On pointed Hopf algebras with non-abelian group

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Plan of the talk.

I. Introduction.

II. Method of classification.

III. Pointed Hopf algebras with non-abelian group.

V. The Weyl group of a semisimple Yetter-Drinfeld module.

IV. Discarding infinite-dimensional pointed Hopf algebras with non-abelian group.

I. Introduction.

Some invariants.

$G(H) := \{x \in H - 0 : \Delta(x) = x \otimes x\}$. = group of *group-likes*.

$H_0 := \sum C$, C simple subcoalgebras of H =:*coradical* of H.

$\mathbb{C}G(H) \subseteq H_0$; H is *pointed non coss.* = *"pointed"* if $\mathbb{C}G(H) = H_0 \neq H$.

$H_{j+1} := \{x \in H : \Delta(x) \in H_j \otimes H + H \otimes H_0\}$.

$H_0 \subseteq H_1 \subseteq \cdots \subseteq H_j \subseteq H_{j+1} \subseteq \ldots$ is the *coradical filtration* of H.

Example.

$\mathfrak{g}$ Lie algebra

Γ group acting by automorphisms on $\mathfrak{g}$

$H = U(\mathfrak{g}) \rtimes \mathbb{C}\Gamma$ = cocommutative Hopf algebra

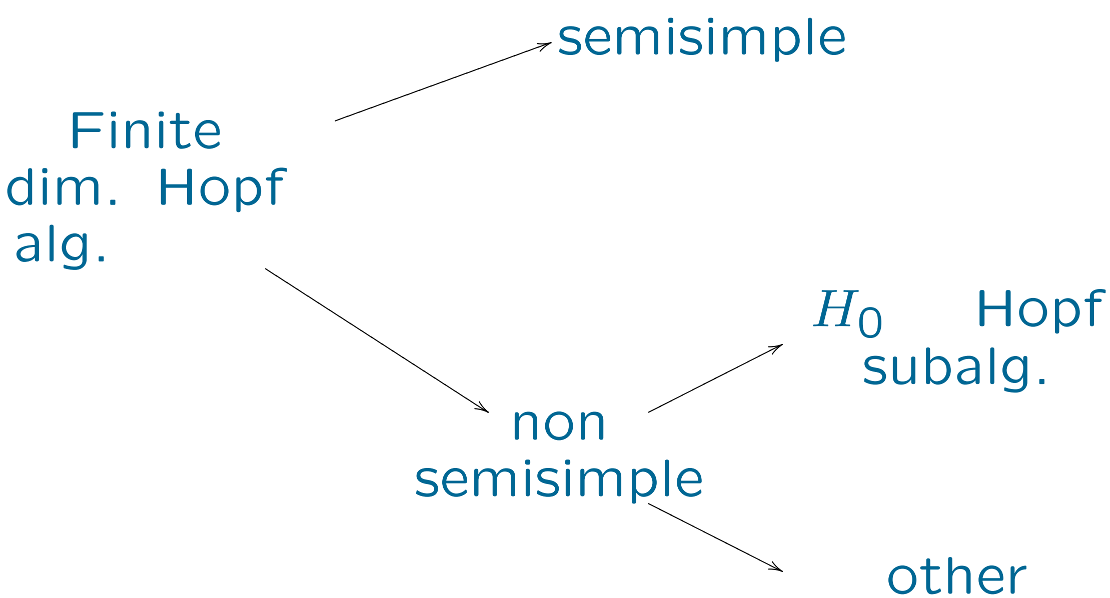
(Kostant-Cartier-Milnor-Moore:
any cocommutative Hopf algebra like this)

