EXTENSIONS OF HOPF ALGEBRAS

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Preface

These notes are based on a series of lectures at the FaMAF, University of Córdoba, in October 1997.

To solve a (specially concrete) problem in Hopf algebra theory, we are often faced with a problem of classifying some Hopf algebra extensions. In this course, I intended to show some methods of computing those extensions in special form. Following the lecutres, these notes are divided into two independent parts. Part I discusses at basic level the extensions arising from finite groups, while Part II sketches quickly some recent results of mine on the extensions arising from Lie algebras.

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I. Hopf algebra extensions arising from finite groups

Throughout in these notes, we work over a fixed ground $\mathbf{k}$, unless otherwise stated.

1. **Crossed products of a group.** Let Γ be a group (with unit 1), R a commutative algebra over $\mathbf{k}$. Let $\rightharpoonup : \Gamma \times R \to R$ be an action of Γ giving algebra automorphisms of R, $\sigma : \Gamma \times \Gamma \to R^\times$ a (normalized) 2-cocycle for the Γ-module $R^\times$ (units of R). (Γ-module means $\mathbb{Z}\Gamma$-module, as usual.) Thus, σ satisfies

$$[x \rightharpoonup \sigma(y,z)]\sigma(x,yz) = \sigma(xy,z)\sigma(x,y), \tag{1.1}$$

$$\sigma(x,1) = \sigma(1,x) = 1 \quad \text{(normalizing condition)}, \tag{1.2}$$

where $x, y, z \in \Gamma$.

Let $R *_\sigma \Gamma$ be the algebra defined as follows: $R *_\sigma \Gamma = \bigoplus_{x \in \Gamma} Ru_x$ as $\mathbf{k}$-vector space, and it has the product given by $(\alpha u_x)(\beta u_y) = \alpha(x \rightharpoonup \beta)\sigma(x,y)u_{xy}$, where $\alpha, \beta \in R$, $x, y \in \Gamma$. It has unit u_1.

Definition 1.3. $R *_\sigma \Gamma$ is called the (left) Γ-*crossed product* of Γ over R determined by $\rightharpoonup, \sigma$. The set $\{u_x | x \in \Gamma\}$ is called a *basis*. The notion of right Γ-crossed product over R determined by an action $\leftharpoonup : R \times \Gamma \to R$ from the right and a 2-cocycle $\tau : \Gamma \times \Gamma \to R^\times$ for the right Γ-module $R^\times$ is defined analogously.

Remark 1.4. 1. If $\rightharpoonup, \sigma$ are both trivial, then $R *_\sigma \Gamma = R\Gamma$, the group ring.
2. If σ is trivial, then $R *_\sigma \Gamma = R \rtimes \Gamma$, the semidirect product.
3. $R = Ru_1$ is a subalgebra of $R *_\sigma \Gamma$. It is central iff $\rightharpoonup$ is trivial. In this case, $R *_\sigma \Gamma = R^t\Gamma$, the twisted group ring.

Note that $R *_\sigma \Gamma$ is a Γ-graded algebra with x-component Ru_x.

Proposition 1.5. *$R *_\sigma \Gamma$ is characterized as the Γ-graded algebra which contains an invertible element in each component and whose 1-component is R.*

Proof. In fact, the 1-component of $R *_\sigma \Gamma$ is R and each x-component Ru_x contains the invertible element u_x, with inverse

$$(u_x)^{-1} = \sigma^{-1}(x, x^{-1})u_{x^{-1}}.$$

Conversely, suppose that a Γ-graded algebra $\tilde{R} = \bigoplus_{x \in \Gamma} \tilde{R}_x$ has these properties. Chose for every $x \in \Gamma$ an invertible element u_x in the x-component. We may suppose $u_1 = 1$. Then $\tilde{R}_x = (\tilde{R}_x u_x^{-1})u_x \subseteq Ru_x \subseteq \tilde{R}_x$, whence $\tilde{R}_x = Ru_x$ and further $\tilde{R} = \bigoplus_{x \in \Gamma} Ru_x$. Define $\rightharpoonup$ and σ by

$$x \rightharpoonup \alpha = u_x \alpha u_x^{-1}, \tag{1.6}$$

$$\sigma(x,y) = u_x u_y u_{xy}^{-1}. \tag{1.7}$$

Then

$$(ru_x)(su_y) = r(u_x s u_{x^{-1}})u_x u_y u_{xy}^{-1} u_{xy} = r(x \rightharpoonup s)\sigma(x,y)u_{xy},$$

whence $\tilde{R} = R *_\sigma \Gamma$. Here the associativity of $\tilde{R}$ and the condition $u_1 = 1$ require Γ to act by algebra automorphisms, and σ to be a normalized 2-cocycle. $\square$

The definition of $\rightharpoonup$ above is independent of the choice of the basis $\{u_x\}$ since R is commutative: we say that $\tilde{R}$ *gives rise* to $\rightharpoonup$. It is clear that $R *_\sigma \Gamma$ gives rise to the originally given $\rightharpoonup$.

We want to classify all Γ-crossed products over R.

Definition 1.8. We say that $R *_\sigma \Gamma$ is *equivalent* to $R *_\tau \Gamma$ (and denote $R *_\sigma \Gamma \sim R *_\tau \Gamma$) if there is a map (necessarily an isomorphism) $f : R *_\sigma \Gamma \to R *_\tau \Gamma$ of Γ-graded algebras preserving each element in R, i.e. such that the following diagram commutes

$$\begin{tikzcd} R \arrow[r, equal] \arrow[d] & R \arrow[d] \\ R *_\sigma \Gamma \arrow[r, "f"] & R *_\tau \Gamma \end{tikzcd}$$

If $R *_\sigma \Gamma \sim R *_\tau \Gamma$, then these give rise to the same action (this is why the action $\rightharpoonup$ is not indicated in the notation $R *_\sigma \Gamma$.)

Fix an action $\rightharpoonup : \Gamma \times R \to R$, and define

$$\mathcal{E}(\Gamma, R, \rightharpoonup) := \left\{ \begin{array}{l} \text{equivalence classes of all the } \Gamma\text{-crossed} \\ \text{products over } R \text{ giving rise to } \rightharpoonup \end{array} \right\}.$$

Proposition 1.9. *The map $\sigma \mapsto R *_\sigma \Gamma$ induces a bijection*

$$H^2(\Gamma, R^\times) \simeq \mathcal{E}(\Gamma, R, \rightharpoonup),$$

where $H^2(\Gamma, R^\times)$ is the second cohomology group of Γ with coefficients in $R^\times$, i.e. the group (with respect to the point-wise product) of 2-cocycles modulo 2-coboundaries.

A 2-cocycle σ is called a 2-*coboundary* if there is a map

$$\nu : \Gamma \to R^\times \quad \text{with } \nu(1) = 1$$

such that

$$\sigma(x, y) = \partial\nu(x, y) := [x \rightharpoonup \nu(y)]\nu^{-1}(xy)\nu(x) \quad (x, y \in \Gamma).$$

Proof. It is left as an exercise. Hint: for f in Definition 1.8, set $\nu(x) = f(u_x)(u'_x)^{-1}$ and show $\sigma = \tau\partial(\nu)$. □

We wonder which product $\mathcal{E}(\Gamma, R, \rightharpoonup)$ inherits from $H^2(\Gamma, R^\times)$. Let $\tilde{R}$, $\tilde{R}'$ be Γ-crossed products over R giving rise to a fixed action $\rightharpoonup$. Then $\tilde{R} \otimes \tilde{R}'$ is a $\Gamma \times \Gamma$-crossed product over $R \otimes R$ ($\otimes = \otimes_{\mathbf{k}}$). Let $\tilde{R} \square \tilde{R}'$ denote the sum of all (x, x)-components ($x \in \Gamma$) in $\tilde{R} \otimes \tilde{R}'$, that is,

$$\tilde{R} \square \tilde{R}' := \bigoplus_{x \in \Gamma} \tilde{R}_x \otimes \tilde{R}'_x,$$

a Γ-crossed product over $R \otimes R$, which gives rise to the diagonal action of $\rightharpoonup$. Now, take $R \otimes R \to R$ the multiplication map, which is a morphism of Γ-modules. This endows $R \otimes R$ with a structure of an R-module. Form then the base extension

$$\tilde{R} \cdot \tilde{R}' := R \otimes_{R \otimes R} (\tilde{R} \square \tilde{R}').$$

This gives rise to $\rightharpoonup$, since

$$(u_x \otimes u'_x)(\alpha \otimes 1)(u_{x^{-1}} \otimes u'_{x^{-1}}) = (x \rightharpoonup \alpha) \otimes (x \rightharpoonup 1) = x \rightharpoonup \alpha.$$

This is the *Baer product* of $\tilde{R}$ and $\tilde{R}'$.

Corollary 1.10. *The Baer product induces a product on $\mathcal{E}(\Gamma, R, \rightharpoonup)$. The bijection in Proposition 1.9 preserves the product. Hence $\mathcal{E}(\Gamma, R, \rightharpoonup)$ forms an abelian group.* □

Exercise 1.11. *The tensor product $\tilde{R} \otimes_R \tilde{R}'$ of the left R-modules ${}_R\tilde{R}$ and ${}_R\tilde{R}'$ is naturally a $\Gamma \times \Gamma$ graded vector space with 1-component R. Show that $\tilde{R} \cdot \tilde{R}'$ is naturally identified with the direct sum of all (x, x)-components in $\tilde{R} \otimes_R \tilde{R}'$, where $x \in \Gamma$.*

2. **Presentation by generators and relations.** Let R be a commutative $\mathbf{k}$-algebra. An *R-ring* is an algebra $\tilde{R}$ (over $\mathbf{k}$) together with an algebra map $R \to \tilde{R}$ (we do not impose the image of R to be central in $\tilde{R}$). For example, $R *_\sigma \Gamma$ with the inclusion $R \hookrightarrow R *_\sigma \Gamma$, is an R-ring. It is often useful to present the R-ring $R *_\sigma \Gamma$ by generators and relations. As an example, suppose $\Gamma = C_n =< x >$, the cyclic group of order n (> 1) with generator x. Let $\tilde{R} = R *_\sigma C_n$ for some $\rightharpoonup, \sigma$. Define

$$\phi(\alpha) := x \rightharpoonup \alpha \ (\alpha \in R), \quad \xi := u_x^n.$$

We have then that

$$\phi \in \mathrm{Aut}(R), \quad \phi^n = \mathrm{id}; \quad \xi \in R^\times, \quad \phi(\xi) = \xi. \tag{2.1}$$

Moreover, the R-ring $\tilde{R}$ has a generator $u = u_x$ and satisfies

$$u\alpha = \phi(\alpha)u \ (\alpha \in R); \quad u^n = \xi 1. \tag{2.2}$$

Lemma 2.3. *These are defining relations for $\tilde{R}$.*

Proof. Denote by $\tilde{R}'$ the R-ring generated by u with relations (2.2). As an algebra, $\tilde{R}$ is generated by u and all elements in R with the relations (2.2) plus $\alpha\beta =$ the product of α and β in R $(\alpha, \beta \in R)$.

There is a unique R-ring map $f : \tilde{R}' \to \tilde{R}$ such that $f(u) = u_x$, and it is easy to see that this is surjective. Since

$$\tilde{R} = R1 \oplus Ru_x \oplus \cdots \oplus Ru_x^{n-1},$$

there is a unique left R-linear map $g : \tilde{R} \to \tilde{R}'$ such that $g(u_x^i) = u^i$ $(0 \le i < n)$. Clearly $f \circ g = \mathrm{id}$. Regard $\tilde{R}$ as a left $\tilde{R}'$-module along f, i.e., by $\alpha'\alpha = f(\alpha')\alpha$ $(\alpha' \in \tilde{R}',\ \alpha \in \tilde{R})$. Then one sees that g is left $\tilde{R}'$-linear. This, combined with $(g \circ f)(1) = 1$, yields that $g \circ f = \mathrm{id}$, and hence that f is an isomorphism. $\square$

Corollary 2.4. *Given ϕ, ξ which satisfy* (2.1)*, the R-ring $\tilde{R}'$ defined as above is a C_n-crossed product over R with x^i-component Ru^i. Conversely, every C_n-crossed product is obtained in this way.*

Proof. The second statement follows from the Lemma, and hence we have to prove the first. In the proof above, replace $\tilde{R}$ by

$$M = Rm_0 \oplus Rm_1 \oplus \cdots \oplus Rm_{n-1},$$

a free left R-module. Define an action of u on M by

$$u(\alpha m_i) = \begin{cases} \phi(\alpha) m_{i+1} & (0 \le i < n-1) \\ \phi(\alpha)\xi m_0 & (i = n-1) \end{cases},$$

where $\alpha \in R$. One sees that, for $\alpha, \beta \in R$, $0 \le i < n$,

$$u(\beta(\alpha m_i)) = \phi(\beta) u(\alpha m_i),$$
$$(u \circ \cdots \circ u)(\alpha m_i) = \phi^n(\alpha)\xi m_i = \xi \alpha m_i \quad (n \text{ times } u).$$

Hence the representation $R \to \mathrm{End}_k(M)$ is extended to an R-ring map $\tilde{R}' \to \mathrm{End}_k(M)$ so that M gives rise to a left $\tilde{R}'$-module. There is a unique left $\tilde{R}'$-linear map $f : \tilde{R}' \to M$ such that $f(1) = m_0$, while there is a unique left R-linear map $g : M \to \tilde{R}'$ such that $g(m_i) = u^i$ $(0 \le i < n)$. Clearly $f \circ g = \mathrm{id}$. One sees that g is left $\tilde{R}'$-linear and $(g \circ f)(1) = 1$, whence f is an isomorphism. We have

$$\tilde{R}' = R1 \oplus Ru \oplus \cdots \oplus Ru^{n-1},$$

a C_n-crossed product over R. $\square$

We now fix an action $\rightharpoonup : C_n \times R \to R$ of algebra automorphisms. Let ξ be in the subgroup $(R^\times)^{C_n}$ of C_n-invariants in $R^\times$. We write $\tilde{R}_\xi$ for the C_n-crossed product $\tilde{R}'$ over R defined for $\phi := \rightharpoonup$ and ξ.

Proposition 2.5. *The map $\xi \mapsto \tilde{R}_\xi$ induces an isomorphism*

$$(R^\times)^{C_n}/N(R^\times) \simeq \mathcal{E}(C_n, R, \rightharpoonup),$$

where $N(R^\times)$ denotes the image of the norm map

$$N : R^\times \to (R^\times)^{C_n},\ N(\eta) = \prod_{i=0}^{n-1} (x^i \rightharpoonup \eta).$$

Proof. It follows from the Corollary that the correspondence gives a surjection $(R^\times)^{C_n} \to \mathcal{E}(C_n, R, \rightharpoonup)$, which is seen to preserve the product. Suppose that $f : \tilde{R}_\xi \to \tilde{R}_{\xi'}$ gives an equivalence. There exists $\eta \in R^\times$ such that $f(u) = \eta u'$. Then

$$\xi = f(u^n) = f(u)^n = (\eta u')^n = N(\eta)\xi'.$$

Conversely, if $\xi = N(\eta)\xi'$ for some $\eta \in R^\times$, the unique R-ring map $f : \tilde{R}_\xi \to \tilde{R}_{\xi'}$ given by $f(u) = \eta u'$, gives an equivalence. $\square$

Exercise 2.6. *Prove the Proposition by using the following result on group cohomologies:*

$$H^2(C_n, M) \simeq M^{C_n}/N(M).$$

3. **Extensions of finite-dimensional Hopf algebras.** To motivate what follows, we think about Hopf algebras as being "morally" equivalent to groups in a contravariant way, as follows[1]

$$
\begin{array}{ccccccc}
A & A & A & & G & G & G \\
\Delta\downarrow & \varepsilon\downarrow & \mathcal{S}\downarrow & \rightsquigarrow & \uparrow\text{prod.} & \uparrow\text{unit} & \uparrow\text{inverse} \\
A\otimes A & \mathbf{k} & A & & G\times G & \{1\} & G
\end{array}
$$

$$\text{a Hopf subalgebra } (K\subset A) \leftrightsquigarrow \text{ a quotient group } (G\twoheadrightarrow \bar{G})$$

$$\text{a quotient Hopf algebra } (A\twoheadrightarrow \bar{A}) \leftrightsquigarrow \text{ a subgroup } (F\subset G)$$

Note that the kernel of $G\twoheadrightarrow\bar{G}$ is the equalizer of the maps $G\rightrightarrows G\times\bar{G}$ defined by $x\mapsto(x,\bar{x})$, $x\mapsto(x,\bar{1})$. Consequently, the (left) cokernel of $K\hookrightarrow A$ should be defined to be the coequalizer of the linear maps $A\otimes K\rightrightarrows A$ given by $a\otimes b\mapsto ab$, $a\otimes b\mapsto a\varepsilon(b)$; that is, A/AK^+, where $K^+=\ker(\varepsilon:K\to\mathbf{k})$. (This is not necessarily a quotient Hopf algebra, but a quotient left A-module coalgebra.)

Note that the right cokernel G/F of $F\hookrightarrow G$ is the coequalizer of the maps $G\times F\rightrightarrows G$ defined by $(x,y)\mapsto xy$, $(x,y)\mapsto x$. Consequently, the right kernel $A^{\mathrm{co}\bar{A}}$ of $A\twoheadrightarrow\bar{A}$ should be defined to be the equalizer of the linear maps $A\rightrightarrows A\otimes\bar{A}$ given by $a\mapsto a_{(1)}\otimes\overline{a_{(2)}}$, $a\mapsto a\otimes\bar{1}$ (this is a left coideal subalgebra). The left kernel ${}^{\mathrm{co}\bar{A}}A$ is defined analogously.

Lemma-Definition 3.1. *Let* $(A)=K\xrightarrow{\iota}A\xrightarrow{\pi}H$ *be a sequence of finite-dimensional Hopf algebras and Hopf algebra maps, and suppose that* ι *is an injection and* π *is a surjection (we often suppose that* ι *is an inclusion and* π *a quotient map). The following are equivalent:*

1. $A/K^+A=H$,
2. $A/AK^+=H$,
3. $K=A^{\mathrm{co}H}$,
4. $K={}^{\mathrm{co}H}A$.

Such a sequence (A) *is called an* extension of H by K.

Proof. $1\Rightarrow 2$. Apply the antipode (bijective) to each side of $A/K^+A=H$ to get $\mathcal{S}(A)/\mathcal{S}(A)\mathcal{S}(K^+)=\mathcal{S}(H)$. Then $A/AK^+=H$. Similarly, $2\Rightarrow 1$.

