On the Nichols algebra of a semisimple Yetter-Drinfeld module

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Plan of the talk.

I. Overview.

II. Braided vector spaces and Nichols algebras.

III. On the Nichols algebra of a semisimple Yetter-Drinfeld module.

I. Overview.

In the early 80's, Drinfeld and Jimbo introduced quantized enveloping algebras:

$\mathfrak{g}$ simple Lie algebra,

Cartan matrix $(a_{ij})_{1\leq i,j\leq\theta}$,

$(d_1, \dots, d_\theta)$, $d_i \in \{1, 2, 3\}$, $d_i a_{ij} = d_j a_{ji}$

q not a root of 1

$U_q(\mathfrak{g}) = \mathbb{C}\langle k_1^{\pm 1}, \ldots, k_\theta^{\pm 1}, e_1, \ldots, e_\theta, f_1, \ldots, f_\theta \rangle$ with relations:

$$k_i k_j = k_j k_i, \quad k_i k_i^{-1} = k_i^{-1} k_i = 1,$$
$$k_i e_j k_i^{-1} = q^{d_i a_{ij}} e_j,$$
$$k_i f_j k_i^{-1} = q^{-d_i a_{ij}} f_j,$$
$$\sum_{l=0}^{1-a_{ij}} (-1)^l \begin{bmatrix} 1-a_{ij} \\ l \end{bmatrix}_{q_i} e_i^{1-a_{ij}-l} e_j e_i^l = 0 \qquad (i \neq j),$$
$$\sum_{l=0}^{1-a_{ij}} (-1)^l \begin{bmatrix} 1-a_{ij} \\ l \end{bmatrix}_{q_i} f_i^{1-a_{ij}-l} f_j f_i^l = 0 \qquad (i \neq j)$$
$$e_i f_j - q^{-d_i a_{ij}} f_j e_i = \delta_{ij}(1 - k_i^2), i < j, i \nsim j$$

This is a Hopf algebra with $\Delta(k_i) = k_i \otimes k_i$, $\Delta(e_i) = k_i \otimes e_i + e_i \otimes 1$, $\Delta(f_i) = 1 \otimes f_i + k_i^{-1} \otimes f_i$.

$U_q(\mathfrak{g}) = \mathbb{C}\langle k_1^{\pm 1}, \ldots, k_\theta^{\pm 1}, e_1, \ldots, e_\theta, f_1, \ldots, f_\theta \rangle$ with relations:

$$
\begin{aligned}
&k_i k_j = k_j k_i, \quad k_i k_i^{-1} = k_i^{-1} k_i = 1, \\
&k_i e_j k_i^{-1} = q^{d_i a_{ij}} e_j, \\
&k_i f_j k_i^{-1} = q^{-d_i a_{ij}} f_j, \\
&\mathrm{ad}_c(e_i)^{1-a_{ij}}(e_j) = 0, \quad i \neq j \\
&\mathrm{ad}_c(f_i)^{1-a_{ij}}(f_j) = 0, \quad i \neq j \\
&e_i f_j - q^{-d_i a_{ij}} f_j e_i = \delta_{ij}(1 - k_i^2), i < j, i \nsim j.
\end{aligned}
$$

Here $\mathrm{ad}_c(e_i)(e_j) = e_i e_j - q^{d_i a_{ij}} e_j e_i$.

$U_q^+(\mathfrak{g}) = \mathbb{C}\langle e_1, \ldots, e_\theta \rangle$ with relations:

$$\mathrm{ad}_c(e_i)^{1-a_{ij}}(e_j) = 0, \quad i \neq j$$

This is not a Hopf algebra but it is a braided Hopf algebra, or Hopf algebra in a braided tensor category

$$\Delta(e_i) = 1 \otimes e_i + e_i \otimes 1.$$

It turns out that this is braided Hopf algebra of a very special sort– a Nichols algebra.