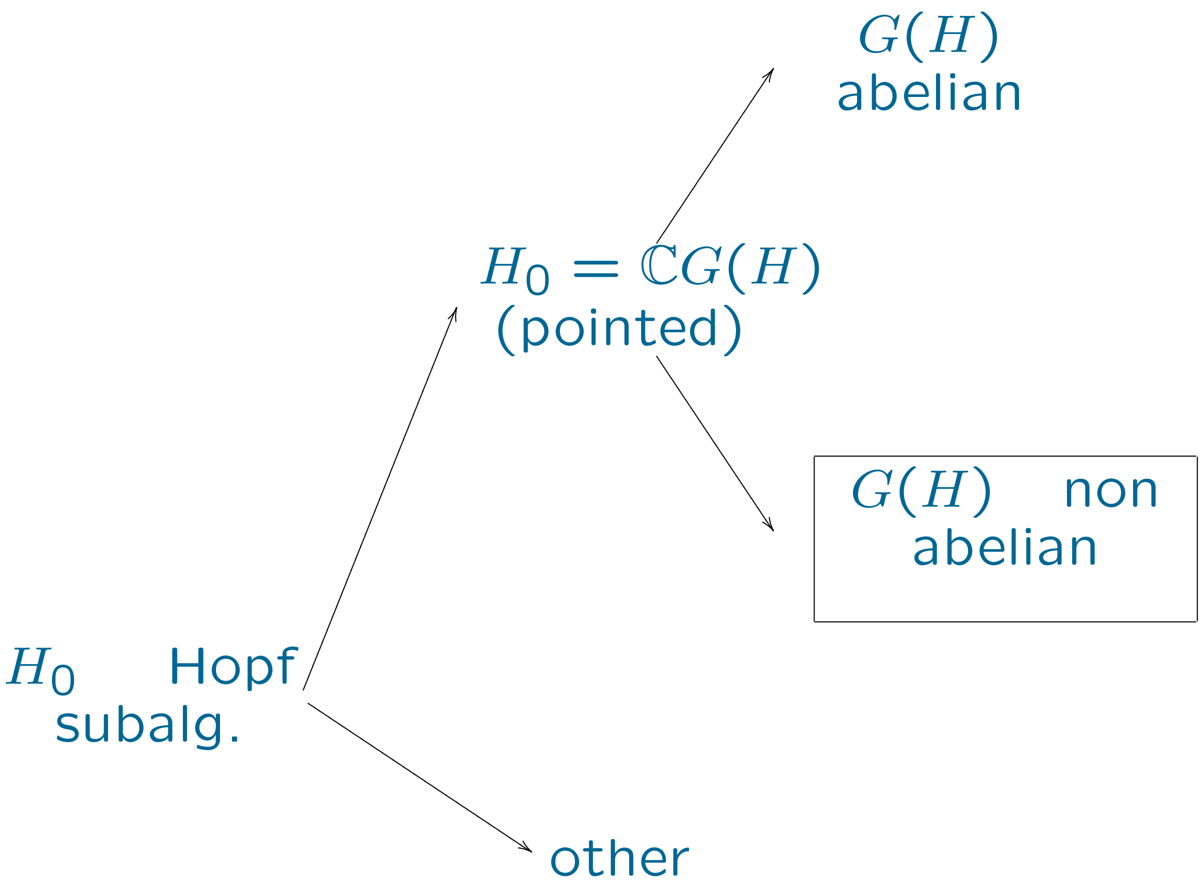
$H_0 = \mathbb{C}\Gamma$.

$H_j = U_j(\mathfrak{g}) \rtimes \mathbb{C}\Gamma$.

Problem. Classify finite dim. Hopf algs.

Approach: relative position of the coradical.




H_0 Hopf subalg.
$H_0 = \mathbb{C}G(H)$ (pointed)
other
$G(H)$ abelian
$G(H)$ non abelian

II. Method of classification.

N. A. and H.-J. Schneider,
Pointed Hopf Algebras, MSRI Publications **43** (2002), 1-68, Cambridge Univ. Press.

H pointed (more generally, H_0 Hopf subalgebra)

$0 = H_{-1} \subseteq H_0 \subseteq H_1 \subseteq \cdots \subseteq H_j \subseteq H_{j+1} \subseteq \ldots$

coradical filtration of H.

$$\operatorname{gr} H := \oplus_{n\geq 0} H_n/H_{n-1} \simeq \mathbb{C}\Gamma \# R,$$

(Radford-Majid)

$R = \oplus_{n\geq 0} R^n$, $\quad R(n) = R \cap H_n / H_{n-1}$, $\quad R' = \mathbb{C} < R(1) > \subseteq R$.