$1\Rightarrow 3$. There is an isomorphism $A\otimes A\simeq A\otimes A$ given by $a\otimes b\mapsto ab_{(1)}\otimes b_{(2)}$, with inverse $a\otimes b\mapsto a\mathcal{S}(b_{(1)})\otimes b_{(2)}$. One sees that this induces an isomorphism $A\otimes_K A\simeq A\otimes A/K^+A$. If $A/K^+A=H$, it follows that

$$
\begin{aligned}
A^{\mathrm{co}H} &= \{a\in A|a_{(1)}\otimes\pi(a_{(2)})=a\otimes\pi(1)\}\\
&= \{a\in A|1\otimes a=a\otimes 1 \text{ in } A\otimes_K A\}\\
&= K
\end{aligned}
$$

since K, a Frobenius algebra, is a left (and right) K-module-direct summand in A.

$2\Rightarrow 4$. Similar.

$4\Rightarrow 2$. Dualizing (A), we have $K^*\xleftarrow{\iota^*}A^*\xleftarrow{\pi^*}H^*$. Suppose 4, then K is the equalizer of $A\rightrightarrows H\otimes A$, whence K^* is the coequalizer of $A^*\leftleftarrows H^*\otimes A^*$, i.e. $K^*=A^*/(H^*)^+A^*$. From the proof of "1⇒3", $H^*=(A^*)^{\mathrm{co}K^*}$, which implies 2.

$3\Rightarrow 1$. Similar. □

Proposition 3.2. *Every extension* $(A)=K\xrightarrow{\iota}A\xrightarrow{\pi}H$ *of finite-dimensional Hopf algebras is cleft in the following sense: there exists an isomorphism* $\theta:A\xrightarrow{\simeq}K\otimes H$ *of left* K*-modules and right* H*-comodules which preserves unit and counit (*A *is a left* K*-module along* ι *and a right* H*-comodule along* π*.) For such* θ*, we define*

$$\phi:H\to A,\quad \phi(h)=\theta^{-1}(1\otimes h).$$

[1] This correspondence is a functor if for instance we restrict to commutative finite-dimensional Hopf algebras from one side and finite groups from the other, provided that $\mathbf{k}$ is algebraically closed.

This is a right H-colinear map which is invertible under the convolution product and preserves unit and counit. We define also

$$\gamma : A \to K, \quad \gamma(a) = (id \otimes \varepsilon)\theta(a).$$

This is a left K-linear map, which is invertible and preserves unit and counit.

Proof. See [Sch, Thm 2.4] or [MD, Thm 3.5]. □

Let F, G be finite groups, and consider the special case where

$$H = \mathbf{k}F, \text{ the group-like Hopf algebra,}$$
$$K = \mathbf{k}^G = (\mathbf{k}G)^*, \text{ the dual Hopf algebra of } \mathbf{k}G.$$

Thus $K = \bigoplus_{s \in G} \mathbf{k}e_s$, where $\{e_s\}$ is the dual basis of $\{s\}$, and this forms a commutative Hopf algebra with the following structure

$$e_s e_t = \delta_{st} e_s, \text{ unit} = \sum_{s \in G} e_s,$$
$$\Delta(e_s) = \sum_{t \in G} e_t \otimes e_{t^{-1}s}, \ \varepsilon(e_s) = \delta_{1s},$$
$$\mathcal{S}(e_s) = e_{s^{-1}}.$$

Proposition 3.3. *Let $(A) = \mathbf{k}^G \xrightarrow{\iota} A \xrightarrow{\pi} \mathbf{k}F$ be an extension and take θ, ϕ, γ as in the previous Proposition. Set $u_x = \phi(x)$ $(x \in F)$, $v_s = \gamma^*(s)$ $(s \in G)$. Then*

1. *A is a left F-crossed product over $\mathbf{k}^G$ with basis $\{u_x\}_{x \in F}$ (in particular, $u_1 = 1$).*
2. *A^* is a right G-crossed product over $\mathbf{k}^F$ with basis $\{v_s\}_{s \in G}$.*
3. *$< e_t u_x, v_s e_y >= \delta_{ts}\delta_{xy}$ $(x, y \in F, \ s, t \in G)$.*

Proof. 1. Being a right $\mathbf{k}F$-comodule algebra with structure $\rho = (\mathrm{id} \otimes \pi) \circ \Delta$, A is an F-graded algebra with x-component $A_x = \{a \in A | \rho(a) = a \otimes x\}$ $(x \in F)$. Since ϕ is $\mathbf{k}F$-colinear, $u_x \in A_x$. Since ϕ is invertible, $u_x \in A^\times$. Since $\phi(1) = 1$, $u_1 = 1$.

2. Note that $(A^*) = \mathbf{k}G \xleftarrow{\iota^*} A^* \xleftarrow{\pi^*} \mathbf{k}^F$ is an extension, and that $\gamma^* : \mathbf{k}G \to A^*$ is unit preserving, left $\mathbf{k}G$-colinear and invertible. The proof then works as in 1.
3. $< e_t u_x, v_s e_y >=< \theta^*(e_t \otimes x), \theta^{-1}(s \otimes e_y) >< e_t, s >< x, e_y >= \delta_{st}\delta_{xy}$.

□

Corollary-Definition 3.4. *Let (A), (A') be extensions of $\mathbf{k}F$ by $\mathbf{k}^G$, and suppose that there is a Hopf algebra map $f : A \to A'$ making the following diagram commutative*

$$\begin{array}{ccccc} \mathbf{k}^G & \longrightarrow & A & \longrightarrow & \mathbf{k}F \\ \| & & {\scriptstyle f}\downarrow & & \| \\ \mathbf{k}^G & \longrightarrow & A' & \longrightarrow & \mathbf{k}F \end{array}$$

Then, f is an isomorphism. In this case, (A) and (A') are said to be equivalent.

Proof. This holds since f is in particular a map of F-graded algebras preserving each element in $\mathbf{k}^G$. □

Exercise 3.5. *Generalize the Corollary to arbitrary extensions of finite-dimensional Hopf algebras (or more generally to cleft extensions of Hopf algebras).*

4. **Extensions of $\mathbf{k}F$ by $\mathbf{k}^G$.** Let A be a vector space which has the properties 1–3 of Proposition 3.3. Then the crossed products A, A^* give rise to actions by algebra automorphisms

$$\rightharpoonup: F\times \mathbf{k}^G \to \mathbf{k}^G, \quad \leftharpoonup: \mathbf{k}^F \times G \to \mathbf{k}^F$$

respectively. These are induced from actions of permutations

$$\triangleleft : G\times F \to G, \quad \triangleright : G\times F\to F$$

so that

$$x \rightharpoonup e_s = e_{s\triangleleft x^{-1}},\ e_x \leftharpoonup s = e_{s^{-1}\triangleright x} \quad (x\in F,\ s\in G).$$

The product of A is given, with respect to some basis $\{u_x\}_{x\in F}$, by means of a 2-cocycle

$$\sigma : F\times F \to (\mathbf{k}^G)^\times = \mathrm{Map}(G, \mathbf{k}^\times).$$

This is viewed as a map

$$\sigma : G\times F\times F\to \mathbf{k}^\times,$$

whose value is written as $\sigma(s; x, y)$. Then the (normalized) 2-cocycle condition is discribed as follows

$$\begin{cases} \sigma(s\triangleleft x; y, z)\sigma(s; x, yz) = \sigma(s; xy, z)\sigma(s; x, y) \\ \sigma(s; 1, x) = 1 = \sigma(s; x, 1)\end{cases} \quad (s\in G,\ x, y, z\in F). \tag{4.1}$$

The product of A is given by

$$(e_s u_x)(e_t u_y) = e_s(x\rightharpoonup e_t)u_x u_y = e_s(x\rightharpoonup e_t)\sigma(x, y)u_{xy} \tag{4.2}$$

$$= e_s\delta_{s,t\triangleleft x^{-1}}\sigma(x, y)u_{xy} = \delta_{s\triangleleft x,t}\sigma(s; x, y)e_s u_{xy}. \tag{4.3}$$

Similarly, the product of A^* is given by

$$(v_s e_x)(v_t e_y) = v_{st}e_y\delta_{x,t\triangleright y}\tau(s, t; y),$$

where $\tau : G\times G\times F\to \mathbf{k}^\times$ is a map satisfying

$$\begin{cases} \tau(st, p; x)\tau(s, t; p\triangleright x) = \tau(t, p; x)\tau(s, tp; x) \\ \tau(s, 1; x) = 1 = \tau(1, s; x)\end{cases} \quad (x\in F,\ s, t, p\in G). \tag{4.4}$$

We want to recover now the coalgebra structure of A from the algebra structure of A^*. It follows from 3 in Proposition 3.3 that the left $\mathbf{k}G$-comodule structure of A^* is dualized exactly to the left $\mathbf{k}^G$-module structure of A, i.e., $\mathbf{k}^G\otimes A\to A$ is dual to $(\iota^*\otimes \mathrm{id})\Delta_{A^*} : A^*\to \mathbf{k}G\otimes A^*$. Hence Δ_A, ε_A are the left $\mathbf{k}^G$-linear maps ($A\otimes A$ is a left $\mathbf{k}^G$-module through its comultiplication, and $\mathbf{k}$ is a left $\mathbf{k}^G$-module through its counit) determined by

$$\Delta(u_x) = \sum_{s,t\in G}\tau(s, t; x)e_s u_{t\triangleright x}\otimes e_t u_x, \tag{4.5}$$

$$\varepsilon(u_x) = 1 \tag{4.6}$$

Proposition 4.7. 1. *A is a bialgebra iff the following holds for all $x, y\in F,\ s, t\in G$.*

$$\sigma(st; x, y)\tau(s, t; xy) = \sigma(s; t\triangleright x, (t\triangleleft x)\triangleright y)\sigma(t; x, y)\tau(s, t; x)\tau(s\triangleleft(t\triangleright x), t\triangleleft x; y) \tag{4.8}$$

$$\sigma(1; x, y) = 1 = \tau(s, t; 1) \tag{4.9}$$

$$\begin{cases} s\triangleright xy = (s\triangleright x)((s\triangleleft x)\triangleright y) \\ st\triangleleft x = (s\triangleleft(t\triangleright x))(t\triangleleft x)\end{cases} \tag{4.10}$$

2. *If these hold, A is a Hopf algebra. Moreover, it forms an extension of $\mathbf{k}F$ by $\mathbf{k}^G$ together with*

$$\iota : \mathbf{k}^G\to A, \quad \iota(e_s) = e_s u_1,$$

$$\pi : A\to \mathbf{k}F, \quad \pi(e_s u_x) = \delta_{1,s}x$$

We write $A = \mathbf{k}^G *_{\sigma,\tau}\mathbf{k}F$ (with $\triangleright$, $\triangleleft$ kept in mind), and denote the extension by $(\mathbf{k}^G *_{\sigma,\tau}\mathbf{k}F)$.

Proof. 1. Δ, ε are algebra maps iff the following hold for all $x, y \in F,\ s \in G$.

$$\Delta(u_1) = u_1 \otimes u_1 \tag{4.11}$$
$$\Delta(u_x u_y) = \Delta(u_x)\Delta(u_y) \tag{4.12}$$
$$\Delta(u_x e_s) = \Delta(u_x)\Delta(e_s) \tag{4.13}$$
$$\varepsilon(u_x u_y) = \varepsilon(u_x)\varepsilon(u_y) \tag{4.14}$$
$$\varepsilon(u_x e_s) = \varepsilon(u_x)\varepsilon(e_s) \tag{4.15}$$

We see that

(4.11) $\iff$ the right equality in (4.9) holds and $t \triangleright 1 = 1$ $(t \in G)$.
(4.12) $\iff$ (4.8) and the first equation in (4.10).
(4.13) $\iff$ the second equation in (4.10) holds.
(4.14) $\iff$ the left equality in (4.9) holds.
(4.15) $\iff$ $1 \triangleleft x = 1$ $(x \in F)$.

In fact, we see for example for (4.12) that

$$LHS = \sum_{s,t \in G} \sigma(st; x, y)\tau(s, t; xy) e_s u_{t\triangleright xy} \otimes e_t u_{xy},$$

$$RHS = \sum_{s,t,\bar{s},\bar{t} \in G} \tau(s, t; x)\tau(\bar{s}, \bar{t}; y) e_s u_{t\triangleright x} e_{\bar{s}} u_{\bar{t}\triangleright y} \otimes e_t u_x e_{\bar{t}} u_y$$

(we take $\bar{s} = s \triangleleft (t \triangleright x)$, $\bar{t} = t \triangleleft x$ since otherwise the summand would vanish)

$$= \sum_{s,t \in G} \tau(s, t; x)\tau(s \triangleleft (t \triangleright x), t \triangleleft x; y)\sigma(s; t \triangleright x, (t \triangleleft x) \triangleright y)\sigma(t; x, y) e_s u_{(t\triangleright x)((t\triangleleft x)\triangleright y)} \otimes e_t u_{xy}.$$

These yield the second " $\iff$ ". Note that the conditions

$$s \triangleright 1 = 1\ (s \in G), \quad 1 \triangleleft x = 1\ (x \in F)$$

follow from (4.10). Then Part 1 follows.

2. The linear map $\mathcal{S} : A \to A$ determined by

$$\mathcal{S}(e_s u_x) = \sigma((s \triangleright x)^{-1}, s \triangleright x; (s \triangleleft x)^{-1})^{-1} \tau(s^{-1}, s; x)^{-1} e_{(s\triangleleft x)^{-1}} u_{(s\triangleright x)^{-1}}$$

gives the antipode. See [Ho, Prop 3.13]. □

Lemma-Definition 4.16. *The maps*

$$G \xleftarrow{\triangleleft} G \times F \xrightarrow{\triangleright} F$$

give actions of permutations and satisfy 4.10 iff the cartesian product $F \times G$ forms a group with respect to the product defined by

$$(x, s)(y, t) = (x(s \triangleright y), (s \triangleleft y)t)$$

whose unit is $(1, 1)$. In this case (F, G) is called a matched pair*, and the group $F \times G$ is denoted by $F \bowtie G$.*

Proof. Straightforward. □

If (F, G) is a matched pair, then $F = F \times \{1\}$ and $G = \{1\} \times G$ are subgroups of $F \bowtie G$ such that the product map $F \times G \to F \bowtie G$ is a bijection. Conversely, if F and G are subgroups of a group Σ such that the product map $\mu : F \times G \to \Sigma$ is a bijection, then (F, G) forms a matched pair in a unique way so that $\mu : F \bowtie G \to \Sigma$ is an isomorphism. The actions are defined by the equation $sx = (s \triangleright x)(s \triangleleft x)$ for $s \in G$ and $x \in F$.

It follows from Proposition 3.3 that every extension (A) of $\mathbf{k}F$ by $\mathbf{k}^G$ is equivalent to some $(\mathbf{k}^G *_{\sigma,\tau} \mathbf{k}F)$. The actions $\triangleright, \triangleleft$ associated with $\mathbf{k}^G *_{\sigma,\tau} \mathbf{k}F$ are independent of choice of bases. (In other words, if $(\mathbf{k}^G *_{\sigma,\tau} \mathbf{k}F) \sim (\mathbf{k}^G *_{\sigma',\tau'} \mathbf{k}F)$ then the associated actions are the same.)

Definition 4.17. In this case, (A) is said to be *associated with* the matched pair $(F,G) = (F,G,\triangleright,\triangleleft)$. For a fixed matched pair (F,G), we denote by

$$\mathrm{Opext}(\mathbf{k}F, \mathbf{k}^G)$$

the set of the equivalence classes of all extensions of $\mathbf{k}F$ by $\mathbf{k}^G$ associated with (F,G).

5. **Cohomological description of Opext**$(\mathbf{k}F, \mathbf{k}^G)$. We now want to describe $\mathrm{Opext}(\mathbf{k}F, \mathbf{k}^G)$ by some cohomology group.

Let us fix a matched pair (F,G), and denote $\Sigma = F \bowtie G$. Let

$$B_{\cdot} = 0 \leftarrow \mathbb{Z} \xleftarrow{\varepsilon} B_0 \xleftarrow{d_1} B_1 \xleftarrow{d_2} B_2 \leftarrow \cdots$$

be the normalized bar resolution of the trivial left F-module $\mathbb{Z}$. This means

$$B_n = \oplus_{1\neq x_i\in F}\mathbb{Z}F[x_1|\dots|x_n] \quad \text{(a free left } \mathbb{Z}F\text{-module)},$$

$$d_n[x_1|\dots|x_n] = x_1[x_2|\dots|x_n] + \Big(\sum_{i=1}^{n-1}(-1)^i[x_1|\dots|x_ix_{i+1}|\dots|x_n]\Big) + (-1)^n[x_1|\dots|x_{n-1}] \quad (F\text{-linear}),$$

$$\varepsilon : B_0 = \mathbb{Z}F \to \mathbb{Z},\ \varepsilon(x) = 1 \quad (x \in F).$$

We define an action $G \curvearrowright B_n$ of $\mathbb{Z}$-linear endomorphisms by

$$s(x[x_1|\dots|x_n]) = (s\triangleright x)[(s\triangleleft x)\triangleright x_1|(s\triangleleft xx_1)\triangleright x_2|\dots|(s\triangleleft xx_1\cdots x_{n-1})\triangleright x_n].$$

One sees then that each B_n is a left G-module and further a left Σ-module and that d_n, ε turn out to be Σ-linear. Similarly, let

$$B'_{\cdot} = 0 \leftarrow \mathbb{Z} \xleftarrow{\varepsilon} B'_0 \xleftarrow{d'_1} B'_1 \xleftarrow{d'_2} B'_2 \leftarrow \cdots$$

be the normalized bar resolution of the trivial *right* G-module $\mathbb{Z}$. We make it into a resolution of right Σ-modules by defining the action $B'_n \curvearrowleft F$ given by

$$([s_n|\dots|s_1]s)x = [s_n\triangleleft(s_{n-1}\cdots s_1s\triangleright x)|\dots|s_2\triangleleft(s_1s\triangleright x)|s_1\triangleleft(s\triangleright x)](s\triangleleft x).$$

We further regard $B'_{\cdot}$ as a resolution of *left* Σ-modules by twisting the action via the inverse.

We now tensor $B'_{\cdot}$ and $B_{\cdot}$ over $\mathbb{Z}$ to obtain the double complex

$$\begin{array}{ccccccc}
 & & \vdots & & \vdots & & \\
 & & \downarrow{\scriptstyle d'_2\otimes\mathrm{id}} & & \downarrow{\scriptstyle -d'_2\otimes\mathrm{id}} & & \\
B_{\cdot\cdot} & = & B'_1\otimes B_0 & \xleftarrow{\mathrm{id}\otimes d_1} & B'_1\otimes B_1 & \longleftarrow & \cdots \\
 & & \downarrow{\scriptstyle d'_1\otimes\mathrm{id}} & & \downarrow{\scriptstyle -d'_1\otimes\mathrm{id}} & & \\
 & & B'_0\otimes B_0 & \xleftarrow{\mathrm{id}\otimes d_1} & B'_0\otimes B_1 & \longleftarrow & \cdots ,
\end{array}$$

where we applied the usual sign trick, changing the sign of the differentials in odd columns.

