LIFTING OF NICHOLS ALGEBRAS OF TYPE A_2 AND POINTED HOPF ALGEBRAS OF ORDER p^4

NICOLÁS ANDRUSKIEWITSCH
UNIVERSIDAD DE CÓRDOBA
CÓRDOBA, ARGENTINA AND HANS-JÜRGEN SCHNEIDER
UNIVERSITÄT MÜNCHEN
MÜNCHEN, GERMANY

Abstract. We compute all finite-dimensional Hopf algebras whose coradical is the group algebra of an abelian group Γ and such that $\mathrm{gr}(A) \cong \mathfrak{B}(V)\#\mathbf{k}[\Gamma]$, V a 2-dimensional Yetter-Drinfeld module over Γ of type A_2. We classify pointed Hopf algebras of order p^4 up to isomorphism and show that the number of quasi-isomorphism types is finite.

0. Introduction

We shall work over an algebraically closed field $\mathbf{k}$ of characteristic 0. Let p be a prime number. Several classification results are known for Hopf algebras of order p^N. Semisimple Hopf algebras of order p^N have always a non-trivial central group-like element; so they can be obtained recursively by an extension process. The explicit description of all isomorphism types for p^2 and p^3 is actually known [9].
We deal in this paper with *pointed* Hopf algebras of dimension p^N. A Hopf algebra with coradical $\mathbf{k}[\mathbb{Z}/(2)]$ has dimension 2^N and for each N there is exactly one such Hopf algebra up to isomorphisms [11, Th. 4.2.1]. Pointed Hopf algebras of order 8, 16 and 32 are respectively classified in [14], [6], [8]. ¿From now on we assume that p is odd. The only known non-semisimple Hopf algebras of dimension p^2 are pointed [1] and it is easy to see that they are Taft algebras. Pointed non-semisimple Hopf algebras of dimension p^3 are classified in [2], [5], [12]. Infinite families of pointed non-semisimple Hopf algebras of dimension p^4 were constructed in [2], [4], [7]. In this paper we complete the classification of all pointed non-semisimple Hopf algebras of dimension p^4.
If A is a pointed Hopf algebra of dimension p^4, then we know by [3, §7] $\mathrm{gr}(A)$, the graded Hopf algebra associated to the coradical filtration. We have to describe all possible liftings A of $\mathrm{gr}(A)$. There are two main cases: $\mathrm{gr}(A) \cong \mathfrak{B}(V)\#\mathbf{k}[\Gamma]$, and V is a Yetter-Drinfeld module over Γ of (1) type A_1 or $A_1 \times A_1$, or (2) type A_2 over the cyclic group of order p. The liftings of type a product of copies of A_1 are determined in [2]. We describe explicitly all the isomorphism types in Section 2. In Section 3 of this paper we compute all liftings of type A_2 over any finite abelian group Γ, if the corresponding root of 1 is odd and greater than 3. We conclude that a pointed Hopf algebra of

dimension p^4 with coradical of dimension p is necessarily coradically graded, and we obtain the classification of pointed Hopf algebras of dimension p^4 in Theorem 4.1.
In the appendix we show that although the number of isomorphism types of pointed Hopf algebras of dimension p^4 is infinite, the number of quasi-isomorphism types is in fact finite.

1. Preliminaries

We use standard notation for Hopf algebras. Our references for the theory of Hopf algebras are [10], [13].
We shall follow the method developed in [2], [3]. Let us briefly recall the main ideas. Given a pointed Hopf algebra A with coradical $A_0 = \mathbf{k}[\Gamma]$, the graded coalgebra associated to the coalgebra filtration is in fact a graded Hopf algebra which we denote by $\operatorname{gr}(A)$. In fact, $\operatorname{gr}(A)$ is a *coradically graded* Hopf algebra, and can be obtained as a bosonization $R\#\mathbf{k}[\Gamma]$, where R is a braided Hopf algebra in the category of Yetter-Drinfeld modules over Γ. Now we know from [3, §7] the classification of all possible coradically graded Hopf algebras of order p^4:

Theorem 1.1. *Let p be an odd prime number. Let A be a pointed non-semisimple Hopf algebras of order p^4 with coradical $\mathbf{k}[\Gamma]$. Then the associated graded algebra $\operatorname{gr}(A)$ is isomorphic to $R\#\mathbf{k}[\Gamma]$, where*

(A). R is a quantum line if Γ has order p^3;

(B). R is a quantum line or plane if Γ has order p^2;

(C). $R = \mathfrak{B}(V)$, where V has Dynkin diagram A_2 if Γ has order p, where $p = 3$ or $p-1$ is divisible by 3.

To obtain our result we have to determine all possible Hopf algebras A with $\operatorname{gr}(A)$ as in (A), (B) or (C) of Theorem 2.1. For (A) and (B), the description follows from [2, Theorem 2]; see Section 2. We treat the case (C) in Section 3.
In principle it is easy to describe all possible isomorphisms between liftings of $\mathfrak{B}(V)\#\mathbf{k}[\Gamma]$ when the Yetter-Drinfel'd module V is of finite Cartan type (once the defining relations are known). Let (a_{ij}) be a Cartan matrix of finite type of size $\theta \times \theta$, Γ a finite abelian group, $g_1, \dots, g_\theta \in \Gamma$, $\chi_1, \dots, \chi_\theta \in \widehat{\Gamma}$ such that for all i, j
$$\chi_i(g_j)\chi_j(g_i) = \chi_i(g_i)^{a_{ij}}.$$
We define $q_i = \chi_i(g_i)$ and assume that the orders N_i of all q_i are odd and > 1. Let V be a Yetter-Drinfel'd module over $\mathbf{k}[\Gamma]$ with a basis $x_1, \dots, x_\theta$ such that
$$h \cdot x_i = \chi_i(h)x_i \quad \text{and} \quad \delta(x_i) = g_i \otimes x_i$$
for all $h \in \Gamma$ and all i. Let A be a pointed Hopf algebra with $\operatorname{gr}(A) \cong \mathfrak{B}(V)\#\mathbf{k}[\Gamma]$. We choose elements a_i in the first term of the coradical filtration of A such that the image of $x_i\#1$ in $\operatorname{gr}(A)$ is the canonical image of a_i in A_1/A_0. Then A is generated as an algebra by Γ and the elements a_i. We consider another such Yetter-Drinfel'd module W over $\mathbf{k}[\widetilde{\Gamma}]$ of Cartan type $(\widetilde{a}_{ij})$ of size $\widetilde{\theta} \times \widetilde{\theta}$ with elements $h_j \in \widetilde{\Gamma}$, characters η_j and corresponding

basis elements w_j. Let B be a lifting of $\mathfrak{B}(W)\#\mathbf{k}[\widetilde{\Gamma}]$ with corresponding elements b_j.

