \otimes y\rangle = \langle \zeta, \phi(x) \otimes \phi(y)\rangle, \quad x, y \in C.$$

(b): *Given two C-comodules M and N, $S_{M,N}$ coincides with the braiding in $\mathcal{M}^H$.*

If a CQT-Hopf algebra (H, ζ) satisfies (a) and (b), then we can extend the map of (a) to a CQT-bialgebra map $\phi : H(R) \to H$ which verifies the following: for any $M \in \mathcal{M}^C$, if δ_H and $\delta_{H(R)}$ are the structure maps given by (b), then

$$\delta_H = (\mathrm{id} \otimes \phi) \circ \delta_{H(R)}.$$

Proof. This follows easily from Theorem 3.5.4, Lemma 3.5.1 and Corollary 3.5.1. □

3.6. **The Hopf algebra $K(R)$.** We preserve the notation above. As $d \in G(A(R))$,

$$\begin{aligned}\epsilon(d-1) =& \epsilon(d) - \epsilon(1) = 0,\\ \Delta(d-1) =& d \otimes d - 1 \otimes 1\\ =& d \otimes (d-1) + (d-1) \otimes 1.\end{aligned}$$

Hence $K(R) := A(R)/\langle d-1\rangle$ inherits the bialgebra structure of $A(R)$. Furthermore, we consider $\wedge_S(V)$ as $K(R)$-comodule algebra by "corestriction". The proof of the following theorem is completely analogous to that of Theorem 3.5.4; we will omit it.

Theorem 3.6.1. *Assume that S verifies the Braid equation, the Hecke condition, $F_1(p)$, and that ${}^tS$ verifies $F_{p-1}(p)$. Then $K(R)$ is a CQT-Hopf algebra.* □

It is well-known, and evident, that a generator t_{ij} vanishes in the quotient $K(R)$ whenever it does not commute with d.

3.7. **The Frobenius conditions.** Let V be a finite dimensional vector space, $C = (\operatorname{End} V)^*$, $R : V \otimes V \to V \otimes V$ a linear map, $S = \tau R$. We assume that S is a solution of the Braid equation which satisfies the Hecke condition with label q. In this section, we study Frobenius conditions on the graded algebras $\operatorname{Sym}_S V = T(V)/\langle \operatorname{Im}(S-q)\rangle = T(V)/\langle \operatorname{Ker}(S+1)\rangle$ and $\wedge_S V = T(V)/\langle \operatorname{Im}(S+1)\rangle = T(V)/\langle \operatorname{Ker}(S-q)\rangle$. We shall prove that under very weak conditions these graded algebras are Frobenius. In particular, this alleviates the hypothesis in Theorems 3.5.4, 3.6.1. The idea of our argument appears essentially already in (NICHOLS 1978).

We first point out the relation between the notions of "Frobenius algebra" in the sense of, respectively, (MANIN 1988) and, say, (CURTIS & REINER 1988). For this, let us adopt following the provisional notation:

A graded algebra Λ is *graded Frobenius* of rank p if, for a fixed p, $\dim \Lambda^p = 1$ and the multiplication $\Lambda^{p-j} \times \Lambda^j \xrightarrow{\triangle} \Lambda^p$ is non-degenerate for any j, $0 \le j \le p$ and $\Lambda^j = 0$, for $j > p$.

On the other hand, an algebra A is *Frobenius* if there exists an isomorphism of left A-modules $T : A \to A^*$, where A, resp. A^*, is considered as left module by left multiplication, resp. by the transpose of the right multiplication. See (CURTIS & REINER 1988), Chapter IX. In particular, A is finite dimensional. It follows from the bijectivity of T that the bilinear form

$$(a, b) := \langle T(b), a\rangle = \langle T(1), ab\rangle \tag{23}$$

is non-degenerate; and by construction, it is associative: $(ac, b) = (a, cb)$ for all $a, b, c \in A$. Conversely, given a non-degenerate associative bilinear form $(,)$ in a finite dimensional algebra A, the map T defined by (23) is an isomorphism of left A-modules. This equivalence implies, in particular, that A is Frobenius if and only if A^{op} is Frobenius; that is, if and only if there exists an isomorphism of right A-modules $T : A_A \to (A^*)_A$.

Lemma 3.7.1. *Let $A = \oplus_{j\geq 0} A^j$ be a graded finite dimensional algebra such that $A^0 = \mathbb{C}$. Then A is Frobenius if and only if is graded Frobenius.*

Proof. We shall grade A^* by $A^* = \oplus_{j\geq 0} (A^*)^j$, where $(A^*)^j$ is isomorphic by restriction to $\left(A^j\right)^*$ and is orthogonal to A^i for $i \neq j$. Let us assume first that A is graded Frobenius; note that $A^0 = \mathbb{C}$ is part of the requirements. Let $\mu \in A^p - 0$, $\lambda \in (A^*)^p$ such that $\langle \lambda, \mu\rangle = 1$ and let $T : A \to A^*$ be the linear map given by $\langle T(a), b\rangle = \langle \lambda, ba\rangle$. Then we see without trouble that T is an isomorphism of A-modules.

Conversely, let $T : A \to A^*$ be an isomorphism of A-modules. Since A is finite dimensional, there exists $p \in \mathbb{N}$ such that $A^p \neq 0$ and $A^{p+j} = 0$ for all $j > 0$. Let $T(1) = \sum_j T(1)_j$, where $T(1)_j \in (A^*)^j$. We shall see that $[a, b] = \langle T(1)_p, ab\rangle$ is non-degenerate.

First we claim that $T(1)_p \neq 0$ and that the restriction $[,] : A^p \times A^0 \to \mathbb{C}$ is non-degenerate.

Indeed, let $0 \neq a \in A^p$. If $b \in \oplus_{j\geq 1} A^j$, then $ab = 0$. Hence there exists $b \in A^0$ such that $0 \neq \langle T(1), ab\rangle = \langle T(1)_p, ab\rangle$. That is, $[,]$ induces a monomorphism $A^p \to \left(A^0\right)^*$ and this is an isomorphism because $\dim A^0 = 1$.

The (left) radical of the bilinear form $[,]$

$$\operatorname{rad} A = \{a \in A : \langle T(1)_p, ab\rangle = 0 \quad \forall b \in A\}$$

is a graded subspace of A. Assume that $\operatorname{rad} A \neq 0$ and let $0 \neq a \in \operatorname{rad} A$ homogeneous of degree t with t maximal. We claim that

$$\langle T(1), ab\rangle = 0 \quad \forall b \in A^s, \quad \forall s > 0. \tag{24}$$

Indeed, if $s > p - t$, $ab = 0$; if $0 < s \leq p - t$ and $ab \neq 0$ then $\langle T(1)_p, abx\rangle \neq 0$ for some x by the maximality of t, but this is not possible. Let then $b \in A$ such that $\langle T(1), ab\rangle = 1$; it follows from the preceding analysis that $b \in A^0$. Let $d \in A^p$ such that $\langle T(1)_p, d\rangle = 1$. We claim now that

$$\langle T(1), (ab - d)x\rangle = 0, \quad \forall x \in A^s, \quad \forall 0 \leq s \leq p.$$

Indeed, for $s = 0$ this follows from the choice of d and for $s > 0$ from (24). This implies that $ab - d = 0$, and hence $t = p$, a possibility that we already excluded. Therefore $\operatorname{rad} A = 0$ and $[,]$ is non-degenerate. □

Now we want to show that finite dimensional braided Hopf algebras, *i.e.* Hopf algebras in some braided category, are Frobenius. (See (MAJID 1995), or the article (ANDRUSKIEWITSCH & GRAÑA 1999) in this volume for generalities on braided Hopf algebras). This question was already discussed in several works ((LYUBASHENKO 1995), (TAKEUCHI 1999),
(**?**)) and the argument offered is a variation of the usual one for Hopf algebras: a fundamental theorem for Hopf modules in braided categories is established, and this implies that the space of left integrals in a finite dimensional braided Hopf algebra has dimension one. The braided category we are interested in is $\mathcal{M}^{A(R)}$, *cf.* Theorem 3.5.2; this category is not rigid. We could not then directly invoke *strictu senso* the

above mentioned works, so we include a complete proof which is, nevertheless, standard. We thank H.-J. Schneider for clarifying discussions on this point.

Let (B, ρ) be a CQT-bialgebra; $\mathcal{M}^B$ is a braided category with braiding $C_{M,N} : M \otimes N \to N \otimes M$ given by (17).