G. Lusztig, *Introduction to quantum groups*, Birkhäuser, 1993.

M. Rosso, C.R.A.S. (Paris) **320** (1995). Invent. Math. **133** (1998).

P. Schauenburg, Comm. in Algebra **24** (1996), pp. 2811–2823.

INPUT: braided vector space (W, c) → OUTPUT: Nichols algebra $\mathfrak{B}(W)$

(W, c) braided vector space: $c \in GL(W \otimes W)$

$$(c \otimes \mathrm{id})(\mathrm{id} \otimes c)(c \otimes \mathrm{id}) = (\mathrm{id} \otimes c)(c \otimes \mathrm{id})(\mathrm{id} \otimes c)$$

Example:

$$W = \mathbb{C}v_1 \oplus \cdots \oplus \mathbb{C}v_\theta, \ c(v_i \otimes v_j) = q^{d_i a_{ij}} v_j \otimes v_i \rightsquigarrow \mathfrak{B}(W) = U_q^+(\mathfrak{g})$$

Motivation. A.-Schneider, 1998: essential tool in the classification of pointed Hopf algebras over $\mathbb{C}$.

H pointed Hopf algebra with group

$$\Gamma = \{x \in H - 0 : \Delta(x) = x \otimes x\}.$$

(Pointed $\equiv \mathbb{C}\Gamma$ is the largest cosemisimple subcoalgebra of H)

$\rightsquigarrow$ (W, c) braided vector space of special type (a Yetter-Drinfeld module over Γ)

$\rightsquigarrow$ Nichols algebra $\mathfrak{B}(W)$

Problems. Given a Yetter-Drinfeld module V over Γ

- When $\dim \mathfrak{B}(W) < \infty$? Or, when $\text{GK-dim}\, \mathfrak{B}(W) < \infty$?

- If so, give a formula for $\dim \mathfrak{B}(W)$, or $\text{GK-dim}\, \mathfrak{B}(W)$.

- Also, give presentation by generators and relations of $\mathfrak{B}(W)$.

Example. Assume Γ is an abelian group

A Yetter-Drinfeld module W over $\Gamma \equiv \Gamma$-graded Γ-module W

Assume W is semisimple as a Γ-module (always the case if Γ is finite). Then W is of diagonal type:

$$W = \mathbb{C}v_1 \oplus \cdots \oplus \mathbb{C}v_\theta, \qquad c(v_i \otimes v_j) = q_{ij} v_j \otimes v_i$$

Braided vector space of diagonal type.

$\exists$ basis $v_1, \dots, v_\theta$, $(q_{ij})_{1 \le i,j \le \theta}$ in $\mathbb{C}^\times$:

$$c(v_i \otimes v_j) = q_{ij} v_j \otimes v_i, \quad \forall i, j$$

Theorem. $1 \neq q_{ii}$ roots of 1. $\Rightarrow \dim \mathfrak{B}(W) < \infty$ classified.

I. Heckenberger, *Classification of arithmetic root systems*, `http://arxiv.org/abs/math.QA/0605795`.

Braided vector space of Cartan type. This is diagonal type plus

$\exists\ (a_{ij})_{1\le i,j\le\theta}$ generalized Cartan matrix

$$q_{ij}q_{ji} = q_{ii}^{a_{ij}}.$$

Theorem. (W, c) Cartan type, $1 \neq q_{ii}$ root of 1.
dim $\mathfrak{B}(W) < \infty \iff (a_{ij})$ of finite type.

There are formulas for dim and presentations by gens and rels. (including the quantum Serre relations)

N. A. & H.-J. Schneider, *Finite quantum groups and Cartan matrices*, Adv. Math. **154** (2000), 1-45.

I. Heckenberger, *The Weyl groupoid of a Nichols algebra of diagonal type*, Invent. Math. **164**, 175–188 (2006).

Braided vector space of diagonal type such that q_{ii} are NOT roots of 1.

Theorem. GK-dim $\mathfrak{B}(W) < \infty \iff (W, c)$ Cartan type and (a_{ij}) of finite type.

M. Rosso, Invent. Math. **133** (1998) assuming $q_{ii} > 0$.
N. A. and I. Angiono, `arxiv:math/0703924` using techniques of Heckenberger.

There are formulas for GK-dim and presentations by gens and rels. (essentially the quantum Serre relations!)