Lemma 1.2. *Assume for all $i \neq j$, $g_i \neq g_j$ or $\chi_i \neq \chi_j$. (In particular this holds if for all $i \neq j$, the order of q_i does not divide $2 - a_{ij}$). Let $\Phi : A \to B$ be a Hopf algebra isomorphism. Then $\theta = \widetilde{\theta}$, $q_i^{a_{ij}} = q_i^{\widetilde{a}_{ij}}$ for all i, j and there are a group isomorphism $\phi\colon\ \Gamma \to \widetilde{\Gamma}$, a permutation σ and non-zero scalars α_i such that*

$$\Phi|\Gamma = \phi, \quad \Phi(a_i) = \alpha_i b_{\sigma(i)}, \quad \phi(g_i) = h_{\sigma(i)}, \quad \chi_i = \eta_{\sigma(i)}\phi, \quad 1 \leq i \leq \theta.$$

Proof. Φ induces a Hopf algebra isomorphism $\mathrm{gr}(A) \to \mathrm{gr}(B)$. By [3, Proposition 6.3] the assertion holds on the graded level since by our assumption the Yetter-Drinfel'd module V is a direct sum of pairwise non-isomorphic one-dimensional submodules $\mathbf{k}x_i$. Thus for all i, $\Phi(a_i) - \alpha_i b_{\sigma(i)}$ is an element in the group algebra which must be zero since the group is acting on it via the non-trivial character $\eta_{\sigma(i)}$. □

Conversely, data ϕ, σ and α_i as in Lemma 1.2 define a Hopf algebra isomorphism Φ by $\Phi|\Gamma = \phi$ and $\Phi(a_i) = \alpha_i b_{\sigma(i)}$ if all the relations between the a_i are respected. This gives a condition on the scalars α_i which has to be worked out in each case.

In the case of quantum linear spaces, that is if $a_{ij} = 0$ for all $i \neq j$, this is very easy. By [2, Section 5], a lifting A of a quantum linear space is given by scalars $\mu_i \in \{0, 1\}$ for all i, $\lambda_{ij} \in \mathbf{k}$ for all $i < j$ such that $\mu_i = 0$ if $g_i^{N_i} = 1$ or $\chi_i^{N_i} \neq 1$, and $\lambda_{ij} = 0$ if $g_i g_j = 1$ or $\chi_i\chi_j \neq 1$. The relations we have to check are

$$a_i^{N_i} = \mu_i(1 - g_i^{N_i}) \quad \text{for all } i, \quad a_j a_i = \chi_i(g_j)a_i a_j + \lambda_{ij}(1 - g_i g_j) \quad \text{for all } i < j.$$

If B is another such lifting with data $\widetilde{\mu}_i, \widetilde{\lambda}_{ij}$, then A is isomorphic to B if and only if there are σ and α_i as in Lemma 1.2 satisfying for all i and $i < j$,

$$\alpha_i^{N_i}\widetilde{\mu}_{\sigma(i)} = \mu_i;$$

$$\alpha_i\alpha_j\widetilde{\lambda}_{\sigma(i)\sigma(j)} = \lambda_{ij} \text{ if } \sigma(i) < \sigma(j); \ -\chi_i(g_j)\alpha_i\alpha_j\widetilde{\lambda}_{\sigma(j)\sigma(i)} = \lambda_{ij} \text{ if } \sigma(i) > \sigma(j).$$

2. The braided Hopf algebra R is a quantum linear space

Let p be an odd prime number. In order to classify pointed Hopf algebras of order p^4 we describe all liftings of quantum linear lines over a group Γ of order p^3 and of quantum lines or planes over Γ of order p^2. The following list contains all such Hopf algebras, and the remarks at the end of the last section show that Hopf algebras in two different cases of this list are non-isomorphic. We leave the details of the proof to the reader.

2.1 Index p

Here Γ is a group of order p^3. Since R is a quantum line, there exist $g \in \Gamma$ central and a character $\chi \in \widehat{\Gamma}$ such that $\chi(g)$ has order p. Note that Γ is necessarily abelian: indeed, the center of a non-abelian group of order p^3 is contained in its commutator, so $\chi(g) = 1$ for any $g \in \Gamma$ central, $\chi \in \widehat{\Gamma}$.

There are three abelian groups of order p^3 which we write as $\langle u_1, \dots, u_r\rangle$, $1 \le r \le 3$. Then A can be presented by generators u_i, $1 \le i \le r$, a, and relations

$$
\begin{aligned}
&u_i u_j = u_j u_i \text{ for all } 1 \le i, j \le r;\\
&u_1^{p^3} = 1 \text{ if } r = 1;\ u_1^{p^2} = 1,\ u_2^p = 1 \text{ if } r = 2;\\
&u_i^p = 1 \text{ for all } i \text{ if } r = 3.\\
&a^p = \mu(1 - g^p), \text{ with } \mu \text{ either } 0 \text{ or } 1;\\
&hah^{-1} = \chi(h)a,\ h \in \Gamma.
\end{aligned}
\tag{1}
$$

If $\mu = 1$, then $g^p \neq 1$ and $\chi^p = 1$. The value of μ is invariant under isomorphism. The Hopf algebra structure of A is determined by

$$\Delta(h) = h \otimes h, \qquad \Delta(a) = a \otimes 1 + g \otimes a, \quad h \in \Gamma.$$

Up to isomorphism there are various subcases according to the order of g.

If $\Gamma = \langle u_1\rangle$, we have three cases where $g = u_1^{p^s}$ and $\chi(u_1) = \xi$, ξ a root of unity of order p^{s+1}, $0 \le s \le 2$. In each case, if η is another root of unity of the same order p^{s+1}, then the corresponding Hopf algebras are isomorphic if and only if there is an integer i with $\xi = \eta^i$ and $i \equiv 1 \bmod p^{3-s}$ (hence $\xi = \eta$ in case $s = 0, 1$).

If $\Gamma = \langle u_1, u_2\rangle$, we take $g = u_1^p$ and $\chi(u_1) = \xi$, $\chi(u_2) = 1$, ξ a root of unity of order p^2 (we get an isomorphic Hopf algebra only if we replace ξ by ξ^i, $i \equiv 1 \bmod p$); or $g = u_2$ and $\chi(u_1) = \rho$, $\chi(u_2) = \xi$, ξ a primitive p-th root of unity and $\rho = 1$ or an a priori given root of unity of order p^2; or $g = u_1$ and $\chi(u_1) = \xi$, $\chi(u_2) = 1$, ξ a p-th root of unity. In the last two cases the Hopf algebras are pairwise non-isomorphic.

If $\Gamma = \langle u_1, u_2, u_3\rangle$, we take $g = u_1$ and $\chi(u_1) = \xi$, ξ a primitive p-th root of unity, $\chi(u_2) = 1, \chi(u_3) = 1$. These Hopf algebras are pairwise non-isomorphic.

