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Isomorphism classes and automorphisms of finite Hopf algebras of type A_n

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1. Introduction

In [2] we classified a large class of finite-dimensional pointed Hopf algebras up to isomorphism. However the following problem was left open for Hopf algebras of type A, D or E_6, that is whose Cartan matrix is connected and allows a non-trivial automorphism of the corresponding Dynkin diagram. In this case we described the isomorphisms between two such Hopf algebras with the same Cartan matrix only implicitly. The problem is whether it is possible to compute the isomorphisms in terms of the defining families of parameters.

In the present paper we solve this problem for type A. To our surprise there are closed formulas for these isomorphisms. They are based on an action of the non-trivial automorphism σ of the Dynkin diagram on the parameter spaces of the Hopf algebras of type A.

The Hopf algebras $u(\mathcal{D}, \mu)$ of type A_n can be defined as follows. For more details and references to the literature we refer to our survey paper [1]. Let $n \geq 2$ and $(a_{ij})_{1\leq i,j\leq n}$ the Cartan matrix of type A_n in the form

$$a_{ij} = \begin{cases} 2, & \text{if } i = j, \\ -1, & \text{if } |i-j| = 1, \\ 0, & \text{if } |i-j| > 1. \end{cases} \tag{1.1}$$

Let Γ be a finite abelian group, $g_i \in \Gamma$ and χ_i characters of Γ for all $1 \leq i \leq n$. Define $q_{ij} = \chi_j(g_i), 1 \leq i, j \leq n$. Then

$$\mathcal{D} = \mathcal{D}(\Gamma, (g_i)_{1\leq i\leq n}, (\chi_i)_{1\leq i\leq n}, (a_{ij})_{1\leq i,j\leq n})$$

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is a *datum of Cartan type* if there is a root of unity q of order $N > 1$ in k such that

$$q_{ii} = q \text{ for all } 1 \leq i \leq n, \text{ and} \tag{1.2}$$

$$q_{ij}q_{ji} = \begin{cases} q^{-1}, & \text{if } |i-j| = 1, \\ 1, & \text{if } |i-j| > 1. \end{cases} \tag{1.3}$$

For simplicity we assume that N is *odd*. The case when N is even could be treated in the same way.

Let Φ^+ be the positive roots of the root system of type A_n, and let k^{Φ^+} be the set of all families $\mu = (\mu_{ij})_{1 \leq i < j \leq n+1}$ of scalars in k. A *family of root vector parameters* for $\mathcal{D}$ is a family $\mu \in k^{\Phi^+}$ satisfying the following two conditions.

(R1) $\mu_{ij} = 0$ for all $1 \leq i < j \leq n+1$ with $(g_i g_{i+1} \cdots g_{j-1})^N = 1$.

(R2) $\mu_{ij} = 0$ for all $1 \leq i < j \leq n+1$ with $(\chi_i \chi_{i+1} \cdots \chi_{j-1})^N \neq 1$.

In (2.9) we associate to any family $\mu \in k^{\Phi^+}$ satisfying (R2) a family $(u_{ij}(\mu))_{1 \leq i < j \leq n+1}$ of elements in the group algebra $k[\Gamma]$. If μ satisfies (R2) we can always normalize it such that μ becomes a family of root vector parameters without changing the elements $u_{ij}(\mu)$. This normalization process is discussed in Lemma 2.2. The Hopf algebra $u(\mathcal{D}, \mu)$ is generated as an algebra by the group Γ, that is, by generators of Γ satisfying the relations of the group, and $x_1, \dots, x_n$, with the relations:

(*Action of the group*)	$g x_i g^{-1} = \chi_i(g) x_i,$ for all $i,$ and all $g \in \Gamma$,
(*Serre relations*)	$\operatorname{ad}_c(x_i)^{1-a_{ij}}(x_j) = 0,$ for all $i \neq j$,
(*Root vector relations*)	$x_{ij}^N = u_{ij}(\mu),$ for all $1 \leq i < j \leq n+1$.

The coalgebra structure is given by

$$\Delta(x_i) = g_i \otimes x_i + x_i \otimes 1, \quad \Delta(g) = g \otimes g, \text{ for all } 1 \leq i \leq \theta, g \in \Gamma.$$

The Serre relations are the deformed Serre relations where

$$(\operatorname{ad}_c(x_i))(x_{j_1} \cdots x_{j_s}) = x_i x_{j_1} \cdots x_{j_s} - q_{ij_1} \cdots q_{ij_s} x_{j_1} \cdots x_{j_s} x_i, s \geq 1,$$

is the braided adjoint action. The root vectors x_{ij} are iterated braided commutators. They are defined in (2.3).

The non-trivial automorphism of the Dynkin diagram of A_n is the permutation $\sigma \in \mathbb{S}_n$ defined by $\sigma(i) = n - i + 1$ for all $1 \leq i \leq n$. For each

$\mathcal{D}$ we have an action of σ on the parameter spaces by an explicitly defined morphism of affine algebraic varieties

$$\sigma^{\mathcal{D}} : k^{\Phi^+} \to k^{\Phi^+}, \mu \mapsto (\sigma^{\mathcal{D}}_{ij}(\mu))_{1 \leq i < j \leq n+1}.$$

The polynomials $\sigma^{\mathcal{D}}_{ij}(\mu)$ of degree $j-i$ are defined in (4.4). By Theorem 2 they define an isomorphism of affine algebraic varieties between the subspaces of all elements of k^{Φ^+} satisfying (R2) for $\mathcal{D}$ resp. for $\mathcal{D}^\sigma$. Here

$$\mathcal{D}^\sigma = \mathcal{D}(\Gamma, (g_{\sigma(i)})_{1 \leq i \leq n}, (\chi_{\sigma(i)})_{1 \leq i \leq n}, (a_{ij})_{1 \leq i,j \leq n}).$$

In Corollary 4.4 we show

$$u(\mathcal{D}^\sigma, \sigma^{\mathcal{D}}(\mu)) \cong u(\mathcal{D}, \mu)$$

for all $\mu \in k^{\Phi^+}$ satisfying (R2).

Our main result is Theorem 5.1, where we compute all Hopf algebra isomorphisms between two Hopf algebras $u(\mathcal{D}', \mu')$ and $u(\mathcal{D}, \mu)$ of type A_n. The polynomials $\sigma^{\mathcal{D}}_{ij}$ play an important role in this theorem. The first essential steps in the proof of Theorem 5.1 is Theorem 3.1, where we compute the basis representation of the N-th powers of the "reverse root vectors" in the usual PBW-basis formed by the root vectors. The second essential step is Theorem 4.3, where we prove that the images of the N-th powers of the reverse root vectors in $u(\mathcal{D}, \mu)$ are the elements $u^{\mathcal{D}^\sigma}_{ij}(\sigma^{\mathcal{D}}(\mu))$.

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2. Finite Hopf algebras of type A_n

2.1. Diagrams of type A_n, root vectors, and reverse root vectors

Let $n \geq 2$ and $(a_{ij})_{1 \leq i,j \leq n}$ the Cartan matrix of type A_n in the form (1.1). Let $\mathbb{Z}[I]$ be the free abelian group with basis $\alpha_1, \ldots, \alpha_n$. The Weyl group W of (a_{ij}) is the subgroup of $\mathrm{Aut}(\mathbb{Z}[I])$ generated by the simple reflections $s_1, \ldots, s_n$ defined by $s_i(\alpha_j) = \alpha_j - a_{ij}\alpha_i$ for all $1 \leq i, j \leq n$. The root system Φ of (a_{ij}) is defined by $\Phi = \cup_{i=1}^n W(\alpha_i)$. It has the basis $\alpha_1, \ldots, \alpha_n$, and the positive roots with respect to this basis are the elements

$$\alpha_{ij} = \sum_{l=i}^{j-1} \alpha_l, 1 \leq i < j \leq n+1.$$

Let w_0 be the longest element in W. We choose the reduced representation

$$w_0 = s_1 s_2 \cdots s_n s_1 s_2 \cdots s_{n-1} s_1 s_2 \cdots s_{n-2} \cdots s_1$$

of w_0 of length $p = \frac{n(n+1)}{2}$. The corresponding convex ordering of the positive roots

$$\beta_l = s_{i_1} \cdots s_{i_{l-1}}(\alpha_{i_l}), 1 \leq l \leq p,$$

is the lexicographic ordering, that is,

$$\alpha_{12} < \alpha_{13} < \cdots < \alpha_{1,n+1} < \alpha_{23} < \cdots \alpha_{2,n+1} < \cdots < \alpha_{n,n+1}.$$

Let Γ be a finite abelian group,

$$\mathcal{D} = \mathcal{D}(\Gamma, (g_i)_{1\leq i\leq n}, (\chi_i)_{1\leq i\leq n}, (a_{ij})_{1\leq i,j\leq n})$$

a Cartan datum and define $q_{ij} = \chi_j(g_i), 1 \leq i, j \leq n$. Then the Cartan condition $q_{ij}q_{ji} = q_{ii}^{a_{ij}}, 1 \leq i, j \leq n$, is equivalent to the following: There is a root of unity q of order $N > 1$ in k such that (1.2) and (1.3) hold.

For simplicity we assume that N is *odd*; as we said, this is not essential and the case when N is even could be treated in the same way.

Let $V \in {}^\Gamma_\Gamma\mathcal{YD}$ with basis $x_i \in V_{g_i}^{\chi_i}$, $1 \leq i \leq n$; that is $g \cdot x_i = \chi_i(g)x_i$ for all $g \in \Gamma$, and $\delta(x_i) = g_i \otimes x_i$. Then

$$R = R(\mathcal{D}) = k\langle x_1, \dots, x_n \mid \mathrm{ad}_c(x_i)^{1-a_{ij}}(x_j) = 0, \forall 1 \leq i, j \leq n, i \neq j\rangle$$

is a Hopf algebra in the braided category ${}^\Gamma_\Gamma\mathcal{YD}$. For $x, y \in R$ we define the braided commutator

$$[x, y]_c = xy - \mu c(x \otimes y),$$

where c denotes the braiding and μ the multiplication map of R. As in [1, (6-7) and (6-8)] we define root vectors x_{ij}, $1 \leq i < j \leq n+1$, in R inductively by

$$x_{i,i+1} = x_i \text{ for all } 1 \leq i \leq n, \tag{2.1}$$

$$x_{ij} = [x_{i,i+1}, x_{i+1,j}]_c \text{ for all } 1 \leq i < j \leq n+1, j - i > 1. \tag{2.2}$$

Then

$$x_{ij} = [x_{il}, x_{lj}]_c \text{ for all } 1 \leq i < l < j \leq n+1, \tag{2.3}$$

and the root vectors x_{ij} in the lexicographic order define a PBW-basis of R [1, Theorem (6.13)].