Exercise 5.1. *Show that each left Σ-module $B'_q \otimes B_p$ is free with basis $[s_q|\dots|s_1]\otimes[x_1|\dots|x_p]$, where $1 \neq x_i \in F$, $1 \neq s_i \in G$.*

We regard $\mathbf{k}^\times$ as trivial left Σ-module and apply $\mathrm{Hom}_\Sigma(-,\mathbf{k}^\times)$ to $B_{..}$ to obtain the double cochain complex

$$\begin{array}{cccccc} & & \vdots & & \vdots & \\ & & \uparrow & & \uparrow & \\ D^{\cdot\cdot} & = & D^{01} & \longrightarrow & D^{11} & \longrightarrow \cdots \\ & & \uparrow & & \uparrow & \\ & & D^{00} & \longrightarrow & D^{10} & \longrightarrow \cdots . \end{array}$$

By Exercise 5.1, $D^{pq} = \mathrm{Hom}_\Sigma(B'_q \otimes_{\mathbb{Z}} B_p, \mathbf{k}^\times)$ is identified with

$$\mathrm{Map}_+(G^q \times F^p, \mathbf{k}^\times) := \left\{ \begin{array}{l} \text{the abelian group of the maps } f : G^q \times F^p \to \mathbf{k}^\times \text{ such that} \\ f(s_1, \dots, s_1; x_1, \dots x_p) = 1 \text{ if either } s_q, \dots, s_1, x_1, \dots \text{ or } x_p = 1 \end{array} \right\}.$$

By deleting the first row and the first column from $D^{\cdot\cdot}$, we obtain

$$\begin{array}{cccccc} & & \vdots & & \vdots & \\ & & \uparrow & & \uparrow & \\ C^{\cdot\cdot} & = & \mathrm{Map}_+(G^2 \times F^1, \mathbf{k}^\times) & \xrightarrow{\partial} & \mathrm{Map}_+(G^2 \times F^2, \mathbf{k}^\times) & \longrightarrow \cdots \\ & & {\scriptstyle\partial'}\uparrow & & {\scriptstyle\partial'}\uparrow & \\ & & \mathrm{Map}_+(G^1 \times F^1, \mathbf{k}^\times) & \xrightarrow{\partial} & \mathrm{Map}_+(G^1 \times F^2, \mathbf{k}^\times) & \longrightarrow \cdots . \end{array}$$

Let $f \in \mathrm{Map}_+(G^q \times F^p, \mathbf{k}^\times)$. Then ∂f, $\partial' f$ are given as follows:

$$\begin{aligned} \partial f(s_q, \dots, s_1; x_1, \dots, x_{p+1}) = \\ & f(s_q \triangleleft (s_{q-1} \cdots s_1 \triangleright x_1), \dots, s_2 \triangleleft (s_1 \triangleright x_1), s_1 \triangleleft x_1; x_2, \dots, x_{p+1}) \\ & \times \prod_{i=1}^{p} f(s_1, \dots, s_1; x_1, \dots x_i x_{i+1}, \dots, x_{p+1})^{(-1)^i} \\ & \times f(s_q, \dots, s_1; x_1, \dots x_p)^{(-1)^{p+1}}; \\ \partial' f(s_{q+1}, \dots, s_1; x_1, \dots, x_p)^{(-1)^p} = \\ & f(s_{q+1}, \dots, s_2; s_1 \triangleright x_1, (s_1 \triangleleft x_1) \triangleright x_2, \dots, (s_1 \triangleleft x_1 \cdots x_{p-1}) \triangleright x_p) \\ & \times \prod_{i=1}^{q} f(s_{q+1}, \dots, s_{i+1} s_i, \dots, s_1; x_1, \dots, x_p)^{(-1)^i} \\ & \times f(s_q, \dots, s_1; x_1, \dots, x_p)^{(-1)^{q+1}}. \end{aligned}$$

(To see that ∂f is as above, note that $[s_q|\dots|s_1] \otimes x_1[x_2|\dots|x_{p+1}] = x_1([s_1|\dots|s_1]x_1 \otimes [x_2|\dots|x_{p+1}])$.)

The maps $\sigma : G \times F^2 \to \mathbf{k}^\times$, $\tau : G^2 \times F \to \mathbf{k}^\times$ satisfy (4.1), (4.4), (4.8) and (4.9) iff (σ, τ) is a 1-cocycle in the total complex $\mathrm{Tot}(C^{\cdot\cdot})$. Therefore we obtain by Proposition 4.7 Part 2 the map $Z^1(\mathrm{Tot}(C^{\cdot\cdot})) \to \mathrm{Opext}(\mathbf{k}F, \mathbf{k}^G)$ given by $(\sigma, \tau) \mapsto$ the equivalence class of $(\mathbf{k}^G *_{\sigma,\tau} \mathbf{k}F)$.

Proposition 5.2. *This map induces a bijection*

$$H^1(\mathit{Tot}(C^{\cdot\cdot})) \simeq \mathit{Opext}(\mathbf{k}F, \mathbf{k}^G).$$

Proof. It follows from Proposition 4.7 Part 1 that the map is a surjection. Let (σ, τ), (σ', τ') be 1-cocycles. If ν is a 0-cochain such that $\sigma' = \sigma \partial \nu$, $\tau' = \tau \partial' \nu$, then $e_s u'_x \mapsto \nu(s; x) e_s u_x$ gives an equivalence $(\mathbf{k}^G *_{\sigma',\tau'} \mathbf{k}F) \sim (\mathbf{k}^G *_{\sigma,\tau} \mathbf{k}F)$. Conversely, any equivalence is given in this way by some ν, in which case $\sigma' = \sigma \partial \nu$, $\tau' = \tau \partial' \nu$ by a simple computation. $\square$

We wonder now which product $\mathrm{Opext}(\mathbf{k}F, \mathbf{k}^G)$ inherits from $H^1(\mathrm{Tot}(C^{\cdot\cdot}))$. Let (A), (A') be extensions associated with the fixed matched pair (F, G). By forming the Baer product of the left F-crossed products A, A' over $\mathbf{k}^G$, we obtain the $\mathbf{k}^G$-ring $A \cdot A'$. It follows from the right version of Exercise 1.11 that this is the dual of the Baer product $A^* \cdot A'^*$ of the right G-crossed products A^*, A'^* over $\mathbf{k}^F$. Hence $A \cdot A'$ is a coalgebra giving a coalgebra map onto $\mathbf{k}F$. If we present A, A' in the form $A = (\mathbf{k}^G *_{\sigma,\tau} \mathbf{k}F)$, $A' = (\mathbf{k}^G *_{\sigma',\tau'} \mathbf{k}F)$, we see that $A \cdot A' = \mathbf{k}^G *_{\sigma\sigma',\tau\tau'} \mathbf{k}F$, and hence obtain an extension $(A \cdot A')$ associated with (F, G).

Corollary 5.3. *The product defined by*

$$(A) \cdot (A') = (A \cdot A')$$

induces a product in Opext$(\mathbf{k}F, \mathbf{k}^G)$. Then the bijection in Proposition 5.2 preserves product and hence Opext$(\mathbf{k}F, \mathbf{k}^G)$ forms an abelian group. □

$\mathrm{Opext}(\mathbf{k}F, \mathbf{k}^G)$ is called the Opext group associated with the matched pair (F, G). Its unit is the element $(\mathbf{k}^G *_{1,1} \mathbf{k}F)$, which is called *the split extension* (since in this case ι has a coalgebra splitting and π has an algebra splitting) and is denoted simply by $(\mathbf{k}^G * \mathbf{k}F)$. We denote by $\mathrm{Aut}(\mathbf{k}^G * \mathbf{k}F)$ the group of the auto-equivalences of $(\mathbf{k}^G * \mathbf{k}F)$.

Exercise 5.4. *Let ν be a 0-cocycle in Tot$(C^{\cdot\cdot})$. Show that $e_s u_x \mapsto \nu(s; x) e_s u_x$ gives an auto-equivalence of $(\mathbf{k}^G * \mathbf{k}F)$. Prove that this gives an isomorphism $H^0(\mathrm{Tot}(C^{\cdot\cdot})) \simeq \mathrm{Aut}(\mathbf{k}^G * \mathbf{k}F)$. (Hint: see the proof of Proposition 5.2.)*

Exercise 5.5. *Let (F, G) be a matched pair and define new actions $F \xleftarrow{\triangleleft'} F \times G \xrightarrow{\triangleright'} G$ by*

$$x \triangleleft' s = (s^{-1} \triangleright x^{-1})^{-1}, \quad x \triangleright' s = (s^{-1} \triangleleft x^{-1})^{-1}.$$

Then, $(G, F, \triangleright', \triangleleft')$ is a matched pair (arising from the factorization $(s, x) \mapsto (s^{-1})^{\mathrm{op}} (x^{-1})^{\mathrm{op}}$, which gives $G \times F \xrightarrow{\simeq} (F \bowtie G)^{\mathrm{op}}$), and hence the Opext group Opext$(\mathbf{k}G, \mathbf{k}^F)$ is defined. Show that the map $(A) \mapsto (A^*)$ gives an isomorphism Opext$(\mathbf{k}F, \mathbf{k}^G) \simeq$ Opext$(\mathbf{k}G, \mathbf{k}^F)$.

Exercise 5.6. *Define the category of all matched pairs (F, G). Show that $(F, G) \mapsto$ Opext$(\mathbf{k}F, \mathbf{k}^G)$ gives a group-valued functor. (This implies that, if $(F, G) \simeq (F', G')$, then Opext$(\mathbf{k}F, \mathbf{k}^G) \simeq$ Opext$(\mathbf{k}F', \mathbf{k}^{G'})$.)*

6. **Sample computation I.** The trivial action $\triangleright : G \times F \to F$ and an action $\triangleleft : G \times F \to G$ of group-automorphisms make (F, G) into a matched pair, so that $F \bowtie G = F \ltimes G$, the semi-direct product.

As an example, we suppose

$$F = C_2 = \langle x \mid x^2 = 1 \rangle,$$
$$G = C_n \times C_n = \langle s, t \mid s^n = 1 = t^n, \ ts = st \rangle,$$

with $\triangleright$ trivial and $\triangleleft$ given by

$$s^i t^j \triangleleft x = s^j t^i \quad (i, j \in \mathbb{Z}/n).$$

Let us consider the matched pair $(C_2, C_n \times C_n)$ with these actions.

Theorem 6.1. *If $(\mathbf{k}^\times)^n = \mathbf{k}^\times$, then the Opext group associated with the matched pair just defined is given by*

$$\mathrm{Opext}(\mathbf{k}C_2, \mathbf{k}^{C_n \times C_n}) \simeq \boldsymbol{\mu}_n(\mathbf{k}),$$

where $\boldsymbol{\mu}_n(\mathbf{k})$ is the group of n-th roots of 1 in $\mathbf{k}$.

Proof. Define $K := \mathbf{k}^{C_n \times C_n} = \bigoplus_{i,j \in \mathbb{Z}/n} \mathbf{k} e_{ij}$, where $\{e_{ij}\}$ is the dual basis of $\{s^i t^j\}$. Let $\zeta \in \boldsymbol{\mu}_n(\mathbf{k})$, and define an extension (A_ζ) associated with the matched pair as follows: first, define $A = A_\zeta$ to be the K-ring generated by u with relations

$$u^2 = \sum_{i,j} \zeta^{ij} e_{ij}, \ u e_{ij} = e_{ji} u.$$

Then, by the first part of Corollary 2.4,

$$A = K1 \oplus Ku \quad \text{(a free left } K\text{-module)}.$$

Next, define K-ring maps $\Delta : A \to A \otimes A,\ \varepsilon : A \to \mathbf{k}$ by

$$\Delta(u) = \sum_{ijpq} \zeta^{jp} e_{ij} u \otimes e_{pq} u,\ \varepsilon(u) = 1.$$

To show that Δ is well defined, we see that

$$\begin{aligned}\Delta(u)^2 &= \sum_{\substack{ijpq\\IJPQ}} \zeta^{jp+JP} e_{ij} e_{JI} u^2 \otimes e_{pq} e_{QP} u^2\\ &= \sum_{ijpq} \zeta^{jp+iq} e_{ij} u^2 \otimes e_{pq} u^2\\ &= \sum_{ijpq} \zeta^{(i+p)(j+q)} e_{ij} \otimes e_{pq} = \Delta(u^2),\end{aligned}$$

and similarly that

$$\Delta(u)\Delta(e_{ij}) = \Delta(e_{ji})\Delta(u).$$

Further, to show that (A, Δ, ε) is a bialgebra, we see that

$$\begin{aligned}(\mathrm{id} \otimes \Delta) \circ \Delta(u) &= \sum_{ijpq} \zeta^{jp} e_{ij} u \otimes \Delta(e_{pq} u)\\ &= \sum_{ijabrs} \zeta^{ja+jr+br} e_{ij} u \otimes e_{ab} u \otimes e_{rs} u\\ &= \sum_{pqrs} \zeta^{qr} \Delta(e_{pq} u) \otimes e_{rs} u = (\Delta \otimes \mathrm{id}) \circ \Delta(u),\end{aligned}$$

and similarly that

$$(\mathrm{id} \otimes \varepsilon) \circ \Delta(u) = 1 = (\varepsilon \otimes \mathrm{id}) \circ \Delta(u).$$

It is straightforward to verify that the anti-algebra map $\mathcal{S} : A \to A$ determined by $\mathcal{S}(u) = u,\ \mathcal{S}(e_{ij}) = e_{-i,-j}$ gives an antipode of $A = A_\zeta$. Finally we see that

$$(A_\zeta) = K \xrightarrow{\iota} A_\zeta \xrightarrow{\pi} \mathbf{k}C_2$$

is an extension associated with the matched pair, where ι is the natural map and π is the Hopf algebra map determined by

$$\pi(e_{ij}) = \varepsilon(e_{ij})1,\ \pi(u) = x.$$

Clearly, (A_1) is the split extension. It is easy to see that $(A_\zeta) \cdot (A_{\zeta'}) \sim (A_{\zeta\zeta'})$. Moreover, if $(A_\zeta) \sim (A_1)$ then A_ζ is cocommutative, and so $\zeta = 1$. Thus the map

$$\boldsymbol{\mu}_n(\mathbf{k}) \to \mathrm{Opext}(\mathbf{k}C_2, K), \qquad \zeta \mapsto \text{ class of } (A_\zeta)$$

is a monomorphism.

It remains to show that any (A) is equivalent to some (A_ζ). There exist bases $u_1 = 1,\ u_x$ in A and $v_1 = 1,\ v_s,\ v_t, \dots$ in A^* with the properties (1)–(3) in Proposition 3.3. We denote

$$\alpha = (v_s)^n, \quad \beta = (v_t)^n, \quad \gamma = v_t^{-1} v_s^{-1} v_t v_s,$$

which are units in $\mathbf{k}^{C_2}$. Let $\{e_0, e_1\} \subset \mathbf{k}^{C_2}$ be the dual basis of $\{1, x\} \subset \mathbf{k}C_2$. Then α has the form

$$\alpha = e_0 + ce_1 \quad (c \in \mathbf{k}^\times)$$

(the coefficient of e_0 equals 1 since $< v_s, 1 >= 1$). Since $(\mathbf{k}^\times)^n = \mathbf{k}^\times$ by assumption, there exists an n-th root $\sqrt[n]{c}$ of c in $\mathbf{k}^\times$. By replacing v_s by $v_s(e_0 + (\sqrt[n]{c})^{-1} e_1)$, we may suppose that $\alpha = 1$, and similarly that

$\beta = 1$. By computing $v_t \underbrace{v_s \cdots v_s}_{n}$ in the two ways (using α from one side, and using γ n times from the other), it follows that $\gamma^n = 1$, and so that $\gamma = e_0 + \zeta e_1$ for some $\zeta \in \boldsymbol{\mu}_n(\mathbf{k})$. We may suppose that

$$v_{s^i t^j} = v_s^i v_t^j \quad (i, j \in \mathbb{Z}/n).$$

Then we see that

$$< \Delta(u_x), v_s^i v_t^j e_k \otimes v_s^p v_t^q e_l >=< u_x, \delta_{kl} v_s^{i+p} v_t^{j+q} \gamma^{jp} e_k >= \begin{cases} \zeta^{jp} & (k = l = 1) \\ 0 & (\text{otherwise}), \end{cases}$$

and so that

$$\Delta(u_x) = \sum_{ijpq} \zeta^{jp} e_{ij} u_x \otimes e_{pq} u_x.$$

Write $\delta = u_x^2$ ($\in K^\times$). Since $u_x \delta = \delta u_x$, δ is of the form $\delta = \sum_{ij} c_{ij} e_{ij}$, where $c_{ij} = c_{ji} \in \mathbf{k}^\times$, $c_{00} = 1$. Since $\Delta(u_x)^2 = \Delta(\delta)$, we have

$$\sum_{ijpq} \zeta^{iq+jp} c_{ij} c_{pq} e_{ij} \otimes e_{pq} = \sum_{ijpq} c_{i+p,j+q} e_{ij} \otimes e_{pq},$$

and hence

$$c_{i+p,j+q} = \zeta^{iq+jp} c_{ij} c_{pq} \quad (i, j, p, q \in \mathbb{Z}/n).$$

Let $\xi = c_{10} = c_{01}$. Then, $\xi \in \boldsymbol{\mu}_n(\mathbf{k})$ and

$$\begin{aligned} c_{ij} &= \zeta^{ij} c_{i0} c_{0j} = \zeta^{ij} \underbrace{c_{10} \cdots c_{10}}_{i} \underbrace{c_{01} \cdots c_{01}}_{j} \\ &= \zeta^{ij} \xi^{i+j} \quad (i, j \in \mathbb{Z}/n). \end{aligned}$$

The K-ring map $f : A_\zeta \to A$ determined by

$$f(u) = (\sum_{ij \in \mathbb{Z}/n} \xi^{-i} e_{ij}) u_x$$

gives an equivalence $(A_\zeta) \sim (A)$. In fact, f is well defined since

$$f(u)^2 = \sum_{ij} \xi^{-i-j} e_{ij} u_x^2 = \sum_{ij} \zeta^{ij} e_{ij}.$$

We see further that

$$\Delta(f(u)) = \sum_{ijpq} \zeta^{jp} \xi^{-i-p} e_{ij} u_x \otimes e_{pq} u_x = (f \otimes f) \circ \Delta(u),$$

and that f is compatible with ι, π. $\square$

Exercise 6.2. *Let $p < q$ be primes such that $q \not\equiv 1 \bmod p$. Prove that either if $(\mathbf{k}^\times)^p = \mathbf{k}^\times$ or if $(\mathbf{k}^\times)^q = \mathbf{k}^\times$, then any extension of $\mathbf{k}C_p$ by $\mathbf{k}^{C_q}$ is equivalent to the trivial one $\mathbf{k}^{C_q} \to \mathbf{k}^{C_q} \otimes \mathbf{k}C_p \to \mathbf{k}C_p$.*