2.2 Index p^2, R a quantum line
There exist $g \in \Gamma$ and a character $\chi \in \widehat{\Gamma}$ such that $\chi(g)$ has order p^2. Necessarily, Γ is isomorphic to $\mathbb{Z}/(p^2)$. Then there exists a primitive p^2-th root of unity $\xi \in \mathbf{k}$ such that A can be presented by generators g, a and relations

$$
\begin{aligned}
&g^{p^2} = 1,\\
&a^{p^2} = 0;\\
&gag^{-1} = \xi a.
\end{aligned}
\tag{2}
$$

The Hopf algebra structure of A is determined by

$$\Delta(g) = g \otimes g, \qquad \Delta(a) = a \otimes 1 + g \otimes a.$$

That is, A is isomorphic to a Taft algebra of order p^4. For different roots of unity the Hopf algebras are non-isomorphic.

2.3 Index p^2, R a quantum plane and $\Gamma \simeq \mathbb{Z}/(p) \times \mathbb{Z}/(p)$

Then there exist $g_1, g_2 \in \Gamma$ and characters $\chi_1, \chi_2 \in \widehat{\Gamma}$ such that $\chi_1(g_1)$ and $\chi_2(g_2)$ have order p, $\chi_1(g_2)\chi_2(g_1) = 1$. A can be presented by generators u_1, u_2, a_1, a_2 and relations

$$
\begin{aligned}
&u_i^p = 1,\ i = 1,2,\ u_1u_2 = u_2u_1;\\
&a_i^p = 0,\ i = 1,2;\\
&ha_ih^{-1} = \chi_i(h)a_i,\ h \in \Gamma,\ i = 1,2;\\
&a_1a_2 - \chi_2(g_1)a_2a_1 = \lambda(1 - g_1g_2),\ \text{with } \lambda \text{ either } 0 \text{ or } 1.
\end{aligned}
\tag{3}
$$

If $\lambda = 1$, then $\chi_1\chi_2 = 1$ and $g_1g_2 \neq 1$. The value of λ is invariant under isomorphism. The Hopf algebra structure of A is determined by

$$\Delta(u_i) = u_i \otimes u_i, \qquad \Delta(a_i) = a_i \otimes 1 + g_i \otimes a_i, \quad i = 1,2.$$

There are two subcases depending on whether or not g_1, g_2 are linearly independent over $\mathbb{Z}/(p)$. In the first subcase we take $g_i = u_i$ and $\chi_i(u_i) = \xi_i$, ξ_i primitive p-th roots of unity, $1 \leq i \leq 2$ and $\chi_1(u_2) = \rho$, ρ a p-th root of unity. Note that necessarily $\chi_2(u_1) = \rho^{-1}$. If we interchange ξ_1, ξ_2 and replace ρ by ρ^{-1}, then we get isomorphic Hopf algebras, and this is the only possibility to get an isomorphism.

In the second subcase we take $g_1 = u_1, g_2 = u_1^l$, l an integer not divisible by p, $\chi_1(u_1) = \xi, \chi_2(u_1) = \xi^{-l}, \chi_1(u_2) = 1, \chi_2(u_2) = \xi^{\varepsilon}$, ξ a primitive p-th root of unity and $\varepsilon = 0, 1$. The only way to get isomorphic Hopf algebras is to replace l by l^{-1} mod p and ξ by ξ^{-l^2}.

2.4 Index p^2, R a quantum plane and $\Gamma \simeq \mathbb{Z}/(p^2)$

Then there exist $g_1, g_2 \in \Gamma$ and characters $\chi_1, \chi_2 \in \widehat{\Gamma}$ such that $\chi_1(g_1)$ and $\chi_2(g_2)$ have order p, $\chi_1(g_2)\chi_2(g_1) = 1$. There are further possible subcases according to the orders of g_1, g_2, χ_1, and χ_2. It is easy to see that either $\operatorname{ord}(g_i) = p$ and $\operatorname{ord}(\chi_i) = p^2$, $i = 1,2$ (first subcase) or $\operatorname{ord}(g_i) = p^2$ and $\operatorname{ord}(\chi_i) = p$, $i = 1,2$ (second, third and fourth subcases according to the different choices of μ_1 and μ_2).

First subcase. Let ξ be a primitive p^2-th root of unity and integers b, c_1, c_2 integers prime to p such that $p | (bc_1 + c_2)$. A is generated by h, x_1, x_2 with relations

$$
\begin{aligned}
&h^{p^2} = 1\\
&hx_1 = \xi x_1h,\ hx_2 = \xi^b x_2h;\\
&x_1^p = 0,\ x_2^p = 0;\\
&x_2x_1 - \xi^{pbc_1}x_1x_2 = \lambda\left(1 - h^{p(c_1+c_2)}\right),\ \text{where } \lambda \text{ is } 0 \text{ or } 1\\
&(\text{and it could be } 1 \text{ only if } b \equiv -1 \bmod p^2 \text{ and } c_1 + c_2 \not\equiv 0 \bmod p).
\end{aligned}
\tag{4}
$$

carries a Hopf algebra structure given by

$$\Delta(h) = h \otimes h, \qquad \Delta(x_i) = x_i \otimes 1 + h^{pc_i} \otimes x_i, \quad 1 \leq i \leq 2.$$

The only ways to get isomorphic Hopf algebras (except congruence mod p for c_1, c_2 and mod p^2 for b) are replacing c_1, c_2, b, ξ by sc_1, sc_2, b, η where s is an integer prime to p and $\eta^s = \xi$; or by sc_2, sc_1, d, η where s again is prime to p, d is an integer with $bd \equiv 1 \bmod p^2$ and $\eta^s = \xi^b$.

Second subcase. Let q be a primitive p-th root of unity and a, d_1, d_2 integers prime to p such that $p|(ad_1 + d_2)$. A is generated by h, x_1, x_2 with relations

$$\begin{aligned}
&h^{p^2} = 1;\\
&hx_1 = q^{d_1} x_1 h, \ hx_2 = q^{d_2} x_2 h;\\
(5)\quad &x_1^p = 0, \ x_2^p = 0;\\
&x_2 x_1 - q^{d_2} x_1 x_2 = \lambda \left(1 - h^{1+a}\right), \text{ where } \lambda \text{ is 0 or 1}\\
&(\text{and it could be 1 only if } d_1 + d_2 \equiv 0 \bmod p \text{ and } 1 + a \not\equiv 0 \bmod p).
\end{aligned}$$

carries a Hopf algebra structure given by

$$\Delta(h) = h \otimes h, \quad \Delta(x_1) = x_1 \otimes 1 + h \otimes x_1, \quad \Delta(x_2) = x_2 \otimes 1 + h^a \otimes x_2.$$

The only way to get isomorphic Hopf algebras (except congruence mod p for d_1, d_2 and mod p^2 for a) is to replace d_1, d_2, a by $d_2 a, d_1 a, b$ where $ab \equiv 1 \bmod p^2$.