Let R be a braided Hopf algebra in $\mathcal{M}^B$; in particular R is an algebra and a coalgebra. To avoid misunderstandings, we denote the comultiplication of R and the coaction of a right R-comodule M through the following variation of Sweedler's notation:

$$\Delta_R(r) = r^{(1)} \otimes r^{(2)}, \qquad \delta(m) = m^{(0)} \otimes m^{(1)}.$$

The compatibility between the product and the coproduct is expressed by the following formula:

$$(xy)^{(1)} \otimes (xy)^{(2)} = x^{(1)} \left(y^{(1)}\right)_{(0)} \otimes \left(x^{(2)}\right)_{(0)} y^{(2)} \left\langle \rho, \left(x^{(2)}\right)_{(1)} \otimes \left(y^{(1)}\right)_{(1)} \right\rangle. \tag{25}$$

The antipode $\mathcal{S}_R$ is antimultiplicative and anticomultiplicative in a braided sense; it is also a morphism in $\mathcal{M}^B$. This means, respectively, that the following conditions hold for any $x, y \in R$:

$$\mathcal{S}_R(x)_{(0)} \otimes \mathcal{S}_R(x)_{(1)} = \mathcal{S}_R(x_{(0)}) \otimes x_{(1)}, \tag{26}$$

$$\mathcal{S}_R(xy) = \mathcal{S}_R\left(y_{(0)}\right) \mathcal{S}_R\left(x_{(0)}\right) \left\langle \rho, x_{(1)} \otimes y_{(1)} \right\rangle, \tag{27}$$

$$\mathcal{S}_R(x)^{(1)} \otimes \mathcal{S}_R(x)^{(2)} = \left\langle \rho, \left(x^{(1)}\right)_{(1)} \otimes \left(x^{(2)}\right)_{(1)} \right\rangle \mathcal{S}_R\left(\left(x^{(2)}\right)_{(0)}\right) \otimes \mathcal{S}_R\left(\left(x^{(1)}\right)_{(0)}\right). \tag{28}$$

We denote by $\left(\mathcal{M}^B\right)^R$ (resp. $\left(\mathcal{M}^B\right)_R$) the category of R-comodules (resp. modules) *in the category* $\mathcal{M}^B$. That is, the objects are those B-comodules carrying a coaction (resp. an action) of R that is a morphism of B-comodules. For instance the same R with the right regular coaction (resp. action) is an object of $\left(\mathcal{M}^B\right)^R$ (resp. $\left(\mathcal{M}^B\right)_R$).

The tensor product of a right R-module M and N in $\left(\mathcal{M}^B\right)_R$ is again a right R-module via

$$(m \otimes n) \cdot r = m \cdot \left(r^{(1)}\right)_{(0)} \otimes n_{(0)} \cdot r^{(2)} \left\langle \rho, n_{(1)} \otimes \left(r^{(1)}\right)_{(1)} \right\rangle$$

Similarly, the tensor product of a right R-comodule M and N in $\left(\mathcal{M}^B\right)^R$ is again a right R-comodule via

$$\delta(m \otimes n) = m^{(0)} \otimes \left(n^{(0)}\right)_{(0)} \otimes \left(m^{(1)}\right)_{(0)} n^{(1)} \left\langle \rho, \left(m^{(1)}\right)_{(1)} \otimes \left(n^{(0)}\right)_{(1)} \right\rangle.$$

Let us consider the category $\mathcal{M}_R^R = \left(\mathcal{M}^B\right)_R^R$ of right Hopf modules over R (*warning* we do not ask that $M \in \mathcal{M}_R^R$ is in $\mathcal{M}^B$!). These are modules and comodules over R such that the coaction is R-linear: if $M \in \mathcal{M}_R^R$, then

$$\begin{aligned} \delta(m \cdot r) &= m^{(0)} \cdot \left(r^{(1)}\right)_{(0)} \otimes \left(m^{(1)}\right)_{(0)} r^{(2)} \\ &\quad \left\langle \rho, \left(m^{(1)}\right)_{(1)} \otimes \left(r^{(1)}\right)_{(1)} \right\rangle . \end{aligned} \tag{29}$$

It can be shown that the condition (29) is equivalent to the R-colinearity of the action.

For $M \in \mathcal{M}^R$, the space of coinvariants is $M^{\mathrm{co}R} = \{m \in M : \delta(m) = m \otimes 1\}$. If V is a vector space over k, then $V \otimes R$ has a natural structure of Hopf module whose action and coaction are defined on R. Clearly, $(V \otimes R)^{\mathrm{co}R} = V \otimes 1$.

Lemma 3.7.2. "Fundamental Theorem of Hopf modules" *Let (B,ρ) be a CQT-bialgebra, R a braided Hopf algebra in $\mathcal{M}^B$ and $M \in \mathcal{M}_R^R$. Then the map $\psi : M^{\mathrm{co}R} \otimes R \to M$ given by the restriction of the action is a natural isomorphism of Hopf modules, with inverse*

$$\phi : M \to M^{\mathrm{co}R} \otimes R, \quad \phi(m) = m^{(0)} \cdot \mathcal{S}_R(m^{(1)}) \otimes m^{(2)}.$$

Proof. The point to check is the well-definition of the map ϕ; that ψ and ϕ are mutually inverse is not difficult and will be left to the reader. So let $m \in M$; it is enough to show that $(\delta \otimes \mathrm{id}\,)\phi(m) \in M \otimes 1 \otimes R$. We compute:

$$\begin{aligned} &(\delta \otimes \mathrm{id}\,)\phi(m) = \\ &= \left(m^{(0)} \cdot \mathcal{S}_R\left(m^{(1)}\right)\right)^{(0)} \otimes \left(m^{(0)} \cdot \mathcal{S}_R\left(m^{(1)}\right)\right)^{(1)} \\ &\quad \otimes m^{(2)} \\ &= m^{(0)} \cdot \left(\mathcal{S}_R(m^{(2)})^{(1)}\right)_{(0)} \otimes \left(m^{(1)}\right)_{(0)} \left(\mathcal{S}_R(m^{(2)})\right)^{(2)} \\ &\quad \otimes m^{(3)} \left\langle \rho, \left(m^{(1)}\right)_{(1)} \otimes \left(\mathcal{S}_R(m^{(2)})^{(1)}\right)_{(1)} \right\rangle \\ &= m^{(0)} \cdot \left(\mathcal{S}_R\left(m^{(3)}\right)\right)_{(0)} \otimes \\ &\quad \left(m^{(1)}\right)_{(0)} \left(\mathcal{S}_R\left(m^{(2)}\right)\right)_{(0)} \otimes m^{(4)} \\ &\quad \left\langle \rho, \left(m^{(1)}\right)_{(1)} \otimes \left(\mathcal{S}_r\left(m^{(3)}\right)\right)_{(1)} \right\rangle \\ &\quad \left\langle \rho, \left(\mathcal{S}_R\left(m^{(2)}\right)\right)_{(1)} \otimes \left(\mathcal{S}_R m^{(3)}\right)_{(2)} \right\rangle \\ &= m^{(0)} \cdot \left(\mathcal{S}_R\left(m^{(2)}\right)\right)_{(0)} \otimes 1 \otimes m^{(3)} \epsilon_R\left(m^{(1)}\right) \\ &\quad \left\langle \rho, 1 \otimes \left(\mathcal{S}_R\left(m^{(2)}\right)\right)_{(1)} \right\rangle \\ &= m^{(0)} \cdot \left(\mathcal{S}_R\left(m^{(1)}\right)\right)_{(0)} \otimes 1 \otimes m^{(2)}. \end{aligned}$$

Here, the first equality is by definition; the second, by the Hopf module axiom (29) and the coassociativity of the coaction; the third, by (28) and because $\mathcal{S}_R$ is a morphism; the fourth, by (15), the axiom of the antipode and because the multiplication of R is a morphism; the last, by the axiom of the counit, since $\langle \rho, 1 \otimes m \rangle = \epsilon(m)$ by the invertibility of ρ. □

Lemma 3.7.3. *Let (B,ρ) be a CQT-bialgebra, R a finite dimensional braided Hopf algebra in $\mathcal{M}^B$. Let R^* be considered as a right R module by trasposition of the left regular action: $\langle \alpha \cdot x, y \rangle = \langle \alpha, xy \rangle$, $\alpha \in R^*$, $x, y \in R$. If R^* admits a coaction such that*

$$\begin{aligned} &\left\langle \alpha, x^{(1)} \right\rangle \mathcal{S}_R(x^{(2)}) = \\ &\quad \left\langle \alpha^{(0)}, x_{(0)} \right\rangle \left(\alpha^{(1)}\right)_{(0)} \left\langle \rho, \left(\alpha^{(1)}\right)_{(1)} \otimes x_{(1)} \right\rangle , \end{aligned} \tag{30}$$

then it is a Hopf module. In particular, R is a Frobenius algebra.