7. **The Kac exact sequence.** We fix a matched pair (F, G). Recall the double complexes $C^{\cdot\cdot}, D^{\cdot\cdot}$ defined in Section 5. The dimension shifting $C^{\cdot+1,\cdot+1}$ of $C^{\cdot\cdot}$ (i.e. the double complex obtained by replacing each term in the first row and the first column of $D^{\cdot\cdot}$ by the trivial groups 1) is a sub-double complex of $D^{\cdot\cdot}$. The

quotient complex $E^{\cdot\cdot} := D^{\cdot\cdot}/C^{\cdot+1,\cdot+1}$ looks as follows:

$$E^{\cdot\cdot} \quad = \quad \begin{tikzcd}
\vdots & \vdots & \vdots & \\
\mathrm{Map}_+(G^2,\mathbf{k}^\times) \arrow[r] \arrow[u] & 1 \arrow[r] \arrow[u] & 1 \arrow[r] \arrow[u] & \cdots \\
\mathrm{Map}_+(G,\mathbf{k}^\times) \arrow[r] \arrow[u] & 1 \arrow[r] \arrow[u] & 1 \arrow[r] \arrow[u] & \cdots \\
\mathbf{k}^\times \arrow[r] \arrow[u] & \mathrm{Map}_+(F,\mathbf{k}^\times) \arrow[r] \arrow[u] & \mathrm{Map}_+(F^2,\mathbf{k}^\times) \arrow[r] \arrow[u] & \cdots .
\end{tikzcd}$$

The complexes in the edges of this double complex are the standard complexes for computing $H^{\cdot}(F,\mathbf{k}^\times)$ and $H^{\cdot}(G,\mathbf{k}^\times)$, respectively (note that the cohomology groups $H^{\cdot}(-,\mathbf{k}^\times)$ for the left trivial module $\mathbf{k}^\times$ and for the right trivial module $\mathbf{k}^\times$ are the same). Hence,

$$H^n\mathrm{Tot}(E^{\cdot\cdot}) = H^n(F,\mathbf{k}^\times) \oplus H^n(G,\mathbf{k}^\times) \quad (n > 0). \tag{7.1}$$

Recall that $D^{\cdot\cdot}$ is obtained by applying $\mathrm{Hom}_{F\bowtie G}(-,\mathbf{k}^\times)$ to $B_{\cdot\cdot}$. The next Lemma says that $\mathrm{Tot}(B_{\cdot\cdot})$ is a free resolution of the trivial left $F \bowtie G$-module $\mathbb{Z}$ with the augmentation

$$B'_0 \otimes_{\mathbb{Z}} B_0 = \mathbb{Z}G \otimes_{\mathbb{Z}} \mathbb{Z}F \overset{\varepsilon\otimes\varepsilon}{\to} \mathbb{Z}.$$

Hence we have

$$H^n\mathrm{Tot}(D^{\cdot\cdot}) = \mathrm{Ext}^n_{F\bowtie G}(\mathbb{Z},\mathbf{k}^\times) = H^n(F \bowtie G,\mathbf{k}^\times) \quad (n \geq 0). \tag{7.2}$$

Lemma 7.3 (Weil). *In general, let*

$$B_{\cdot\cdot} \quad = \quad \begin{tikzcd}
\vdots \arrow[d] & \vdots \arrow[d] & \\
B_{01} \arrow[d, "d'_{01}"'] & B_{11} \arrow[l, "d_{11}"'] \arrow[d, "d'_{11}"'] & \cdots \arrow[l] \\
B_{00} & B_{10} \arrow[l, "d_{10}"'] & \cdots \arrow[l]
\end{tikzcd}$$

be a double complex with exact columns, and let

$$\bar{B}_{\cdot} = 0 \leftarrow \mathit{Coker}\ d'_{01} \overset{\bar{d}_1}{\leftarrow} \mathit{Coker}\ d'_{11} \overset{\bar{d}_2}{\leftarrow} \mathit{Coker}\ d'_{21} \leftarrow \cdots$$

be the chain complex induced from the bottom row of $B_{\cdot\cdot}$. Then the natural epimorphism $Tot(B_{\cdot\cdot}) \to \bar{B}_{\cdot}$ induces isomorphisms

$$H_n\mathit{Tot}(B_{\cdot\cdot}) \simeq H_n(\bar{B}_{\cdot}) \quad (n \geq 0).$$

Hence, if $H_n(\bar{B}_{\cdot}) = 0$ for all $n > 0$, then $Tot(B_{\cdot\cdot})$ gives a resolution of $H_0(\bar{B}_{\cdot}) = \mathit{Coker}\ \bar{d}_1$.

Proof. See for example [SS, Thm 10.1.1]. □

If we consider the long exact cohomology sequence arising from the short exact sequence

$$0 \to \mathrm{Tot}(C^{\cdot+1,\cdot+1}) \to \mathrm{Tot}(D^{\cdot\cdot}) \to \mathrm{Tot}(E^{\cdot\cdot}) \to 0$$

of cochain complexes and apply Corollary 5.3, Exercise 5.4, (7.1) and (7.2), then the next Theorem follows.

Theorem 7.4 ([K]). *We have the following exact sequence*

$$
\begin{aligned}
0 &\to H^1(F \bowtie G, \mathbf{k}^\times) \xrightarrow{res^1} H^1(F, \mathbf{k}^\times) \oplus H^1(G, \mathbf{k}^\times) \to Aut(\mathbf{k}^G * \mathbf{k}F) \\
&\to H^2(F \bowtie G, \mathbf{k}^\times) \xrightarrow{res^2} H^2(F, \mathbf{k}^\times) \oplus H^2(G, \mathbf{k}^\times) \to Opext(\mathbf{k}F, \mathbf{k}^G) \\
&\to H^3(F \bowtie G, \mathbf{k}^\times) \xrightarrow{res^3} H^3(F, \mathbf{k}^\times) \oplus H^3(G, \mathbf{k}^\times),
\end{aligned}
$$

where res^i $(i = 1, 2, 3)$ are induced from the restriction maps of cohomology groups for $F \subseteq F \bowtie G \supseteq G$.

Exercise 7.5. *Show in the same way as in (7.2) that*

$$
\begin{aligned}
Opext(\mathbf{k}F, \mathbf{k}^G) &= Ext^1_{F \bowtie G}((\mathbb{Z}G)^+ \otimes_{\mathbb{Z}} (\mathbb{Z}F)^+, \mathbf{k}^\times), \\
Aut(\mathbf{k}^G * \mathbf{k}F) &= Hom_{F \bowtie G}((\mathbb{Z}G)^+ \otimes_{\mathbb{Z}} (\mathbb{Z}F)^+, \mathbf{k}^\times),
\end{aligned}
$$

where $(\ )^+$ denotes the kernel of the augmentation ε.

Exercise 7.6. *Solve Exercise 6.2 by using the Kac exact sequence. (Hint: use the facts $H^2(C_n, \mathbf{k}^\times) = \mathbf{k}^\times/(\mathbf{k}^\times)^n$, $H^3(C_n, \mathbf{k}^\times) = \boldsymbol{\mu}_n(\mathbf{k})$.)*

We show now some consequences of the Kac exact sequence.

Proposition 7.7. *The abelian group $Opext(\mathbf{k}F, \mathbf{k}^G)$ is a torsion group. This is finite if $\mathbf{k}$ is algebraically closed.*

Proof. The first statement: this holds since for a finite group Γ of order m, $mH^n(\Gamma, M) = 0$ $(\forall n > 0)$. The second statement: let Γ be a group, and M a trivial Γ-module. The universal coefficient Theorem (see [R, Thm 10.22]) states that

$$H^n(\Gamma, M) \simeq \mathrm{Hom}_{\mathbb{Z}}(H_n(\Gamma, \mathbb{Z}), M) \oplus \mathrm{Ext}^1_{\mathbb{Z}}(H_{n-1}(\Gamma, \mathbb{Z}), M).$$

If we take $M = \mathbf{k}^\times$, where $\mathbf{k}$ is an algebraically closed field, then M is a divisible $\mathbb{Z}$-module, which implies $\mathrm{Ext}^1_{\mathbb{Z}} = 0$, and hence we have

$$H^n(\Gamma, \mathbf{k}^\times) \simeq \mathrm{Hom}_{\mathbb{Z}}(H_n(\Gamma, \mathbb{Z}), \mathbf{k}^\times).$$

If Γ is finite then $H_n(\Gamma, \mathbb{Z})$ $(n > 0)$ is finite (since $\mathbb{Z}$ is a f.g. Γ-module, see [R, Thm 10.29]), and hence $H^n(\Gamma, \mathbf{k}^\times)$ $(n > 0)$ is finite. The statement follows from the Kac exact sequence. $\square$

In general, $\mathrm{Opext}(\mathbf{k}F, \mathbf{k}^G)$ is not necessarily finite, while $\mathrm{Aut}(\mathbf{k}^G * \mathbf{k}F)$ is finite for any $\mathbf{k}$, which follows from the next Proposition.

If R is any commutative ring, we define analogously the split extension $(R^G * RF)$ and the group $\mathrm{Aut}(R^G * RF)$ of its auto-equivalences. We obtain in this way a group functor (i.e., a functor from commutative rings to groups),

$$\mathbf{Aut}(\mathbb{Z}^G * \mathbb{Z}F) : R \mapsto \mathrm{Aut}(R^G * RF).$$

Proposition 7.8. *There is an isomorphism of group functors*

$$\mathbf{Aut}(\mathbb{Z}^G * \mathbb{Z}F) \simeq \boldsymbol{\mu}_{n_1} \times \cdots \times \boldsymbol{\mu}_{n_r} \quad (0 < n_i \in \mathbb{Z}),$$

where $\boldsymbol{\mu}_n$ denotes the group functor of nth roots of unity, and thus $\boldsymbol{\mu}_n(R) = \{\alpha \in R \mid \alpha^n = 1\}$.

Proof. In $C^{\cdot\cdot}$, replace $\mathbf{k}^\times$ by $R^\times$ to obtain the double complex

$$
\begin{array}{ccccccc}
 & & \vdots & & \vdots & & \\
 & & \uparrow & & \uparrow & & \\
C^{\cdot\cdot}(R) & = & C^{01}(R) & \longrightarrow & C^{11}(R) & \longrightarrow & \cdots \\
 & & {\scriptstyle \partial'^{00}(R)}\uparrow & & \uparrow & & \\
 & & C^{00}(R) & \xrightarrow{\partial^{00}(R)} & C^{10}(R) & \longrightarrow & \cdots,
\end{array}
$$

where $C^{pq}(R) = \mathrm{Map}_+(G^{q+1} \times F^{p+1}, R^\times)$. Then each C^{pq} gives rise to a group functor, and ∂, ∂' are natural transformations. It can be seen, as in Exercise 5.4, that

$$\mathbf{Aut}(\mathbb{Z}^G * \mathbb{Z}F) \simeq \mathrm{Ker}\partial^{00} \cap \mathrm{Ker}\partial'^{00}.$$

We define $X_{pq} = \{(s_{q+1}, \dots, s_1; x_1, \dots, x_{p+1}) \mid 1 \neq s_i \in G,\ 1 \neq x_i \in F\}$. Then C^{pq} is represented by the group ring $\mathbb{Z}M_{pq}$ of the free abelian group M_{pq} on X_{pq}, that is,

$$C^{pq}(R) = \mathrm{Hom}_{ring}(\mathbb{Z}M_{pq}, R),$$

where the right hand-side is a group under the convolution product. Since ∂, ∂' are induced from group maps between the M_{pq}'s (which are induced from maps between the $X_{pq} \cup \{1\}$'s), we see that

$$\mathbf{Aut}(\mathbb{Z}^G * \mathbb{Z}F) \simeq \mathrm{Hom}_{ring}(\mathbb{Z}M, R),$$

where M is the cokernel of some group map $M_{01} \oplus M_{10} \to M_{00}$. Clearly, M is finitely generated abelian. Since one sees, as in Proposition 7.7, that $\mathrm{Aut}(\mathbb{C}^G * \mathbb{C}F)$ is finite, it follows that M is a torsion group, that is,

$$M \simeq \mathbb{Z}/n_1 \oplus \cdots \oplus \mathbb{Z}/n_r \quad (0 < n_i \in \mathbb{Z}),$$

for some $0 < n_i \in \mathbb{Z}$, which yields the result. □

8. **Sample Computation II.** Let $\Sigma = \mathbb{S}_n$ $(n \geq 3)$, $a = (12\dots n) \in \mathbb{S}_n$. Σ has subgroups

$$F = C_n = < a >, \quad G = \mathbb{S}_{n-1} = \{s \in \mathbb{S}_n \mid s(n) = n\}.$$

The factorization $F \times G = \Sigma$ makes $(F, G) = (C_n, \mathbb{S}_{n-1})$ into a matched pair. Let $s \in G$ and $1 \leq i \leq n$. Since $a^{-s(i)} s a^i \in G$, it follows that

$$s \triangleright a^i = a^{s(i)} \quad (\text{while } s \triangleright 1 = 1).$$

The other action $\triangleleft$ is trivial if $n = 3$. If $n > 3$ it is not trivial, but cannot be expressed easily. Concerning the Opext group associated with the matched pair just obtained, we have:

Theorem 8.1.

$$Opext(\mathbf{k}C_n, \mathbf{k}^{\mathbb{S}_{n-1}}) = \begin{cases} \mathbf{k}^\times/(\mathbf{k}^\times)^n & (3 \leq n \neq 4) \\ \mathbf{k}^\times/(\mathbf{k}^\times)^8 & (n = 4) \end{cases}$$

□

Corollary 8.2. *If* $(\mathbf{k}^\times)^n = \mathbf{k}^\times$, $Opext(\mathbf{k}C_n, \mathbf{k}^{\mathbb{S}_{n-1}}) = 0$.

Let us sketch the proof. We have first the following Lemma, which is slightly weaker than the Corollary above.

Lemma 8.3. *If* $(\mathbf{k}^\times)^2 = \mathbf{k}^\times$ *and* $(\mathbf{k}^\times)^n = \mathbf{k}^\times$, *then* $Opext(\mathbf{k}C_n, \mathbf{k}^{\mathbb{S}_{n-1}}) = 0$.

If $n = 3$, this is shown in the way of Section 6 (without the assumption $(\mathbf{k}^\times)^2 = \mathbf{k}^\times$).

Suppose $n \geq 4$. We apply the Kac exact sequence. It is well known that $H^2(C_n, \mathbf{k}^\times) = \mathbf{k}^\times/(\mathbf{k}^\times)^n = 0$. Under the assumption $(\mathbf{k}^\times)^2 = \mathbf{k}^\times$, we have that

$$H^2(\mathbb{S}_n, \mathbf{k}^\times) = \begin{cases} \boldsymbol{\mu}_2(\mathbf{k}) & (n \geq 4) \\ 0 & (n = 3). \end{cases}$$

Moreover, $\mathrm{res} : H^2(\mathbb{S}_n, \mathbf{k}^\times) \to H^2(\mathbb{S}_{n-1}, \mathbf{k}^\times)$ is surjective (in fact, this is the identity morphism if $n > 4$). By the Kac exact sequence, it is enough to show that

$$\mathrm{res}^3 : H^3(\mathbb{S}_n, \mathbf{k}^\times) \to H^3(C_n, \mathbf{k}^\times) \oplus H^3(\mathbb{S}_{n-1}, \mathbf{k}^\times)$$

is injective.

Claim 8.4. *We suppose* $(\mathbf{k}^\times)^2 = \mathbf{k}^\times$. *Let* $Z = Ker(res^3)$.

1. *If* $n = 5$ *or* $n \geq 7$, *then* $Z = 0$, *so that* res^3 *is injective.*
2. *If* $n = 4$, *then* $Z \simeq \boldsymbol{\mu}_4(\mathbf{k})$.

3. *If* $n = 6$, *then* $Z \simeq \boldsymbol{\mu}_2(\mathbf{k})$.

Even if $n = 4$ or 6, one can see that each element in Z vanishes through res : $H^3(\mathbb{S}_n, \mathbf{k}^\times) \to H^3(C_n, \mathbf{k}^\times)$, which implies Lemma 8.3.

Proof of Claim 8.4 (sketch). Regard $M = \mathrm{Map}(\mathbb{S}_n/\mathbb{S}_{n-1}, \mathbf{k}^\times)$ as a $\mathbb{Z}$-module under the point-wise product and further as a right $\mathbb{S}_n$-module with the induced action by $\mathbb{S}_n$ on $\mathbb{S}_n/\mathbb{S}_{n-1}$ from the left. By Shapiro's Lemma (see [R, Thm 10.32]), we have

$$H^{\cdot}(\mathbb{S}_{n-1}, \mathbf{k}^\times) \simeq H^{\cdot}(\mathbb{S}_n, \mathrm{Map}(\mathbb{S}_n/\mathbb{S}_{n-1}, \mathbf{k}^\times)).$$

Since we have the identification

$$\{1, 2, \ldots, n\} = \mathbb{S}_n/\mathbb{S}_{n-1}, \quad i \longleftrightarrow a^i$$

of left $\mathbb{S}_n$-sets, M is identified with the $\mathbb{Z}$-module $P = \mathbf{k}^\times \times \ldots \times \mathbf{k}^\times$ (n times $\mathbf{k}^\times$) with the right $\mathbb{S}_n$-action

$$(c_1, \ldots, c_n)s = (c_{s(1)}, \ldots, c_{s(n)}) \quad (c_i \in \mathbf{k}^\times,\ s \in \mathbb{S}_n).$$

Since res : $H^{\cdot}(\mathbb{S}_n, \mathbf{k}^\times) \to H^{\cdot}(\mathbb{S}_{n-1}, \mathbf{k}^\times)$ is identified with the map $H^{\cdot}(\mathbb{S}_n, \mathbf{k}^\times) \to H^{\cdot}(\mathbb{S}_n, P)$ induced from the diagonal map $\delta : \mathbf{k}^\times \to P$, we have the exact sequence

$$H^2(\mathbb{S}_n, \mathbf{k}^\times) \xrightarrow{\text{res}} H^2(\mathbb{S}_{n-1}, \mathbf{k}^\times) \to H^2(\mathbb{S}_n, Q) \to H^3(\mathbb{S}_n, \mathbf{k}^\times) \xrightarrow{\text{res}} H^3(\mathbb{S}_{n-1}, \mathbf{k}^\times) \to H^3(\mathbb{S}_n, Q),$$

where $Q = \mathrm{Coker}\ \delta$. Since the first res is surjective (as seen before), we have that

$$Z \simeq H^2(\mathbb{S}_n, Q).$$

In general, for a fixed right $\mathbb{S}_n$-module N, $H^2(\mathbb{S}_n, N)$ is in $1-1$ correspondence with the equivalence classes of the group extensions $N \to E \to \mathbb{S}_n$ giving rise to the given $\mathbb{S}_n$-action on N. In [B], Blackburn classified such extensions in terms of some data associated with each extension (such as the α, β, γ in the proof of Theorem 6.1). By applying his results, we see that, if char $\mathbf{k} = 2$ and $n \geq 5$, then $Z = 0$. Using the same results, Kleshchev and Premet [KP] computed $H^2(\mathbb{S}_n, I^*)$, where I^* is the right $\mathbb{S}_n$-module obtained by modifying the definition of Q, replacing $\mathbf{k}^\times$ by $\mathbb{Z}/p$ with p a prime. We suppose $p = 2$ and apply the result. Suppose char $\mathbf{k} \neq 2$. One sees that if $n \geq 5$ the map

$$H^2(\mathbb{S}_n, I^*) \to H^2(\mathbb{S}_n, Q) \quad (\simeq Z)$$

induced from $\mathbb{Z}/2 = \{\pm 1\} \hookrightarrow \mathbf{k}^\times$ is surjective. Since it is shown in [KP] that if $n = 5$ or $n \geq 7$ then $H^2(\mathbb{S}_n, I^*) = 0$, we have that $Z = 0$ in this case. In other cases, we compute $H^2(\mathbb{S}_n, Q)$ by modifying or correcting results in [KP] and reach the claim. $\square$

For the (sketchy) proof of Theorem 8.1, let H, K be finite-dimensional Hopf algebras. All (cleft) extensions (A) of H by K form an algebraic system whose structure is given by the linear maps

$$A \otimes A \xrightarrow{\text{prod}} A,\ \mathbf{k} \xrightarrow{\text{unit}} A, \ldots, K \xrightarrow{\iota} A,\ A \xrightarrow{\pi} H.$$

The base extension $(A \otimes R)$ is naturally defined. For a field extension $\mathbf{l}/\mathbf{k}$, we say that (A) is an $\mathbf{l}/\mathbf{k}$-form of a fixed extension (A_0) if $(A \otimes \mathbf{l}) \sim (A_0 \otimes \mathbf{l})$.