Third subcase.
Let q be a primitive p-th root of unity and a, d_1, d_2 integers prime to p such that $p|(ad_1 + d_2)$. A is generated by h, x_1, x_2 with relations

$$\begin{aligned}
&h^{p^2} = 1;\\
&hx_1 = q^{d_1} x_1 h, \ hx_2 = q^{d_2} x_2 h;\\
(6)\quad &x_1^p = 1 - h^p, \ x_2^p = 0;\\
&x_2 x_1 - q^{d_2} x_1 x_2 = \lambda \left(1 - h^{1+a}\right), \text{ where } \lambda \text{ is 0 or 1}\\
&(\text{and it could be 1 only if } d_1 + d_2 \equiv 0 \bmod p \text{ and } 1 + a \not\equiv 0 \bmod p).
\end{aligned}$$

carries a Hopf algebra structure given by

$$\Delta(h) = h \otimes h, \quad \Delta(x_1) = x_1 \otimes 1 + h \otimes x_1, \quad \Delta(x_2) = x_2 \otimes 1 + h^a \otimes x_2.$$

Here the only way to get isomorphic Hopf algebras is to replace d_1, d_2 by mod p congruent elements and a by a mod p^2 congruent element.

In the first three subcases, λ is invariant under isomorphism.

Fourth subcase.
It is only here that we get an infinite family of non-isomorphic Hopf algebras. Let q be a primitive p-th root of unity and a, d_1, d_2 integers prime to p such

that $p|(ad_1+d_2)$. A is generated by h, x_1, x_2 with relations

$$
\begin{array}{ll}
 & h^{p^2}=1;\\
 & hx_1=q^{d_1}x_1h,\ \ hx_2=q^{d_2}x_2h;\\
(7) & x_1^p=1-h^p,\ \ x_2^p=1-h^{pa};\\
 & x_2x_1-q^{d_2}x_1x_2=\lambda\,(1-h^{1+a})\,,\text{where } \lambda\in\mathbf{k}\\
 & (\text{and it could be non zero only if } d_1+d_2\equiv 0 \bmod p \text{ and}\\
 & 1+a\not\equiv 0 \bmod p).
\end{array}
$$

carries a Hopf algebra structure given by

$$\Delta(h)=h\otimes h,\quad \Delta(x_1)=x_1\otimes 1+h\otimes x_1,\quad \Delta(x_2)=x_2\otimes 1+h^a\otimes x_2.$$

Hopf algebras defined by d_1, d_2, a, λ and $\widetilde{d}_1, \widetilde{d}_2, \widetilde{a}, \widetilde{\lambda}$ are isomorphic if and only if there exists a scalar $\alpha\in\mathbf{k}$ such that $\alpha^p=1$ and (1) $\lambda=\alpha\widetilde{\lambda}$, $\widetilde{d}_i\equiv d_i \bmod p$, $1\le i\le 2$, $\widetilde{a}\equiv a \bmod p^2$, or (2) $\lambda=-\alpha\widetilde{\lambda}$, $\widetilde{d}_2\equiv d_1a \bmod p$, $\widetilde{d}_1\equiv d_2a \bmod p$ and $a\widetilde{a}\equiv 1 \bmod p^2$.

3. Nichols algebras of type A_2 and their liftings

We fix a finite abelian group Γ, elements $g_1, g_2\in\Gamma$ and characters $\chi_1,\chi_2\in\widehat{\Gamma}$ of Γ, an odd integer $p>1$ (which is not assumed to be prime in this section) and a primitive p-th root of unity q in $\mathbf{k}$ such that

$$q=\chi_1(g_1)=\chi_2(g_2),\quad \chi_1(g_2)\chi_2(g_1)=q^{-1}. \tag{8}$$

Thus the braiding $b_{ij}=\chi_j(g_i), 1\le i,j\le 2$, is of type A_2.
If C is a coalgebra, we denote by $G(C)$ the set of group-like elements of C. If $g,\ h\in G(C)$, then we let $P_{g,h}(C)=\{x\in C\mid \Delta(x)=g\otimes x+x\otimes h\}$ be the space of all (g,h)-primitive elements. If A is a Hopf algebra and $\Gamma=G(A)$ is abelian, then Γ acts on each $P_{g,h}(A)$ by conjugation, and if χ is a character of Γ, then we define $P_{g,h}(A)^{\chi}=\{a\in P_{g,h}(A)\mid uau^{-1}=\chi(u)a$, for all $u\in\Gamma\}$.

Lemma 3.1. *Let A be a Hopf algebra with coradical $\mathbf{k}[\Gamma]$ and $a_1,a_2\in A$ such that*

$$\Delta(a_i)=g_i\otimes a_i+a_i\otimes 1,\quad ha_ih^{-1}=\chi_i(h)a,\quad \text{for all}\quad h\in\Gamma,\quad 1\le i\le 2.$$

Define $c=a_1a_2-\chi_2(g_1)a_2a_1$. Then
 (1) $\Delta(c)=g_1g_2\otimes c+c\otimes 1+(1-q^{-1})a_1g_2\otimes a_2$.
 (2) $ca_1-\chi_1(g_2)a_1c\in P_{g_1^2g_2,1}(A)^{\chi_1^2\chi_2}$.
 (3) $a_2c-\chi_1(g_2)ca_2\in P_{g_1g_2^2,1}(A)^{\chi_1\chi_2^2}$.
 (4) $a_i^p\in P_{g_i^p,1}(A)^{\chi_i^p}, 1\le i\le 2$.
 (5) Assume the Serre relations

$$ca_1-\chi_1(g_2)a_1c=0,\ \text{ and } a_2c-\chi_1(g_2)ca_2=0.$$

Then

$$\Delta(c^p)=g_1^pg_2^p\otimes c^p+c^p\otimes 1+(q-1)^p\chi_1(g_2)^{\frac{p(p-1)}{2}}a_1^pg_2^p\otimes a_2^p.$$

Proof. (1) follows easily from the assumptions. (2) and (3) follow by direct computation using also (8).
(4) follows from the q-binomial formula: let $u_1 = a_i \otimes 1, u_2 = g_i \otimes a_i$. Then

$$u_2 u_1 = g_i a_i \otimes a_i = q a_i g_i \otimes a_i = q u_1 u_2.$$

Hence by the q-binomial formula,

$$\Delta(a_i)^p = (u_1 + u_2)^p = u_1^p + u_2^p = g_i^p \otimes a_i^p + a_i^p \otimes 1,$$

since q is a p-th root of unity.
(5) Let $u_1 = c \otimes 1, u_2 = (1-q^{-1})a_1 g_2 \otimes a_2$ and $u_3 = g_1 g_2 \otimes c$. Then

$$\begin{aligned} u_2 u_1 &= (1-q^{-1})a_1 g_2 c \otimes a_2 \\ &= (1-q^{-1})\chi_1(g_2)\chi_2(g_2) a_1 c g_2 \otimes a_2, \\ &= (1-q^{-1})\chi_1(g_2) q \chi_1(g_2)^{-1} c a_1 g_2 \otimes a_2, \\ &\qquad \text{since } a_1 c = \chi_1(g_2)^{-1} c a_1, \\ &= q u_1 u_2. \end{aligned}$$