Proof. We have to show that R^* verifies (29), for this it is enough to show that

$$\left\langle (\alpha \cdot x)^{(0)}, y_{(0)} \right\rangle \left((\alpha \cdot x)^{(1)}\right)_{(0)} \left\langle \rho, \left((\alpha \cdot x)^{(1)}\right)_{(1)} \otimes y_{(1)} \right\rangle = \left\langle \alpha^{(0)} \cdot \left(x^{(1)}\right)_{(0)}, y_{(0)} \right\rangle \left(\left(\alpha^{(1)}\right)_{(0)} x^{(2)}\right)_{(0)} \left\langle \rho, \left(\left(\alpha^{(1)}\right)_{(0)} \right.\right.$$

for all $\alpha \in R^*,\ x, y \in R$.
Starting from the right hand side of (31) we get

$$\begin{aligned}
&\left\langle \alpha^{(0)} \cdot \left(x^{(1)}\right)_{(0)}, y_{(0)} \right\rangle \left(\left(\alpha^{(1)}\right)_{(0)} x^{(2)} \right)_{(0)} \\
&\left\langle \rho, \left(\left(\alpha^{(1)}\right)_{(0)} x^{(2)} \right)_{(1)} \otimes y_{(1)} \right\rangle \\
&\left\langle \rho, \left(\alpha^{(1)}\right)_{(1)} \otimes \left(x^{(1)}\right)_{(1)} \right\rangle = \\
&\quad = \left\langle \alpha^{(0)} \cdot \left(x^{(1)}\right)_{(0)}, y_{(0)} \right\rangle \left(\alpha^{(1)}\right)_{(0)} \left(x^{(2)}\right)_{(0)} \\
&\quad \left\langle \rho, \left(\alpha^{(1)}\right)_{(1)} \left(x^{(2)}\right)_{(1)} \otimes y_{(1)} \right\rangle \\
&\quad \left\langle \rho, \left(\alpha^{(1)}\right)_{(2)} \otimes \left(x^{(1)}\right)_{(1)} \right\rangle \\
&\quad \overset{(15)}{=} \left\langle \alpha^{(0)}, \left(x^{(1)}\right)_{(0)} y_{(0)} \right\rangle \left(\alpha^{(1)}\right)_{(0)} \left(x^{(2)}\right)_{(0)} \\
&\quad \left\langle \rho, \left(\alpha^{(1)}\right)_{(1)} \otimes y_{(1)} \right\rangle \left\langle \rho, \left(x^{(2)}\right)_{(1)} \otimes y_{(2)} \right\rangle \\
&\quad \left\langle \rho, \left(\alpha^{(1)}\right)_{(2)} \otimes \left(x^{(1)}\right)_{(1)} \right\rangle \\
&\quad \overset{(16)}{=} \left\langle \alpha^{(0)}, \left(x^{(1)}\right)_{(0)} y_{(0)} \right\rangle \left(\alpha^{(1)}\right)_{(0)} \left(x^{(2)}\right)_{(0)} \\
&\quad \left\langle \rho, \left(\alpha^{(1)}\right)_{(1)} \otimes \left(x^{(1)}\right)_{(1)} y_{(1)} \right\rangle \\
&\quad \left\langle \rho, \left(x^{(2)}\right)_{(1)} \otimes y_{(2)} \right\rangle \\
&\quad = \left\langle \alpha^{(0)}, \left(x^{(1)} y_{(0)}\right)_{(0)} \right\rangle \left(\alpha^{(1)}\right)_{(0)} \left(x^{(2)}\right)_{(0)} \\
&\quad \left\langle \rho, \left(\alpha^{(1)}\right)_{(1)} \otimes \left(x^{(1)} y_{(0)}\right)_{(1)} \right\rangle \\
&\quad \left\langle \rho, \left(x^{(2)}\right)_{(1)} \otimes y_{(1)} \right\rangle \\
&\quad = \left\langle \alpha, \left(x^{(1)} y_{(0)}\right)^{(1)} \right\rangle \mathcal{S}_R \left(\left(x^{(1)} y_{(0)}\right)^{(2)} \right) \left(x^{(2)}\right)_{(0)} \\
&\quad \left\langle \rho, \left(x^{(2)}\right)_{(1)} \otimes y_{(1)} \right\rangle \\
&\quad = \left\langle \alpha, x^{(1)} \left(\left(y_{(0)}\right)^{(1)} \right)_{(0)} \right\rangle \\
&\quad \mathcal{S}_R \left(\left(x^{(2)}\right)_{(0)} \left(y_{(0)}\right)^{(2)} \right) \left(x^{(3)}\right)_{(0)} \\
&\quad \left\langle \rho, \left(x^{(2)}\right)_{(1)} \otimes \left(\left(y_{(0)}\right)^{(1)} \right)_{(1)} \right\rangle \\
&\quad \left\langle \rho, \left(x^{(3)}\right)_{(1)} \otimes y_{(1)} \right\rangle
\end{aligned}$$

$$
\begin{aligned}
&= \left\langle \alpha, x^{(1)} \left(\left(y_{(0)} \right)^{(1)} \right)_{(0)} \right\rangle \mathcal{S}_R \left(\left(\left(y_{(0)} \right)^{(2)} \right)_{(0)} \right) \\
&\quad \mathcal{S}_R \left(\left(x^{(2)} \right)_{(0)} \right) \left(x^{(3)} \right)_{(0)} \\
&\quad \left\langle \rho, \left(x^{(2)} \right)_{(1)} \otimes \left(\left(y_{(0)} \right)^{(2)} \right)_{(1)} \right\rangle \\
&\quad \left\langle \rho, \left(x^{(2)} \right)_{(2)} \otimes \left(\left(y_{(0)} \right)^{(1)} \right)_{(1)} \right\rangle \\
&\quad \left\langle \rho, \left(x^{(3)} \right)_{(1)} \otimes y_{(1)} \right\rangle \\
&\overset{(16)}{=} \left\langle \alpha, x^{(1)} \left(\left(y_{(0)} \right)^{(1)} \right)_{(0)} \right\rangle \mathcal{S}_R \left(\left(\left(y_{(0)} \right)^{(2)} \right)_{(0)} \right) \\
&\quad \mathcal{S}_R \left(\left(x^{(2)} \right)_{(0)} \right) \left(x^{(3)} \right)_{(0)} \\
&\quad \left\langle \rho, \left(x^{(2)} \right)_{(1)} \otimes \left(\left(y_{(0)} \right)^{(1)} \right)_{(1)} \left(\left(y_{(0)} \right)^{(2)} \right)_{(1)} \right\rangle \\
&\quad \left\langle \rho, \left(x^{(3)} \right)_{(1)} \otimes y_{(1)} \right\rangle \\
&= \left\langle \alpha, x^{(1)} \left(\left(y_{(0)} \right)^{(1)} \right)_{(0)} \right\rangle \mathcal{S}_R \left(\left(\left(y_{(0)} \right)^{(2)} \right)_{(0)} \right) \\
&\quad \left(\mathcal{S}_R \left(x^{(2)} \right) \right)_{(0)} \left(x^{(3)} \right)_{(0)} \\
&\quad \left\langle \rho, \left(\mathcal{S}_R \left(x^{(2)} \right) \right)_{(1)} \otimes \right. \\
&\quad \left. \left(\left(y_{(0)} \right)^{(1)} \right)_{(1)} \left(\left(y_{(0)} \right)^{(2)} \right)_{(1)} \right\rangle \\
&\quad \left\langle \rho, \left(x^{(3)} \right)_{(1)} \otimes y_{(1)} \right\rangle \\
&= \left\langle \alpha, x^{(1)} \left(y_{(0)} \right)^{(1)} \right\rangle \mathcal{S}_R \left(\left(y_{(0)} \right)^{(2)} \right) \\
&\quad \left(\mathcal{S}_R \left(x^{(2)} \right) \right)_{(0)} \left(x^{(3)} \right)_{(0)} \\
&\quad \left\langle \rho, \left(\mathcal{S}_R \left(x^{(2)} \right) \right)_{(1)} \otimes y_{(1)} \right\rangle \\
&\quad \left\langle \rho, \left(x^{(3)} \right)_{(1)} \otimes y_{(2)} \right\rangle \\
&\overset{(15)}{=} \left\langle \alpha, x^{(1)} \left(y_{(0)} \right)^{(1)} \right\rangle \mathcal{S}_R \left(\left(y_{(0)} \right)^{(2)} \right) \\
&\quad \left(\mathcal{S}_R \left(x^{(2)} \right) x^{(3)} \right)_{(0)} \\
&\quad \left\langle \rho, \left(\mathcal{S}_R \left(x^{(2)} \right) \right)_{(1)} \left(x^{(3)} \right)_{(1)} \otimes y_{(1)} \right\rangle
\end{aligned}
$$

$$= \left\langle \alpha, x^{(1)} \left(y_{(0)}\right)^{(1)} \right\rangle \mathcal{S}_R \left(\left(y_{(0)}\right)^{(2)} \right)$$
$$\left(\mathcal{S}_R \left(x^{(2)} \right) x^{(3)} \right)_{(0)}$$
$$\left\langle \rho, \left(\mathcal{S}_R \left(x^{(2)} \right) x^{(3)} \right)_{(1)} \otimes y_{(1)} \right\rangle$$
$$= \left\langle \alpha, x \left(y_{(0)}\right)^{(1)} \right\rangle \mathcal{S}_R \left(\left(y_{(0)}\right)^{(2)} \right) (\mathcal{S}_R((1)))_{(0)}$$
$$\left\langle \rho, (\mathcal{S}_R((1)))_{(1)} \otimes y_{(1)} \right\rangle$$
$$= \left\langle \alpha, x \left(y_{(0)}\right)^{(1)} \right\rangle \mathcal{S}_R \left(\left(y_{(0)}\right)^{(2)} \right)$$
$$(\mathcal{S}_R((1)))_{(0)}\, \epsilon(y_{(1)})$$
$$= \left\langle \alpha, x y^{(1)} \right\rangle \mathcal{S}_R \left(y^{(2)} \right)$$
$$= \left\langle \alpha \cdot x, y^{(1)} \right\rangle \mathcal{S}_R \left(y^{(2)} \right)$$
$$= \left\langle (\alpha \cdot x)^{(0)}, y_{(0)} \right\rangle \left((\alpha \cdot x)^{(1)} \right)_{(0)}$$
$$\left\langle \rho, \left((\alpha \cdot x)^{(1)} \right)_{(1)} \otimes y_{(1)} \right\rangle$$