Lemma 8.3 implies that every extension (A) associated with the matched pair $(F, G) = (C_n, \mathbb{S}_{n-1})$ is a $\bar{\mathbf{k}}/\mathbf{k}$-form of the split extension $(A_0) = (\mathbf{k}^G * \mathbf{k}F)$, where $\bar{\mathbf{k}}$ is the algebraic closure of $\mathbf{k}$. Conversely, such a $\bar{\mathbf{k}}/\mathbf{k}$-form is associated with (F, G). Hence the faithfully flat descent Theorem A.5 gives a bijection

$$H^1(\bar{\mathbf{k}}/\mathbf{k}, \mathbf{Aut}(\mathbf{k}^G * \mathbf{k}F)) \simeq \mathrm{Opext}(\mathbf{k}F, \mathbf{k}^G), \tag{8.5}$$

where H^1 denotes the Amitsur 1st cohomology group and $\mathbf{Aut}(\mathbf{k}^G * \mathbf{k}F)$ denotes the $\mathbf{k}$-group functor (functor from commutative $\mathbf{k}$-algebras to groups) defined by

$$R \mapsto \mathrm{Aut}(R^G * RF).$$

This arises from $\mathbf{Aut}(\mathbb{Z}^G * \mathbb{Z}F)$ defined over $\mathbb{Z}$. This bijection is seen to be a group isomorphism (see the proof just below).

Proposition 8.6.

$$Aut(\mathbb{Z}^{\mathbb{S}_{n-1}} * \mathbb{Z}C_n) \simeq \begin{cases} \boldsymbol{\mu}_n & (3 \leq n \neq 4) \\ \boldsymbol{\mu}_8 & (n = 4). \end{cases}$$

Proof. By Proposition 7.8, it is enough to show

$$\mathrm{Aut}(\mathbb{C}^{\mathbb{S}_{n-1}} * \mathbb{C}C_n) \simeq \begin{cases} \mathbb{Z}_n & (3 \leq n \neq 4) \\ \mathbb{Z}_8 & (n = 4). \end{cases}$$

The Kac exact sequence applies in the case where $\mathbf{k} = \mathbb{C}$. For a group Γ and a trivial Γ-module N, we have

$$H^1(\Gamma, N) \simeq \mathrm{Hom}_{group}(\Gamma, N).$$

Hence, $H^1(\mathbb{S}_n, \mathbb{C}^\times) \simeq \mathbb{Z}/2$ (generated by sign) and $H^1(C_n, \mathbb{C}^\times) \simeq \mathbb{Z}/n$. The morphism res^1 is identified with $\mathbb{Z}/2 \to \mathbb{Z}_n \oplus \mathbb{Z}_2$ given by

$$1 \mapsto (0, 1)\ (n \text{ odd}), \quad 1 \mapsto (\frac{n}{2}, 1)\ (n \text{ even}).$$

In any case, Coker $(\mathrm{res}^1) \simeq \mathbb{Z}/n$. On the other hand, as seen before,

$$\mathrm{Ker}(\mathrm{res}^2) \simeq \begin{cases} 0 & (3 \leq n \neq 4) \\ \mathbb{Z}/2 & (n = 4). \end{cases}$$

The result follows immediately, if $3 \leq n \neq 4$. If $n = 4$, there are two posibilities:

$$\mathrm{Aut}(\mathbb{C}^{\mathbb{S}_3} * \mathbb{C}C_4) \simeq \mathbb{Z}_8 \text{ or } \mathbb{Z}_2 \oplus \mathbb{Z}_4.$$

But, since we can find an element of order 8 in the group, we have $\mathrm{Aut}(\mathbb{C}^{\mathbb{S}_3} * \mathbb{C}C_4) \simeq \mathbb{Z}_8$. □

The proof of Theorem 8.1 completes, if we prove that the bijection (8.5) is a group map. In fact for any matched pair (F, G), there is a natural injection [2]

$$H^1(\bar{\mathbf{k}}/\mathbf{k}, \mathbf{Aut}(\mathbf{k}^G * \mathbf{k}F)) \to \mathrm{Opext}(\mathbf{k}F, \mathbf{k}^G) \tag{8.7}$$

which is seen to be a group map as follows. Note that

$$\begin{array}{rcl} H^0\mathrm{Tot}(C^{\cdot\cdot}(\bar{\mathbf{k}} \otimes \bar{\mathbf{k}})) & \simeq & \mathrm{Aut}((\bar{\mathbf{k}} \otimes \bar{\mathbf{k}})^G * (\bar{\mathbf{k}} \otimes \bar{\mathbf{k}})F) \\ & & \cup \\ & & Z^1(\bar{\mathbf{k}}/\mathbf{k}, \mathbf{Aut}(\mathbf{k}^G * \mathbf{k}F)), \end{array}$$

where Z^1 denotes the Amitsur 1-cocycles. Denote by $C^{\cdot\cdot}(\bar{\mathbf{k}}/\mathbf{k})$ the double complex obtained by replacing $\mathbf{k}^\times$ in $C^{\cdot\cdot}$ by $\bar{\mathbf{k}}^\times/\mathbf{k}^\times$. Then, the map $\bar{\mathbf{k}}^\times/\mathbf{k}^\times \to (\bar{\mathbf{k}} \otimes \bar{\mathbf{k}})^\times$ given by $x\mathbf{k}^\times \mapsto x \otimes x^{-1}$ induces an isomorphism

$$H^0\mathrm{Tot}(C^{\cdot\cdot}(\bar{\mathbf{k}}/\mathbf{k})) \simeq Z^1(\bar{\mathbf{k}}/\mathbf{k}, \mathbf{Aut}(\mathbf{k}^G * \mathbf{k}F).$$

Now the connecting map

$$H^0\mathrm{Tot}(C^{\cdot\cdot}(\bar{\mathbf{k}}/\mathbf{k})) \to H^1\mathrm{Tot}(C^{\cdot\cdot}(\mathbf{k})) \tag{8.8}$$

arising from the short exact sequence $0 \to \mathbf{k}^\times \to \bar{\mathbf{k}}^\times \to \bar{\mathbf{k}}^\times/\mathbf{k}^\times \to 0$ is identified with

$$Z^1(\bar{\mathbf{k}}/\mathbf{k}, \mathbf{Aut}(\mathbf{k}^G * \mathbf{k}F)) \to \mathrm{Opext}(\mathbf{k}F, \mathbf{k}^G),$$

which induces (8.7).

It is possible to describe explicitly, by using (8.8), the extensions associated with $(C_n, \mathbb{S}_{n-1})$.

[2]This is naturally extended to the exact sequence

$$0 \to H^1(\bar{\mathbf{k}}/\mathbf{k}, \mathbf{Aut}(\mathbf{k}^G * \mathbf{k}F)) \to \mathrm{Opext}(\mathbf{k}F, \mathbf{k}^G) \to \mathrm{Opext}(\bar{\mathbf{k}}F, \bar{\mathbf{k}}^G).$$

A. **Appendix. The faithfully flat descent Theorem.** Let $\mathbf{k}$ be a commutative base ring, and write $\otimes = \otimes_{\mathbf{k}}$. Let $\boldsymbol{G}$ be a $\mathbf{k}$-group functor. Let

$$R \underset{d_2}{\overset{d_1}{\rightrightarrows}} R \otimes R \overset{d_1}{\underset{d_3}{\overset{d_2}{\Rrightarrow}}} R \otimes R \otimes R$$

be a diagram of commutative $\mathbf{k}$-algebras, where d_i is the $\mathbf{k}$-algebra map inserting 1 in the i place (e.g., $d_3(x \otimes y) = x \otimes y \otimes 1$). We apply $\boldsymbol{G}$ and obtain the diagram of groups

$$\boldsymbol{G}(R) \underset{\delta_2}{\overset{\delta_1}{\rightrightarrows}} \boldsymbol{G}(R \otimes R) \overset{\delta_1}{\underset{\delta_3}{\overset{\delta_2}{\Rrightarrow}}} \boldsymbol{G}(R \otimes R \otimes R),$$

where $\delta_i = \boldsymbol{G}(d_i)$.

Definition A.1. An element ϕ in $\boldsymbol{G}(R \otimes R)$ is called an *Amitsur* 1*-cocycle for* $\boldsymbol{G}$ *in* $R/\mathbf{k}$, if

$$\delta_2(\phi) = \delta_3(\phi)\delta_1(\phi).$$

The group $\boldsymbol{G}(R)$ acts on the set $Z^1(R/\mathbf{k}, \boldsymbol{G})$ of such 1-cocycles by

$$\gamma \cdot \phi = \delta_2(\gamma)\phi\delta_1(\gamma)^{-1} \quad (\gamma \in \boldsymbol{G}(R),\ \phi \in Z^1).$$

The quotient set $\boldsymbol{G}(R)\backslash Z^1(R/\mathbf{k}, \boldsymbol{G})$ is denoted by $H^1(R/\mathbf{k}, \boldsymbol{G})$, and is called the *Amitsur* 1*st cohomology set for* $\boldsymbol{G}$ *in* $R/\mathbf{k}$. If $\boldsymbol{G}$ is abelian (i.e., each $\boldsymbol{G}(R)$ is abelian), then Z^1, H^1 are abelian groups.

Example A.2. *Suppose that* $\mathbf{k}$ *is a field and let* $\bar{\mathbf{k}}$ *be its algebraic closure. Let* $\boldsymbol{G} = \boldsymbol{\mu}_n$ *be the* $\mathbf{k}$*-group functor of* n*th roots of unity. We have an isomorphism*

$$\mathbf{k}^\times/(\mathbf{k}^\times)^n \simeq H^1(\bar{\mathbf{k}}/\mathbf{k}, \boldsymbol{\mu}_n)$$

given by (*class of* $x \in \mathbf{k}$) $\mapsto$ (*class of* $\sqrt[n]{x} \otimes (\sqrt[n]{x})^{-1}$).

Proof. See [W, Sect. 18.2]. $\square$

Fix an algebraic system (e.g., module, algebra, Hopf algebra, cleft Hopf algebraic extension, ...) whose structure is given by linear maps between tensor products so that the base extension is naturally defined. Denote by $\mathcal{C}_{\mathbf{k}}$ the category of its objects defined over $\mathbf{k}$ (e.g., the category of $\mathbf{k}$-modules, the category of $\mathbf{k}$-algebras, ...). Let $A \in \mathcal{C}_{\mathbf{k}}$ and let R be a commutative $\mathbf{k}$-algebra. Then $A \otimes R \in \mathcal{C}_R$. Fix an object $C \in \mathcal{C}_{\mathbf{k}}$.

Definition A.3. An object A in $\mathcal{C}_{\mathbf{k}}$ is called an $R/\mathbf{k}$*-form of* C, if

$$A \otimes R \simeq C \otimes R \quad \text{in } \mathcal{C}_R.$$

Definition A.4. Let $\operatorname{Aut} C$ be the $\mathbf{k}$-group functor defined by

$$\operatorname{Aut} C : R \mapsto \operatorname{Aut}_R(C \otimes R) := \{\text{automorphisms of } C \otimes R \text{ in } \mathcal{C}_R\}.$$

It is called the *automorphism group functor* of C.

Theorem A.5. *If* R *is faithfully flat over* $\mathbf{k}$ *(this means* $R \neq 0$*, if* $\mathbf{k}$ *is a field), then there is a natural bijection*

$$H^1(R/\mathbf{k}, \mathit{Aut} C) \simeq \{\textit{isomorphism classes of all } R/\mathbf{k}\textit{-forms of } C\}.$$

Proof (sketch). For $\phi \in Z^1(R/\mathbf{k}, \operatorname{Aut} C)$, we define

$$A = \{\sum_i c_i \otimes x_i \in C \otimes R | \sum_i \phi(c_i \otimes 1 \otimes x_i) = \sum c_i \otimes x_i \otimes 1 \text{ in } C \otimes R \otimes R\}.$$

One sees that A is an object in $\mathcal{C}_{\mathbf{k}}$ with the structure induced from $C \otimes R$, and further that it is an $R/\mathbf{k}$-form of C. The correspondence $\phi \mapsto A$ induces the bijection. See [W, Sect. 17.6] for a detailed proof. $\square$

II. Hopf algebra extensions arising from Lie algebras

1. **Generalities on Hopf algebra extensions.** Let H be a cocommutative Hopf algebra, K a commutative Hopf algebra.

Definition 1.1. A *cleft (Hopf algebra) extension of* H *by* K is a sequence $(A) = K \xrightarrow{\iota} A \xrightarrow{\pi} H$ of Hopf algebras and Hopf algebra maps such that there exists a left K-linear and right H-colinear isomorphism $\theta : A \xrightarrow{\cong} K \otimes H$, where A is regarded as a left K-module along ι and as a right H-comodule along π. In this case, ι is necessarily an injection and π a surjection. Furthermore, θ can be chosen to be unit and counit-preserving. Two cleft extensions $(A), (A')$ of H by K are said to be *equivalent* if there is a Hopf algebra map (necessarily an isomorphism) $f : A \to A'$ which makes the following diagram commute

$$\begin{array}{ccccc} K & \longrightarrow & A & \longrightarrow & H \\ \| & & \downarrow{\scriptstyle f} & & \| \\ K & \longrightarrow & A' & \longrightarrow & H. \end{array}$$

Definition 1.2. The pair (H, K) together with linear maps

$$\rightharpoonup : H \otimes K \to K \text{ and } \rho : H \to H \otimes K,\ \rho(x) = \sum x_H \otimes x_K$$

is called a *Singer pair*[3], if K is a left H-module algebra under $\rightharpoonup$, if H is a right K-comodule coalgebra under ρ, and if

1. $\rho(xy) = \sum \rho(x_{(1)})(y_H \otimes (x_{(2)} \rightharpoonup y_K))$,
2. $\Delta(x \rightharpoonup p) = \sum (x_{(1)H} \rightharpoonup p_{(1)}) \otimes x_{(1)K}(x_{(2)} \rightharpoonup p_{(2)})$

for $x, y \in H,\ p \in K$. We regard $(\rightharpoonup, \rho)$ as the *structure* of the Singer pair $(H, K, \rightharpoonup, \rho)$.

A cleft extension (A) of H by K gives rise to a structure of a Singer pair on (H, K) as follows. First we choose an isomorphism θ as in Def. 1.1 and define a right H-colinear map $\phi : H \to A$ and a left K-linear map $\gamma : A \to K$ by

$$\phi(x) = \theta^{-1}(1 \otimes x)\ (x \in H), \qquad \gamma(a) = (\mathrm{id} \otimes \varepsilon)\theta(a)\ (a \in A).$$

These are convolution-invertible morphisms (see [Sch, Thm 2.4] or [MD, Thm 3.5]). Next, we define linear maps $\rightharpoonup : H \otimes K \to K$, $\rho : H \to H \otimes K$ by the equations

$$\iota(x \rightharpoonup p) = \sum \phi(x_{(1)})\iota(p)\phi^{-1}(x_{(2)}) \quad (x \in H, p \in K),$$
$$(\rho \circ \pi)(a) = \sum \pi(a_{(2)}) \otimes \gamma^{-1}(a_{(1)})\gamma(a_{(3)}) \quad (a \in A),$$

where $()^{-1}$ indicates the convolution-inverse. Finally, we can see that $\rightharpoonup, \rho$ are independent of choice of θ and that they give a structure of a Singer pair on (H, K). In this case, (A) is said to be *associated with* the Singer pair (H, K). We fix now a Singer pair $(H, K, \rightharpoonup, \rho)$. We denote by

$$\mathrm{Opext}(H, K) = \mathrm{Opext}(H, K, \rightharpoonup, \rho)$$

the set of all cleft extensions associated with it.

We define the product $(A) \cdot (A')$ of two cleft extensions associated with the fixed (H, K) as follows. First, we take the tensor product $A \otimes_K A'$ of the *left* K-modules A, A'. Next, we define the bi-tensor product $A \otimes_K^H A'$ to be the equalizer of the two right H-coactions $A \otimes_K A' \rightrightarrows A \otimes_K A' \otimes H$ arising from the right H-coactions on A and A'. Now, the Hopf algebra $A \otimes A'$ induces on $A \otimes_K^H A'$ a Hopf algebra structure, and we form thus a cleft extension $(A \otimes_K^H A')$ associated with the fixed (H, K), where ι, π are the natural maps. (If we take first the cotensor product $A \square_H A'$ of the *right* H-comodules A, A and then form the coequalizer of the two left K-actions on $A \square_H A'$, we obtain a naturally equivalent cleft extension.) We define then the product by

$$(A) \cdot (A') = (A \otimes_K^H A').$$

[3]In literature, this is usually called an *abelian matched pair*, see [Si]. We would propose here this term to avoid confusions with the notion defined in Definition 3.1.