In the same way we see $u_3 u_2 = q u_2 u_3$ using the relation $a_2 c = \chi_1(g_2) c a_2$.
Finally

$$u_3 u_1 = g_1 g_2 c \otimes c = (\chi_1\chi_2)(g_1 g_2) c g_1 g_2 \otimes c = q u_1 u_3,$$

since $(\chi_1\chi_2)(g_1g_2) = \chi_1(g_1)\chi_1(g_2)\chi_2(g_1)\chi_2(g_2) = q$ by (8).
Hence by the q-binomial formula,

$$\Delta(c^p) = (u_1 + u_2 + u_3)^p = (u_1 + u_2)^p + u_3^p = u_1^p + u_2^p + u_3^p.$$

This proves (5) since

$$(a_1 g_2)^p = \chi_1(g_2)^{\frac{p(p-1)}{2}} a_1^p g_2^p.$$

□

To compute all liftings of type A_2 we first list the following properties of our elements g_1, g_2, χ_1, χ_2 satisfying (8).

Lemma 3.2. *(1) $g_1^2 g_2 \neq g_i$ and $g_1 g_2^2 \neq g_i$, $1 \leq i \leq 2$.*
(2) If $p > 3$, then $\chi_1^2\chi_2 \neq \varepsilon, \chi_1\chi_2^2 \neq \varepsilon$.
(3) If Γ is cyclic of order 3 and $\chi_1^2\chi_2 = \varepsilon$ or $\chi_1\chi_2^2 = \varepsilon$, then $g_1^2 g_2 = 1 = g_1 g_2^2$.

Proof. (1) Assume $g_1^2 g_2 = g_2$. Then $g_1^2 = 1$, and we get the contradiction $\chi_1(g_1)^2 = 1$. If $g_1^2 g_2 = g_1$, then $g_1 g_2 = 1$, and we see from (8) that $\chi_2(g_1) = 1$, and $\chi_1(g_2) = 1$. By (8) this implies $q = 1$ which is impossible. In the same way we see that $g_1 g_2^2 \neq g_i$.
(2) Assume $\chi_1^2\chi_2 = \varepsilon$. Then $\chi_1\chi_2 = \chi_1^{-1}$, and $\chi_1^{-1}(g_1)\chi_1(g_2) = 1$ and $\chi_1^{-1}(g_2)\chi_2(g_1) = 1$ by (8). Hence $q = \chi_1(g_2) = \chi_2(g_1)$ which implies by (8) the contradiction $q^3 = 1$. In the same way we prove $\chi_1\chi_2^2 \neq \varepsilon$.
(3) We consider the case $\chi_1^2\chi_2 = \varepsilon$. Since $\widehat{\Gamma} \cong \Gamma$ is of order 3, we get $\chi_1 = \chi_2$. Then $\chi_1(g_1^2 g_2) = \chi_1(g_1)\chi_2(g_2)\chi_1(g_2) = 1$ by (8). Therefore $g_1^2 g_2 = 1$. Hence $g_1 = g_2$, and $g_1 g_2^2 = 1$. □

We also need

Lemma 3.3. *Let Γ be any group, $g, h \in \Gamma, \alpha \in \mathbf{k}$ and $x \in \mathbf{k}[\Gamma]$. Then the following are equivalent:*
(1) $\Delta(x) = gh \otimes x + x \otimes 1 + \alpha(1-g)h \otimes (1-h)$.
(2) $x = \alpha(1-h) + \lambda(1-gh)$ for some $\lambda \in \mathbf{k}$.

Proof. Both statements mean that $x - \alpha(1-h)$ is $(gh, 1)$-primitive in the group algebra $\mathbf{k}[\Gamma]$. □

Lemma 3.4. *Let x, y, z be elements in an algebra, α, β scalars in $\mathbf{k}$ and n a natural number. Assume $yx = \alpha xy + z$.*
(1) Asssume $zx = \beta xz$. Then $yx^n = \alpha^n x^n y + (\sum_{i=0}^{n-1} \alpha^i \beta^{n-1-i}) x^{n-1} z$, and if $\alpha \neq \beta$ and $\alpha^n = \beta^n$, then $(\alpha xy + z)x^{n-1} = \alpha^n x^n y$.
(2) Assume $yz = \beta zy$. Then $y^n x = \alpha^n xy^n + (\sum_{i=0}^{n-1} \alpha^i \beta^{n-1-i}) zy^{n-1}$, and if $\alpha \neq \beta$ and $\alpha^n = \beta^n$, then $y^{n-1}(\alpha xy + z) = \alpha^n xy^n$.

Proof. The first formula in (1) is easily proved by induction and the second formula follows from the first since $\sum_{i=0}^{n-1} \alpha^i \beta^{n-1-i} = 0$. (2) is dual to (1). □

Definition 3.5. *Define*

$$\begin{aligned} U^+ := \mathbf{k}\langle x_1, z, x_2 \mid x_2 x_1 &= \chi_2(g_1)^{-1} x_1 x_2 - \chi_2(g_1)^{-1} z, \\ z x_1 &= \chi_1(g_2) x_1 z, \\ x_2 z &= \chi_1(g_2) z x_2 \rangle \end{aligned}$$

as a Hopf algebra in the category of left Yetter-Drinfel'd modules over $\mathbf{k}[\Gamma]$ with primitive elements x_1, x_2 and action and coaction of Γ given by

$$h \cdot x_i = \chi_i(h) x_i, \quad \text{for all} \quad h \in \Gamma, \quad \delta(x_i) = g_i \otimes x_i, \quad 1 \leq i \leq 2.$$

Let $U := U^+ \# \mathbf{k}[\Gamma]$ be the Radford biproduct. Identifying $y \# u = yu$, $y \in U^+, u \in \mathbf{k}[\Gamma]$ we have in U

$$\Delta(x_i) = g_i \otimes x_i + x_i \otimes 1, \quad h x_i h^{-1} = \chi_i(h) x_i, \quad \text{for all} \quad h \in \Gamma, 1 \leq i \leq 2.$$

In quantum group theory our U is usually called $U^{\geq 0}$. Note that U^+ is a well-defined braided Hopf algebra since by Lemma 3.1 the Serre relations are primitive in the free algebra generated by the primitive elements x_1, x_2. It follows easily from the diamond lemma that U^+ has the PBW-basis $x_1^i z^j x_2^l$, $i, j, l \geq 0$.
Then $V := \mathbf{k}x_1 + \mathbf{k}x_2 \subset U^+$ is a Yetter-Drinfel'd module over $\mathbf{k}[\Gamma]$ of type A_2.