By Lemma 3.7.2, $R^* \simeq R$ as right modules and R is Frobenius. □

Lemma 3.7.4. *Let (B, ρ) be a CQT-bialgebra, $R = \oplus_{j \geq 0} R^j$ a graded braided Hopf algebra in $\mathcal{M}^B$. Let us assume that $R^0 = \mathbb{C}$, that R is generated in degree 1, that $\dim R^1$ is finite, that there exists an integer p such that $R^p \neq 0$, $R^{p+1} = 0$ and a coaction satisfiying relation (30). Then R is a Frobenius algebra of rank p.*

Proof. Because R is generated in degree 1, $R^{p+1} = 0$ implies $R^{p+s} = 0$ for any positive s. Hence R is finite dimensional. By Lemma 3.7.3, R is Frobenius. By Lemma 3.7.1 R is a Frobenius algebra of rank p. □

Now we pass to the objects of our interest. We shall concentrate on $\operatorname{Sym}_S V$, since $\wedge_S V = \operatorname{Sym}_{\widetilde{S}} V$ for $\widetilde{S} = -q^{-1} S$. By Proposition 3.1.1, $\operatorname{Sym}_S V$ is an $A(R)$-comodule algebra. By the following Lemma, a particular case of (Majid 1995), Thm. 10.2.1, $\operatorname{Sym}_S V$ is indeed a braided Hopf algebra in $\mathcal{M}^{A(R)}$.

Lemma 3.7.5. *There is a unique braided Hopf algebra structure on $\operatorname{Sym}_S V$ whose comultiplication is determined by*

$$\Delta(v) = v \otimes 1 + 1 \otimes v, \qquad v \in V. \tag{32}$$

Proof. It is not difficult to see that Δ extends to a morphism of algebras, where the product on $\operatorname{Sym}_S V \otimes \operatorname{Sym}_S V$ is given by $(u \otimes m)(n \otimes z) = \langle \rho, m_{(1)} \otimes n_{(1)} \rangle u n_{(0)} \otimes m_{(0)} z$. The existence of the antipode follows from a Lemma of Takeuchi – see (Montgomery 1993), Lemma 5.2.10 – since the coradical of $\operatorname{Sym}_S V$ is $\mathbb{C}$. □

Remark 3.7.1. It is not true that any $\operatorname{Sym}_S V$ bears a coalgebra coaction on its dual satisfiying (30); take for instance $S = \mathrm{Id}$. In the affirmative case we say that S is *admissible.* In (Gurevich 1991) the notion of a *closed* Symmetry is introduced. We conjecture that S closed implies S admissible.

Proposition 3.7.1. *If S is admissible, $\mathrm{Sym}_S^{p+1}V = 0$ and $\mathrm{Sym}_S^p V \neq 0$, then $\mathrm{Sym}_S V$ is a Frobenius algebra of rank p.*

Proof. This follows at once from Lemmas 3.7.4 and 3.7.5. □

As explained above, the analogous result holds for the "quantum exterior algebra". Explicitly, and combining with Theorem 3.3.1, we have

Corollary 3.7.1. *If S is admissible, $\wedge_S^{p+1}V = 0$ and $\wedge_S^p V \neq 0$, then $\wedge_S V$ and $\wedge_{^tS}(V^*)$ are Frobenius algebras of rank p.* □

We do not know the full characterization of the "quantum exterior algebras" satisfying the "weak" Frobenius conditions as in the hypothesis of Theorem 3.3.2 (b). In this direction, we can offer the following result.

Lemma 3.7.6. *Let $R = \oplus_{j\geq 0} R^j$ be a graded algebra generated in degree one such that $R^j \neq 0$ for all j. If $F_{p-1}(p)$ holds, then $\dim R^{p+h} = 1$ for all $h \geq 0$. More precisely, there exists $v \in R^1$ such that $R^{p+h} = \mathbb{C}v^{p+h}$, for all $h \geq 0$.*

Proof. Let $\mu \in R^p - 0$. As R is generated in degree one, $R^{p+1} = \mu R^1 = R^1\mu$. So, there exists $v \in R^1$ such that $v\mu \neq 0$. By condition $F_{p-1}(p)$, there exists $u \in R^{p-1}$ such that $\mu = vu$. Hence, for any $w \in R^1$, we have

$$\mu w = vuw = cv\mu, \qquad \text{for some } c \in \mathbb{C},$$

since $uw \in R^p$. This shows that $R^{p+1} = \mathbb{C}v\mu$.

We claim now that $R^{p+h} = \mathbb{C}v^h\mu$, $h > 0$. This implies in turn that $v^h \neq 0$ for all h. In particular, $v^p \in R^p - 0$; so we can take $\mu = v^p$ and conclude the Lemma.

We prove the claim by recurrence on h; the case $h = 1$ was already treated. If $R^{p+h} = \mathbb{C}v^h\mu$, then by the case $h = 1$ we have for any $w \in R^1$

$$v^h\mu w = cv^{h+1}\mu \qquad \text{for some } c \in \mathbb{C}.$$

As $R^{p+h+1} = R^{p+h}R^1$, the claim follows. □

Remark 3.7.2. The condition $F_j(p)$ for a graded algebra R is equivalent to the condition $F_{p-j}(p)$ for R^{op}. Thus, we could replace the hypothesis of the Lemma by $F_1(p)$ and the conclusion will still be true.

4. COMPACT QUANTUM GROUPS $H(R)$ AND $K(R)$

4.1. $H(R)$ **is a $*$-Hopf algebra.** Let C be a $\circ$-coalgebra, A its dual $*$-algebra with respect to the involution (1). We extend $*$ to T^nA in the usual way: $(T_1 \otimes \cdots \otimes T_n)^* = T_1^* \otimes \cdots \otimes T_n^*$. On the other hand, $T(C)$ is a $\circ$-bialgebra where $\circ : T(C) \to T(C)$ is the natural extension of $\circ : C \to C$. Let $R \in A \otimes A$.

Proposition 4.1.1. *We assume that*

$$R = \tau(R^*). \tag{33}$$

Then the preceding involution induces a $\circ$-bialgebra structure on $A(R)$.

Proof. Let $\zeta = \tau(\eta R) - R\eta \in J_R$. Then

$$\begin{aligned} \zeta^\circ &= \tau(R^*\eta^\circ) - \eta^\circ R^* \\ &= \tau(R^*)\tau(\eta^\circ) - \tau(\tau(\eta^\circ)\tau(R^*)) \in J_{\tau(R^*)}. \end{aligned}$$

Hence $(J_R)^\circ = J_R$. □

We assume now V is a finite dimensional vector space provided with a fixed hermitian form. Then $A = \mathrm{End}\,(V)$ is a $*$-algebra via $(T^*(u)|v) = (u|T(v))$, $T \in A$, $u, v \in V$; and $C = A^*$ is a $\circ$-coalgebra by (1).

Let $S = \tau R$. Then R satisfies (33) if and only if S is selfadjoint. Indeed, as $\tau(R) = \tau R \tau$, we have $R = \tau(R^*) = \tau R^* \tau$ iff $\tau R = R^* \tau$, but this is $S = S^*$.

We assume in what follows that S is selfadjoint and verifies the Braid equation and the Hecke condition $(S-q)(S+1) = 0$ with $q \neq 0$, $q^m \neq 1$ if $m \geq 2$, or $q = 1$. Note that $q \in \mathbb{R}$. Let $I = \mathrm{Im}\,(S+1) = \ker(S-q)$, $\wedge_S(V) = T(V)/\langle I \rangle$ as before, and let $\pi_\wedge : T(V) \to \wedge_S(V)$ be the canonical projection.

The hermitian form on V induces an hermitian form on $T(V)$ by:

1. $(v|w) = 0$ if $v \in T^n V$, $w \in T^m V$ and $n \neq m$.
2. $(v_1 \otimes \cdots \otimes v_n | w_1 \otimes \cdots \otimes w_n) = (v_1|w_1) \cdots (v_n|w_n)$, $v_i, w_i \in V$, $i = 1, \cdots, n$.