In addition we define inductively reverse root vectors x_{ji}, $1 \leq i < j \leq n+1$, in R by

$$x_{i+1,i} = x_i \text{ for all } 1 \leq i \leq n, \tag{2.4}$$

$$x_{ji} = [x_{j,j-1}, x_{j-1,i}]_c \text{ for all } 1 \leq i < j \leq n+1, j - i > 1. \tag{2.5}$$

Again it follows that

$$x_{ji} = [x_{jl}, x_{li}]_c \text{ for all } 1 \leq i < l < j \leq n+1. \tag{2.6}$$

Thus for all $1 \leq i < j \leq n+1$, x_{ij} is any bracketing of the elements $x_i, x_{i+1}, \dots, x_{j-1}$ in this order, and x_{ji} is any bracketing of the reverse sequence $x_{j-1}, x_{j-2}, \dots, x_i$.

2.2. Root vector parameters and normalization

For any positive root we define elements in the group and characters of the group by

$$g_{ij} = \prod_{i \leq l < j} g_l, \chi_{ij} = \prod_{i \leq l < j} \chi_l, 1 \leq i < j \leq n+1. \tag{2.7}$$

A *family of root vector parameters* for $\mathcal{D}$ is a family $\mu = (\mu_{ij})_{1 \leq i < j \leq n+1}$ of scalars $\mu_{ij} \in k$ satisfying the following two conditions.

(R1) $\mu_{ij} = 0$ for all $1 \leq i < j \leq n+1$ with $g_{ij}^N = 1$.

(R2) $\mu_{ij} = 0$ for all $1 \leq i < j \leq n+1$ with $\chi_{ij}^N \neq 1$.

For all $1 \leq i < j \leq n+1$ let

$$I_{ij} = \{(i_1, \dots, i_r) \mid r \geq 2, i = i_1 < i_2 < \dots < i_r = j\}.$$

We denote the set of all families $\mu = (\mu_{ij})_{1 \leq i < j \leq n+1}$ of elements in k by k^{Φ^+}.

For any $\mu \in k^{\Phi^+}$ we define for all $1 \leq i < j \leq n+1$ scalars

$$\mu(i_1, \dots, i_r) = \mu_{i_1 i_2} \cdots \mu_{i_{r-1} i_r} \text{ for all } (i_1, \dots, i_r) \in I_{ij}, \tag{2.8}$$

and elements in the group algebra

$$u_{ij}^{\mathcal{D}}(\mu) = \sum_{(i_1, \dots, i_r) \in I_{ij}} (q-1)^{N(r-1)} \mu(i_1, \dots, i_r)(1 - g_{i_{r-1} i_r}^N). \tag{2.9}$$

Thus

$$\begin{aligned} u_{ij}^{\mathcal{D}}(\mu) = \mu_{ij}(1 - g_{ij}^N) \\ + \sum_{i<p<j} \Big(\sum_{(i_1, \dots, i_r) \in I_{ip}} (q-1)^{N(r-1)} \mu(i_1, \dots, i_r) \Big) \mu_{pj}(1 - g_{pj}^N). \end{aligned}$$

We will write $u_{ij}(\mu) = u_{ij}^{\mathcal{D}}(\mu)$ when $\mathcal{D}$ is fixed. Recall that $q = \chi_i(g_i)$ for all $1 \leq i \leq n$ also depends on $\mathcal{D}$. It is easy to see that the family

$(u_{ij}(\mu))_{1\le i<j\le n+1}$ of elements in the group algebra can inductively be defined by

$$u_{ij}(\mu) = \mu_{ij}(1-g_{ij}^N) + \sum_{i<p<j} (q-1)^N \mu_{ip} u_{pj}(\mu), 1 \le i < j \le n+1. \tag{2.10}$$

This definition agrees with the inductive definition in [1, Theorem 6.18] when μ is a family of root vector parameters for $\mathcal{D}$. There we defined

$$\begin{aligned} C_{kl}^j &= (1-q^{-1})^N \chi_{kl}(g_{lj})^{\frac{N(N-1)}{2}}, i \le k < l \le j, \\ u_{ij}(\mu) &= \mu_{ij}(1-g_{ij}^N) + \sum_{i<p<j} C_{ip}^j \, \mu_{ip} u_{pj}(\mu), 1 \le i < j \le n+1. \end{aligned}$$

Since N is odd it follows from (R2) that $C_{kl}^j \mu_{kl} = (q-1)^N \mu_{kl}$ for all $i \le k < l \le j$. Thus both definitions do agree.

Lemma 2.1. *Let $\mu \in k^{\Phi^+}$.*

1. *Suppose μ satisfies* (R2). *Then*

$$u_{ij}(\mu) = 0 \textit{ for all } 1 \le i < j \le n+1 \textit{ with } \chi_{ij}^N \ne 1. \tag{R3}$$

2. *Suppose μ satisfies* (R1) *and* (R3). *Then μ satisfies* (R2)*, that is, μ is a family of root vector parameters for $\mathcal{D}$.*

PROOF: This follows by induction on $j-i$ from (2.10) since for all $i<p<j$ the inequality $\chi_{ij}^N \ne 1$ implies that $\chi_{ip}^N \ne 1$ or $\chi_{pj}^N \ne 1$. $\square$

By [1, Theorem 6.18] the families $(u_{ij}(\mu))_{1\le i<j\le n+1}$ are exactly the solutions of the equations

$$\Delta(u_{ij}) = u_{ij} \otimes 1 + g_{ij}^N \otimes u_{ij} + \sum_{i<p<j} (q-1)^N u_{ip} g_{pj}^N \otimes u_{pj} \tag{2.11}$$

in $k[\Gamma] \otimes k[\Gamma]$ for all $1 \le i < j \le n+1$. This characterization of the $u_{ij}(\mu)$ is used to prove the next lemma. It shows how to "normalize" an arbitrary sequence μ so that (R1) is satisfied.

Lemma 2.2. *Let $\mu \in k^{\Phi^+}$. Then there is exactly one family $\mu' \in k^{\Phi^+}$ satisfying* (R1) *such that*

$$u_{ij}(\mu) = u_{ij}(\mu') \textit{ for all } 1 \le i < j \le n+1. \tag{2.12}$$

If μ satisfies (R2) *then μ' is a family of root vector parameters for $\mathcal{D}$.*

PROOF: Let $u_{ij} = u_{ij}(\mu)$ for all $1 \leq i < j \leq n+1$. We define the elements μ'_{ij} by induction on $j-i$. Let $j = i+1$. Let

$$\mu'_{i,i+1} = \begin{cases} \mu_{i,i+1}, & \text{if } g^N_{i,i+1} \neq 1, \\ 0, & \text{if } g^N_{i,i+1} = 1. \end{cases}$$

Then (2.12) for $(i, i+1)$ holds since $u_{i,i+1}(\mu) = \mu_{i,i+1}(1 - g^N_{i,i+1})$.

Let $k > 1$. Suppose we have already defined μ'_{ij} whenever $j - i \leq k-1$ such that (2.12) holds if $j - i \leq k-1$. Let $1 \leq i < j \leq n+1$ and assume that $j - i = k$. If $g^N_{ij} = 1$, then we define $\mu'_{ij} = 0$. If $g^N_{ij} \neq 1$, we define $\mu'_{ij} \in k$ to be the unique scalar satisfying

$$u_{ij} = \mu'_{ij}(1 - g^N_{ij}) + \sum_{i<p<j} (q-1)^N \mu'_{ip} u_{pj}. \tag{2.13}$$

The existence of the scalar μ'_{ij} follows from the argument in the induction step of the proof of [1, Theorem 6.18].

Thus we have shown the existence of the family μ'. Uniqueness follows easily by induction on $j - i$ from (2.10).

Suppose that μ satisfies (R2). Then μ satisfies (R3) by Lemma 2.1 (1), and μ' satisfies (R2) by Lemma 2.1 (2). $\square$

For any $\mu \in k^{\Phi^+}$ we define

$$\nu^{\mathcal{D}}(\mu) = \mu', \quad \nu^{\mathcal{D}}_{ij}(\mu) = \mu'_{ij} \text{ for all } 1 \leq i < j \leq +1, \tag{2.14}$$

where μ' is the family constructed from μ in Lemma 2.2. We call $\nu^{\mathcal{D}}(\mu)$ the *normalization* of μ.

The elements $\mu'_{ij} = \nu^{\mathcal{D}}_{ij}(\mu)$ can be computed inductively. Let $1 \leq i < j \leq n+1$ and assume that $g^N_{ij} \neq 1$. Then we have $u_{ij}(\mu) = u_{ij}(\mu')$. We replace $u_{ij}(\mu)$ and $u_{ij}(\mu')$ by the right hand sides of (2.9) and collect all terms with coefficient $(1 - g^N_{ij})$. This gives the equality

$$\begin{aligned} \mu_{ij} &+ \sum_{\substack{i<p<j \\ g^N_{ip}=1}} \sum_{(i_1,\dots,i_r)\in I_{ip}} (q-1)^{N(r-1)} \mu(i_1,\dots,i_r)\mu_{pj} \\ &= \mu'_{ij} + \sum_{\substack{i<p<j \\ g^N_{ip}=1}} \sum_{(i_1,\dots,i_r)\in I_{ip}} (q-1)^{N(r-1)} \mu'(i_1,\dots,i_r)\mu'_{pj}, \end{aligned} \tag{2.15}$$

where $\mu'(i_1, \dots, i_r)$ is defined in (2.8) for μ'. Thus μ'_{ij} is a function of μ and $\mu'_{ab}, b - a < j - i$. In particular, we see that $\nu^{\mathcal{D}}_{ij}$ is a polynomial in the variables (μ_{ij}) with coefficients in $\mathbb{Z}[q]$. The polynomial $\nu_{ij}^{\mathcal{D}}$ depends on $\mathcal{D}$, more precisely on $q = \chi_i(g_i)$ and on the numbers

$$d_{ab} = \begin{cases} 1, & \text{if } g_{ab}^N \neq 1, \\ 0, & \text{if } g_{ab}^N = 1. \end{cases}, \qquad b - a \leq j - i.$$

For example assume that $g_{i,i+2}^N \neq 1$. Then

$$\mu'_{i,i+2} = \begin{cases} \mu_{i,i+2}, & \text{if } g_i^N \neq 1, \\ \mu_{i,i+2} + \mu(i, i+1, i+2), & \text{if } g_i^N = 1. \end{cases}$$

2.3. The Hopf algebras $u(\mathcal{D}, \mu)$ and isomorphisms

As in [1, 2] we define for any family of root vector parameters μ for $\mathcal{D}$ a finite-dimensional Hopf algebra by

$$u(\mathcal{D}, \mu) = R(\mathcal{D}) \# k[\Gamma] / (x_{ij}^N - u_{ij}(\mu) \mid 1 \leq i < j \leq n+1). \tag{2.16}$$

We can extend this definition to all $\mu \in k^{\Phi^+}$ satisfying (R2) since by Lemma 2.2 then $\nu^{\mathcal{D}}(\mu)$ is a family of root vector parameters, and $u_{ij}(\mu) = u_{ij}(\nu^{\mathcal{D}}(\mu))$ for all $1 \leq i < j \leq n+1$. By [1, Theorem 6.25] a finite-dimensional pointed Hopf algebra A is of the form $A \cong u(\mathcal{D}, \mu)$ if and only if $\mathrm{gr}(A) \cong u(\mathcal{D}, 0) \# k[\Gamma]$, where $\mathrm{gr}(A)$ is the graded Hopf algebra associated to the coradical filtration of A.