Assume next that Γ is a finite group (not necessarily abelian)

A Yetter-Drinfeld module W over Γ is then semisimple:

$$W = V_1 \oplus \cdots \oplus V_\theta,$$

where each V_i is "irreducible" and

$$c(V_i \otimes V_j) = V_j \otimes V_i.$$

Suggestion. Split the research on the Nichols algebra $\mathfrak{B}(W)$ into two parts:

- To study $\mathfrak{B}(V)$ for all V irreducible;

- consider the V_i's as "fat points" of a generalized Dynkin diagram and, assuming the knowledge of the Nichols algebras $\mathfrak{B}(V_i)$, to describe the Nichols algebra $\mathfrak{B}(W)$ as a "gluing" of the various Nichols subalgebras $\mathfrak{B}(V_i)$ along the generalized Dynkin diagram.

$\mathfrak{B}(V)$ for V irreducible:

- A few known examples where $\dim \mathfrak{B}(V) < \infty$, computed "by hand" or with help of computers.

- A criterium to discard some V such that $\dim \mathfrak{B}(V) = \infty$.

- very difficult to deal with the remaining V.

In this talk I will report progress concerning the second part.

II. Braided vector spaces and Nichols algebras.

(W, c) braided vector space: $c \in GL(W \otimes W)$,
$(c \otimes \mathrm{id})(\mathrm{id} \otimes c)(c \otimes \mathrm{id}) = (\mathrm{id} \otimes c)(c \otimes \mathrm{id})(\mathrm{id} \otimes c)$

Definition. (Artin, 1948). The braid group in n trends is the quotient $\mathbb{B}_n$ of the free group in $T_1, T_2, \dots, T_{n-1}$ by the defining relations

(B1) $T_i T_{i+1} T_i = T_{i+1} T_i T_{i+1}$, for all i.

(B2) $T_i T_j = T_j T_i$, if $|i - j| \geq 2$.

By definition there is a surjective map of groups $\pi : \mathbb{B}_n \to \mathbb{S}_n$.

Remark. (W, c) braided vector space.

Define $c_i \in \mathrm{Aut}(W^{\otimes n})$ by

$$c_i = \mathrm{id}_{W^{\otimes (i-1)}} \otimes c \otimes \mathrm{id}_{W^{\otimes (n-i-1)}}.$$

Then $c_1, \dots, c_{n-1}$ satisfy the relations (B1), (B2).

Thus we have a representation $\rho_n : \mathbb{B}_n \to \mathrm{Aut}(W^{\otimes n})$.

Examples. • The usual flip $\tau : W \otimes W \to W \otimes W$, $\tau(x \otimes y) = y \otimes x$ satisfies the braid equation.

• Let $W = W_0 \oplus W_1$ be a super vector space. If $x \in W_i$, $|x| = i$. The super flip $s : W \otimes W \to W \otimes W$, $s(x \otimes y) = (-1)^{|x||y|} y \otimes x$ satisfies the braid equation.

• Let Γ be a group, $\mathcal{C} \subseteq \Gamma$ a conjugacy class, $W = \oplus_{g \in \mathcal{C}} \mathbb{C}g$. Then $c : W \otimes W \to W \otimes W$, $c(g \otimes h) = ghg^{-1} \otimes g$ satisfies the braid equation.

Yetter-Drinfeld module W over a finite group Γ.

Irreducibles: $\mathcal{C}$ a conjugacy class in Γ; fix $s \in \mathcal{C}$; let (ρ, V) irred. repr. of Γ^s.

$$M(\mathcal{C}, \rho) = \mathrm{Ind}_{\Gamma^s}^{\Gamma} = \mathbb{C}\mathcal{C} \otimes V.$$

Any finite-dimensional Yetter-Drinfeld module W over Γ is a direct sum of different $M(\mathcal{C}_i, \rho_i)$'s (Dijkgraaf, Pasquier, Roche).

Definition. (Nichols, 1978; Woronowicz, 1988). If (W, c) is a braided vector space, then the Nichols algebra is

$$\mathfrak{B}(W) = \oplus_{n \in \mathbb{N}_0} \mathfrak{B}^n(W),$$

where

$$\mathfrak{B}^n(W) = T^n(W) / \ker \sum_{\sigma \in \mathbb{S}_n} \rho_n(S(\sigma)).$$

Here:

ρ_n is the representation of $\mathbb{B}_n$ induced by c
S is the Matsumoto section (of sets), $S : \mathbb{S}_n \to \mathbb{B}_n$ preserving the length.