Proposition 4.1.2. *$T(V)$ is a $\circ$-comodule algebra over $T(C)$.*

Proof. We have to prove

$$\left(v_{(0)}|w\right) v_{(1)} = \left(v|w_{(0)}\right) \left(w_{(1)}\right)^\circ, \ v, w \in T(V). \tag{34}$$

We can assume that $v, w \in T^n V$ and proceed by induction on n. The case $n = 1$ is straightforward. For the general case, we assume further that $v = v_1 \otimes \cdots \otimes v_n$, $w = w_1 \otimes \cdots \otimes w_n$. Hence

$$\begin{aligned}
&\left(v_{(0)}|w\right) v_{(1)} = \\
&= \left((v_1 \otimes \cdots \otimes v_n)_{(0)} | w_1 \otimes \cdots \otimes w_n\right) \\
&\qquad (v_1 \otimes \cdots \otimes v_n)_{(1)} \\
&= \left(v_{1(0)} \otimes \cdots \otimes v_{n(0)} | w_1 \otimes \cdots \otimes w_n\right) \\
&\qquad v_{1(1)} \otimes \cdots \otimes v_{n(1)} \\
&= \left(v_{1(0)}|w_1\right) \cdots \left(v_{n(0)}|w_n\right) v_{1(1)} \otimes \cdots \otimes v_{n(1)} \\
&= \left(v_{1(0)}|w_1\right) v_{1(1)} \otimes \cdots \otimes \left(v_{n(0)}|w_n\right) v_{n(1)} \\
&= \left(v_1|w_{1(0)}\right) \left(w_{1(1)}\right)^\circ \otimes \cdots \otimes \left(v_n|w_{n(0)}\right) \left(w_{n(1)}\right)^\circ \\
&= \left(v_1 \otimes \cdots \otimes v_n | w_{1(0)} \otimes \cdots \otimes w_{n(0)}\right) \\
&\qquad \left(w_{1(1)}\right)^\circ \otimes \cdots \otimes \left(w_{n(1)}\right)^\circ \\
&= \left(v_1 \otimes \cdots \otimes v_n | w_{1(0)} \otimes \cdots \otimes w_{n(0)}\right) \\
&\qquad \left(w_{1(1)} \otimes \cdots \otimes w_{n(1)}\right)^\circ \\
&= \Big(v_1 \otimes \cdots \otimes v_n | \left(w_1 \otimes \cdots \otimes w_n\right)_{(0)}\Big) \\
&\qquad \Big(\left(w_1 \otimes \cdots \otimes w_n\right)_{(1)}\Big)^\circ \\
&= \left(v|w_{(0)}\right) \left(w_{(1)}\right)^\circ .
\end{aligned}$$

Here we used that δ and $\circ$ are multiplicative. □

From this point, we assume that the hermitian form is an *inner product, i.e.* it is positive definite.

Proposition 4.1.3. *The inner product*

$$(\ |\ ) : T^n V \times T^n V \to \mathbb{C}$$

is $H_q(n)$-invariant. Moreover, $(T^nV)_- = (I^n)^\perp$.

Proof. The first statement is clear: if $1 \leq i \leq n-1$ then

$$\begin{aligned}(\sigma_i \cdot v|w) &= \big(S^i(v)|w\big)\\ &= \big(v|(S^i)^*(w)\big)\\ &= \big(v|(S^*)^i(w)\big)\\ &= \big(v|S^i(w)\big)\\ &= (v|\sigma_i \cdot w).\end{aligned}$$

Let now $v \in (T^nV)_-$ and $u \in T^nV$. Then

$$\begin{aligned}\big(v\,\big|\big(S^i+1\big)\,(u)\big) &= (v|(\sigma_i+1)\cdot u)\\ &= ((\sigma_i+1)^*\cdot v|u)\\ &= ((\sigma_i+1)\cdot v|u)\\ &= (\sigma_i\cdot v+v|u)\\ &= (-v+v|u) = 0.\end{aligned}$$

Thus $(T^nV)_- \subset \big(\mathrm{Im}\,\big(S^i+1\big)\big)^\perp = \big(I^{n,i}\big)^\perp$, $i = 1, \cdots, n-1$; since $I^n = \sum_{i=1}^{n-1} I^{n,i}$, we have

$$(T^nV)_- \subset (I^n)^\perp.$$

The claim follows because $(T^nV)_-$ and $(I^n)^\perp$ have the same dimension (see Proposition 3.3.2). □

Observe that the orthogonal projection from T^nV onto $(I^n)^\perp$ is given by the action of $M_- \in H_q(n)$. Indeed, if $v \in T^nV$, then $M_- \cdot v \in (T^nV)_- = (I^n)^\perp$ and $v - M_- \cdot v \in I^n$ since $\pi_\wedge(v - M_- \cdot v) = \pi_\wedge(v) - M_- \cdot \pi_\wedge(v) = 0$ (recall that $H_q(n)$ acts on $\wedge_S^n V$ by the sign).

Therefore, we can provide $\wedge_S(V)$ with an inner product such that the different homogeneous components are orthogonal to each other; and given on $\wedge_S^n(V)$ by identification with $(T^nV)_- = (I^n)^\perp$. That is, $(v + I^n|w + I^n) = (M_- \cdot v|M_- \cdot w)$, $v, w \in T^nV$.

Proposition 4.1.4. **(a):** *$\wedge_S(V)$ is a $\circ$-comodule algebra over $A(R)$.*
(b): *If in addition S verifies $F_0(p)$ and $d \in A(R)$ is given by Proposition 3.1.2, then $d^\circ = d$.*

Proof. (a). We have to prove

$$\begin{aligned}&\Big((u+<I>)_{(0)}\,|v+<I>\Big)\,(u+<I>)_{(1)} =\\ &\qquad\Big(u+<I>|\,(v+<I>)_{(0)}\Big)\,\Big((v+<I>)_{(1)}\Big)^\circ,\end{aligned} \tag{35}$$

$u, v \in T(V)$. We can assume that $u, v \in T^nV$; then

$$
\begin{aligned}
&\left((u+I^n)_{(0)}\,|v+I^n\right)(u+I^n)_{(1)} = \\
&= \left(u_{(0)}+I^n|v+I^n\right)\left(u_{(1)}+J^n\right) \\
&= \left(M_-\cdot u_{(0)}|M_-\cdot v\right)\left(u_{(1)}+J^n\right) \\
&= \left(M_-\cdot u_{(0)}|M_-\cdot v\right)u_{(1)}+J^n \\
&= \left((M_-\cdot u)_{(0)}|M_-\cdot v\right)(M_-\cdot u)_{(1)}+J^n \\
&= \left(M_-\cdot u|(M_-\cdot v)_{(0)}\right)\left((M_-\cdot v)_{(1)}\right)^\circ+J^n \\
&= \left(M_-\cdot u|(M_-\cdot v)_{(0)}\right)\left((M_-\cdot v)_{(1)}+J^n\right)^\circ \\
&= \left(M_-\cdot u|(M_-\cdot v)_{(0)}\right)\left((M_-\cdot v)_{(1)}\right)^\circ+J^n \\
&= \left(M_-\cdot u|M_-\cdot v_{(0)}\right)\left(v_{(1)}\right)^\circ+J^n \\
&= \left(M_-\cdot u|M_-\cdot v_{(0)}\right)\left(v_{(1)}+J^n\right)^\circ \\
&= \left(u+I^n|(v+I^n)_{(0)}\right)\left((v+I^n)_{(1)}\right)^\circ.
\end{aligned}
$$

We used that the action of $H_q(n)$ and the coaction of $A(R)$ in T^nV commute.

(b). By Proposition 3.1.2,

$$(\mu|\mu)d = \left(\mu_{(0)}|\mu\right)\mu_{(1)} = \left(\mu|\mu_{(0)}\right)\left(\mu_{(1)}\right)^\circ = (\mu|\mu)d^\circ.$$

Since $(\ |\ )$ is an inner product on $\wedge_S^p V$, $(\mu|\mu) > 0$ and thus $d = d^\circ$. □

Proposition 4.1.5. *Assume that S is selfadjoint and verifies the Braid equation, the Hecke condition, $F_1(p)$ and $F_{p-1}(p)$. Then $H(R) = A(R)_d$ is a $*$-Hopf algebra. Furthermore, $\wedge_S(V)$ is a $\circ$-comodule algebra over $H(R)$.*

Proof. By Lemma 3.1.1, and because $\overline{S} = {}^tS$, ${}^tS$ verifies $F_j(p)$ whenever S verifies $F_j(p)$. We can apply Theorem 3.5.4 to conclude that $H(R)$ is a Hopf algebra. Furthermore, the last Proposition shows that $H(R)$ inherits the $\circ$-bialgebra structure from $A(R)$; explicitly

$$\left(\frac{h}{d^n}\right)^\circ = \frac{h^\circ}{d^n}.$$

The first assertion follows now from Lemma 2.0.1 and the second, from the first and Proposition 4.1.4. □

We can now state our main result.