Let $\rho \in \mathbb{S}_n$ be a *diagram automorphism* of (a_{ij}), that is,

$$a_{ij} = a_{\rho(i)\rho(j)} \text{ for all } 1 \leq i, j \leq n.$$

Then $\rho = \mathrm{id}$ or $\rho = \sigma$, where

$$\sigma(i) = n - i + 1 \text{ for all } 1 \leq i \leq n. \tag{2.17}$$

As in [2, Theorem 7.5] let

$$\mathcal{D}^\rho = \mathcal{D}(\Gamma, (g_i^\rho)_{1 \leq i \leq n}, (\chi_i^\rho)_{1 \leq i \leq n}, (a_{ij})_{1 \leq i,j \leq n})$$

be the Cartan datum with $g_i^\rho = g_{\rho(i)}, \chi_i^\rho = \chi_{\rho(i)}$ for all $1 \leq i \leq n$. Let $V^\rho \in {}_\Gamma^\Gamma\mathcal{YD}$ with basis $x_i^\rho \in (V^\rho)_{g_{\rho(i)}}^{\chi_{\rho(i)}}$ for all $1 \leq i \leq n$. Then

$$F^\rho : R(\mathcal{D}^\rho) \to R(\mathcal{D}), \quad x_i^\rho \mapsto x_{\rho(i)} \text{ for all } 1 \leq i \leq n,$$

defines an isomorphism of braided Hopf algebras in ${}_\Gamma^\Gamma\mathcal{YD}$. For all $1 \leq i < j \leq n+1$, we denote the root vector of α_{ij} in $R(\mathcal{D}^\rho)$ by x_{ij}^ρ.

Let $\mathcal{D}' = \mathcal{D}(\Gamma', (g_i')_{1\le i\le n}, (\chi_i')_{1\le i\le n}, (a_{ij})_{1\le i,j\le n})$ be another Cartan datum with finite abelian group Γ' and the same Cartan matrix of type A_n as $\mathcal{D}$. Let $\varphi : \Gamma' \to \Gamma$ be a group isomorphism, $\rho \in \mathbb{S}_n$ a diagram automorphism of (a_{ij}) and $s = (s_i)_{1\le i\le n}$ a family of non-zero elements in k. Let

$$s_{ij} = \prod_{i\le l<j} s_l \text{ for all } 1 \le i < j \le n+1. \tag{2.18}$$

Let $\pi : R(\mathcal{D})\#k[\Gamma] \to u(\mathcal{D}, \mu)$ be the canonical projection.

The triple $(\varphi, \rho, (s_i))$ is called an *isomorphism* from $(\mathcal{D}', \mu')$ to $(\mathcal{D}, \mu)$ if the following conditions are satisfied:

$$\varphi(g_i') = g_{\rho(i)}, \chi_i' = \chi_{\rho(i)}\varphi \text{ for all } 1 \le i \le n. \tag{2.19}$$

$$\varphi(u_{ij}^{\mathcal{D}'}(\mu')) = s_{ij}^N \pi(F^\rho(x_{ij}^\rho)^N) \text{ for all } 1 \le i < j \le n+1. \tag{2.20}$$

Let $\operatorname{Isom}(\mathcal{D}', \mu'), (\mathcal{D}, \mu)$ be the set of all isomorphisms from $(\mathcal{D}', \mu')$ to $(\mathcal{D}, \mu)$. For Hopf algebras A', A we denote by $\operatorname{Isom}(A', A)$ the set of all Hopf algebra isomorphisms from A' to A. Then

Theorem 2.3. *[2, Theorem 7.2] The map*

$$\operatorname{Isom}((\mathcal{D}', \mu'), (\mathcal{D}, \mu)) \to \operatorname{Isom}(u(\mathcal{D}', \mu'), u(\mathcal{D}, \mu))$$

given by $(\varphi, \rho, (s_i)) \mapsto F$*, where* $F(x_i') = s_i x_{\rho(i)}$ *and* $F(g') = \varphi(g')$ *for all* $1 \le i \le \theta$ *and* $g' \in \Gamma'$*, is bijective.*

The main result in this paper is the explicit computation of the set $\operatorname{Isom}((\mathcal{D}', \mu'), (\mathcal{D}, \mu))$. In Section 3 we first compute the elements $F^\sigma(x_{ij}^\sigma)^N$ in terms suitable for our purpose. The next lemma shows that these elements are reverse root vectors. This lemma also allows to derive (2.6) from (2.3).

Lemma 2.4. *For all* $1 \le i < j \le n+1$*,*

$$F^\sigma(x_{ij}^\sigma) = x_{n-i+2,n-j+2}.$$

Proof: This follows by induction on $j - i$. Suppose that $j = i+1$. Then $x_{ij} = x_i$ and $F^\sigma(x_{ij}^\sigma) = x_{\sigma(i)} = x_{n-i+1} = x_{n-i+2,n-j+2}$.

If $j - i \ge 2$, then

$$\begin{aligned} F^\sigma(x_{ij}^\sigma) &= F^\sigma([x_i^\sigma, x_{i+1,j}^\sigma]_{c^\sigma}) \\ &= [x_{\sigma(i)}, F^\sigma(x_{i+1,j}^\sigma)]_c \\ &= [x_{n-i+1}, x_{n-i+1,n-j+2}]_c && \text{(by induction)} \\ &= x_{n-i+2,n-j+2} && \text{(by (2.5))}. \end{aligned}$$

□

3. The reverse root vectors

In the next theorem we compute the basis representation of the N-th powers of the reverse root vectors in the standard PBW-basis.

As in the last section we fix a diagram $\mathcal{D}$ of Cartan type A_n and let $R = R(\mathcal{D})$. For all $1 \le i < j \le n+1$ we define

$$\tau_{ij} = \prod_{i \le k < l < j} q_{lk}^N. \tag{3.1}$$

$$\tau(i_1, \dots, i_r) = \tau_{i_1 i_2} \tau_{i_2 i_3} \cdots \tau_{i_{r-1} i_r}, \text{ for all } (i_1, \dots, i_r) \in I_{ij}. \tag{3.2}$$

Note that $\tau_{ij} = \prod_{i<l<j} \chi_{il}^N(g_l)$. We write $\tau_{ij}^{\mathcal{D}}$ instead of τ_{ij} if we want to emphasize the datum $\mathcal{D}$.

Theorem 3.1. *Assume that $1 \le i < j \le n+1$. For all $(i_1, \dots, i_r) \in I_{ij}$ define*

$$t(i_1, \dots, i_r) = (-1)^{j-i-r+1} (q-1)^{N(r-2)} \tau(i_1, \dots, i_r)^{-\frac{N-1}{2}} \tau_{ij}^{\frac{N+1}{2}}.$$

Then

$$x_{ji}^N = \sum_{(i_1, \dots, i_r) \in I_{ij}} t(i_1, \dots, i_r) x_{i_1 i_2}^N x_{i_2 i_3}^N \cdots x_{i_{r-1} i_r}^N. \tag{3.3}$$

The proof of Theorem 3.1 will be done after Lemma 3.8.

To compute the coefficients $t(i_1, \dots, i_r)$ we first change the notation using characteristic functions. We can assume that $j - 2 \ge i$ since $x_{i+1,i} = x_i$. For natural numbers $k < l$ let $[k, l] = \{k, k+1, \dots, l\}$.

Let E_{ij} be the set of all functions $e : [i, j-2] \to \mathbb{N}$ with values in $\{0, 1\}$. We consider the bijection

$$\Omega : I_{ij} \to E_{ij}$$

given for all $(i_1, \dots, i_r) \in I_{ij}$ and $l \in [i, j-2]$ by

$$\Omega(i_1, \dots, i_r)(l) = \begin{cases} 1, & \text{if } l \in \{i_2 - 1, \dots, i_{r-1} - 1\}, \\ 0, & \text{otherwise.} \end{cases}$$

For any $e \in E_{ij}$ define

$$|e| = \#\{l \mid i \le l \le j-2, e(l) = 0\}.$$

If $e = \Omega(i_1, \dots, i_r)$ then

$$|e| = j - i - r + 1. \tag{3.4}$$

The constant function in E_{ij} with value 1 resp. 0 will be denoted by (1) resp. (0). Thus $(1) = \Omega(i, i+1, \dots, j)$ and $(0) = \Omega(i, j)$.

For $e, f \in E_{ij}$ we write $e \le f$ if for all $i \le l \le j-2, e(l) = 0$ implies $f(l) = 0$.