Theorem 3.6. *Let $\mu_1, \mu_2 \in \{0, 1\}$ and $\lambda \in \mathbf{k}$ be scalars such that*

$$\mu_i = 0, \quad if \quad g_i^p = 1 \ or \ \chi_i^p \neq \varepsilon, \quad 1 \leq i \leq 2. \tag{9}$$

$$\lambda = 0, \quad if \quad g_1^p g_2^p = 1 \ or \ \chi_1^p \chi_2^p \neq \varepsilon. \tag{10}$$

Define $u_i := \mu_i(1 - g_i^p)$, $1 \leq i \leq 2$, and $u := \alpha(1 - g_2^p) + \lambda(1 - g_1^p g_2^p)$ in $\mathbf{k}[\Gamma]$, where $\alpha := \mu_1 \mu_2 (q-1)^p$. Then

$$A := U/(x_1^p - u_1, z^p - u, x_2^p - u_2)$$

is a quotient Hopf algebra of U *of dimension* $\operatorname{ord}(\Gamma)p^3$, $\mathfrak{B}(V) \cong U^+/(x_1^p, z^p, x_2^p)$, *and*

$$\operatorname{gr}(A) \cong \mathfrak{B}(V)\#\mathbf{k}[\Gamma] \cong U/(x_1^p, z^p, x_2^p).$$

Proof. 1) We first show that A is a quotient Hopf algebra of U. By Lemma 3.1 (4), $x_i^p - u_i$ is $(g_i^p, 1)$-primitive, since u_i is $(g_i^p, 1)$-primitive by construction. Let $\sigma := (q-1)^p \chi_1(g_2)^{\frac{p(p-1)}{2}}$. Note that $\mu_1\mu_2\sigma = \alpha$ since p is odd and $\chi_1^p = \varepsilon$ if $\mu_1 \neq 0$. By Lemma 3.1 (5) and Lemma 3.3 (1),

$$\Delta(z^p) = g_1^p g_2^p \otimes z^p + \sigma x_1^p g_2^p \otimes x_2^p + z^p \otimes 1$$

and

$$\Delta(u) = g_1^p g_2^p \otimes u + \sigma u_1 g_2^p \otimes u_2 + u \otimes 1.$$

Hence

$$\Delta(z^p - u) = g_1^p g_2^p \otimes (z^p - u) + \sigma x_1^p g_2^p \otimes (x_2^p - u_2) + \sigma(x_1^p - u_1) g_2^p \otimes u_2 + (z^p - u) \otimes 1.$$

Thus $(x_1^p - u_1, z^p - u, x_2^p - u_2)$ is a biideal of U, and A is a quotient Hopf algebra of U since U is pointed.

2) Now we show that $\dim A = \operatorname{ord}(\Gamma)p^3$. Let a_1, c, a_2 be the canonical images of x_1, z, x_2 in A. Then the elements $h a_1^i c^j a_2^l$, $h \in \Gamma$, $i, j, l \geq 0$ form a $\mathbf{k}$-basis of A. This can be seen using the diamond lemma. As in the proof of [2, Proposition 5.2] we use the ordering $h_1 < \ldots < h_s < a_1 < c < a_2$ where $h_1, \ldots, h_s$ are generators of the group Γ. If M is a monomial in $h_1, \ldots, a_2$ we let $\Phi(M)$ be its $\{a_1, c, a_2\}$-part. If M, N are monomials we define $M < N$ if $\Phi(M) < \Phi(N)$ in the standard degree-lexicographic order or $\Phi(M) = \Phi(N)$ and M is smaller than N in the standard order. The relations of A are the relations of Γ, the relations of U, namely for all $h \in \Gamma$, $1 \leq i \leq 2$,

$$\begin{aligned}
a_2 a_1 &= \chi_2(g_1)^{-1} a_1 a_2 - \chi_2(g_1)^{-1} c, \\
c a_1 &= \chi_1(g_2) a_1 c, \\
a_2 c &= \chi_1(g_2) c a_2, \\
a_i h &= \chi_i(h)^{-1} h a_i, \\
c h &= (\chi_1\chi_2)(h)^{-1} h c,
\end{aligned}$$

and the additional relations $a_1^p = u_1, \quad c^p = u, \quad a_2^p = u_2$.

We only need to consider overlaps involving the additional relations. In the overlapping pair $(a_2 a_1^p, (a_2 a_1) a_1^{p-1})$ the first element is reduced to $a_2 u_1$. The second element is reduced to $(\chi_2(g_1)^{-1} a_1 a_2 - \chi_2(g_1)^{-1} c) a_1^{p-1}$. By Lemma 3.4 (1) this expression is equal to $\chi_2(g_1)^{-p} a_1^p a_2 = \chi_2(g_1)^{-p} u_1 a_2$. To see that we have reached the same normal form we need $a_2 u_1 = \chi_2(g_1)^{-p} u_1 a_2$. This equality holds in case $u_1 = 0$. If $u_1 \neq 0$, we have $\chi_1^p = \varepsilon$ by (10), hence by (9) also $\chi_2(g_1)^p = 1$, and the desired equality again follows. In the same way we treat $(a_2 a_1^p, a_2^{p-1}(a_2 a_1))$ using Lemma 3.4 (2).

The remaining overlaps

$$(a_i h, a_i^{p-1}(a_i h)), \quad (c^p h, c^{p-1} c h), \quad (c a_1^p, (c a_1) a_1^{p-1}), \quad (c^p a_1, c^{p-1}(c a_1)),$$
$$(a_2^p c, a_2^{p-1}(a_2 c)), \quad (a_2 c^p, (a_2 c) c^{p-1}), \quad (a_i^p a_i^r, a_i^r a_i^p) \quad \text{and} \quad (c^p c^r, c^r c^p)$$

are easily reduced to the same normal form since respectively

$$u_i = \chi_i(h)^{-1}u_i,\ u = (\chi_1\chi_2)(h)^{-p}u,\ cu_1 = \chi_1(g_2)^p u_1 c,\ ua_1 = \chi_1(g_2)^p a_1 u,$$

$$u_2 c = \chi_1(g_2)^p c u_2, \quad a_2 u = \chi_1(g_2) u a_2, \quad u_1 a_i = a_i u_1, \quad \text{and} \quad uc = cu.$$