Theorem 4.1.1. *Let V be a finite dimensional vector space provided with an inner product and let $S : V\otimes V \to V\otimes V$ be a linear automorphism. Assume that S is selfadjoint and verifies the Braid equation, the Hecke condition, $F_1(p)$ and $F_{p-1}(p)$. Then $H(R) = A(R)_d$ is a Compact Linear Quantum Group.* □

Proof. This follows from Theorems 2.0.1, 3.5.4, Corollary 3.5.2 and Proposition 4.1.5. □

4.2. $K(R)$ **is a $*$-Hopf algebra.** We keep the notation of the preceding Subsection. The following Theorem can be proved with arguments analogous to those given above.

Theorem 4.2.1. *Let V be a finite dimensional vector space provided with an inner product and let $S: V\otimes V \to V\otimes V$ be a linear automorphism. Assume that S is selfadjoint and verifies the Braid equation, the Hecke condition, $F_1(p)$ and $F_{p-1}(p)$. Then*

1. *$K(R)=A(R)/\langle d-1\rangle$ is a $*$-Hopf algebra.*
2. *$\wedge_S(V)$ is a $\circ$-comodule algebra over $K(R)$.*
3. *$K(R)$ is a Compact Linear Quantum Group.* □

5. EXAMPLES

We apply now the results of the preceding Section to show some new examples of CQG. Let $V=\mathbb{C}^m$ with the standard inner product (denoted by $\cdot$) and let $\{e_1,\cdots,e_m\}$ be the canonical basis of V. We identify as usual $\mathrm{End}\,(V\otimes V)$ with $M_{m^2\times m^2}(\mathbb{C})$ via $S(e_i\otimes e_j)=\sum_{kl} S_{ij}^{kl} e_k\otimes e_l$ for $S=\left(S_{kl}^{ij}\right)\in M_{m^2\times m^2}(\mathbb{C})$; then $S^*={}^t\overline{S}$. Because of Theorems 4.1.1 and 4.2.1, we have to find hermitian matrices $S\in M_{m^2\times m^2}(\mathbb{C})$ which verify the Braid equation, the Hecke condition, $F_1(p)$ and $F_{p-1}(p)$. We shall look for such matrices among those described in (Gurevich 1991).

We first recall some general remarks from (Gurevich 1991). Let S be a solution of the Braid equation that verifies $F_0(p)$. Then there exists $v_{i_1\cdots i_p}$ such that

$$\left\{\sum v_{i_1\cdots i_p}\, e_{i_1}\otimes\cdots\otimes e_{i_p}\right\} \text{ is a basis of } (T^pV)_-. \tag{36}$$

Let $u_{i_1\cdots i_p}$ be defined by

$$\begin{aligned} &M_-^p\left(e_{i_1}\otimes\cdots\otimes e_{i_p}\right)=\\ &u_{i_1\cdots i_p}\left(\sum v_{j_1\cdots j_p}\, e_{j_1}\otimes\cdots\otimes e_{j_p}\right); \end{aligned}\tag{37}$$

from (36) and (37) we obtain

$$\sum u_{i_1\cdots i_p}\, v_{i_1\cdots i_p}=1. \tag{38}$$

Also, if $\mu:=\sum v_{i_1\cdots i_p}\, e_{i_1}\wedge\cdots\wedge e_{i_p}$ then $\{\mu\}$ is a basis of $\wedge_S^p V$. It follows from (37) that

$$e_{i_1}\wedge\cdots\wedge e_{i_p}=u_{i_1\cdots i_p}\,\mu. \tag{39}$$

Since $\delta(\mu)=\mu\otimes d$, we have

$$d=\sum u_{i_1\cdots i_p}\, v_{j_1\cdots j_p}\, t_{i_1j_1}\cdots t_{i_pj_p}. \tag{40}$$

From now on, *we shall assume that the rank of S is $p=2$*; we fix $u=(u_{i_1i_2})$ and $v=(v_{i_1i_2})$.

Proposition 5.0.1. *Let S be a solution of the Braid equation satisfying the Hecke condition with label $q\in\mathbb{C}$, where $q\neq 0$ and $q^n\neq 1$ if $n\geq 2$ or $q=1$.*

(a) *Assume that S satisfies $F_1(2)$. Then the matrices u and v given by (36) and (37) satisfy*

$$S_{kl}^{ij}=q\,\delta_{ik}\,\delta_{jl}-(1+q)\,v_{ij}\,u_{kl}, \tag{41}$$

$$\begin{aligned} &\left[q\,\mathrm{id}-(1+q)^2\left(u\,v\,{}^tu\,{}^tv\right)\right]_{il} v_{rn}\,u_{jk}=\\ &\left[q\,\mathrm{id}-(1+q)^2\left({}^tu\,{}^tv\,u\,v\right)\right]_{kn} v_{lr}\,u_{ij}, \end{aligned}\tag{42}$$

$$\sum_{ij} u_{ij}\,v_{ij}=1. \tag{43}$$

Conversely, let $u, v \in M_{m\times m}(\mathbb{C})$ such that (42) and (43) hold. Then S defined by (41) is a solution of the Braid equation satisfying the Hecke condition with label q and $F_1(2)$.

(b) *Assume that S satisfies $F_1(2)$ and let u, v be as above. Then $\wedge_S^n V = 0$ if $n > 2$ if and only if*

$$u\, v\, {}^t\!u\, {}^t\!v = q\,(1+q)^{-2}\,\mathrm{id}\,. \tag{44}$$

(b) *Let $u, v \in M_{m\times m}(\mathbb{C})$ and set*

$$z = (1+q)\, v\, {}^t\!u. \tag{45}$$

If u, v verify (43) and (44), then z and v and satisfy

$$q\, {}^t\!z^{-1} = v^{-1}\, z\, v, \tag{46}$$

$$\operatorname{tr} z = 1+q. \tag{47}$$

Conversely, if $z, v \in M_{m\times m}(\mathbb{C})$ are invertible and fulfill (46) and (47), then $u = \frac{1}{1+q}\,{}^t\!\left(v^{-1}\, z\right)$ and v satisfy (43) and (44).

(d) *If $z, v \in M_{m\times m}(\mathbb{C})$ are invertible and fulfill (46) and (47), then $\operatorname{tr} z = 1+q$ and its Jordan form has, together with any cell corresponding to an eigenvalue α, an analogous cell with eigenvalue $q\,\alpha^{-1}$ (with the same multiplicity).*

Conversely, if z is invertible, $\operatorname{tr} z = 1+q$ and its Jordan form has such blocks, then there exists an invertible matrix v such that z and v verify (46) and (47).

Parts **b**, **c** and **d** of this Proposition are contained in (Gurevich 1991).

Proof. The proofs of **c** and **d** are straightforward and we leave them to the reader. We prove **b**; the proof of **a** is similar. Assume that S satisfies $F_1(2)$ and $\wedge_S^n V = 0$ if $n > 2$. Let $v = (v_{ij})$ and $u = (u_{ij})$ be matrices fulfilling (36) and (37). The relation (43) is nothing but (38). The relation (41) follows from (37) and

$$M_-^2 = \frac{1}{1+q}(q - S). \tag{48}$$

By Proposition 3.2.1 (2),

$$M_-^2\ \left(M_-^2\right)_2\ M_-^2 = q\,(1+q)^{-2} M_-^2; \tag{49}$$

writing this in coordinates we obtain (44). Notice that (44) implies, clearly, (42).

Conversely, let u and v verify (43) and (44) and let S be defined by (41). Then S is a solution of the Braid equation satisfying the Hecke condition with label $q \in \mathbb{C}$, $F_1(2)$ and $\wedge_S^n V = 0$ if $n > 2$.

Indeed, working in coordinates, we see that the Braid equation is equivalent to

$$\begin{aligned}
&\left(q\left((q+1)\sum_{ab} u_{ab}\, v_{ab} - q\right)\mathrm{id} - \right.\\
&\left.(1+q)^2\left(u\, v\, {}^t\!u\, {}^t\!v\right)\right)_{il} v_{rn}\, u_{jk} =\\
&\left(q\left((q+1)\sum_{ab} u_{ab}\, v_{ab} - q\right)\mathrm{id} - \right.\\
&\left.(1+q)^2\left({}^t\!u\, {}^t\!v\, u\, v\right)\right)_{kn} v_{lr}\, u_{ij}, \qquad i, j, k, l, r, n.
\end{aligned}$$

Hence (44) and (43) imply that S verifies the Braid equation. The Hecke condition follows immediatly from (43). If $\mu = \sum_{kl} v^{kl}\, e_k \wedge e_l$, then $e_i \wedge e_j = u_{ij}\,\mu$. Hence S verifies $F_1(2)$.