Lemma 3.2. *Let $1 \le i < j \le n+1, j-i \ge 2$, and $f \in E_{ij}, (1) \ne f$. Then*

$$\sum_{e \in E_{ij}, e \le f} (-1)^{|e|} = 0, \tag{3.5}$$

$$\sum_{(i_1,\dots,i_r) \in I_{ij}} (-1)^r = 0. \tag{3.6}$$

Proof: Since $f \ne (1)$ we can choose an index l with $f(l) = 0$. Then $\{e \in E_{ij} \mid e \le f\}$ is the disjoint union of elements e with $e(l) = 0$ and with $e(l) = 1$, and (3.5) is obvious. To prove (3.6) we consider the case of (3.5) with $f = (0)$. Then $\sum_{e \in E_{ij}} (-1)^{|e|} = 0$, and (3.6) follows from the bijection Ω and (3.4). □

For all $e = \Omega(i_1, \dots, i_r) \in E_{ij}$ let

$$\begin{aligned}
\tau_e &= (q-1)^{N|e|} (\tau_{i_1 i_2} \tau_{i_2 i_3} \cdots \tau_{i_{r-1} i_r})^{\frac{N-1}{2}}, \\
t_e &= (-1)^{|e|} \tau_e^{-1} (q-1)^{N(j-i-1)} \tau_{ij}^{\frac{N+1}{2}}, \\
x_e^N &= x_{i_1 i_2}^N x_{i_2 i_3}^N \cdots x_{i_{r-1} i_r}^N.
\end{aligned}$$

Note that $t_e = t(i_1, \dots, i_r)$ if $e = \Omega(i_1, \dots, i_r)$. This follows from the definitions using (3.4). Hence (3.3) in Theorem 3.1 can be restated as

$$x_{ji}^N = \sum_{e \in E_{ij}} t_e x_e^N. \tag{3.7}$$

The idea of the proof of Theorem 3.1 is to project R onto skew-polynomial rings R_e, one for each $e \in E_{ij}$. Before we begin the proof we establish some technical results on these projections.

Definition 3.3. For any $e \in E_{ij}$, let R_e be the algebra generated by x_i, x_{i+1}, …, x_{j-1} with relations

$$x_l x_{l+1} - q_{l,l+1} x_{l+1} x_l = 0, \text{ if } e(l) = 1, i \le l \le j-2, \tag{3.8}$$

$$x_{l+1} x_l - q_{l+1,l} x_l x_{l+1} = 0, \text{ if } e(l) = 0, i \le l \le j-2, \tag{3.9}$$

$$x_k x_l - q_{kl} x_l x_k = 0, \text{ if } i \le k, l \le j-1, |k-l| \ge 2. \tag{3.10}$$

Lemma 3.4. *For any $f \in E_{ij}$, the natural projection*

$$\pi_f : R \to R_f, \pi_f(x_l) = \begin{cases} x_l, & \text{if } i \le l \le j-1, \\ 0, & \text{otherwise}, \end{cases}, 1 \le l \le n,$$

is a well-defined algebra map, and for all $i \leq u < v \leq j, v-u \geq 2$,

$$\pi_f(x_{uv}) = 0, \text{ if } f(l) = 1 \text{ for some } u \leq l < l+2 \leq v, \tag{3.11}$$

$$\pi_f(x_{vu}) = 0, \text{ if } f(l) = 0 \text{ for some } u \leq l < l+2 \leq v. \tag{3.12}$$

PROOF: The Serre relations can be reformulated according to the following identities

$$-q_{l+1,l}\mathrm{ad}_c(x_l)^2(x_{l+1}) = x_l[x_{l+1}, x_l]_c - q_{l,l+1}[x_{l+1}, x_l]_c x_l, \tag{3.13}$$

$$-q_{l,l+1}\mathrm{ad}_c(x_{l+1})^2(x_l) = x_{l+1}[x_l, x_{l+1}]_c - q_{l+1,l}[x_l, x_{l+1}]_c x_{l+1}, \tag{3.14}$$

in the free algebra $k\langle x_1, \dots, x_n\rangle$ for all $1 \leq l \leq n-1$. Hence both Serre relations $\mathrm{ad}_c(x_l)^2(x_{l+1}) = 0$ and $\mathrm{ad}_c(x_{l+1})^2(x_l) = 0$ hold in R_f for all $i \leq l < j-1$, since $[x_l, x_{l+1}]_c = 0$ by (3.8) if $f(l) = 1$, and $[x_{l+1}, x_l]_c = 0$ by (3.9) if $f(l) = 0$. Thus π_f is well-defined.

To prove (3.11) note that by (2.3)

$$x_{uv} = \begin{cases} [x_{ul}, x_{l,l+2}]_c, & \text{if } u < l < l+2 = v, \\ [x_{ul}, [x_{l,l+2}, x_{l+2,v}]_c]_c, & \text{if } u < l < l+2 < v, \end{cases}$$

and that $\pi_f(x_{l,l+2}) = 0$ by definition of π_f. In the same way (3.12) follows from (2.6). □

We note the following obvious rule in skew polynomial rings.

Lemma 3.5. *Let* $x_1, \dots, x_m$ *be elements in an algebra such that*

$$x_l x_k = p_{lk} x_k x_l \text{ for all } k < l,$$

where $p_{kl} \in k$ *for all* $k < l$. *Then for any natural number* N,

$$(x_1 \cdots x_m)^N = \prod_{k<l} p_{lk}^{\frac{N(N-1)}{2}} x_1^N \cdots x_m^N.$$

□

Lemma 3.6. *Let* $f \in E_{ij}$ *and* $i \leq k < l \leq j$ *with* $k \leq l-2$. *Suppose that* $f(k) = f(k+1) = \cdots = f(l-2) = 0$. *Then*

$$\pi_f(x_{kl}^N) = (q-1)^{N(l-k-1)} \tau_{kl}^{\frac{N-1}{2}} x_k^N x_{k+1}^N \cdots x_{l-1}^N.$$

Proof: We first prove by induction on $l-k$ that

$$\pi_f(x_{kl}) = (1-q^{-1})^{l-k-1} x_k x_{k+1} \cdots x_{l-1}. \tag{3.15}$$

Suppose that $l = k+2$. Then

$$\begin{aligned} \pi_f(x_{k,k+2}) &= x_k x_{k+1} - q_{k,k+1} x_{k+1} x_k \\ &= x_k x_{k+1} - q_{k,k+1} q_{k+1,k} x_k x_{k+1} && \text{(by (3.9) and } f(k)=0) \\ &= (1-q^{-1}) x_k x_{k+1} && \text{(by (1.3)).} \end{aligned}$$

This proves (3.15) for $l-k=2$.

For the induction step let $l-k>2$. We obtain by induction

$$\begin{aligned} \pi_f(x_{kl}) &= [x_k, x_{k+1,l}]_c \\ &= [x_k, (1-q^{-1})^{l-k-2} x_{k+1} x_{k+2} \cdots x_{l-1}]_c \\ &= (1-q^{-1})^{l-k-2} (x_k x_{k+1} \cdots x_{l-1} \\ &\qquad - q_{k,k+1} q_{k,k+2} \cdots q_{k,l-1} x_{k+1} x_{k+2} \cdots x_{l-1} x_k). \end{aligned}$$

Since $x_{k+1} x_{k+2} \cdots x_{l-1} x_k = q_{l-1,k} q_{l-2,k} \cdots q_{k+1,k} x_k x_{k+1} \cdots x_{l-1}$ in R_f by $f(k)=0$, (3.9) and (3.10), and

$$q_{k,k+1} q_{k,k+2} \cdots q_{k,l-1} q_{l-1,k} q_{l-2,k} \cdots q_{k+1,k} = q^{-1}$$

by (1.3), equation (3.15) for k,l follows.

Since $f(k) = f(k+1) = \cdots = f(l-2) = 0$, (3.9) and (3.10) imply by Lemma 3.5 that

$$(x_k x_{k+1} \cdots x_{l-1})^N = \prod_{k \le \mu < \nu < l} q_{\nu,\mu}^{\frac{N(N-1)}{2}} x_k^N x_{k+1}^N \cdots x_{l-1}^N.$$

Hence Lemma 3.6 follows from (3.15) by taking N-th powers. Note that $(1-q^{-1})^N = (q-1)^N$ since $q^N = 1$. $\square$

Lemma 3.7. *Let $e, f \in E_{ij}$. Then*

$$\pi_f(x_e^N) = \begin{cases} \tau_e\, x_i^N x_{i+1}^N \cdots x_{j-1}^N, & \text{if } e \le f, \\ 0, & \text{otherwise.} \end{cases}$$

Proof: Let $e = \Omega(i_1, \ldots, i_r)$. Suppose that $e \not\le f$, that is $f(l) = 1$ and $e(l) = 0$ for some $i \le l \le j-2$. Then $l+1 \notin \{i_1, \ldots, i_r\}$, since $e(l) = 0$. Hence there is an index s with $i_s \le l < l+2 \le i_{s+1}$. Since $f(l) = 1$, it follows by (3.11) that $\pi_f(x_{i_s, i_{s+1}}) = 0$, and thus $\pi_f(x_e^N) = 0$.

Now assume $e \le f$. Since π_f is an algebra map, it is enough to show for all $1 \le s < r$ that

$$\pi_f(x^N_{i_s,i_{s+1}}) = (q-1)^{N(i_{s+1}-i_s-1)}\,\tau^{\frac{N-1}{2}}_{i_s,i_{s+1}}\,x^N_{i_s}x^N_{i_s+1}\cdots x^N_{i_{s+1}-1}. \tag{3.16}$$

Note that by (3.4) $\sum_{s=1}^{r-1}(i_{s+1}-i_s-1) = j-i-r+1 = |e|$.

If $i_s+1 = i_{s+1}$, then $x_{i_s,i_{s+1}} = x_{i_s}$ and (3.16) is obvious.