The product makes the $\mathrm{Opext}(H,K)$ into an abelian group, which is called the *Opext group associated with the Singer pair* (H,K).

A cohomological description for the Opext group is given by the following double complex

$$\begin{array}{ccccc}
 & \vdots & & \vdots & \\
 & \partial'\uparrow & & \partial'\uparrow & \\
C_H^{\cdot\cdot} = & \mathrm{Reg}_+(H, K^{\otimes 2}) & \xrightarrow{\partial} & \mathrm{Reg}_+(H^{\otimes 2}, K^{\otimes 2}) & \xrightarrow{\partial} \cdots \\
 & \partial'\uparrow & & \partial'\uparrow & \\
 & \mathrm{Reg}_+(H, K) & \xrightarrow{\partial} & \mathrm{Reg}_+(H^{\otimes 2}, K) & \xrightarrow{\partial} \cdots ,
\end{array}$$

whose precise definition will be given in Section 5. Here we remark only that Reg_+ denotes a certain subgroup of the abelian group Reg of (convolution) invertible linear maps. Let $\sigma : H\otimes H \to K$, $\tau : H \to K\otimes K$ be invertible linear maps which form a 1-cocycle in the total complex $\mathrm{Tot}(C_H^{\cdot\cdot})$. Then the vector space $K\otimes H$, with the structure given by

$$\begin{aligned}
\text{product } & (p\#x)(q\#y) = \sum p(x_{(1)} \rightharpoonup q)\sigma(x_{(2)}\otimes y_{(1)})\#x_{(3)}y_{(2)},\\
\text{unit } & 1\#1,\\
\text{counit } & \varepsilon(p\#x) = \varepsilon(p)\varepsilon(x),\\
\text{coproduct } & \Delta(p\#x) = \sum (p_{(1)}x_{(1)I}\#x_{(2)H})\otimes(p_{(2)}x_{(1)II}x_{(2)K}\#x_{(3)}),
\end{aligned}$$

where $\tau(x) = \sum x_I \otimes x_{II}$, forms a Hopf algebra. We denote it by $K\#_{\sigma,\tau}H$. We obtain furthermore a cleft extension $(K\#_{\sigma,\tau}H)$ associated with the fixed (H,K), where $\iota(x) = 1\#x$ and $\pi(p\#x) = \varepsilon(p)x$.

Proposition 1.3. *The correspondence* $(\sigma,\tau)\mapsto(K\#_{\sigma,\tau}H)$ *induces an isomorphism*

$$H^1\,Tot(C_H^{\cdot\cdot}) \simeq Opext(H,K).$$

Suppose that σ and τ are both trivial, i.e., $\sigma(x\otimes y) = \varepsilon(x)\varepsilon(y)1$ and $\tau(x) = \varepsilon(x)1\otimes 1$. Then $K\#_{\sigma,\tau}H$ is denoted simply by $K\#H$, and is called the *bismash product* constructed from $(H,K,\rightharpoonup,\rho)$. The cleft extension $(K\#H)$ is called the *split extension*, which represents the unit of the Opext group. caca Define

$$\mathrm{Aut}(K\#H) = \text{ the group of the auto-equivalences of } (K\#H).$$

For a 0-cocycle $\nu : H \to K$ in $\mathrm{Tot}(C_H^{\cdot\cdot})$, the map $p\#x \mapsto \sum p\nu(x_{(1)})\#x_{(2)}$, $K\#H \to K\#H$ gives an auto-equivalence of $(K\#H)$.

Proposition 1.4. *The correspondence just obtained gives an isomorphism*

$$H^0\,Tot(C_H^{\cdot\cdot}) \simeq Aut(K\#H).$$

Exercise 1.5. *Show that the group $Aut(K\#_{\sigma,\tau}H)$ of the auto-equivalences of $(K\#_{\sigma,\tau}H)$, where (σ,τ) is any 1-cocycle in $Tot(C_H^{\cdot\cdot})$, is naturally isomorphic to $Aut(K\#H)$. (Hint: apply $\left(-\otimes_K^H(K\#_{\sigma',\tau'}H)\right)$ to an element in $Aut(K\#_{\sigma,\tau}H)$. Then one obtains an element in $Aut(K\#_{\sigma\sigma',\tau\tau'}H)$.)*

Let F,G be groups, where G is finite (and F may be infinite) and consider the special case where

$$H = \mathbf{k}F, \quad K = \mathbf{k}^G.$$

Note that the module actions $\rightharpoonup : H\otimes K \to K$ are in 1–1 correspondence with the actions of permutations $\triangleleft : G\times F \to G$, so that $x \rightharpoonup e_s = e_{s\triangleleft x^{-1}}$ $(x\in F, s\in G)$. The comodule coactions $\rho : H \to H\otimes K$ are in 1–1 correspondence with the actions of permutations $\triangleright : G\times F \to F$, so that $\rho(x) = \sum_{s\in G}(s\triangleright x)\otimes e_s$ $(x\in F)$. Furthermore, $\rightharpoonup,\rho$ give a structure of a Singer pair on (H,K) iff the correspondence $\triangleright,\triangleleft$ give a structure of a matched pair on (F,G). Replace "associated with a Singer pair $(\mathbf{k}F,\mathbf{k}^G)$" by "associated with the corresponding matched pair (F,G)", and regard invertible linear maps $\sigma : \mathbf{k}F\otimes\mathbf{k}F \to \mathbf{k}^G$, $\tau : \mathbf{k}F \to \mathbf{k}^G\otimes\mathbf{k}^G$

as maps $\sigma : G \times F \times F \to \mathbf{k}^\times$, $\tau : G \times G \times F \to \mathbf{k}^\times$. Then the results mentioned so far are specialized to those in Sections 4–5 of Part I, where the "classical" notation $\mathbf{k}^G *_{\sigma,\tau} \mathbf{k}F$ is used for $\mathbf{k}^G \#_{\sigma,\tau} \mathbf{k}F$. Without the assumption for F to be finite, the results in those sections hold true, except Exercise 5.5 and Theorem 7.4 (the Kac exact sequence).

Exercise 1.6. *Let*

$$F = \text{ the free abelian group generated by two elements } x, y,$$
$$G = C_2 =< s >,$$

and let (F, G) be the matched pair defined by the trivial action $\triangleleft : G \times F \to G$ and the action of group-automorphisms $\triangleright : G \times F \to F$ determined by

$$s \triangleright x^i y^j = x^j y^i \quad (i, j \in \mathbb{Z}).$$

Show that, if $(\mathbf{k}^\times)^2 = \mathbf{k}^\times$, the Opext group associated with the matched pair (F, G) or with the corresponding Singer pair $(\mathbf{k}F, \mathbf{k}^G)$, is the isomorphism to $\mathbf{k}^\times$.

2. **Lie bialgebra extensions.** Let us define the Opext group for Lie bialgebra extensions and present a Lie bialgebra-version of the Kac exact sequence.

Let $\mathfrak{f}, \mathfrak{g}$ be Lie algebras of finite dimension.

Definition 2.1. The pair $(\mathfrak{f}, \mathfrak{g})$, together with linear maps $\mathfrak{g} \xleftarrow{\triangleleft} \mathfrak{g} \otimes \mathfrak{f} \xrightarrow{\triangleright} \mathfrak{f}$, is called a *matched pair of Lie algebras* if $\mathfrak{f}$ is a left $\mathfrak{g}$-Lie module under $\triangleright$, $\mathfrak{g}$ is a right $\mathfrak{f}$-Lie module under $\triangleleft$ and

1. $s \triangleright [x, y] = [s \triangleright x, y] + [x, s \triangleright y] + (s \triangleleft x) \triangleright y - (s \triangleleft y) \triangleright x$,
2. $[s, t] \triangleleft x = [s, t \triangleleft x] + [s \triangleleft x, t] + s \triangleleft (t \triangleright x) - t \triangleleft (s \triangleright x)$

for $x, y \in \mathfrak{f}$, $s, t \in \mathfrak{g}$.

These conditions hold iff the direct sum $\mathfrak{f} \oplus \mathfrak{g}$ of vector spaces forms a Lie algebra, denoted by $\mathfrak{f} \bowtie \mathfrak{g}$, under the bracket defined by

$$[x \oplus s, y \oplus t] = ([x, y] + s \triangleright y - t \triangleright x) \oplus ([s, t] + s \triangleleft y - t \triangleleft x).$$

In this case, we denote this Lie algebra by $\mathfrak{f} \bowtie \mathfrak{g}$. If $(\mathfrak{f}, \mathfrak{g})$ is a matched pair, $\mathfrak{f} = \mathfrak{f} \oplus 0$ and $\mathfrak{g} = 0 \oplus \mathfrak{g}$ are Lie subalgebras of $\mathfrak{f} \bowtie \mathfrak{g}$ such that $\mathfrak{f} \oplus \mathfrak{g} = \mathfrak{f} \bowtie \mathfrak{g}$ (as vector spaces). Conversely, if $\mathfrak{f}$ and $\mathfrak{g}$ are Lie subalgebras of a Lie algebra L such that $\mathfrak{f} \oplus \mathfrak{g} = L$ then $(\mathfrak{f}, \mathfrak{g})$ forms a matched pair with the structure maps determined by

$$[s, x] = s \triangleright x \oplus s \triangleleft x \quad (s \in \mathfrak{g}, x \in \mathfrak{f}),$$

so that $\mathfrak{f} \bowtie \mathfrak{g} = L$ (as Lie algebras).

We say that a finite-dimensional vector space $\mathfrak{l}$ is a *Lie coalgebra* with co-bracket $\delta : \mathfrak{l} \to \mathfrak{l} \otimes \mathfrak{l}$ if the dual vector space $\mathfrak{l}^*$ is a Lie algebra with bracket $\delta^* : \mathfrak{l}^* \otimes \mathfrak{l}^* = (\mathfrak{l} \otimes \mathfrak{l})^* \to \mathfrak{l}^*$. $\mathfrak{l}$ is called a *Lie bialgebra* if $\mathfrak{l}$ is a Lie algebra with bracket $[,]$ and a Lie coalgebra with co-bracket δ satisfying

$$\delta[a, b] = \sum [a, b_{[1]}] \otimes b_{[2]} + \sum b_{[1]} \otimes [a, b_{[2]}] + \sum [a_{[1]}, b] \otimes a_{[2]} + \sum a_{[1]} \otimes [a_{[2]}, b] \tag{2.2}$$

for $a, b \in \mathfrak{l}$, where $\delta(a) = \sum a_{[1]} \otimes a_{[2]}$. Any Lie algebra (resp. Lie coalgebra) of finite dimension is a Lie bialgebra with zero co-bracket (resp. zero bracket). We regard $\mathfrak{f}$ as a Lie bialgebra with zero co-bracket. Naturally, $\mathfrak{g}^*$ is a Lie coalgebra, which is regarded as a Lie bialgebra with zero bracket.

Definition 2.3. The pair $(\mathfrak{f}, \mathfrak{g}^*)$, together with linear maps $\rightharpoonup : \mathfrak{f} \otimes \mathfrak{g}^* \to \mathfrak{g}^*$ and $\rho : \mathfrak{f} \to \mathfrak{f} \otimes \mathfrak{g}^*$, $\rho(x) = \sum x_{[0]} \otimes x_{[1]}$ gives rise to $(\mathfrak{f}, \mathfrak{g}, \triangleright, \triangleleft)$, where

$$s \triangleright x = \sum x_{[0]} < s, x_{[1]} >, \quad < s \triangleleft x, s^* >=< s, x \rightharpoonup s^* >, \qquad (x \in \mathfrak{f}, s \in \mathfrak{g}, s^* \in \mathfrak{g}^*).$$

We say that $(\mathfrak{f}, \mathfrak{g}^*)$ is a *Singer pair of Lie bialgebras* if $(\mathfrak{f}, \mathfrak{g}, \triangleright, \triangleleft)$ forms a matched pair of Lie algebras.

Definition 2.4. A *Lie bialgebra extension of* $\mathfrak{f}$ *by* $\mathfrak{g}^*$ is a sequence $(\mathfrak{l}) = \mathfrak{g}^* \to \mathfrak{l} \to \mathfrak{f}$ of Lie bialgebras and Lie bialgebra maps such that $0 \to \mathfrak{g}^* \to \mathfrak{l} \to \mathfrak{f} \to 0$ is an exact sequence of vector spaces. Two such extensions $(\mathfrak{l}), (\mathfrak{l}')$ are *equivalent*, if there is a Lie bialgebra map (necessarily, an isomorphism) $f : \mathfrak{l} \to \mathfrak{l}'$ which makes the following diagram commute

$$\begin{tikzcd}
\mathfrak{g}^* \arrow[r] \arrow[d, equal] & \mathfrak{l} \arrow[r] \arrow[d, "f"'] & \mathfrak{f} \arrow[d, equal] \\
\mathfrak{g}^* \arrow[r] & \mathfrak{l}' \arrow[r] & \mathfrak{f}\,.
\end{tikzcd}$$

Each Lie bialgebra extension $(\mathfrak{l})$ of $\mathfrak{f}$ by $\mathfrak{g}^*$ gives rise to structure maps $\rightharpoonup : \mathfrak{f} \otimes \mathfrak{g}^* \to \mathfrak{g}^*$, $\rho : \mathfrak{f} \to \mathfrak{f} \otimes \mathfrak{g}^*$ of a Singer pair on $(\mathfrak{f}, \mathfrak{g}^*)$ as follows. First, $(\mathfrak{l})$ may be identified as a vector space extension with the trivial $\mathfrak{g}^* \to \mathfrak{g}^* \oplus \mathfrak{f} \to \mathfrak{f}$. We define $\rightharpoonup$ by

$$x \rightharpoonup s^* = [x, s^*] \quad (x \in \mathfrak{f}, s^* \in \mathfrak{g}^*),$$

where the bracket is of $\mathfrak{l} = \mathfrak{g}^* \oplus \mathfrak{g}$. Next, we define $\leftharpoonup : \mathfrak{f}^* \oplus \mathfrak{g} \to \mathfrak{f}^*$ in $\mathfrak{l}^* = \mathfrak{g} \oplus \mathfrak{f}^*$ by

$$x^* \leftharpoonup s = [x^*, s] \quad (x^* \in \mathfrak{f}^*, s \in \mathfrak{g})$$

and put $\rho = (\leftharpoonup)^*$. Then one sees that $\rightharpoonup, \rho$ give a structure of a Singer pair on $(\mathfrak{f}, \mathfrak{g}^*)$ which is independent of the way of identification $\mathfrak{l} = \mathfrak{g}^* \oplus \mathfrak{f}$. We say that $(\mathfrak{l})$ is *associated with* the Singer pair $(\mathfrak{f}, \mathfrak{g}^*, \rightharpoonup, \rho)$.

Let now $(\mathfrak{f}, \mathfrak{g}^*, \rightharpoonup, \rho)$ be a Singer pair of Lie bialgebras. We denote by $\mathrm{Opext}(\mathfrak{f}, \mathfrak{g}^*) = \mathrm{Opext}(\mathfrak{f}, \mathfrak{g}, \rightharpoonup, \rho)$ the set of equivalence classes of all Lie bialgebra extensions associated with it. For such two extensions $(\mathfrak{l}), (\mathfrak{l}')$, we define the product $(\mathfrak{l}) \Diamond (\mathfrak{l}')$ as follows

$$\begin{tikzcd}
(\mathfrak{l} \oplus \mathfrak{l}') & = & \mathfrak{g}^* \oplus \mathfrak{g}^* \arrow[r] & \mathfrak{l} \oplus \mathfrak{l}' \arrow[r] & \mathfrak{f} \oplus \mathfrak{f} \\
(\mathfrak{l}_0) & = & \mathfrak{g}^* \oplus \mathfrak{g}^* \arrow[r] \arrow[u, equal] \arrow[d] & \mathfrak{l}_0 \arrow[r] \arrow[u, hook] \arrow[d] \arrow[ur, phantom, "\text{p.b.}"] & \mathfrak{f} \arrow[u, hook] \arrow[d, equal] \\
(\mathfrak{l}_1) & = & \mathfrak{g}^* \arrow[r] \arrow[ur, phantom, "\text{p.o.}"] & \mathfrak{l}_1 \arrow[r] & \mathfrak{f}\,.
\end{tikzcd}$$

We construct first of all the Lie bialgebra extension $(\mathfrak{l} \oplus \mathfrak{l}')$. Next, we regard $\mathfrak{f}$ as a sub-vector space of $\mathfrak{f} \oplus \mathfrak{f}$ via $x \mapsto x \oplus x$, and let $\mathfrak{l}_0$ be the pull-back of $\mathfrak{f}$ along $\mathfrak{l} \oplus \mathfrak{l}' \to \mathfrak{f} \oplus \mathfrak{f}$, which is seen to be a Lie sub-bialgebra of $\mathfrak{l} \oplus \mathfrak{l}'$. Then we obtain the Lie bialgebra extension $(\mathfrak{l}_0)$. Finally, regard $\mathfrak{g}^*$ as a quotient vector space of $\mathfrak{g}^* \oplus \mathfrak{g}^*$ via $s^* \oplus t^* \mapsto s^* + t^*$, and let $\mathfrak{l}_1$ be the push-out of $\mathfrak{g}^*$ along $\mathfrak{g}^* \oplus \mathfrak{g}^* \to \mathfrak{l}_0$, which is seen to be a quotient Lie bialgebra of $\mathfrak{l}_0$. We define then $(\mathfrak{l}) \Diamond (\mathfrak{l}')$ to be the Lie bialgebra extension $(\mathfrak{l}_1)$ just obtained, which is associated with the fixed Singer pair. (A naturally equivalent extension is obtained by forming first the push-out and then the pull back.) The product $\Diamond$ makes $\mathrm{Opext}(\mathfrak{f}, \mathfrak{g}^*)$ into an abelian group, which is called the *Opext group associated with the Singer pair.*

Let $(\mathfrak{l})$ be as above and identify $\mathfrak{l} = \mathfrak{g}^* \oplus \mathfrak{f}$. Since $\mathfrak{l}$ is in particular a Lie algebra extension, it follows as is well known that the bracket of $\mathfrak{l}$ is defined by

$$[s^* \oplus x, t^* \oplus y] = ((x \rightharpoonup t^*) - (y \rightharpoonup s^*) + \sigma(x \wedge y)) \oplus [x, y],$$

where $\sigma : \mathfrak{f} \wedge \mathfrak{f} \to \mathfrak{g}^*$ is some 2-cocycle for the left $\mathfrak{f}$-Lie module $(\mathfrak{g}^*, \rightharpoonup)$. Dually, the bracket of $\mathfrak{l}^* = \mathfrak{g} \oplus \mathfrak{f}^*$ is described by using some 2-cocycle for the right $\mathfrak{g}$-Lie module $(\mathfrak{f}^*, \rho^*)$. Regard σ, τ as linear maps

$$\sigma : \mathfrak{g} \otimes (\mathfrak{f} \wedge \mathfrak{f}) \to \mathbf{k}, \quad \tau : (\mathfrak{g} \wedge \mathfrak{g}) \otimes \mathfrak{f} \to \mathbf{k}.$$

The 2-cocycle conditions for σ, τ and the compatibility condition (2.2) for $\mathfrak{l}$ can be joined by the condition for (σ, τ) to be a 1-cocycle in the total complex $\mathrm{Tot}(C_L^{\cdot\cdot})$ of the double complex

$$\begin{array}{ccccccc} & & \vdots & & \vdots & & \\ & & {\scriptstyle \partial'}\uparrow & & {\scriptstyle \partial'}\uparrow & & \\ C_L^{\cdot\cdot} & = & \mathrm{Hom}((\wedge^2\mathfrak{g}) \otimes \mathfrak{f}, \mathbf{k}) & \xrightarrow{\partial} & \mathrm{Hom}((\wedge^2\mathfrak{g}) \otimes (\wedge^2\mathfrak{f}), \mathbf{k}) & \xrightarrow{\partial} & \cdots \\ & & {\scriptstyle \partial'}\uparrow & & {\scriptstyle \partial'}\uparrow & & \\ & & \mathrm{Hom}(\mathfrak{g} \otimes \mathfrak{f}, \mathbf{k}) & \xrightarrow{\partial} & \mathrm{Hom}(\mathfrak{g} \otimes (\wedge^2\mathfrak{f}), \mathbf{k}) & \xrightarrow{\partial} & \cdots, \end{array}$$

whose differentials will be defined in Section 5. Conversely, a 1-cocycle (σ, τ) in $\mathrm{Tot}(C_L^{\cdot\cdot})$ constructs on the vector space $\mathfrak{g}^* \oplus \mathfrak{f}$, as above, a Lie bialgebra extension associated with the Singer pair, which is denoted by $(\mathfrak{g}^* {\blacktriangleright\!\!\triangleleft}_{\sigma,\tau} \mathfrak{f})$. Its co-bracket δ is the dual of the bracket on $\mathfrak{g} \oplus \mathfrak{f}^*$ given by τ.