From (44) we deduce (49); via (48) and since $\left(M_-^2\right)^2 = M_-^2$, we conclude that

$$q\, M_-^2 - M_-^2\, S^2\, M_-^2 = q\,(1+q)^{-1} M_-^2. \tag{50}$$

Now $\wedge_S^n V = 0$ if $n > 2$ because of (50) and Proposition 3.2.1 (1). □

Let S be a solution of the Braid equation satisfying the Hecke condition with label $q \in \mathbb{C}$ and $F_1(2)$. We observe that $S^* = S$ if and only if q is real and

$$v_{ij}\, u_{kl} = \overline{u_{ij}\, v_{kl}}, \quad i, j, k, l. \tag{51}$$

Indeed, if $S^* = S$ then q is real because it is an eigenvalue of S. Now (51) follows from (41).

We shall denote by x_i the rows of an arbitrary matrix x.

The preceding Proposition reduced the search of solution S of rank 2 to some invertible matrices z. We show now that to find matrices S which verify the Braid equation, the Hecke condition, $F_1(2)$, $\wedge_S^n V = 0$ if $n > 2$ and $S^* = S$, it is enough to have the following data:

1. w, y unitary matrices,
2. $a_1, \cdots, a_m$ positive scalars,
3. $\sigma \in \mathcal{S}_m$ such that $\sigma^2 = \mathrm{id}$,

which verify

$$a_i\, a_{\sigma(i)} = a_j\, a_{\sigma(j)}, \ \forall i, j \tag{52}$$

$$\left(a_{\sigma(j)} - a_i\right) y_i \cdot \overline{w_j} = 0, \ \forall i, j. \tag{53}$$

The solution S we are looking for is explicitly given by

$$S_{kl}^{ij} = q\, \delta_{ik}\, \delta_{jl} - (1+q)\, r\, v_{ij}\, \overline{v_{kl}}, \tag{54}$$

where q, r and v are related to the above data by

$$r = \left(\sum_l a_l^2\right)^{-1}, \qquad \frac{\sqrt{q}}{1+q} = a_1\, a_{\sigma(1)}\, r,$$
$$v_{ij} = \sum_k \overline{w_{ki}}\, a_k\, y_{kj}.$$

Theorem 5.0.2. *Let S be an hermitian matrix which verifies the Braid equation, the Hecke condition with label q, $F_1(2)$ and $\wedge_S^n V = 0$ if $n > 2$. Let z be a matrix corresponding to S via Proposition 5.0.1. Then*

1. *The label q is positive and z is an hermitian matrix. Hence $z = {}^t\overline{w}\, \mathrm{Diag}(\alpha_1, \cdots, \alpha_m)\, w$, where w is a unitary matrix, and $\alpha_1, \cdots, \alpha_m$ are positive scalars. Moreover, there exists a permutation $\sigma \in \mathbb{S}_m$ such that $\sigma^2 = \mathrm{id}$ and*

$$\alpha_{\sigma(i)} = q \alpha_i^{-1}, \ i = 1, \cdots, m. \tag{55}$$

2. *If v verifies (36), let $x = wv$, $a_i = ||x_i||$, $y_i = a_i^{-1} x_i$ and y the matrix of rows $y_1, \cdots, y_m$. Then y is a unitary matrix and (52), (53) hold.*

Conversely, let y and w be two unitary matrices, $a_1, \cdots, a_m$ positive scalars and $\sigma \in \mathbb{S}_m$ such that $\sigma^2 = \mathrm{id}$. We suppose that (52) and (53) hold. Let q, $\alpha_1, \cdots, \alpha_m$ and the $m \times m$ matrix z be given by

$$\begin{aligned} \frac{\sqrt{q}}{1+q} &= \frac{a_1\, a_{\sigma(1)}}{\sum_l a_l^2} \left(= \frac{a_i\, a_{\sigma(i)}}{\sum_l a_l^2},\ \forall i \right), && (56) \\ \alpha_i \sum_l a_l^2 &= (1+q) a_i^2,\ i = 1, \cdots, m, && (57) \\ z &= w^{-1} \mathrm{Diag}(\alpha_1, \cdots, \alpha_m) w. && (58) \end{aligned}$$

Then z verifies the hypothesis of Proposition 5.0.1 **d** *and also (51) holds. Hence the corresponding matrix S is an hermitian matrix which verifies the Braid equation, the Hecke condition with label q, $F_1(2)$ and $\wedge_S^n V = 0$ if $n > 2$.*

Proof. Let u, v be a pair of matrices corresponding to S via Proposition 5.0.1 **b**. From (51), we have

$$\left(v\,{}^t u\right)_{ij} = \sum_k v_{ik}\, u_{jk} = \overline{\sum_k u_{ik}\, v_{jk}} = \overline{(u\,{}^t v)}_{ij}$$

Then $v\,{}^t u = \overline{u\,{}^t v} = {}^t\overline{(v\,{}^t u)}$, and from (45) we have $z = {}^t\overline{z}$; hence there exists an unitary matrix w such that

$$w\, z\, {}^t\overline{w} = \mathrm{Diag}(\alpha_1, \cdots, \alpha_m), \tag{59}$$

where $\alpha_1, \cdots, \alpha_m \in \mathbb{R}$ are the eigenvalues of z.
By Proposition 5.0.1 **d**, there exists $\sigma \in \mathbb{S}_m$, such that $\sigma^2 = \mathrm{id}$ and σ verifies (55); we can assume $\sigma^2 = \mathrm{id}$ because the map $\alpha \mapsto q\alpha^{-1}$ from
$\{\alpha_1, \cdots, \alpha_m\}$ onto itself is an involutive bijective map.
It follows from (51) that $v_{ij}\, u_{ij} = \overline{v_{ij}\, u_{ij}}$, i, j; then $v_{ij}\, u_{ij} \in \mathbb{R}$. Observe that (51) together with $v \neq 0$ implies $u_{ij} = 0$ whenever $v_{ij} = 0$. Then there exists $r_{ij} \in \mathbb{R}$ such that

$$u_{ij} = r_{ij} \overline{v_{ij}},\ i, j \tag{60}$$

if $v_{ij} = 0$ we take $r_{ij} = 0$. Now from (51) and (60) we have

$$(r_{ij} - r_{kl})\, \overline{v_{ij}}\, v_{kl} = 0.$$

Then there exists $r \in \mathbb{R}^*$ such that $u = r\,\overline{v}$. Let $x = wv$, then from (43) and since w is unitary, we have

$$r = \left(\sum_i ||v_i||^2 \right)^{-1} = \left(\sum_i ||x_i||^2 \right)^{-1} \in \mathbb{R}^+. \tag{61}$$

From (45) we have $z = (1+q)\, r\, v\, {}^t\overline{v}$, and then

$$\mathrm{Diag}(\alpha_1, \cdots, \alpha_m) = (1+q)\, r\, x\, {}^t\overline{x},$$

or equivalently

$$\begin{aligned} x_i \cdot x_j &= 0 \text{ if } i \neq j, && (62) \\ r(1+q) ||x_i||^2 &= \alpha_i,\ i = 1, \cdots, m. && (63) \end{aligned}$$

The relations (63) imply sign(α_i)=sign$(1+q)$, $\forall i$ and from (47) we have

$$1 + q = \sum_i \alpha_i = \sum_i q\, \alpha_{\sigma(i)}^{-1} = q \sum_i \alpha_i^{-1}.$$

Hence we have $q > 0$, and then $\alpha_i > 0$, $\forall i$. Let be $a_i = ||x_i||$, $y_i = a_i^{-1} x_i$ and y the matrix of rows $y_1, \cdots, y_m$. The equation (62) implies that y is a unitary matrix. The equations (55) and (63) force

$$\frac{a_i\, a_{\sigma(i)}}{\sum_l a_l^2} = \frac{\sqrt{q}}{1+q}, \; i = 1, \cdots, m. \tag{64}$$

Thus (52) follows. By (55), (59) and (63) we have

$$\begin{aligned} q\, {}^t z^{-1} &= v^{-1} z v \Leftrightarrow \\ \Leftrightarrow \quad & \left(x\, {}^t w\right) q \operatorname{Diag}\left(\alpha_1^{-1}, \cdots, \alpha_m^{-1}\right) = \\ & \operatorname{Diag}(\alpha_1, \cdots, \alpha_m) \left(x\, {}^t w\right) \\ \Leftrightarrow \quad & \left(\alpha_{\sigma(j)} - \alpha_i\right) \left(x\, {}^t w\right)_{ij} = 0, \; \forall i, j \\ \Leftrightarrow \quad & r(1+q) \left(||x_{\sigma(j)}||^2 - ||x_i||^2\right) x_i \cdot \overline{w_j} = 0, \; \forall i, j \\ \Leftrightarrow \quad & \left(a_{\sigma(j)} - a_i\right) y_i \cdot \overline{w_j} = 0, \; \forall i, j. \end{aligned}$$

Hence (53) holds. This finishes the proof of the first part.