If $i_s \le i_{s+1}-2$, then $e(l)=0$ for all $i_s \le l \le i_{s+1}-2$ by definition of the function Ω. Hence $f(l)=0$ for all $i_s \le l \le i_{s+1}-2$ since $e \le f$, and (3.16) follows from Lemma 3.6. □

Lemma 3.8.

$$\pi_{(1)}(x^N_{ji}) = t_{(1)}x^N_i x^N_{i+1}\cdots x^N_{j-1}.$$

PROOF: We first prove by induction on $j-i$ that

$$\pi_{(1)}(x_{ji}) = (q-1)^{j-i-1}\left(\prod_{i\le k<l<j} q_{lk}\right) x_i x_{i+1}\cdots x_{j-1}. \tag{3.17}$$

Suppose that $j = i+2$. Then

$$x_{i+1}x_i = q^{-1}_{i,i+1}x_i x_{i+1} \text{ in } R_{(1)}. \tag{3.18}$$

Hence

$$\begin{aligned}\pi_{(1)}(x_{i+2,i}) &= x_{i+1}x_i - q_{i+1,i}x_i x_{i+1}\\ &= (q^{-1}_{i,i+1} - q_{i+1,i})x_i x_{i+1}\\ &= (q-1)q_{i+1,i}x_i x_{i+1} \qquad \text{(since } q_{i,i+1}q_{i+1,i} = q^{-1} \text{ by (1.3)).}\end{aligned}$$

For the induction step let $j-i>2$. Then by induction

$$\begin{aligned}\pi_{(1)}(x_{ji}) &= [x_{j-1}, x_{j-1,i}]_c\\ &= [x_{j-1}, (q-1)^{j-i-2}\Big(\prod_{i\le k<l<j-1} q_{lk}\Big) x_i x_{i+1}\cdots x_{j-2}]_c\\ &= (q-1)^{j-i-2}\Big(\prod_{i\le k<l<j-1} q_{lk}\Big)(x_{j-1}x_i x_{i+1}\cdots x_{j-2}\\ &\qquad - q_{j-1,i}q_{j-1,i+1}\cdots q_{j-1,j-2}x_i x_{i+1}\cdots x_{j-2}x_{j-1}).\end{aligned}$$

Since in $R_{(1)}$

$$\begin{aligned}x_{j-1}x_{j-2} &= q^{-1}_{j-2,j-1}x_{j-2}x_{j-1}\\ &= q_{j-1,j-2}q\,x_{j-2}x_{j-1},\end{aligned} \tag{3.19}$$

and hence

$$x_{j-1}x_ix_{i+1}\cdots x_{j-2} = q_{j-1,i}q_{j-1,i+1}\cdots q_{j-1,j-2}qx_ix_{i+1}\cdots x_{j-1},$$

it follows that

$$\begin{aligned}\pi_{(1)}(x_{ji}) &= (q-1)^{j-i-2}\prod_{i\le k<l<j-1} q_{lk}\prod_{k=i}^{j-2} q_{j-1,k}(q-1)x_ix_{i+1}\cdots x_{j-1}\\ &= (q-1)^{j-i-1}\prod_{i\le k<l<j} q_{lk}x_ix_{i+1}\cdots x_{j-1}.\end{aligned}$$

This finishes the proof of (3.17).

By (3.8), (3.10) and Lemma 3.5

$$(x_ix_{i+1}\cdots x_{j-1})^N = q^{\frac{N(N-1)}{2}(j-i-1)}\Big(\prod_{i\le k<l<j} q_{lk}^{\frac{N(N-1)}{2}}\Big)x_i^Nx_{i+1}^N\cdots x_{j-1}^N.$$

Hence Lemma 3.8 follows from (3.17) by taking N-th powers. Note that $q^{\frac{N(N-1)}{2}} = 1$ since N is odd by assumption. □

We now prove Theorem 3.1.

Proof: Since the root vectors x_{ij} in the lexicographic order define a PBW-basis of R, there are uniquely determined coefficients $\widetilde{t}_e \in k, e \in E_{ij}$, with

$$x_{ji}^N = \sum_{e\in E_{ij}} \widetilde{t}_e x_e^N, \tag{3.20}$$

cf. [2, Th. 2.6 (2)]– and compare with [1, Lemma 6.9]. By (3.7) we have to show that $\widetilde{t}_e = t_e$ for all $e \in E_{ij}$.

To prove that $\widetilde{t}_{(1)} = t_{(1)}$, we apply $\pi_{(1)}$ to both sides of (3.20). For all $(1) \neq e \in E_{ij}$ we see from Lemma 3.7 that $\pi_{(1)}(x_e^N) = 0$, since $e \not\leq (1)$. Hence $\pi_{(1)}(\sum_{e\in E_{ij}} \widetilde{t}_e x_e^N) = \widetilde{t}_{(1)}x_i^Nx_{i+1}^N\cdots x_{j-i}^N$, and $\widetilde{t}_{(1)} = t_{(1)}$ by Lemma 3.8.

Let $(1) \neq f \in E_{ij}$. Then $f(l) = 0$ for some $i \le l \le j-2$ and $\pi_f(x_{ji}) = 0$ by (3.12). Hence applying π_f to both sides of (3.20) and using Lemma 3.7 we obtain $0 = \sum_{e\le f} \widetilde{t}_e\tau_e x_i^Nx_{i+1}^N\cdots x_{j-1}^N$, hence

$$\sum_{e\le f}\widetilde{t}_e\tau_e = 0. \tag{3.21}$$

Note that by definition

$$t_f = (-1)^{|f|}\tau_f^{-1}t_{(1)} \text{ for all } f \in E_{ij}.$$

To finish the proof of the theorem we therefore show by induction on $|f|$ that

$$\widetilde{t}_f \tau_f = (-1)^{|f|} t_{(1)} \text{ for all } f \in E_{ij}. \tag{3.22}$$

Suppose that $|f| = 0$. Then $f = (1)$. Since $\widetilde{t}_{(1)} = t_{(1)}$ and $\tau_{(1)} = 1$, (3.22) follows for $f = (1)$.

For the induction step we note that $|e| < |f|$ for all $e \leq f, e \neq f$. Hence we get by induction from (3.21) for all $f \neq (1)$

$$\widetilde{t}_f \tau_f = - \sum_{e \leq f, e \neq f} \widetilde{t}_e \tau_e = - \sum_{e \leq f, e \neq f} (-1)^{|e|} t_{(1)}. \tag{3.23}$$

By (3.5) $\sum_{e \leq f} (-1)^{|e|} = 0$ for $f \neq (1)$, and (3.22) follows from (3.23). □

By the same proof and (3.15), (3.17) but without taking N-th powers we get the basis representation of x_{ji}.

Theorem 3.9. *Assume that $1 \leq i < j \leq n+1$. Then*

$$x_{ji} = (-q)^{j-i-1} \Big(\prod_{i \leq k < l < j} q_{lk} \Big) \sum_{(i_1, \dots, i_r) \in I_{ij}} (q^{-1} - 1)^{r-2} x_{i_1 i_2} \cdots x_{i_{r-1} i_r}.$$

□

4. The action of the diagram automorphism on root vector parameters

As in Section 2 let

$$\mathcal{D} = \mathcal{D}(\Gamma, (g_i)_{1 \leq i \leq n}, (\chi_i)_{1 \leq i \leq n}, (a_{ij})_{1 \leq i,j \leq n})$$

be a datum with finite abelian group Γ and Cartan matrix (1.1) of Type A_n. Recall that σ denotes the non-trivial diagram automorphism of (a_{ij}) given by (2.17). In this section we will construct for each $\mu \in k^{\Phi^+}$ satisfying (R2) a family $\sigma^{\mathcal{D}}(\mu) \in k^{\Phi^+}$ satisfying (R2) such that the isomorphism

$$F^\sigma \# \mathrm{id} : R(\mathcal{D}^\sigma) \# k[\Gamma] \to R(\mathcal{D}) \# k[\Gamma]$$

induces an isomorphism $u(\mathcal{D}^\sigma, \sigma^{\mathcal{D}}(\mu)) \to u(\mathcal{D}, \mu)$. We begin with a technical lemma to simplify the constants $\tau(i_1, \dots, i_r)$ in Theorem 3.1 when they appear as factors of certain root vector parameters.

Lemma 4.1. *Let $1 \leq i < j \leq n+1, (l_1, \dots, l_m) \in I_{ij}, (k_1, \dots, k_r) \in I_{1m}$. Let μ be a family of root vector parameters for $\mathcal{D}$. Then*

$$\mu(l_1, \dots, l_m) \tau_{ij} = \mu(l_1, \dots, l_m) \tau(l_{k_1}, l_{k_2}, \dots, l_{k_r}).$$

PROOF: The lemma is trivial if one of the factors $\mu_{l_k l_{k+1}}$ of $\mu(l_1, \dots, l_m)$ is zero. Assume that $\mu_{l_k l_{k+1}} \neq 0$ for all $1 \leq k < m$. Then $(\chi_{l_k l_{k+1}})^N = 1$ for all $1 \leq k < m$. In particular $(\chi_{l_{k_1} l_{k_s}})^N = 1$ for all $2 \leq s \leq r$. Therefore

$$\begin{aligned}\tau_{ij} &= \prod_{i<l<j} (\chi_{il})^N(g_l) \\ &= \prod_{l_{k_1}<l<l_{k_2}} (\chi_{l_{k_1} l})^N(g_l) \prod_{l_{k_2}\leq l<l_{k_3}} (\chi_{l_{k_1} l})^N(g_l) \cdots \prod_{l_{k_{r-1}}\leq l<l_{k_r}} (\chi_{l_{k_1} l})(g_l) \\ &= \tau_{l_{k_1} l_{k_2}} \cdots \tau_{l_{k_{r-1}} l_{k_r}} = \tau(l_{k_1}, l_{k_2}, \dots, l_{k_r}),\end{aligned}$$

since $(\chi_{l_{k_1} l})^N = (\chi_{l_{k_1} l_{k_s}})^N (\chi_{l_{k_s} l})^N = (\chi_{l_{k_s} l})^N$ for all $2 \leq s < r$ and $l_{k_s} \leq l$. $\Box$

We also need the following identity in group algebras.

Lemma 4.2. *Let G be a group, $m \geq 2$ and h_{st} elements in G for all $1 \leq s < t \leq m$. Assume*

$$h_{rs} h_{s,s+1} = h_{r,s+1} \text{ for all } 1 \leq r \leq 2, 1 \leq s < m.$$

Then

$$\begin{aligned}&\sum_{(k_1,\dots,k_r)\in I_{1m}} (-1)^r (1 - h_{k_{r-1}k_r}) \\ &= \sum_{(k_1,\dots,k_r)\in I_{1m}} (-1)^r (1 - h_{k_1,k_1+1}) \cdots (1 - h_{k_{r-1},k_{r-1}+1}).\end{aligned}$$

PROOF: Let

$$S_m = \sum_{(k_1,\dots,k_r)\in I_{1m}} (-1)^r (1 - h_{k_1,k_1+1}) \cdots (1 - h_{k_{r-1},k_{r-1}+1}).$$

The Lemma is true for $m = 2$. We show by induction on $m > 2$ that

$$S_m = h_{2m} - h_{1m} \text{ if } m > 2. \tag{4.1}$$

This holds for $m = 3$ since

$$S_3 = -(1 - h_{12})(1 - h_{23}) + 1 - h_{12} = (1 - h_{12})h_{23} = h_{23} - h_{13}.$$

The induction step follows from

$$\begin{aligned}S_{m+1} &= \sum_{\substack{(k_1,\dots,k_r)\in I_{1,m+1} \\ k_{r-1}=m}} (-1)^r (1 - h_{k_1,k_1+1}) \cdots (1 - h_{k_{r-1},k_{r-1}+1}) \\ &\quad + \sum_{\substack{(k_1,\dots,k_r)\in I_{1,m+1} \\ k_{r-1}<m}} (-1)^r (1 - h_{k_1,k_1+1}) \cdots (1 - h_{k_{r-1},k_{r-1}+1}) \\ &= -S_m(1 - h_{m,m+1}) + S_m \\ &= S_m h_{m,m+1}.\end{aligned}$$