III. On the Nichols algebra of a semisimple Yetter-Drinfeld module. (AHS).

Γ finite group,

$V_j = M(\mathcal{C}_j, \rho_j)$, $1 \leq j \leq \theta$,

$W = \oplus_{1 \leq j \leq \theta} V_j$.

There is an "adjoint" action of $\mathfrak{B}(W)$ on itself.

Fix i, $1 \leq i \leq \theta$.

L_j := $\mathrm{ad}_c\, \mathfrak{B}(V_i)$-submod. gen. by V_j, $j \neq i$;
L_j is a graded subspace of $\mathfrak{B}(W)$.

Theorem 1. If $\dim L_j < \infty$, then $L_j^{m_{ij}}$ is also an irreducible Yetter-Drinfeld module over Γ. Here m_{ij} = top degree of L_j.

$a_{ij} = 1 - m_{ij}$, $i \neq j$; $L_i^{-1} = V_i^*$ and $a_{ii} = 2$.
(a_{ij}) is a generalized Cartan matrix.

Note again the quantum Serre relation: $\mathrm{ad}_c\, V_i^{1-a_{ij}}(V_j) = 0$

$\mathfrak{B}(W) \supset \mathcal{K} :=$ subalgebra generated by L_j, $i \neq j$.

$$W_i' = \oplus_{1 \leq j \leq \theta} L_j^{1-a_{ij}}.$$

Here $L_j^{1-a_{ij}}$ are irreducible by Theorem 1.

Theorem 2. $\mathcal{K} \# \mathfrak{B}({}^*V_i) \simeq \mathfrak{B}(W_i')$, $\dim \mathfrak{B}(W) = \dim \mathfrak{B}(W_i')$.

Definition. W is standard if $W \simeq W_i'$ for all i

Theorem 3. W is standard, $\dim \mathfrak{B}(W) < \infty$

$\implies$ (a_{ij}) is of finite type.

Application.

Let $\mathcal{A}(\mathbb{S}_3, \mathcal{O}_2^3, \lambda)$ be the algebra presented by generators e_t, $t \in T := \{(12), (23)\}$, and a_σ, $\sigma \in \mathcal{O}_2^3$; with relations

$$e_t e_s e_t = e_s e_t e_s, \quad e_t^2 = 1, \quad s \neq t \in T; \tag{1}$$

$$e_t a_\sigma = -a_{t\sigma t} e_t \qquad t \in T, \sigma \in \mathcal{O}_2^3; \tag{2}$$

$$a_\sigma^2 = 0, \qquad \sigma \in \mathcal{O}_2^3; \tag{3}$$

$$a_{(12)} a_{(23)} + a_{(23)} a_{(13)} + a_{(13)} a_{(12)} = \lambda(1 - e_{(12)} e_{(23)}); \tag{4}$$

$$a_{(12)} a_{(13)} + a_{(13)} a_{(23)} + a_{(23)} a_{(12)} = \lambda(1 - e_{(23)} e_{(12)}). \tag{5}$$

Set $e_{(13)} = e_{(12)} e_{(23)} e_{(12)}$. Then $\mathcal{A}(\mathbb{S}_3, \mathcal{O}_2^3, \lambda)$ is a Hopf algebra of dimension 72 with comultiplication determined by

$$\Delta(a_\sigma) = a_\sigma \otimes 1 + e_\sigma \otimes a_\sigma, \quad \Delta(e_t) = e_t \otimes e_t, \qquad \sigma \in \mathcal{O}_2^3, t \in T. \tag{6}$$

Theorem. (AHS, using previous work with Milinski, Graña, Zhang). Let H be a finite dimensional pointed Hopf algebra with $G(H) \simeq \mathbb{S}_3$. Then either

- $H \simeq \mathbb{C}\mathbb{S}_3$, or
- $H \simeq \mathfrak{B}(\mathcal{O}_2^3, \mathrm{sgn})\#\mathbb{C}\mathbb{S}_3$, or
- $H \simeq \mathcal{A}(\mathbb{S}_3, \mathcal{O}_2^3, 1)$.

This is the second non-abelian group having the classification finished.

The first is $\mathbb{A}_5$– that admits no finite-dimensional pointed Hopf algebra out of the group algebra (A.–Fantino, Rev. Un. Math. Arg., `http://arxiv.org/abs/math.QA/0702559`).