Let us prove now the converse implication. The solutions of the equation (56) are real and positive and we can choose for q any of them (see Remark 5.0.1 (iii) below). Let x be the matrix of rows $a_i\, y_i$, $i = 1, \cdots, m$. It is clear that the matrices z and x are invertible. From (57) we get

$$\alpha_i\, \alpha_{\sigma(i)} \left(\sum_l a_l^2\right)^2 = (1+q)^2\, a_i^2\, a_{\sigma(i)}^2;$$

via (56) we have (55). The equation (47) follows immediately from (57). From (53) we obtain

$$(1+q) \left(\sum_l a_l^2\right)^{-1} \left(a_{\sigma(j)}^2 - a_i^2\right) a_i\, y_i \cdot \overline{w_j} = 0,$$

and from (57) we conclude

$$\left(\alpha_{\sigma(j)} - \alpha_i\right) \left(x\, {}^t w\right)_{ij} = 0.$$

Now (55) implies

$$\begin{aligned} & \left(x\, {}^t w\right) q \operatorname{Diag}\left(\alpha_1^{-1}, \cdots, \alpha_m^{-1}\right) = \\ & \operatorname{Diag}(\alpha_1, \cdots, \alpha_m) \left(x\, {}^t w\right) \end{aligned}$$

which immediately gives (46). Hence z verifies the hypothesis of Proposition 5.0.1 **d**.

Let $v = w^{-1} x$ and $r = \left(\sum_l a_l^2\right)^{-1}$. Since w is unitary,

$$(1+q)\, v\, {}^t(r\, \overline{v}) = w^{-1} \left((1+q)\, r\, x\, {}^t \overline{x}\right) w. \tag{65}$$

On the other hand, y also is unitary and by (57), we have

$$(1+q)\, r\, x\, {}^t \overline{x} = \operatorname{Diag}(\alpha_1, \cdots, \alpha_m). \tag{66}$$

Putting together (65), (66), (58) and (45) we see that $u = r\overline{v}$ and therefore (51) holds. □

Remark 5.0.1. *(i).* To apply Theorem 5.0.2, it is useful to recall that a permutation σ verifies $\sigma^2 = \mathrm{id}$ if and only if σ is a product of disjoint transpositions.

(ii). In the situation described in the first part of Theorem 5.0.2, suppose that z is in Jordan form. Then we can take $w = \mathrm{id}$, $v = x$ and (53) is equivalent to

$$\left(a_{\sigma(j)} - a_i\right) v_{ij} = 0, \; i, j = 1, \cdots, m. \tag{67}$$

Hence $v_{ij} = v_{ji} = 0$ whenever $a_{\sigma(i)} \neq a_j$. Observe that the a_i's may not be different and then $\sigma(i) \neq j$ does not necessarily imply $v_{ij} = v_{ji} = 0$.

(iii). If we combine (54) with (8), we see that the corresponding algebra $A(R)$ is generated by t_{ij}, $1 \leq i, j \leq m$ with defining relations

$$\overline{v_{ij}} \sum_{pn} v_{pn}\, t_{kp}\, t_{ln} = v_{kl} \sum_{pn} \overline{v_{pn}}\, t_{pi}\, t_{nj}, \qquad i, j, k, l. \tag{68}$$

By (40) the "quantum" determinant is given by

$$d = r \sum_{ijkl} \overline{v_{ij}}\, v_{kl}\, t_{ik}\, t_{jl}. \tag{69}$$

Therefore, the Hopf algebras $H(R)$ and $K(R)$ do not depend on which solution q of (56) is chosen.

Now we consider the data described above in two extreme cases: when the a_i's are all different and when they are all equal.

Case when the a_i's are all different.

From (53) we have that y_i is orthogonal with $\overline{w_{\sigma(j)}}$, if $i \neq \sigma(j)$. As y and w are unitary matrices we get $y_i = \beta_i\, \overline{w_{\sigma(i)}}$, with $|\beta_i| = 1$, $i = 1, \cdots, m$.

In this case, the data to obtain our desired solutions S is the following:

1. a unitary matrix w;
2. $\beta_1, \cdots, \beta_m$, complex numbers in the unitary circle;
3. $a_1, \cdots, a_m$, different real and positive numbers;
4. a permutation $\sigma \in \mathbb{S}_m$ such that $\sigma^2 = \mathrm{id}$;

they have to verify only (52).

Explicitly,

$$\begin{aligned} v_{ij} &= \textstyle\sum_k a_k\, \beta_k\, \overline{w_{ki}\, w_{\sigma(k)\, j}}, \qquad r = \left(\sum_l a_l^2\right)^{-1}, \\ S_{kl}^{ij} &= q\, \delta_{ik}\, \delta_{jl} - (1+q)\, r\, v_{ij}\, \overline{v_{kl}} \end{aligned}$$

with q defined by (56). If we take $w = \mathrm{id}$ then $v_{ij} = a_i\, \beta_i\, \delta_{i\sigma(j)}$.

Case when the a_i's are all equal.

In this case (52) and (53) are trivially verified; (56), (57) and (41) become

$$\begin{aligned} 0 &= q^2 + (2 - m^2)q + 1, \\ \alpha_i &= \frac{1+q}{m}, \qquad \forall i, \\ v &= {}^t w\, y, \\ S_{kl}^{ij} &= q\, \delta_{ki}\, \delta_{lj} - \tfrac{1+q}{m}\, v_{ij}\, \overline{v_{kl}}, \ \forall i, j, k, l. \end{aligned} \tag{70}$$

Observe that S and q do not depend on the common value of the a_i's. In this case we need only two unitary matrices y and w with no relations. S and q are then determined by (70). We have $q = 1$ if $m = 2$ and two distinct real and positive values of q if $m > 2$.

Let us discuss in detail the case $m = 2$. We preserve the notation above.

We suppose $w = \mathrm{id}$ *i. e.* that z is in Jordan form.

Assume first that $a_1 = a_2$. We already know that $q = 1$. Then S has the form

$$S = \begin{pmatrix} |b|^2 & -a\,\overline{b} & -a\,\overline{c} & -a\,\overline{d} \\ -b\,\overline{a} & |a|^2 & -b\,\overline{c} & -b\,\overline{d} \\ -c\,\overline{a} & -c\,\overline{b} & |a|^2 & -c\,\overline{d} \\ -d\,\overline{a} & -d\,\overline{b} & -d\,\overline{c} & |b|^2 \end{pmatrix}, \text{ with}$$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in U(2).$$

Assume next that $a_1 \neq a_2 \in \mathbb{R}^+$ and let $\beta_1, \beta_2 \in \mathbb{C}$, $|\beta_1| = |\beta_2| = 1$. In this case q verifies

$$\frac{\sqrt{q}}{1+q} = \frac{a_1\,a_2}{a_1^2 + a_2^2}$$

and then q is equal to $(a_2/a_1)^2$ or $(a_1/a_2)^2$. If $q = (a_1/a_2)^2$ we take $y = a_1/a_2$ and $\gamma = \beta_1/\beta_2$. Then

$$S = \begin{pmatrix} y^2 & 0 & 0 & 0 \\ 0 & 0 & -\gamma\,y & 0 \\ 0 & -\overline{\gamma}\,y & y^2 - 1 & 0 \\ 0 & 0 & 0 & y^2 \end{pmatrix}. \tag{71}$$

If $q = (a_2/a_1)^2$ we take $y = a_2/a_1$ and $\gamma = \beta_2/\beta_1$. Then the corresponding matrix is $\tau\,S\,\tau$. In both cases is $q = y^2$ with $y \in \mathbb{R}^+$, $y \neq 1$, $\gamma \in \mathbb{C}$, $|\gamma| = 1$, and $d = t_{11}\,t_{22} + (y\,\gamma)^{-1}\,t_{12}\,t_{21}$ in the first case and $d = t_{11}\,t_{22} + y\,\gamma\,t_{12}\,t_{21}$ in the second.

Centrality of the quantum determinant

Let S be a solution of the Braid equation verifying the Hecke condition and such that $\wedge_S V$ is Frobenius of rank p.

Let us define

$$M_{ij} = \sum u_{l_1 \cdots l_{p-1} i}\, v^{j l_1 \cdots l_{p-1}} \tag{72}$$

$$N_{ij} = \sum u_{i l_1 \cdots l_{p-1}}\, v^{l_1 \cdots l_{p-1} j} \tag{73}$$

Proposition 5.0.2. *(*GUREVICH *1991).* $M\,N = q^{p-1}\,[p]_q^{-2}\mathrm{id}$.

Corollary 5.0.1. *$d \in Z(A(R))$ if and only if M are N are scalar matrices.*

It is not difficult to determine when the determinant is central if the rank is 2. Indeed, we have $N = u\,v$ and $M = {}^t(v\,u)$. If S is also hermitian, then $u = r\,\overline{v}$, $r \in \mathbb{R}^+$ and $d \in Z(A(R))$ if and only if $v\,\overline{v}$ is a scalar matrix. If for instance the a_i's are all different, it is enough to take the β_i's such that

$$\frac{\beta_i}{\beta_{\sigma(i)}} = \frac{\beta_j}{\beta_{\sigma(j)}}, \qquad 1 \leq i, j \leq m.$$

If, in particular, $m = 2$ then $\gamma = \pm 1$ in (71).

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