3) The braided Hopf algebra $\mathfrak{B}(V)$ is a quotient of the free algebra $T(V)$ generated by the primitive elements x_1, x_2 with action and coaction of Γ as in V. Define $z := x_1x_2 - \chi_2(g_1)x_2x_1$ in the free algebra. By Lemma 3.1 applied to the biproduct $T(V)\#\mathbf{k}[\Gamma]$ it follows that the elements $zx_1 - \chi_1(g_2)x_1z$, $x_2z = \chi_1(g_2)zx_2$ and x_1^p, x_2^p are primitive in $T(V)$. Since $P(\mathfrak{B}(V)) = V$ by definition of $\mathfrak{B}(V)$, these elements are mapped onto 0 in $\mathfrak{B}(V)$. Then it follows from Lemma 3.1 (5) that also z^p is primitive in $\mathfrak{B}(V)$ hence 0. Thus $\mathfrak{B}(V)$ is a braided Hopf algebra quotient of $U^+/(x_1^p, z^p, x_2^p)$. We know that $\mathfrak{B}(V)$ has dimension p^3 [3]. Hence $\mathfrak{B}(V) \cong U^+/(x_1^p, z^p, x_2^p)$ since the last algebra also has dimension p^3 for example by the proof in 2). Finally consider the Hopf algebra map $U \to \mathrm{gr}(A)$ which maps x_i onto the residue class $\bar{a}_i$ of a_i in $\mathrm{gr}(A)(1) = A_1/A_0$, $1 \le i \le 2$, z onto $\bar{c}$ in A_2/A_1 and h onto h for the elements in the group Γ. This map is surjective since A is generated by a_1, a_2 and Γ, and x_1^p, x_2^p, z^p are in the kernel since $\bar{a_1}^p, \bar{a_2}^p, \bar{c}^p$ are all 0 since by definition a_1^p, a_2^p, c^p are in the group algebra $A_0 = \mathbf{k}[\Gamma]$. Hence

$$U/(x_1^p, x_2^p, z^p) \cong \mathfrak{B}(V)\#\mathbf{k}[\Gamma] \cong \mathrm{gr}(A).$$

□

As an example of dimension p^5 consider $\Gamma = \langle g \rangle$ cyclic of order p^2. Let q be a primitive p-th root of unity and b, d integers such that $bd \equiv 1, b + d \equiv -1 \mod p$. Such integers exist if $p = 3$ or $p \equiv 1 \mod 3$ and -3 is a square mod p by the quadratic reciprocity law. Define $g_1 := g, g_2 := g^b$, and $\chi_1(g) := q, \chi_2(g) := q^d$. Then we can choose in Theorem 3.6 $\mu_1 = 1, \mu_2 = 1$ and λ arbitrary in $\mathbf{k}$.

Theorem 3.7. *Let A be a pointed Hopf algebra with*

$$\mathrm{gr}(A) \cong \mathfrak{B}(V)\#\mathbf{k}[\Gamma]$$

where V is of type A_2 as above.
(1) If $p > 3$ or if Γ is cyclic of order 3, then A is one of the Hopf algebras described in Theorem 3.6.
(2) If Γ has exponent p where $p > 3$ is a prime, or Γ is cyclic of order 3, then $A \cong \mathfrak{B}(V)\#\mathbf{k}[\Gamma]$.

Proof. We show (1); (2) follows immediately.
Let $A_0 \subset A_1 \subset \ldots$ be the coradical filtration of A. Then we know from [2, Lemma 5.4] that

$$A_1 = \mathbf{k}[\Gamma] \oplus \bigoplus_{g,h,\chi} P_{g,h}(A)^\chi \tag{11}$$

where g, h run through all elements of Γ and χ runs through all non-trivial characters of Γ. Since $\mathrm{gr}(A) \cong \mathfrak{B}(V)\#\mathbf{k}[\Gamma]$ we conclude for all elements $g \in \Gamma$ and non-trivial characters χ of Γ

$$P_{g,1}(A)^{\chi} \neq 0 \quad \text{if and only if} \quad g = g_i, \chi = \chi_i, \quad i \in \{1,2\}. \tag{12}$$

Choose $a_i \in P_{g_i,1}(A)^{\chi_i}$ with residue class x_i in A_1/A_0, $1 \leq i \leq 2$, and define $c := a_1a_2 - \chi_2(g_1)a_2a_1$. By Lemma 3.1, $ca_1 - \chi_1(g_2)a_1c \in P_{g_1^2g_2,1}(A)^{\chi_1^2\chi_2}$ and $a_2c - \chi_1(g_2)ca_2 \in P_{g_1g_2^2,1}(A)^{\chi_1\chi_2^2}$.

If $\chi_1^2\chi_2$ is non-trivial, then $P_{g_1^2g_2,1}(A)^{\chi_1^2\chi_2} = 0$ by (12) since $g_1^2g_2 \neq g_i$ by Lemma 3.2 (1). In particular, $P_{g_1^2g_2,1}(A)^{\chi_1^2\chi_2} = 0$ if $p > 3$ by Lemma 3.2 (2). If Γ is cyclic of order 3, and $\chi_1^2\chi_2 = \varepsilon$, then $g_1^2g_2 = 1$ by Lemma 3.2 (3). Hence $P_{g_1^2g_2,1}(A) = P_{1,1}(A) = P(A) = 0$ since A is finite-dimensional and the characteristic of $\mathbf{k}$ is 0. Similarly, $P_{g_1g_2^2,1}(A)^{\chi_1\chi_2^2} = 0$. Thus we have shown the Serre relations

$$ca_1 = \chi_1(g_2)a_1c, \quad a_2c = \chi_1(g_2)ca_2.$$

By Lemma 3.1 (4), $a_i^p \in P_{g_i^p,1}(A)^{\chi_i^p}, 1 \leq i \leq 2$. Since $\chi_i^p(g_i^p) = 1$, $1 \leq i \leq 2$, g_i^p and a_i^p commute. Thus by Lemma 3.1 in [2],

$$a_i^p = \mu_i(1 - g_i^p) \quad \text{for some} \quad \mu_i \in \mathbf{k}, 1 \leq i \leq 2.$$

If $\mu_i \neq 0$, $i \in \{1,2\}$, then we can choose $\mu_i = 1$ by replacing a_i by a scalar multiple of a_i. If $\chi_i^p \neq \varepsilon$, and $g_i^p \neq 1$, then μ_i must be zero. From Lemma 3.1 (5) we then obtain

$$\Delta(c^p) = g_1^pg_2^p \otimes c^p + c^p \otimes 1 + \alpha(1 - g_1^p)g_2^p \otimes (1 - g_2^p),$$

where $\alpha := \mu_1\mu_2(q-1)^p\chi_1(g_2)^{\frac{p(p-1)}{2}}$. Then $\alpha = \mu_1\mu_2(q-1)^p$ since p is odd and $\chi_1^p = \varepsilon$ if $\mu_1 \neq 0$. Hence $\Delta(c^p) \in A_0 \otimes A + A \otimes A_0$, and $c^p \in A_1$. Since Γ acts on c^p via the character $\chi_1^p\chi_2^p$, we see from (11) that $c^p \in \mathbf{k}[\Gamma]$ if $\chi_1^p\chi_2^p = \varepsilon$. Hence if $\chi_1^p\chi_2^p = \varepsilon$, then by Lemma 3.3

$$c^p = \alpha(1 - g_2^p) + \lambda(1 - g_1^pg_2^p)$$

for some scalar λ. Assume that $\chi_1^p\chi_2^p$ is non-trivial. Then χ_1^p or χ_2^p is non-trivial, and $a_1^p = 0$ or $a_2^p = 0$ by the above argument. Hence $\mu_1\mu_2 = 0$. Therefore $\alpha = 0$, and $c^p \in P_{g_1^pg_2^p,1}(A)^{\chi_1^p\chi_2^p}$. Thus again by Lemma 3.1 in [2], c^p is a scalar multiple of $1 - g_1^pg_2^p$. □