Exercise 2.5. *Let $\{s_i\}, \{s_i^*\}$ be bases of $\mathfrak{g}, \mathfrak{g}^*$ dual to each other. Show that*

$$\delta(s^* \oplus x) = \delta(s^*) + \sum_i (s_i \triangleright x) \otimes s_i^* - \sum_i s_i^* \otimes (s_i \triangleright x) + \sum_{i,j} \tau((s_i \wedge s_j) \otimes x) s_i^* \otimes s_j^*,$$

where $\triangleright$ is defined from ρ as in Definition 2.3.

Proposition 2.6. *The correspondence $(\sigma, \tau) \mapsto (\mathfrak{g}^* {\blacktriangleright\!\!\triangleleft}_{\sigma,\tau} \mathfrak{f})$ induces an isomorphism*

$$H^1 \mathit{Tot}(C_L^{\cdot\cdot}) \simeq \mathit{Opext}(\mathfrak{f}, \mathfrak{g}^*).$$

□

If σ, τ are both zero maps, we write simply $(\mathfrak{g}^* {\blacktriangleright\!\!\triangleleft} \mathfrak{f})$ for $(\mathfrak{g}^* {\blacktriangleright\!\!\triangleleft}_{\sigma,\tau} \mathfrak{f})$ and call it the *split extension*, which represents the unit of $\mathrm{Opext}(\mathfrak{f}, \mathfrak{g}^*)$. Let

$$\mathrm{Aut}(\mathfrak{g}^* {\blacktriangleright\!\!\triangleleft} \mathfrak{f}) = \text{ the group of the auto-equivalences of } (\mathfrak{g}^* {\blacktriangleright\!\!\triangleleft} \mathfrak{f}).$$

Let $\nu : \mathfrak{g} \otimes \mathfrak{f} \to \mathbf{k}$ be a 0-cocycle in $\mathrm{Tot}(C_L^{\cdot\cdot})$, and regard it as a linear map $\nu : \mathfrak{f} \to \mathfrak{g}^*$. Then the map $s^* \oplus x \mapsto (s^* + \nu(x)) \oplus x$, $\mathfrak{g}^* {\blacktriangleright\!\!\triangleleft} \mathfrak{f} \to \mathfrak{g}^* {\blacktriangleright\!\!\triangleleft} \mathfrak{f}$ gives an auto-equivalence of $(\mathfrak{g}^* {\blacktriangleright\!\!\triangleleft} \mathfrak{f})$.

Proposition 2.7. *The correspondence just obtained gives an isomorphism*

$$H^0 \mathit{Tot}(C_L^{\cdot\cdot}) \simeq \mathit{Aut}(\mathfrak{g}^* {\blacktriangleright\!\!\triangleleft} \mathfrak{f}).$$

□

Exercise 2.8. *Give a result parallel to Excercise 1.5.*

Recall that the Lie algebra $\mathfrak{f} \bowtie \mathfrak{g}$ is constructed from the matched pair $(\mathfrak{f}, \mathfrak{g}, \triangleright, \triangleleft)$ which corresponds to the Singer pair $(\mathfrak{f}, \mathfrak{g}^*, \rightharpoonup, \rho)$.

Theorem 2.9. *We have the following exact sequence*

$$\begin{aligned} 0 &\to H^1(\mathfrak{f} \bowtie \mathfrak{g}, \mathbf{k}) \to H^1(\mathfrak{f}, \mathbf{k}) \oplus H^1(\mathfrak{g}, \mathbf{k}) \to \mathit{Aut}(\mathfrak{g}^* {\blacktriangleright\!\!\triangleleft} \mathfrak{f}) \\ &\to H^2(\mathfrak{f} \bowtie \mathfrak{g}, \mathbf{k}) \to H^2(\mathfrak{f}, \mathbf{k}) \oplus H^2(\mathfrak{g}, \mathbf{k}) \to \mathit{Opext}(\mathfrak{f}, \mathfrak{g}^*) \\ &\to H^3(\mathfrak{f} \bowtie \mathfrak{g}, \mathbf{k}) \to H^3(\mathfrak{f}, \mathbf{k}) \oplus H^3(\mathfrak{g}, \mathbf{k}), \end{aligned}$$

where $H^{\cdot}(-, \mathbf{k})$ indicates the Lie algebra cohomology with coefficients in the trivial Lie module $\mathbf{k}$. □

Corollary 2.10. *If either*

1. *$\mathfrak{f}$ is semisimple and $\rightharpoonup$ is trivial, or*
2. *$\mathfrak{g}$ is semisimple and ρ is trivial,*

then $\mathit{Aut}(\mathfrak{g}^* {\blacktriangleright\!\!\triangleleft} \mathfrak{f})$, $\mathit{Opext}(\mathfrak{f}, \mathfrak{g}^*)$ are both trivial. □

3. **From Hopf algebra extensions to Lie bialgebra extensions.** H, J cocommutative Hopf algebras.

Definition 3.1. The pair (H, J) together with linear maps $J \xleftarrow{\triangleleft} J \otimes H \xrightarrow{\triangleright} H$ is called a *matched pair of Hopf algebras* if H is a left J-module coalgebra under $\triangleright$, J is a right H-module coalgebra under $\triangleleft$ and

1. $s \triangleright xy = \sum (s_{(1)} \triangleright x_{(1)})((s_{(2)} \triangleleft x_{(2)}) \triangleright y)$,
2. $st \triangleleft x = \sum (s \triangleleft (t_{(1)} \triangleright x_{(2)}))(t_{(2)} \triangleleft x_{(2)})$

for $x, y \in H,\ s, t \in J$.

These conditions hold iff the tensor product coalgebra $H \otimes J$ forms a bialgebra (then necessarily a Hopf algebra) with unit $1 \otimes 1$ under the product defined by

$$(x \otimes s)(y \otimes t) = \sum x(s_{(1)} \triangleright y_{(1)}) \otimes (s_{(2)} \triangleleft y_{(2)})t \qquad (x, y \in H,\ s, t \in J).$$

In this case, we denote the Hopf algebra by $H \bowtie J$,

Definition 3.2. A set $\mathcal{J}$ of ideals in J of cofinite dimension is said to be *admissible* if

i. $\mathcal{J}$ is directed downward, i.e., $\forall I_1, I_2 \in \mathcal{J};\ \exists I \in \mathcal{J}$ such that $I \subset I_1 \cap I_2$,
ii. $J^+ := \mathrm{Ker}(\varepsilon : J \to \mathbf{k}) \in \mathcal{J}$,
iii. $\forall I_1, I_2 \in \mathcal{J};\ \exists I \in \mathcal{J}$ such that $\Delta(I) \subset I_1 \otimes J + J \otimes I_2$,
iv. $\forall I \in \mathcal{J};\ \exists I' \in \mathcal{J}$ such that $\mathcal{S}(I') \subset I$.

For a given admissible set $\mathcal{J}$ of ideals, we define

$$J^\circ_{\mathcal{J}} = \bigcup_{I \in \mathcal{J}} I^\perp, \text{ where } I^\perp = \{p \in J^* | p(I) = 0\}.$$

Note that if $\mathcal{J} = \{\text{all cofinite ideals in } J\}$, then $J^\circ_{\mathcal{J}} = J^\circ$, the dual Hopf algebra of J, which is commutative. In general, $J^\circ_{\mathcal{J}}$ is a Hopf subalgebra of J°.

Definition 3.3. A *topological vector space* is a vector space with a topology such that

1. for each $w \in V$, the translation $v \mapsto v + w$ is continuous, and
2. V has a basis of neighbourhoods of 0 consisting of sub-vector spaces, which we call a *topological basis*.

Any vector space is a topological vector space with discrete topology. For a topological vector space V and a discrete vector space Z, we identify $V \otimes Z = \oplus_{\dim Z} V$, the direct sum of $\dim Z$ copies of V, in order to regard $V \otimes Z$ as a topological vector space, denoted by $V \otimes (Z)$, with the direct sum topology. Thus, if $\{V_\lambda\}$ is a topological basis of V and $\{z_\mu\}$ is a (linear) basis of Z, then $V \otimes (Z)$ has a topological basis consisting of all $\oplus_\mu V_\mu \otimes \mathbf{k} z_\mu$, where V_μ is an arbitrary element in $\{V_\lambda\}$.

J is a topological vector space (and is further a topological algebra in the sense of Takeuchi) with topological basis $\mathcal{J}$. We regard $\mathbf{k}$ as a discrete vector space, so that $J^\circ_{\mathcal{J}} = \mathrm{Hom}_c(J, \mathbf{k})$, the continuous linear maps $J \to \mathbf{k}$.

Lemma 3.4. *Let M be a discrete vector space. There is a 1–1 correspondence between the continuous left J-module structure maps $\triangleright : J \otimes (M) \to M$ and the right $J^\circ_{\mathcal{J}}$-comodule structure maps $\rho : M \to M \otimes J^\circ_{\mathcal{J}}$.*

Proof. Note that $M \otimes J^\circ_{\mathcal{J}} \simeq \mathrm{Hom}_c(J, M)$. Then one sees that the correspondence arises from the natural isomorphisms

$$\mathrm{Hom}_c(J \otimes (M), M) \simeq \mathrm{Hom}(M, \mathrm{Hom}_c(J, M)) \simeq \mathrm{Hom}(M, M \otimes J^\circ_{\mathcal{J}}).$$

□

We regard now H as a discrete vector space.

Proposition 3.5. *Suppose that (H, J) is a matched pair whose structure maps $J \xleftarrow{\triangleleft} J \otimes (H) \xrightarrow{\triangleright} H$ are both continuous. The transpose $\rightharpoonup : H \otimes J^* \to J^*$ of $\triangleleft$ stabilizes $J^\circ_{\mathcal{J}}$ and hence induces an action $H \otimes J^\circ_{\mathcal{J}} \to J^\circ_{\mathcal{J}}$, which is denoted by $\rightharpoonup$, too. Furthermore, $(H, J^\circ_{\mathcal{J}}, \rightharpoonup, \rho)$ is a Singer pair of Hopf algebras, where $\rho : H \to H \otimes J^\circ_{\mathcal{J}}$ is the coaction corresponding to $\triangleright$, as in Lemma 3.4.*

We suppose in what follows that char $\mathbf{k} = 0$. Let $\mathfrak{f}, \mathfrak{g}$ be finite dimensional Lie algebras. We consider the case where

$$H = U\mathfrak{f}, \quad J = U\mathfrak{g},$$

the universal enveloping algebras. These are cocommutative Hopf algebras in which elements in $\mathfrak{f}$ or in $\mathfrak{g}$ are primitive.

Proposition 3.6. *There is a natural 1–1 correspondence between the structures of a matched pair of Lie algebras on $(\mathfrak{f}, \mathfrak{g})$ and the structures of a matched pair of Hopf algebras on $(U\mathfrak{f}, U\mathfrak{g})$.* □

Proof. The correspondence is given as follows. If $(\mathfrak{f}, \mathfrak{g})$ is a matched pair, then a matched pair $(U\mathfrak{f}, U\mathfrak{g})$ arises from the factorization $U\mathfrak{f} \otimes U\mathfrak{g} = U(\mathfrak{f} \bowtie \mathfrak{g})$. Conversely, if $(U\mathfrak{f}, U\mathfrak{g})$ is a matched pair, then a matched pair $(\mathfrak{f}, \mathfrak{g})$ arises from the factorization $\mathfrak{f} \oplus \mathfrak{g} = P(U\mathfrak{f} \bowtie U\mathfrak{g})$, the Lie algebra of the primitives in $U\mathfrak{f} \bowtie U\mathfrak{g}$. □

We fix now an admissible set $\mathcal{J}$ of cofinite ideals in $U\mathfrak{g}$ and regard $U\mathfrak{g}$ as a topological vector space with topological basis $\mathcal{J}$. We regard further $U\mathfrak{f}$ as a discrete vector space. Let $U\mathfrak{g} \overset{\triangleleft}{\leftarrow} U\mathfrak{g} \otimes (U\mathfrak{f}) \overset{\triangleright}{\rightarrow} U\mathfrak{f}$ be continuous linear maps with which $(U\mathfrak{f}, U\mathfrak{g})$ is a matched pair of Hopf algebras. Then, by Proposition 3.5 there arises a Singer pair $(U\mathfrak{f}, (U\mathfrak{g})^{\circ}_{\mathcal{J}}, \rightharpoonup, \rho)$ of Hopf algebras. By Proposition 3.6 there arises a matched pair $(\mathfrak{f}, \mathfrak{g}, \triangleright', \triangleleft')$ of Lie bialgebras. The relation between the structures $(\rightharpoonup, \rho)$ and $(\rightharpoonup', \rho')$ is given by the following commutative diagram, where $\omega : (U\mathfrak{g})^{\circ}_{\mathcal{J}} \to \mathfrak{g}^*$ denotes the restriction map.

$$\begin{tikzcd}
\mathfrak{f} \otimes (U\mathfrak{g})^{\circ}_{\mathcal{J}} \arrow[r, hook] \arrow[d, "\mathrm{id} \otimes \omega"'] & U\mathfrak{f} \otimes (U\mathfrak{g})^{\circ}_{\mathcal{J}} \arrow[r, "\rightharpoonup"] & (U\mathfrak{g})^{\circ}_{\mathcal{J}} \arrow[d, "\omega"] \\
\mathfrak{f} \otimes g^* \arrow[rr, "\rightharpoonup'"] & & \mathfrak{g}^*
\end{tikzcd}$$

$$\begin{tikzcd}
U\mathfrak{f} \arrow[r, "\rho"] & U\mathfrak{f} \otimes (U\mathfrak{g})^{\circ}_{\mathcal{J}} \arrow[r, "\mathrm{id} \otimes \omega"] & U\mathfrak{f} \otimes \mathfrak{g}^* \\
\mathfrak{f} \arrow[u, hook] \arrow[rr, "\rho'"] & & \mathfrak{f} \otimes \mathfrak{g}^* \arrow[u, hook]
\end{tikzcd}$$

Theorem 3.7. *Consider the cleft extensions associated with $(U\mathfrak{f}, (U\mathfrak{g})^{\circ}_{\mathcal{J}}, \rightharpoonup, \rho)$ and the Lie bialgebra extensions associated with $(\mathfrak{f}, \mathfrak{g}^*, \rightharpoonup', \rho')$. There are natural group maps*

$$\kappa_0 : Aut((U\mathfrak{g})^{\circ}_{\mathcal{J}} \# U\mathfrak{f}) \to Aut(\mathfrak{g}^* \blacktriangleright\!\!\triangleleft \mathfrak{f}),$$
$$\kappa_1 : Opext(U\mathfrak{f}, (U\mathfrak{g})^{\circ}_{\mathcal{J}}) \to Opext(\mathfrak{f}, \mathfrak{g}^*).$$

If	(H^1) : $H^1(\mathfrak{g}, (U\mathfrak{g})^{\circ}_{\mathcal{J}}) = 0$,	*then κ_0 is an isomorphism.*
If in addition	(H^2) : $H^2(\mathfrak{g}, (U\mathfrak{g})^{\circ}_{\mathcal{J}}) = 0$,	*then κ_1 is an isomorphism.* □

In (H^1) and (H^2), $(U\mathfrak{g})^{\circ}_{\mathcal{J}}$ has the (continuous) left or right $U\mathfrak{g}$-module structure corresponding to the natural right or left $(U\mathfrak{g})^{\circ}_{\mathcal{J}}$-comodule structure. The choice of "left or right" makes no difference between the cohomology groups.

The maps κ_0, κ_1 are given as follows. Identify the Aut groups with the H^0 groups. For a 0-cocycle $\nu : U\mathfrak{f} \to (U\mathfrak{g})^{\circ}_{\mathcal{J}}$ in $\mathrm{Tot}(C^{\cdot\cdot}_H)$, $\kappa_0(\nu)$ is the composite $\mathfrak{f} \hookrightarrow U\mathfrak{f} \xrightarrow{\nu} (U\mathfrak{g})^{\circ}_{\mathcal{J}} \xrightarrow{\omega} \mathfrak{g}^*$. On the other hand, κ_1 is induced from the correspondence

$$((U\mathfrak{g})^{\circ}_{\mathcal{J}} \#_{\sigma,\tau} U\mathfrak{f}) \mapsto (\mathfrak{g}^* \blacktriangleright\!\!\triangleleft_{\bar{\sigma},\bar{\tau}} \mathfrak{f}),$$

where $\bar{\sigma} : \mathfrak{g} \otimes \wedge^2 \mathfrak{f} \to \mathbf{k}$, regarded as a linear map $\wedge^2 \mathfrak{f} \to \mathfrak{g}^*$, is determined by

$$< \bar{\sigma}(x \wedge y), s > = < \sigma(x \otimes y - y \otimes x), s > \qquad (x, y \in \mathfrak{f},\ s \in \mathfrak{g})$$

and $\bar{\tau} : \wedge^2 \mathfrak{g} \otimes \mathfrak{f} \to \mathbf{k}$, regarded as a linear map $\mathfrak{f} \to (\wedge^2 \mathfrak{g})^*$, is determined by

$$< \bar{\tau}(x), s \wedge t > = < \tau(x), s \otimes t - t \otimes s > \qquad (x \in \mathfrak{f},\ s, t \in \mathfrak{g}).$$

Remark 3.8. 1. If $\mathcal{J}$ satisfies

$$\text{i'. } \forall I_1, I_2 \in \mathcal{J};\ \exists I \in \mathcal{J} \text{ such that } I \subset I_1 I_2,$$

(which implies i in Definition 3.2), then (H^1) holds true.