R and R' braided Hopf algebras $\equiv$ Hopf alg. in a braided category.

Essential step: Determine all possible R' s. t. $\dim R' < \infty$. Why?

(V, c) braided vector space: $c \in GL(V \otimes V)$

$(c \otimes \mathrm{id})(\mathrm{id} \otimes c)(c \otimes \mathrm{id}) = (\mathrm{id} \otimes c)(c \otimes \mathrm{id})(\mathrm{id} \otimes c)$

$\rightsquigarrow \mathfrak{B}(V) = \oplus_{n\geq 0} \mathfrak{B}^n(V)$ (Nichols algebra)

In our case, $R' = \mathfrak{B}(V)$, where $V = R^1$!

Nichols algebra: $\mathfrak{B}(V) = \oplus_{n\geq 0}\mathfrak{B}^n(V)$

graded algebra with extra structure

- $\mathfrak{B}^0(V) = \mathbb{C}$, $\mathfrak{B}^1(V) = V$.
- $\mathfrak{B}(V)$ generated by V as algebra.
- $\mathfrak{B}(V)$ is a braided Hopf algebra.
- $P(\mathfrak{B}(V)) = V$.

rank of $\mathfrak{B}(V) = \dim V$

$\mathfrak{B}(V) = T(V)/J$, but J not explicit!

Method:

- Determine all possible R' s. t. $\dim R' < \infty$.

- If $\dim R' < \infty$, then $R' = R$?

Conjecture (A.-Schneider) Any pointed Hopf alg., dim. $< \infty$ is generated by group-like and skew-primitive elements.

- Find all possible H s. t. $\operatorname{gr} H \simeq \mathbb{C}\Gamma \# R$
(Lifting).

Summarizing, H pointed $\rightsquigarrow (V, c)$ braided vector space

$\dim H < \infty \implies \dim \mathfrak{B}(V) < \infty$

Problem: given (V, c) braided vector space arising from H, decide when $\dim \mathfrak{B}(V) < \infty$

Example. $H = U(\mathfrak{g}) \rtimes \mathbb{C}\Gamma$

$(V, c) =$ vector space $\mathfrak{g}$, c usual flip, $R' = S(\mathfrak{g})$

Γ finite abelian group

Braided vector space of diagonal type.

$\exists$ basis $v_1, \dots, v_\theta$, $(q_{ij})_{1 \le i,j \le \theta}$ in $\mathbb{C}^\times$:

$$c(v_i \otimes v_j) = q_{ij} v_j \otimes v_i, \quad \forall i, j$$

Theorem. $1 \neq q_{ii}$ roots of 1. $\Rightarrow$ $\dim \mathfrak{B}(V) < \infty$ classified.

I. Heckenberger, *Classification of arithmetic root systems*, http://arxiv.org/abs/math.QA/0605795.

Braided vector space of Cartan type.

$\exists\ (a_{ij})_{1\le i,j\le\theta}$ generalized Cartan matrix

$$q_{ij}q_{ji} = q_{ii}^{a_{ij}}.$$

Theorem. (V,c) Cartan type, $1 \neq q_{ii}$ root of 1.
$\dim \mathfrak{B}(V) < \infty \iff (a_{ij})$ of finite type.

N. A. & H.-J. Schneider, *Finite quantum groups and Cartan matrices*, Adv. Math. **154** (2000), 1-45.

I. Heckenberger, *The Weyl groupoid of a Nichols algebra of diagonal type*, Invent. Math. **164**, 175–188 (2006).

$\mathfrak{g}$ simple Lie algebra, Cartan matrix $(a_{ij})_{1\leq i,j\leq\theta}$
q root of 1 of order N small quantum group $u_q(\mathfrak{g})$
$=\mathbb{C}\langle k_1,\dots,k_\theta,e_1,\dots,e_\theta,f_1,\dots,f_\theta\rangle$ with relations:

$$\begin{aligned}
&k_ik_j = k_jk_i, \quad k_i^N = 1,\\
&k_ie_jk_i^{-1} = q^{d_ia_{ij}}e_j,\\
&k_if_jk_i^{-1} = q^{-d_ia_{ij}}f_j,\\
&\mathrm{ad}_c(e_i)^{1-a_{ij}}(e_j) = 0, \quad i\neq j\\
&\mathrm{ad}_c(f_i)^{1-a_{ij}}(f_j) = 0, \quad i\neq j\\
&e_if_j - q^{-d_ia_{ij}}f_je_i = \delta_{ij}(1-k_i^2), i<j, i\nsim j\\
&e_\alpha^N = 0, \quad f_\alpha^N = 0,\\
&\Delta(g) = g\otimes g, \Delta(x_i) = g_i\otimes x_i + x_i\otimes 1.
\end{aligned}$$

$u_q(\mathfrak{g})$ is a pointed Hopf algebra of dim. $N^{\dim\mathfrak{g}}$. Here $\mathrm{ad}_c(x_i)(x_j) = x_ix_j - q_{ij}x_jx_i$.

Γ finite abelian group $\longrightarrow$ Cartan datum $\nearrow$ Linking parameter λ

Cartan datum $\searrow$ parameter μ

$\rightsquigarrow$ $u(\mathcal{D}, \lambda, \mu)$ finite dim. Hopf algebra

Hopf algebra $u(\mathcal{D},\lambda,\mu) = \mathbb{C}\langle \Gamma, x_1, \dots, x_\theta \rangle$ with relations:

(*Action of Γ*)	$g x_i g^{-1} = \chi_i(g) x_i,$
(*Serre*)	$\mathrm{ad}_c(x_i)^{1-a_{ij}}(x_j) = 0,$
	$i \neq j, i \sim j$
(*Linking*)	$\mathrm{ad}_c(x_i)(x_j) = \lambda_{ij}(1 - g_i g_j),$
	$i < j, i \not\sim j$
(*Powers root vect.*)	$x_\alpha^{N_J} = u_\alpha(\mu),$
$\Delta(g) = g \otimes g,$	$\Delta(x_i) = g_i \otimes x_i + x_i \otimes 1.$

Here $\mathrm{ad}_c(x_i)(x_j) = x_i x_j - q_{ij} x_j x_i$.

Classification Theorem. N. A. and H.-J. Schneider, *On the classification of finite-dimensional pointed Hopf algebras*, Ann. Math., to appear.

(1) $u(\mathcal{D}, \lambda, \mu)$ is a pointed Hopf alg.,

$\dim u(\mathcal{D}, \lambda, \mu) = \prod_{J \in \mathbb{X}} N_J^{|\Phi_J^+|} |\Gamma|, \qquad G(u(\mathcal{D}, \lambda, \mu)) \simeq \Gamma.$

(2) Let H be a finite dimensional pointed Hopf algebra, $\Gamma = G(H)$. Assume all prime divisors of $|\Gamma|$ are > 7

$\implies \exists \mathcal{D}, \lambda, \mu$: $H \cong u(\mathcal{D}, \lambda, \mu)$.

(3) $u(\mathcal{D}, \lambda, \mu) \cong u(\mathcal{D}', \lambda', \mu') \implies \dots .$

III. Pointed Hopf algebras with non-abelian group.

H pointed, $G = G(H)$ not abelian

Braided vector spaces attached to G

$\mathcal{C}$ a conjugacy class in G, (ρ, V) irred. repr. of G^s, fixed $s \in \mathcal{C}$.

$M(\mathcal{C}, \rho) = \text{Ind}_{G^s}^{G} = \mathbb{C}\mathcal{C} \otimes V$.

Since $s \in Z(G^s)$, by Schur Lemma, s acts by a scalar q_{ss} on V.

The braided vector spaces attached to G are direct sums of different $M(\mathcal{C}_i, \rho_i)$'s (Dijkgraaf, Pasquier, Roche).