On the other hand if $m > 2$ then

$$\begin{aligned}
\sum_{(k_1,\dots,k_r)\in I_{1m}} & (-1)^r(1-h_{k_{r-1}k_r}) \\
&= 1 - h_{1m} + \sum_{1<p<m} \sum_{\substack{(k_1,\dots,k_r)\in I_{1m} \\ k_{r-1}=p}} (-1)^r(1-h_{k_{r-1}k_r}) \\
&= 1 - h_{1m} + \sum_{1<p<m} \sum_{(k_1,\dots,k_r)\in I_{1p}} (-1)^{r-1}(1-h_{pm}) \\
&= 1 - h_{1m} - (1-h_{2m}) \text{ by (3.6)} \\
&= h_{2m} - h_{1m}.
\end{aligned}$$

□

We introduce the notation

$$\widetilde{i} = n - i + 2 \text{ for all } 1 \le i \le n+1$$

for the non-trivial diagram automorphism of A_{n+1}. Note that the map

$$I_{\widetilde{j}\,\widetilde{i}} \to I_{ij}, (i_1,\dots,i_r) \mapsto (\widetilde{i_r},\dots,\widetilde{i_1}), \tag{4.2}$$

is bijective for all $1 \le i < j \le n+1$. Recall that

$$\mathcal{D}^\sigma = \mathcal{D}(\Gamma, (g_i^\sigma)_{1\le i\le n}, (\chi_i^\sigma)_{1\le i\le n}, (a_{ij})_{1\le i,j\le n}),$$

where $g_i^\sigma = g_{\sigma(i)}, \chi_i^\sigma = \chi_{\sigma(i)}$ for all $1 \le i \le n$. Then

$$g_{ij}^\sigma = \prod_{i\le l<j} g_l^\sigma = g_{\widetilde{j}\,\widetilde{i}}, \text{ for all } 1 \le i < j \le n+1, \tag{4.3}$$

since $g_{ij}^\sigma = g_i^\sigma g_{i+1}^\sigma \cdots g_{j-1}^\sigma = g_{n-i+1}g_{n-i}\cdots g_{n-j+2} = g_{\widetilde{j}\,\widetilde{i}}$. For all $\mu \in k^{\Phi^+}$ and all $1 \le i < j \le n+1$ we define

$$\begin{aligned}
\sigma_{ij}^{\mathcal{D}}(\mu) &= \tau_{\widetilde{j}\,\widetilde{i}}(-1)^{j-i+1} \sum_{(i_1,\dots,i_r)\in I_{ij}} (q-1)^{N(r-2)}\mu(\widetilde{i_r},\cdots,\widetilde{i_1}), \\
\sigma^{\mathcal{D}}(\mu) &= (\sigma_{ij}^{\mathcal{D}}(\mu))_{1\le i<j\le n+1}.
\end{aligned} \tag{4.4}$$

Here $\tau_{\widetilde{j}\,\widetilde{i}} = \tau_{\widetilde{j}\,\widetilde{i}}^{\mathcal{D}}$ and q depend on $\mathcal{D}$, or more precisely on the braiding matrix (q_{ij}) of $\mathcal{D}$. Note that $q = \chi_i(g_i) = \chi_i^\sigma(g_i^\sigma)$ for all $1 \le i \le n$.

We will see in the next theorem that $\sigma^{\mathcal{D}}$ defines an isomorphism of affine algebraic varieties between the subspaces of all elements of k^{Φ^+} satisfying (R2) for $\mathcal{D}$ resp. for $\mathcal{D}^\sigma$.

By abuse of notation we will denote the images of the reverse root vectors in the quotient Hopf algebras $u(\mathcal{D},\mu)$ again by x_{ji}.

Theorem 4.3. *Let $\mu \in k^{\Phi^+}$ satisfying* (R2) *for $\mathcal{D}$. Then*

1. $u_{ij}^{\mathcal{D}^\sigma}(\sigma^{\mathcal{D}}(\mu)) = (x_{\tilde{i}\,\tilde{j}})^N$ *for all* $1 \leq i < j \leq n+1$.

2. *The family $\sigma^{\mathcal{D}}(\mu)$ satisfies* (R2) *for $\mathcal{D}^\sigma$ and* $\sigma^{\mathcal{D}^\sigma}(\sigma^{\mathcal{D}}(\mu)) = \mu$.

Proof: (1) Let $1 \leq i < j \leq n+1$. We write

$$\mu' = \sigma^{\mathcal{D}}(\mu).$$

We first compute $u_{ij}^{\mathcal{D}^\sigma}(\mu')$. By (2.9) and (4.3),

$$u_{ij}^{\mathcal{D}^\sigma}(\mu') = \sum_{(i_1,\ldots,i_r)\in I_{ij}} (q-1)^{N(r-2)}\mu'(i_1,\ldots,i_r)(1-(g_{\widetilde{i_r}\,\widetilde{i_{r-1}}})^N).$$

By (4.4) we can write for all $1 \leq i \leq i_{t-1} < i_t \leq j$

$$\mu'_{i_{t-1}i_t} = (-1)^{i_t - i_{t-1}+1}\tau_{\widetilde{i_t}\,\widetilde{i_{t-1}}} \sum_{(l_1^t,\ldots,l_{s_t}^t)\in I_{i_{t-1},i_t}} (q-1)^{N(s_t-2)}\mu(\widetilde{l_{s_t}^t},\ldots,\widetilde{l_1^t}).$$

Hence we obtain

$$\begin{aligned} u_{ij}^{\mathcal{D}^\sigma}(\mu') = &\sum_{(i_1,\ldots,i_r)\in I_{ij}} (q-1)^{N(r-2)}(-1)^{j-i+r-1}\tau(\widetilde{i_r},\ldots,\widetilde{i_1}) \\ &\times \left(\sum_{(l_1^2,\ldots,l_{s_2}^2)\in I_{i_1 i_2}} (q-1)^{N(s_2-2)}\mu(\widetilde{l_{s_2}^2},\ldots,\widetilde{l_1^2})\right) \\ &\quad\vdots \\ &\times \left(\sum_{(l_1^r,\ldots,l_{s_r}^r)\in I_{i_{r-1} i_r}} (q-1)^{N(s_r-2)}\mu(\widetilde{l_{s_r}^r},\ldots,\widetilde{l_1^r})\right) \\ &\times \left(1-(g_{\widetilde{i_r}\,\widetilde{i_{r-1}}})^N\right). \end{aligned} \tag{4.5}$$

On the other hand by Theorem 3.1

$$(x_{\tilde{i}\,\tilde{j}})^N = \sum_{(i_1,\ldots,i_r)\in I_{ij}} t(\widetilde{i_r},\ldots,\widetilde{i_1})u_{\widetilde{i_r}\,\widetilde{i_{r-1}}}\cdots u_{\widetilde{i_2}\,\widetilde{i_1}}.$$

By (2.9) and (4.2) we have for all $1 \leq i \leq i_{t-1} < i_t \leq j$

$$u_{\widetilde{i_t}\,\widetilde{i_{t-1}}} = \sum_{(l_1^t,\dots,l_{s_t}^t)\in I_{i_{t-1},i_t}} (q-1)^{N(s_t-2)}\mu(\widetilde{l_{s_t}^t},\dots,\widetilde{l_1^t})(1-(g_{\widetilde{l_2^t}\,\widetilde{l_1^t}})^N).$$

Again we get a large sum of products as before:

$$\begin{aligned}(x_{\widetilde{i}\,\widetilde{j}})^N = & \sum_{(i_1,\dots,i_r)\in I_{ij}} t(\widetilde{i_r},\dots,\widetilde{i_1}) \\ & \times\left(\sum_{(l_1^2,\dots,l_{s_2}^2)\in I_{i_1 i_2}} (q-1)^{N(s_2-2)}\mu(\widetilde{l_{s_2}^2},\dots,\widetilde{l_1^2})\right) \\ & \qquad \vdots \\ & \times\left(\sum_{(l_1^r,\dots,l_{s_r}^r)\in I_{i_{r-1} i_r}} (q-1)^{N(s_r-2)}\mu(\widetilde{l_{s_r}^r},\dots,\widetilde{l_1^r})\right) \\ & \times\left(1-(g_{\widetilde{l_2^r}\,\widetilde{l_1^r}})^N\right)\cdots\left(1-(g_{\widetilde{l_2^2}\,\widetilde{l_1^2}})^N\right).\end{aligned} \tag{4.6}$$

The point of the proof is to change the order of the summation indices. We have to sum over all sequences

$$(l_1^2,\dots,l_{s_2}^2,\dots,l_1^r,\dots,l_{s_r}^r)\in I_{ij}$$

with $l_1^2 = i_1, l_{s_2}^2 = i_2 = l_1^3,\dots,l_{s_r}^r = i_r$, where each sequence has

$$m = s_2+\dots+s_r - r + 2$$

elements. Equivalently we can start with an arbitrary sequence

$$(l_1,\dots,l_m)\in I_{ij},$$

then take all subsequences $(l_{k_1},\dots,l_{k_r})$, $(k_1,\dots,k_r)\in I_{1,m}$, of $(l_1,\dots,l_m)$ and define $(i_1,\dots,i_r)\in I_{ij}$ by $i_p = l_{k_p}$ for all p, $1\leq p\leq r$. Thus the right hand side of (4.5) becomes

$$\begin{aligned}&\sum_{(l_1,\dots,l_m)\in I_{ij}}\ \sum_{(k_1,\dots,k_r)\in I_{1m}} (-1)^{j-i-r+1}(q-1)^{N(m-2)} \\ &\cdot\mu(\widetilde{l_m},\dots,\widetilde{l_1})\tau(\widetilde{l_{k_r}},\widetilde{l_{k_{r-1}}},\dots,\widetilde{l_{k_1}})(1-(g_{\widetilde{l_{k_r}}\,\widetilde{l_{k_{r-1}}}})^N),\end{aligned} \tag{4.7}$$

and the right hand side of (4.6) becomes

$$\sum_{(l_1,\dots,l_m)\in I_{ij}} \sum_{(k_1,\dots,k_r)\in I_{1m}} (q-1)^{N(m-r)}\mu(\widetilde{l_m},\dots,\widetilde{l_1})t(\widetilde{l_{k_r}},\widetilde{l_{k_{r-1}}},\dots,\widetilde{l_{k_1}}) \tag{4.8}$$
$$\cdot\,(1-(g_{\widetilde{l_{k_{r-1}+1}}\,\widetilde{l_{k_{r-1}}}})^N)\cdots(1-(g_{\widetilde{l_{k_1+1}}\,\widetilde{l_{k_1}}})^N).$$