2. It is not difficult to see that κ_0, κ_1 are not necessarily isomorphisms. Suppose $\mathcal{J} = \{(U\mathfrak{g})^+\}$. Then $(U\mathfrak{g})^\circ_{\mathcal{J}} = \mathbf{k}$, and hence the groups $\mathrm{Aut}((U\mathfrak{g})^\circ_{\mathcal{J}}\#U\mathfrak{f})$, $\mathrm{Opext}(U\mathfrak{f}, (U\mathfrak{g})^\circ_{\mathcal{J}})$ are both trivial. One has only to find a Singer pair $(\mathfrak{f}, \mathfrak{g}^*)$ with trivial structure for which the groups $\mathrm{Aut}(\mathfrak{g}^* \blacktriangleright\!\!\triangleleft \mathfrak{f})$, $\mathrm{Opext}(\mathfrak{f}, \mathfrak{g}^*)$ are not trivial.

Exercise 3.9. *Find such an example.*

Corollary 3.10. *Suppose that (H^1) and (H^2) hold. Construct the Lie algebra $\mathfrak{f} \bowtie \mathfrak{g}$ from the matched pair $(\mathfrak{f}, \mathfrak{g}, \triangleright', \triangleleft')$ of Lie algebras obtained above. Then we have the following exact sequence.*

$$\begin{aligned} 0 &\to H^1(\mathfrak{f} \bowtie \mathfrak{g}, \mathbf{k}) \to H^1(\mathfrak{f}, \mathbf{k}) \oplus H^1(\mathfrak{g}, \mathbf{k}) \to \mathit{Aut}((U\mathfrak{g})^\circ_{\mathcal{J}}\#U\mathfrak{f}) \\ &\to H^2(\mathfrak{f} \bowtie \mathfrak{g}, \mathbf{k}) \to H^2(\mathfrak{f}, \mathbf{k}) \oplus H^2(\mathfrak{g}, \mathbf{k}) \\ &\to \mathit{Opext}(U\mathfrak{f}, (U\mathfrak{g})^\circ_{\mathcal{J}}) \to H^3(\mathfrak{f} \bowtie \mathfrak{g}, \mathbf{k}) \to H^3(\mathfrak{f}, \mathbf{k}) \oplus H^3(\mathfrak{g}, \mathbf{k}). \end{aligned}$$

□

4. **Two special cases.** Suppose as above that char $\mathbf{k} = 0$ and let $\mathfrak{f}, \mathfrak{g}$ be finite-dimensional Lie algebras. We shall consider $\mathcal{J}$ in two cases.

Case 1. $\mathcal{J} = \{\text{all cofinite ideals in } U\mathfrak{g}\}$.

In this case, $(U\mathfrak{g})^\circ_{\mathcal{J}} = U\mathfrak{g}^\circ$, the dual Hopf algebra. We see that any structure maps $U\mathfrak{g} \xleftarrow{\triangleleft} U\mathfrak{g} \otimes U\mathfrak{f} \xrightarrow{\triangleright} U\mathfrak{f}$ of a matched pair on $(U\mathfrak{f}, U\mathfrak{g})$ are continuous. From Propositions 3.5 and 3.6 the next Proposition follows.

Proposition 4.1. *There is a natural injection from the set of structures $(\rightharpoonup', \rho')$ of a Singer pair of Lie algebras on $(\mathfrak{f}, \mathfrak{g}^*)$ to the set of structures $(\rightharpoonup, \rho)$ of a Singer pair of Hopf algebras on $(U\mathfrak{f}, U\mathfrak{g}^\circ)$. This is a bijection if $\mathfrak{g} = [\mathfrak{g}, \mathfrak{g}]$.* □

Remark 4.2. Suppose that $\mathbf{k}$ is algebraically closed. A commutative Hopf algebra is in the form $U\mathfrak{g}^\circ$, where $\mathfrak{g}$ is a finite-dimensional Lie algebra with $\mathfrak{g} = [\mathfrak{g}, \mathfrak{g}]$ iff it is isomorphic to the coordinate Hopf algebra $\mathcal{O}(G)$ of a connected, simply connected affine algebraic group G with $G = [G, G]$.

Since $\mathcal{J}$ is closed under product, it follows by Remark 3.8(1) that $H^1(\mathfrak{g}, U\mathfrak{g}^\circ) = 0$. Furthermore we have

Proposition 4.3 (Schneider).

$$H^2(\mathfrak{g}, U\mathfrak{g}^\circ) = 0.$$

□

Hence the conclusions of Theorem 3.7 and Corollary 3.10 hold true with $(U\mathfrak{g})^\circ_{\mathcal{J}} = U\mathfrak{g}^\circ$, where $(\mathfrak{f}, \mathfrak{g}^*, \rightharpoonup', \rho')$ is an arbitrary Singer pair of Lie bialgebras and $(U\mathfrak{f}, U\mathfrak{g}^\circ, \rightharpoonup, \rho)$ is the corresponding Singer pair of Hopf algebras. We shall have two consequences. The first follows from Corollary 2.10.

Proposition 4.4. *Consider a Singer pair $(U\mathfrak{f}, U\mathfrak{g}^\circ, \rightharpoonup, \rho)$ of Hopf algebras such that either*

1. $\mathfrak{f}$ *is semisimple and* $\rightharpoonup$ *is trivial or*
2. $\mathfrak{g}$ *is semisimple and* ρ *is trivial.*

Then, $\mathit{Aut}(U\mathfrak{g}^\circ\#U\mathfrak{f})$, $\mathit{Opext}(U\mathfrak{f}, U\mathfrak{g}^\circ)$ are both trivial. □

We define

$$\mathrm{Ext}(U\mathfrak{f}, U\mathfrak{g}^\circ) = \left\{ \begin{array}{c} \text{the set of the equivalence classes of} \\ \text{all cleft extensions of } U\mathfrak{f} \text{ by } U\mathfrak{g}^\circ \end{array} \right\},$$

$$\mathrm{Ext}(\mathfrak{f}, \mathfrak{g}^*) = \left\{ \begin{array}{c} \text{the set of the equivalence classes of} \\ \text{all Lie bialgebra extensions of } \mathfrak{f} \text{ by } \mathfrak{g}^* \end{array} \right\}.$$

Since these equal the disjoint union of all the Opext groups, we have

Theorem 4.5. *If $\mathfrak{g} = [\mathfrak{g}, \mathfrak{g}]$, there is a natural bijection*

$$Ext(U\mathfrak{f}, U\mathfrak{g}^\circ) \simeq Ext(\mathfrak{f}, \mathfrak{g}^*).$$

$\square$

Case 2. $\mathcal{J} = \{(U\mathfrak{g}^+)^n | n = 0, 1, 2, \dots\}$.

In this case, $(U\mathfrak{g})^\circ_{\mathcal{J}}$ is the irreducible component of $U\mathfrak{g}^\circ$ containing 1 (the largest irreducible Hopf subalgebra of $U\mathfrak{g}^\circ$), which is denoted by $U\mathfrak{g}'$. We are interested in the case where $\mathfrak{g}$ is nilpotent in view of the following fact.

Proposition 4.6 (Hochschild). *There is a category equivalence between the finite-dimensional nilpotent Lie algebras $\mathfrak{g}$ and the unipotent affine algebraic group schemes G, given by*

$$\mathfrak{g} \mapsto Spec(U\mathfrak{g}'), \quad G \mapsto Lie\ G.$$

$\square$

Proposition 4.7. *Suppose that $\mathfrak{g}$ is nilpotent. There is a natural 1–1 correspondence between the structures $(\rightharpoonup, \rho)$ of a Singer pair of Hopf algebras on $(U\mathfrak{f}, U\mathfrak{g}')$ and the structures $(\rightharpoonup', \rho')$ of a Singer pair of Lie bialgebras on $(\mathfrak{f}, \mathfrak{g}^*)$ such that the action $\triangleright' : \mathfrak{g} \otimes \mathfrak{f} \to \mathfrak{f}$ corresponding to ρ' is nilpotent (i.e., $\exists m > 0$ such that $\mathfrak{g}^m \triangleright' \mathfrak{f} = 0$).* $\square$

Since $\mathcal{J}$ is closed under product, $H^1(\mathfrak{g}, U\mathfrak{g}') = 0$. On the other hand, we have

Proposition 4.8 (Koszul). *If $\mathfrak{g}$ is nilpotent, $H^n(\mathfrak{g}, U\mathfrak{g}') = 0$ for all $n > 0$.* $\square$

Hence, if $\mathfrak{g}$ is nilpotent, the conclusions of Theorem 3.7 and Corollary 3.10 hold true with $(U\mathfrak{g})^\circ_{\mathcal{J}} = U\mathfrak{g}'$, where $(U\mathfrak{f}, U\mathfrak{g}', \rightharpoonup, \rho)$ is an arbitrary Singer pair of Hopf algebras and $(\mathfrak{f}, \mathfrak{g}^*, \rightharpoonup', \rho')$ is the corresponding Singer pair of Lie bialgebras.

Example 4.9. *Let $\mathfrak{f} = \mathbf{k}x$, $\mathfrak{g} = \mathbf{k}s$ be 1-dimensional (abelian) Lie algebras. Then*

$$U\mathfrak{f} = \mathbf{k}[x],\ U\mathfrak{g} = \mathbf{k}[s],\ U\mathfrak{g}' = \mathbf{k}[p],$$

the polynomial algebras with x, s, p primitive, where $<p, s^n> = \delta_{1,n}$. For arbitrary ξ, η in $\mathbf{k}$, the actions $\triangleright, \triangleleft$ determined by

$$s \triangleright x = \xi x, \quad s \triangleleft x = \eta s$$

give a structure of a matched pair of Lie algebras on $(\mathfrak{f}, \mathfrak{g})$. The action $\triangleright$ is nilpotent iff $\xi = 0$. The module action $\rightharpoonup$: $\mathbf{k}[x] \otimes \mathbf{k}[p] \to \mathbf{k}[p]$ arising from $\triangleleft$ is determined by $x \rightharpoonup p^n = n\eta p^n$ $(n = 0, 1, \dots)$. Hence this $\rightharpoonup$ and the trivial coaction $\mathbf{k}[x] \to \mathbf{k}[x] \otimes \mathbf{k}[p]$ exhaust the structures of a matched pair of Hopf algebras on $(\mathbf{k}[x], \mathbf{k}[p])$. Further, for such a pair we have

$$Opext(\mathbf{k}[x], \mathbf{k}[p]) = 0,$$

since obviously $H^1 Tot(C_L^{\cdot\cdot}) = 0$.

Exercise 4.10. *Show that $Aut(\mathbf{k}[p]\#\mathbf{k}[x]) \simeq \mathbf{k}$.*

From Corollary 2.10 we obtain:

Proposition 4.11. *Consider a Singer pair $(U\mathfrak{f}, U\mathfrak{g}', \rightharpoonup, \rho)$ of Hopf algebras, where $\mathfrak{f}$ is semisimple, $\mathfrak{g}$ is nilpotent and $\rightharpoonup$ is trivial. Then, $Aut(U\mathfrak{g}'\#U\mathfrak{f})$, $Opext(U\mathfrak{f}, U\mathfrak{g}')$ are both trivial.* $\square$

There seems so far to be no computational result for the non-trivial Opext groups of this kind. We refer mainly to [K, M1, M2] for Part I, and to [M3] for Part II.

5. **Supplements.**

5.1. Let $(A) = K \xrightarrow{\iota} A \xrightarrow{\pi} H$ be a sequence of Hopf algebras and Hopf algebra maps, where ι is injective and π is surjective. In order to call (A) a (non-necessarily cleft) extension, assume that A is injective as a right H-comodule (via π) and that $K = A^{\mathrm{co}H}$. If H is, in addition, irreducible (for example if H is the universal enveloping algebra of some Lie algebra), then (A) is a cleft extension in the sense of Definition 1.1. In fact, the unit $\mathbf{k} \to A$ is extended to a right H-colinear map $\phi : H \to A$, since A is right H-injective. One sees that ϕ is convolution-invertible since its restriction on the coradical Corad $H = \mathbf{k}$ is. It follows that $p \otimes x \mapsto p\phi(x)$, $K \otimes H \to A$ gives an isomorphism required by the definition.

5.2. Let us give the precise definition of the double complex $C_H^{\cdot\cdot}$ in Section 1. Let $(H, K, \rightharpoonup, \rho)$ be a Singer pair of Hopf algebras. Let $u : \mathbf{k} \to H$, $\varepsilon : K \to \mathbf{k}$ denote the unit of H and the counit of K, respectively. Recall that Reg denotes the invertible maps in the Hom space. The term $\mathrm{Reg}_+(H^{\otimes p}, K^{\otimes q})$ denotes the intersection of the kernels of the following $p+q$ codegeneracy operators ($i = 1, \dots, p; j = 1, \dots, q$):

$$\sigma_i : \mathrm{Reg}(H^{\otimes p}, K^{\otimes q}) \to \mathrm{Reg}(H^{\otimes (p-1)}, K^{\otimes q}), \qquad \sigma_i f = f \circ (1^{\otimes (i-1)} \otimes u \otimes 1^{\otimes (p-i)}),$$
$$\tau_j : \mathrm{Reg}(H^{\otimes p}, K^{\otimes q}) \to \mathrm{Reg}(H^{\otimes p}, K^{\otimes (q-1)}), \qquad \tau_j f = (1^{\otimes (j-1)} \otimes \varepsilon \otimes 1^{\otimes (q-j)}) \circ f.$$

For a left H-module V, regard $\Phi(V) := V \otimes K$ as a left H-module by

$$x(v \otimes p) = (x_{(1)})_H v \otimes (x_{(1)})_K (x_{(2)} \rightharpoonup p),$$

where $x \in H$, $v \otimes p \in V \otimes K$. Apply this construction q times iteratedly to $V = \mathbf{k}$, the trivial left H-module, and obtain the left H-module $\Phi^q(\mathbf{k}) = K^{\otimes q}$. Define $d^i : \mathrm{Reg}_+(H^{\otimes p}, K^{\otimes q}) \to \mathrm{Reg}_+(H^{\otimes (p+1)}, K^{\otimes q})$ $(i = 0, 1, \dots, p+1)$ by

$$\begin{aligned} &d^0 f = (H\text{-action on } \Phi^q(\mathbf{k}) = K^{\otimes q}) \circ (1 \otimes f), \\ &d^i f = f \circ (1^{\otimes (i-1)} \otimes \mathrm{mult} \otimes 1^{\otimes (p-i)}) \quad (1 \le i \le p), \\ &d^{p+1} f = f \otimes \varepsilon. \end{aligned}$$

The differential ∂ is defined by

$$\partial f = d^0 f * d^1 f^{-1} * \dots * d^{p+1} f^{\pm 1},$$

where $*$ denotes the convolution-product.

For a right K-comodule W with structure $w \mapsto w_{(0)} \otimes w_{(1)}$, regard $\Psi(W) := H \otimes W$ as a right K-comodule by

$$x \otimes w \mapsto (x_{(1)})_H \otimes w_0 \otimes (x_{(1)})_K (x_{(2)} \rightharpoonup w_{(1)}).$$

Apply this construction p times iteratedly to $W = \mathbf{k}$, the trivial right K-comodule, and obtain the right K-comodule $\Psi^p(\mathbf{k}) = H^{\otimes p}$. Define $d'^j : \mathrm{Reg}_+(H^{\otimes p}, K^{\otimes q}) \to \mathrm{Reg}_+(H^{\otimes p}, K^{\otimes (q+1)})$ $(j = 0, 1, \dots, q+1)$ by

$$\begin{aligned} &d'^0 f = (f \otimes 1) \circ (K\text{-coaction on } \Psi^p(\mathbf{k}) = H^{\otimes p}), \\ &d'^j f = (1^{\otimes (q-j)} \otimes \Delta \otimes 1^{\otimes (j-1)}) \circ f \quad (1 \le j \le q), \\ &d'^{q+1} f = u \otimes f. \end{aligned}$$

The differential ∂' is defined by

$$\partial' f = [d'^0 f * d'^1 f^{-1} * \dots * d'^{q+1} f^{\pm 1}]^{(-1)^p}.$$

5.3. Finally, let us define the differentials

$$\begin{aligned} &\partial : \mathrm{Hom}((\wedge^q \mathfrak{g}) \otimes (\wedge^p \mathfrak{f}), \mathbf{k}) \to \mathrm{Hom}((\wedge^q \mathfrak{g}) \otimes (\wedge^{p+1} \mathfrak{f}), \mathbf{k}), \\ &\partial' : \mathrm{Hom}((\wedge^q \mathfrak{g}) \otimes (\wedge^p \mathfrak{f}), \mathbf{k}) \to \mathrm{Hom}((\wedge^{q+1} \mathfrak{g}) \otimes (\wedge^p \mathfrak{f}), \mathbf{k}), \end{aligned}$$

as follows:

$$\partial f(s_q,\dots,s_1;x_1,\dots,x_{p+1}) = \sum_{i,j}(-1)^{i+1} f(s_q,\dots,s_j \triangleleft x_i,\dots,s_1;x_1,\dots,\widehat{x_i},\dots,x_{p+1}) + \sum_{i<j}(-1)^{i+j} f(s_q,\dots,s_1;[x_i,x_j],x_1,\dots,\widehat{x_i},\dots,\widehat{x_j},\dots,x_{p+1}),$$

$$(-1)^p\partial' f(s_{q+1},\dots,s_1;x_1,\dots,x_p) = \sum_{i,j}(-1)^{j+1} f(s_{q+1},\dots,\widehat{s_j},\dots,s_1;x_1,\dots,s_j \triangleright x_i,\dots,x_p) + \sum_{i<j}(-1)^{i+j} f(s_{q+1},\dots,\widehat{s_j},\dots,\widehat{s_i},\dots,s_1,[s_j,s_i];x_1,\dots,x_p).$$

Here$\hat{}$ denotes an ommited term.

We refer mainly to [K, M1, M2] for Part I, and to [M3] for Part II.

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