Problem. To classify finite dim. pointed Hopf algs. with $G(H) = G$, the first step is

when $\dim \mathfrak{B}(M(\mathcal{C}_1, \rho_1) \oplus \cdots \oplus M(\mathcal{C}_N, \rho_N)) < \infty$?

when $\dim \mathfrak{B}(M(\mathcal{C}, \rho)) < \infty$?

Assume know
$\dim \mathfrak{B}(M(\mathcal{C}_i, \rho_i)) < \infty$, then?

Example: $\mathcal{C}$ = transpositions in $G = \mathbb{S}_n$, $s = (12)$, $\rho = \mathrm{sgn}$

n	**rk**	**Relations**	$\dim \mathfrak{B}(V)$	**top**
3	3	5 relations in degree 2	$12 = 3.2^2$	$4 = 2^2$
4	6	16 relations in degree 2	576	12
5	10	45 relations in degree 2	8294400	40

$\mathbb{S}_3$, $\mathbb{S}_4$: A. Milinski and H. Schneider, Contemp. Math. **267** (2000), 215–236.
S. Fomin and K. Kirillov, Progr. Math. **172**, Birkhauser, (1999), 146–182.
$\mathbb{S}_5$: [FK], plus web page of M. Graña. `http://mate.dm.uba.ar/~matiasg/`
$\mathbb{S}_n$, $n \geq 6$: open!

IV. The Weyl groupoid of a semisimple Yetter-Drinfeld module.

G finite group (actually holds for arbitrary H Hopf algebra)

$V_j = \mathfrak{B}(M(\mathcal{C}_j, \rho_j)$, $1 \leq j \leq d$. $\dim \mathfrak{B}(V_j) < \infty$.

$\mathbb{V} = \oplus_{1 \leq j \leq d} V_j$, appropriate finiteness hypothesis.

Fix i, $1 \leq i \leq d$.

$$\begin{array}{ccccc} & & & & \mathfrak{B}(V_i) \\ & & & \overset{\iota}{\swarrow} & \| \\ \mathcal{K} := \mathfrak{B}(\mathbb{V})^{\operatorname{co} \mathfrak{B}(V_i)} & \hookrightarrow & \mathfrak{B}(\mathbb{V}) & \xrightarrow[\pi_{\mathfrak{B}(V_i)}]{} & \mathfrak{B}(V_i). \end{array}$$

$\mathcal{K} \supseteq L_j$:= $\mathrm{ad}_c(V_i)$-submod. gen. by V_j, $j \neq i$

L_j is graded, $\dim L_j < \infty$

m_{ij} = top degree of L_j, $a_{ij} = 1 - m_{ij}$

$L_i^{-1} = V_i^*$ and $a_{ii} = 2$.

$$\mathbb{V}' = \oplus_{1 \leq j \leq d} L_j^{1-a_{ij}}.$$

Theorem. (N. A., I. Heckenberger, H.-J. Schneider).
$\mathcal{K}\#\mathfrak{B}({}^*V_i) \simeq \mathfrak{B}(\mathbb{V}')$, $\dim \mathfrak{B}(\mathbb{V}) = \dim \mathfrak{B}(\mathbb{V}')$.

$$\mathbb{V} \xmapsto{\mathcal{R}_i} \mathbb{V}'.$$

$\mathcal{W}$ = groupoid generated by the "reflections" $\mathcal{R}_i$, $1 \leq i \leq d$.

Definition. $\mathbb{V}$ is standard if $\mathbb{V} \simeq \mathbb{V}'$ for all i
$\implies \mathcal{W}$ determines a Coxeter group $\mathcal{W}_0$

Theorem. (AHS). $\mathbb{V}$ is standard, $\dim \mathfrak{B}(\mathbb{V}) < \infty \implies \mathcal{W}_0$ finite.

V. Discarding infinite-dimensional pointed Hopf algebras with non-abelian group.

Strategy. *Given $(\mathcal{C}, \rho)$, find a braided subspace U of $M(\mathcal{C}, \rho)$ of diagonal type. Check if* $\dim \mathfrak{B}(U)$ *is infinite using the above mentioned results. If so, then* $\dim \mathfrak{B}(\mathcal{C}, \rho) = \infty$.