Both expressions (4.7) and (4.8) can be simplified. By Lemma 4.1 we can write in (4.7)

$$\mu(\widetilde{l_m},\dots,\widetilde{l_1})\tau(\widetilde{l_{k_r}},\widetilde{l_{k_{r-1}}},\dots,\widetilde{l_{k_1}}) = \mu(\widetilde{l_m},\dots,\widetilde{l_1})\tau_{\tilde{j}\,\tilde{i}}.$$

Similarly by Lemma 4.1 and since N is odd we have in (4.8)

$$\begin{aligned}
&\mu(\widetilde{l_m},\dots,\widetilde{l_1})t(\widetilde{l_{k_r}},\widetilde{l_{k_{r-1}}},\dots,\widetilde{l_{k_1}})\\
&= \mu(\widetilde{l_m},\dots,\widetilde{l_1})(-1)^{j-i-r+1}(q-1)^{N(r-2)}\tau(\widetilde{l_{k_r}},\widetilde{l_{k_{r-1}}},\dots,\widetilde{l_{k_1}})^{-\frac{N-1}{2}}\tau_{\tilde{j}\,\tilde{i}}^{\frac{N+1}{2}}\\
&= \mu(\widetilde{l_m},\dots,\widetilde{l_1})(-1)^{j-i-r+1}(q-1)^{N(r-2)}\tau_{\tilde{j}\,\tilde{i}}^{-\frac{N-1}{2}}\tau_{\tilde{j}\,\tilde{i}}^{\frac{N+1}{2}}\\
&= \mu(\widetilde{l_m},\dots,\widetilde{l_1})(-1)^{j-i-r+1}(q-1)^{N(r-2)}\tau_{\tilde{j}\,\tilde{i}}.
\end{aligned}$$

After this simplification we finally obtain

$$u_{ij}^{\mathcal{D}^\sigma}(\mu') = (-1)^{j-i+1}\tau_{\tilde{j}\,\tilde{i}} \sum_{(l_1,\dots,l_m)\in I_{ij}} (q-1)^{N(m-2)}\mu(\widetilde{l_m},\dots,\widetilde{l_1}) \tag{4.9}$$
$$\cdot \sum_{(k_1,\dots,k_r)\in I_{1m}} (-1)^r(1-(g_{\widetilde{l_{k_r}}\,\widetilde{l_{k_{r-1}}}})^N),$$

and

$$(x_{\tilde{i}\,\tilde{j}})^N = (-1)^{j-i+1}\tau_{\tilde{j}\,\tilde{i}} \sum_{(l_1,\dots,l_m)\in I_{ij}} (q-1)^{N(m-2)}\mu(\widetilde{l_m},\dots,\widetilde{l_1}) \tag{4.10}$$
$$\cdot \sum_{(k_1,\dots,k_r)\in I_{1m}} (-1)^r(1-(g_{\widetilde{l_{k_{r-1}+1}}\,\widetilde{l_{k_{r-1}}}})^N)\cdots(1-(g_{\widetilde{l_{k_1+1}}\,\widetilde{l_{k_1}}})^N),$$

and (1) follows from Lemma 4.2 with

$$h_{st} = (g_{\widetilde{l_t}\,\widetilde{l_s}})^N \text{ for all } 1\le s<t\le m$$

and for each sequence $(l_1,\dots,l_m)$.

(2) To prove that μ' satisfies (R2) for $\mathcal{D}^\sigma$ let $\chi'_{ij}\neq 1$. Then $\chi_{\tilde{j}\,\tilde{i}}\neq 1$. Hence for all $(i_1,\dots,i_r)\in I_{ij}$ we have $\chi_{\widetilde{i_s}\,\widetilde{i_{s-1}}}\neq 1$ for some $1<s\le r$. Thus $\mu(\widetilde{i_r},\dots,\widetilde{i_1})=0$, and $\mu'_{ij}=0$ by (4.4).

The proof of the equality $\sigma^{\mathcal{D}^\sigma}(\sigma^{\mathcal{D}}(\mu)) = \mu$ is similar to the proof of (1). Let $1 \le i < j \le n+1$. By definition

$$\sigma_{ij}^{\mathcal{D}^\sigma}(\sigma^{\mathcal{D}}(\mu)) = \tau_{\widetilde{j}\,\widetilde{i}}^{\mathcal{D}^\sigma}(-1)^{j-i+1} \sum_{(i_1,\dots,i_r)\in I_{ij}} (q-1)^{N(r-2)} \mu'(\widetilde{i_r},\dots,\widetilde{i_1}).$$

For all $1 \le i \le i_{t-1} < i_t \le j$ we have

$$\mu'_{\widetilde{i_t}\,\widetilde{i_{t-1}}} = (-1)^{i_t - i_{t-1}+1} \tau_{i_{t-1} i_t} \sum_{(l_1^t,\dots,l_{s_t}^t)\in I_{i_{t-1},i_t}} (q-1)^{N(s_t-2)} \mu(l_1^t,\dots,l_{s_t}^t).$$

As before for $u_{ij}^{\mathcal{D}^\sigma}(\mu')$ we now obtain

$$\begin{aligned}\sigma_{ij}^{\mathcal{D}^\sigma}(\sigma^{\mathcal{D}}(\mu)) &= \tau_{\widetilde{j}\,\widetilde{i}}^{\mathcal{D}^\sigma}\tau_{ij} \sum_{(l_1,\dots,l_m)\in I_{ij}} (q-1)^{N(m-2)} \mu(l_1,\dots,l_m) \sum_{(k_1,\dots,k_r)\in I_{1m}} (-1)^r \\ &= \tau_{\widetilde{j}\,\widetilde{i}}^{\mathcal{D}^\sigma}\tau_{ij}\mu_{ij} \qquad \text{(by (3.6))}.\end{aligned}$$

This proves the claim since

$$\tau_{\widetilde{j}\,\widetilde{i}}^{\mathcal{D}^\sigma} = \prod_{\widetilde{j}\le\widetilde{l}<\widetilde{k}<\widetilde{i}} (\chi_{\widetilde{l}}^\sigma(g_{\widetilde{k}}^\sigma))^N = \prod_{i<k<l\le j} (\chi_{l-1}(g_{k-1}))^N = \prod_{i\le k<l<j} \chi_l^N(g_k),$$

hence

$$\tau_{\widetilde{j}\,\widetilde{i}}^{\mathcal{D}^\sigma}\tau_{ij} = \prod_{i\le k<l<j} \chi_l^N(g_k)\chi_k^N(g_l) = \prod_{i\le k<l<j} \chi_k(g_k)^{Na_{kl}} = 1.$$

□

Corollary 4.4. *Let $\mu \in k^{\Phi^+}$ satisfying* (R2). *Then the map*

$$u(\mathcal{D}^\sigma, \sigma^{\mathcal{D}}(\mu)) \to u(\mathcal{D}, \mu)$$

given by $x_i^\sigma \mapsto x_{\sigma(i)}, g \mapsto g, 1 \le i \le n, g \in \Gamma$, is an isomorphism of Hopf algebras.

PROOF: Let $s_i = 1$ for all $1 \le i \le n$. Then the triple $(\mathrm{id}_\Gamma, \sigma, (s_i))$ is an isomorphism from $(\mathcal{D}^\sigma, \sigma^{\mathcal{D}}(\mu))$ to $(\mathcal{D}, \mu)$ by Theorem 4.3 and Lemma 2.4. Hence the claim follows from Theorem 2.3. □

5. Hopf algebra isomorphisms

In this section let

$$\mathcal{D} = \mathcal{D}(\Gamma, (g_i)_{1\le i\le n}, (\chi_i)_{1\le i\le n}, (a_{ij})_{1\le i,j\le n}),$$
$$\mathcal{D}' = \mathcal{D}(\Gamma', (g_i')_{1\le i\le n}, (\chi_i')_{1\le i\le n}, (a_{ij})_{1\le i,j\le n})$$

be data with finite abelian groups Γ and Γ' and the same Cartan matrix (1.1) of Type A_n. As before σ denotes the non-trivial diagram automorphism of (a_{ij}) given by (2.17).

For $s = (s_i)_{1\le i\le n} \in k^n$ and $\mu \in k^{\Phi^+}$ we define

$$s^N = (s_i^N)_{1\le i\le n}$$
$$s\cdot\mu = (s_{ij}\mu_{ij})_{1\le i<j\le n+1}.$$

Recall that

$$s_{ij} = \prod_{i\le l<j} s_l \text{ for all } 1\le i<j\le n+1.$$

Then

$$u_{ij}(s\cdot\mu) = s_{ij}u_{ij}(\mu) \text{ for all } 1\le i<j\le n+1 \tag{5.1}$$

since $s_{ij} = s_{i_1 i_2}\cdots s_{i_{r-1}i_r}$ for all $(i_1,\dots,i_r)\in I_{ij}$.

An isomorphism of data of Cartan type from $\mathcal{D}'$ to $\mathcal{D}$ is a group isomorphism $\varphi:\Gamma'\to\Gamma$ satisfying

$$\varphi(g_i') = g_i, \chi_i' = \chi_i\varphi \text{ for all } 1\le i\le n. \tag{5.2}$$

We write $\varphi:\mathcal{D}' \xrightarrow{\cong} \mathcal{D}$ if φ is an isomorphism from $\mathcal{D}'$ to $\mathcal{D}$.

Note that (5.2) implies for all $1\le i,j\le n+1$ that

$$\chi_j'(g_i') = \chi_j(g_i), \text{ for all } 1\le i,j\le n+1,$$
$$\varphi(g_{ij}') = g_{ij} \text{ for all } 1\le i<j\le n+1.$$

Hence for all $\mu'\in k^{\Phi^+}$

$$\varphi(u_{ij}^{\mathcal{D}'}(\mu')) = u_{ij}^{\mathcal{D}}(\mu') \text{ for all } 1\le i<j\le n+1. \tag{5.3}$$

Let $k^\times = k\setminus\{0\}$ denote the multiplicative group of k. In part (b) of (II) below, recall the definition of $\nu^{\mathcal{D}}$ in (2.14).