M. Graña, Contemp. Math. **267** (2000), pp. 111–134.

If $\dim \mathfrak{B}(\mathcal{C}, \rho) < \infty$, then

- $\deg \rho > 2$ implies $q_{ss} = -1$.
- $\deg \rho = 2$ implies $q_{ss} = -1$, ω_3 or ω_3^2.

Lemma. (A., Zhang). Assume that there exists $\sigma \in G$ such that

$$\sigma s \sigma = s^{-1}.$$

If $\dim \mathfrak{B}(\mathcal{C}, \rho) < \infty$ then $q_{ss} = -1$, $\operatorname{ord} s$ even.

Theorem. (A., Zhang). Let W be the Weyl group of a finite-dimensional semisimple Lie algebra.
If $\pi \in W$ has odd order then $\dim \mathfrak{B}(\mathcal{C}_\pi, \rho) = \infty$ for any $\rho \in \widehat{W^\pi}$.

Definition. $M(\mathcal{C}, \rho)$ is negative if $\deg \rho = 1$, and for all $s, t \in \mathcal{C}$ s. t. $st = ts$,

$$c(s \otimes t) = -t \otimes s.$$

Theorem. (A., Fantino, Zhang). Let $\pi \in \mathbb{S}_n$. Then for any $\rho \in \widehat{W^\pi}$, either

- $\dim \mathfrak{B}(\mathcal{C}_\pi, \rho) = \infty$, or
- $M(\mathcal{C}, \rho)$ is negative.

N. A. and Shouchuan Zhang, *On pointed Hopf algebras associated to some conjugacy classes in* $\mathbb{S}_n$, Proc. Amer. Math. Soc. **135** (2007), 2723-2731.

N. A. and F. Fantino, *On pointed Hopf algebras associated to some conjugacy classes in* $\mathbb{S}_n$, J. Math. Phys 48, 033502 (2007).

N. A., F. Fantino and Shouchuan Zhang, *in preparation.*

Let $\mathcal{A}(\mathbb{S}_3, \mathcal{O}_2^3, \lambda)$ be the algebra presented by generators e_t, $t \in T := \{(12), (23)\}$, and a_σ, $\sigma \in \mathcal{O}_2^3$; with relations

$$e_t e_s e_t = e_s e_t e_s, \quad e_t^2 = 1, \quad s \neq t \in T; \tag{1}$$

$$e_t a_\sigma = -a_{t\sigma t} e_t \qquad t \in T, \sigma \in \mathcal{O}_2^3; \tag{2}$$

$$a_\sigma^2 = 0, \qquad \sigma \in \mathcal{O}_2^3; \tag{3}$$

$$a_{(12)}a_{(23)} + a_{(23)}a_{(13)} + a_{(13)}a_{(12)} = \lambda(1 - e_{(12)}e_{(23)}); \tag{4}$$

$$a_{(12)}a_{(13)} + a_{(13)}a_{(23)} + a_{(23)}a_{(12)} = \lambda(1 - e_{(23)}e_{(12)}). \tag{5}$$

Set $e_{(13)} = e_{(12)}e_{(23)}e_{(12)}$. Then $\mathcal{A}(\mathbb{S}_3, \mathcal{O}_2^3, \lambda)$ is a Hopf algebra of dimension 72 with comultiplication determined by

$$\Delta(a_\sigma) = a_\sigma \otimes 1 + e_\sigma \otimes a_\sigma, \quad \Delta(e_t) = e_t \otimes e_t, \qquad \sigma \in \mathcal{O}_2^3, t \in T. \tag{6}$$

Theorem. (AHS, using previous work with Milinski, Graña, Zhang).

Let H be a finite dimensional pointed Hopf algebra with $G(H) \simeq \mathbb{S}_3$. Then either $H \simeq \mathfrak{B}(\mathcal{O}_2^3, \text{sgn})\#\mathbb{C}\mathbb{S}_3$ or $H \simeq \mathcal{A}(\mathbb{S}_3, \mathcal{O}_2^3, 1)$.