Theorem 5.1. *Let μ and μ' be families of root vector parameters for $\mathcal{D}$ and $\mathcal{D}'$. Then the Hopf algebra isomorphisms $u(\mathcal{D}',\mu') \to u(\mathcal{D},\mu)$ are given by*

(I) $x_i \mapsto s_i x_i$, $g \mapsto \varphi(g)$, $1 \leq i \leq n$, $g \in \Gamma'$, *where*

(a) $\varphi : \mathcal{D}' \xrightarrow{\cong} \mathcal{D}$, *and*

(b) $s = (s_i) \in (k^\times)^n$ *such that* $\mu' = s^N \cdot \mu$,

and

(II) $x_i \mapsto s_i x_{\sigma(i)}$, $g \mapsto \varphi(g)$, $1 \leq i \leq n$, $g \in \Gamma'$, *where*

(a) $\varphi : \mathcal{D}' \xrightarrow{\cong} \mathcal{D}^\sigma$, *and*

(b) $s = (s_i) \in (k^\times)^n$ *such that* $\mu' = s^N \cdot \nu^{\mathcal{D}^\sigma}(\sigma^{\mathcal{D}}(\mu))$.

Proof: By Theorem 2.3 the isomorphisms $u(\mathcal{D}',\mu') \to u(\mathcal{D},\mu)$ are given by $x_i \mapsto s_i x_{\rho(i)}, g \mapsto \varphi(g), 1 \leq i \leq n, g \in \Gamma'$, where $\varphi : \Gamma' \to \Gamma$ is an isomorphism of groups, $\rho = \mathrm{id}$ or $\rho = \sigma$, and $s = (s_i) \in (k^\times)^n$ such that

$$\varphi(g_i') = g_{\rho(i)}, \chi_i' = \chi_{\rho(i)}\varphi \text{ for all } 1 \leq i \leq n. \tag{5.4}$$

$$\varphi(u_{ij}^{\mathcal{D}'}(\mu')) = s_{ij}^N \pi(F^\rho(x_{ij}^\rho)^N) \text{ for all } 1 \leq i < j \leq n+1. \tag{5.5}$$

We have to show that (5.4) and (5.5) are equivalent to (I)(a) and (b) if $\rho = \mathrm{id}$, and to (II)(a) and (b) if $\rho = \sigma$.

By definition (5.4) coincides with (I)(a) if $\rho = \mathrm{id}$ and with (II)(a) if $\rho = \sigma$.

Let $\rho = \mathrm{id}$ and assume (I)(a). Then the left hand side of (5.5) is $\varphi(u_{ij}^{\mathcal{D}'}(\mu')) = u_{ij}^{\mathcal{D}}(\mu')$ by (5.3). For the right hand side of (5.5) we obtain

$$\begin{aligned} s_{ij}^N \pi(F^\rho(x_{ij}^\rho)^N) &= s_{ij}^N u_{ij}^{\mathcal{D}}(\mu) \\ &= u_{ij}^{\mathcal{D}}(s^N \cdot \mu) \quad \text{(by (5.1))}. \end{aligned}$$

Hence (I)(b) and (5.5) are equivalent by the uniqueness in Lemma 2.2.

Let $\rho = \sigma$ and assume (II)(a). Let $1 \leq i < j \leq n+1$. Again it follows from (5.3) applied to $\mathcal{D}' \xrightarrow{\cong} \mathcal{D}^\sigma$ that $\varphi(u_{ij}^{\mathcal{D}'}(\mu')) = u_{ij}^{\mathcal{D}^\sigma}(\mu')$. We have shown in Lemma 2.4 and Theorem 4.3 that

$$\pi(F^\sigma(x_{ij}^\sigma)^N) = u_{ij}^{\mathcal{D}^\sigma}(\sigma^{\mathcal{D}}(\mu)).$$

Hence

$$\begin{aligned} s_{ij}^N \pi(F^\sigma(x_{ij}^\sigma)^N) &= s_{ij}^N u_{ij}^{\mathcal{D}^\sigma}(\sigma^{\mathcal{D}}(\mu)) \\ &= s_{ij}^N u_{ij}^{\mathcal{D}^\sigma}(\nu^{\mathcal{D}^\sigma}(\sigma^{\mathcal{D}}(\mu))) \\ &= u_{ij}^{\mathcal{D}^\sigma}(s^N \cdot \nu^{\mathcal{D}^\sigma}(\sigma^{\mathcal{D}}(\mu))) \quad \text{(by (5.1))}. \end{aligned}$$

Again it follows that (II)(b) and (5.5) are equivalent. $\square$

We know from Theorem 4.3 that $\nu^{\mathcal{D}}(\mu)$ satisfies (R2) for $\mathcal{D}^\sigma$. If we assume (II)(a), then $\nu^{\mathcal{D}}(\mu)$ also satisfies (R2) for $\mathcal{D}'$. Hence the normalization $\nu^{\mathcal{D}^\sigma}(\sigma^{\mathcal{D}}(\mu))$ is a family of root vector parameters for $\mathcal{D}^\sigma$ and $\mathcal{D}'$. In general $\sigma^{\mathcal{D}}(\mu)$ is not a family of root vector parameters for $\mathcal{D}^\sigma$ since (R1) is not necessarily satisfied, and we have to pass to the normalization.

For example

$$\sigma^{\mathcal{D}}_{13}(\mu) = -\tau_{n-1,n+1}(\mu_{n-1,n+1} + (q-1)\mu_{n-1,n}\mu_{n,n+1}),$$

and if $({g^\sigma}_{13})^N = g^N_{n-1,n+1} = 1$, then $\mu_{n-1,n+1} = 0$, but $\mu_{n-1,n}\mu_{n,n+1}$ can be non-zero if $g^N_{n-1} \neq 1, g^N_n \neq 1$ and $\chi^N_{n-1} = 1, \chi^N_n = 1$. As a realization of this situation take $n = 2$ and let $\Gamma \cong \mathbb{Z}/(N^2) \times \mathbb{Z}/(N)$ with generators g of order N^2 and h of order N. Let $\zeta \in k$ be a root of 1 of order N and $q = \zeta^2$. Define g_1, g_2 and characters χ_1, χ_2 by

$$g_1 = gh, g_2 = g^{-1}h, \chi_1(g) = \zeta, \chi_1(h) = \zeta, \chi_2(g) = \zeta^{-2}, \chi_2(h) = 1.$$

Then $\chi_1(g_1) = q = \chi_2(g_2), \chi_1(g_2)\chi_2(g_1) = q^{-1}$, and

$$g_1^N = g^N \neq 1, g_2^N = g^{-N} \neq 1, \chi_1^N = \chi_2^N = 1, \text{ and } g_{13}^N = g_1^N g_2^N = 1.$$

Note that (R1) is trivially satisfied if $g^N_{ij} \neq 1$ for all $1 \leq i < j \leq n+1$.

The next corollary follows immediately from Theorem 5.1.

Corollary 5.2. *Let μ, μ' be families of root vector parameters for $\mathcal{D}$. Then the following are equivalent:*

1. *$u(\mathcal{D}, \mu') \cong u(\mathcal{D}, \mu)$.*
2. *There is a family $s \in (k^\times)^n$ such that*
$$\mu' = \begin{cases} s^N \cdot \mu, & \text{if } \mathcal{D} \ncong \mathcal{D}^\sigma, \\ s^N \cdot \mu \text{ or } s^N \cdot \nu^{\mathcal{D}^\sigma}(\sigma^{\mathcal{D}}(\mu)), & \text{if } \mathcal{D} \cong \mathcal{D}^\sigma. \end{cases}$$

$\square$

Corollary 5.3. *Suppose there are $1 \leq i < j \leq n+1, j - i \geq 2$, such that $g^N_{ij} \neq 1$ and $g^N_l \neq 1, \chi^N_l = 1$ for all $i \leq l < j$. Then the number of isomorphism classes of Hopf algebras of the form $u(\mathcal{D}, \mu)$ is infinite.*

Proof: By our assumption on $\mathcal{D}$ we can consider families of root vector parameters μ, μ' with $\mu_{l,l+1} = \mu'_{l,l+1} = 1$ for all $i \leq l < j$, and with arbitrary elements $\mu_{ij}, \mu'_{ij} \in k$. If $u(\mathcal{D}', \mu') \cong u(\mathcal{D}, \mu)$, then by Corollary 5.2 for all $i \leq l < j$ we have

$$\mu'_{l,l+1} = \begin{cases} s_l^N \mu_{l,l+1}, & \text{if } \mathcal{D} \not\cong \mathcal{D}^\sigma, \\ s_l^N \mu_{l,l+1} \text{ or } s_l^N \mu_{\sigma(l),\sigma(l)+1}, & \text{if } \mathcal{D} \cong \mathcal{D}^\sigma, \end{cases}$$

hence $s_l^N = 1$. Thus $s_{ij}^N = 1$, and again by Corollary 5.2 it follows that

$$\mu'_{ij} = \begin{cases} \mu_{ij}, & \text{if } \mathcal{D} \not\cong \mathcal{D}^\sigma, \\ \mu_{ij} \text{ or } \nu_{ij}^{\mathcal{D}^\sigma}(\sigma^{\mathcal{D}}(\mu)), & \text{if } \mathcal{D} \cong \mathcal{D}^\sigma. \end{cases}$$

Therefore, we obtain infinitely many isomorphism classes of Hopf algebras $u(\mathcal{D}, \mu)$. □

Theorem 5.1 gives the following description of the group of all Hopf algebra automorphisms of $u(\mathcal{D}, \mu)$.

Corollary 5.4. *Let μ be a family of root vector parameters for $\mathcal{D}$. Then* $\mathrm{Hopfaut}(u(\mathcal{D}, \mu))$ *is isomorphic to the subgroup of* $\mathrm{Aut}(\Gamma) \times (k^\times)^n$ *consisting of all pairs $(\varphi, s), \varphi \in \mathrm{Aut}(\Gamma), s \in (k^\times)^n$, where*

$$\varphi : \mathcal{D} \xrightarrow{\cong} \mathcal{D}, \mu = s^N \cdot \mu, \text{ or}$$
$$\varphi : \mathcal{D} \xrightarrow{\cong} \mathcal{D}^\sigma, \mu = s^N \cdot \nu^{D^\sigma}(\sigma^{\mathcal{D}}(\mu)).$$

□

We note the following special case.

Corollary 5.5. *Let μ be a family of root vector parameters for $\mathcal{D}$. Then the group of all Hopf algebra automorphisms of $u(\mathcal{D}, \mu)$ is finite if $\mu_{i,i+1} \neq 0$ for all $1 \leq i \leq n$.* □

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