4. Pointed Hopf algebras of dimension p^4

Let p be an odd prime and $\Gamma = \langle g \rangle$ the cyclic group of order p. We describe Nichols algebras of type A_2 over Γ. Let q be a primitive p-th root of unity in $\mathbf{k}$, b, d integers and

$$g_1 = g, \quad g_2 = g^b, \quad \chi_1(g) = q, \quad \chi_2(g) = q^d.$$

Type A_2 means $b + d \equiv -1 \equiv -bd \bmod p$. Therefore $b^2 + b + 1 \equiv 0 \bmod p$. Hence $p = 3$ and $b \equiv 1 \bmod p$, or $b \not\equiv 1 \bmod p$ is a cubic root of 1. In the last case, b exists if and only if $p \equiv 1 \mod 3$. The corresponding Radford biproduct $U^+\#\mathbf{k}[\Gamma]$ is the Hopf algebra

$$\begin{aligned} U(q,b) := \mathbf{k}\langle g, x_1, z, x_2 \quad | \quad & gx_1g^{-1} = qx_1,\ gx_2g^{-1} = q^{-1-b}x_2, \\ & x_1x_2 - q^{-1-b}x_2x_1 = z,\ zx_1 = q^b x_1 z,\ x_2 z = q^b z x_2, \\ & g^p = 1,\ x_1^p = 0,\ z^p = 0,\ x_2^p = 0\rangle \end{aligned}$$

with g group-like, x_1 $(g,1)$-primitive and x_2 $(g^b,1)$-primitive.

Theorem 4.1. *Let p be an odd prime. Let A be a pointed non-semisimple Hopf algebra of dimension p^4 with coradical $\mathbf{k}[\Gamma]$. Then A is isomorphic to one of the Hopf algebras in the list of Section 3 if Γ has order p^3 or p^2, or to one of the Hopf algebras $U(q,b)$ if Γ has order p, where $p = 3$ or $p \equiv 1 \mod 3$.*

Proof. This follows from Theorem 1.1, Section 2 and Theorem 3.7. □

In Section 2, the Hopf algebras are classified up to isomorphism. We know from [3, Lemma 6.4] that two Hopf algebras $U(q,b)$ and $U(\widetilde{q},\widetilde{b})$ are isomorphic if and only if $\widetilde{q} = q$, and $\widetilde{b} \equiv b \mod p$ or $\widetilde{b} \equiv -1-b \mod p$. Hence there are exactly $2(p-1)$ non-isomorphic Hopf algebras of the form $U(q,b)$ if $p \equiv 1 \mod 3$, and 2 of dimension 81 if $p = 3$.

5. Appendix. Quasi-isomorphism types.

We now prove that there are only a finite number of *quasi-isomorphism types* of pointed Hopf algebras of order p^4.

Twisting the multiplication by a cocycle. Let A be a Hopf algebra and let $\sigma : A \otimes A \to k$ be a normalized cocycle; that is, σ is a convolution invertible linear map such that $\sigma(x,1) = \varepsilon(x) = \sigma(1,x)$ and

$$\sigma(x_{(1)}, y_{(1)})\sigma(x_{(2)}y_{(2)}, z) = \sigma(y_{(1)}, z_{(1)})\sigma(x, y_{(2)}z_{(2)})$$

for all $x, y, z \in A$. Let $._\sigma : A \otimes A \to A$ be the linear map given by

$$x._\sigma y := \sigma(x_{(1)}, y_{(1)})x_{(2)}y_{(2)}\sigma^{-1}(x_{(3)}, y_{(3)}) \tag{13}$$

Let A_σ be the vector space underlying A with the comultiplication of A and the multiplication $._\sigma$; it is again a Hopf algebra.

Definition 5.1. *We shall say that two Hopf algebras A and B are quasi-isomorphic if there exists an isomorphism $\phi :\ A_\sigma \to B$ for some normalized cocycle σ.*

Remarks 5.2. (i). Dually, it is possible to twist the comultiplication via $F \in A \otimes A$; we obtain a new Hopf algebra A^F and a corresponding notion of *co-quasi-isomorphic.*
(ii). If C is a coalgebra, $\widetilde{\sigma} : C \otimes C \to k$ is a normalized cocycle and $\psi : A \to C$ is a coalgebra map, then evidently $\sigma : A \otimes A \to k$, $\sigma = \widetilde{\sigma}(\psi \otimes \psi)$ is a normalized cocycle. If A is pointed with coradical $\mathbf{k}[\Gamma]$ then there exists always coalgebra maps $A \to \mathbf{k}[\Gamma]$; hence, we get normalized cocycles on A from normalized group cocycles on Γ.

(iii). It is known that two finite dimensional Hopf algebras are quasi-isomorphic if and only if their categories of comodules are equivalent as braided categories.

Lifting of quantum linear spaces.
Let Γ be a finite abelian group. We fix a decomposition $\Gamma = \langle y_1\rangle \oplus \dots \oplus \langle y_\nu\rangle$ and we denote by M_ℓ the order of y_ℓ, $1 \le \ell \le \nu$.

Let $g_1, \dots, g_\theta \in \Gamma$, $\chi_1, \dots, \chi_\theta \in \widehat{\Gamma}$ be a datum for quantum linear space; *i. e.*, the following conditions hold

$$\begin{aligned} &q_i := \chi_i(g_i) \neq 1, \text{for all } i \text{ .We denote by } N_i \text{ the order of } q_i. \\ &\chi_j(g_i)\chi_i(g_j) = 1, \quad \text{for all } i \neq j \end{aligned} \tag{14}$$

We abbreviate $\mathcal{R} := \mathcal{R}(g_1, \dots, g_\theta; \chi_1, \dots, \chi_\theta)$ for the quantum linear space as in [2, Section 3]. The integer θ is called the *rank* of $\mathcal{R}$.

A compatible datum $\mathcal{D}$ for Γ and $\mathcal{R}$ consists of

1) a scalar $\mu_i \in \{0, 1\}$ for each i, $1 \le i \le \theta$; it is arbitrary if $g_i^{N_i} \neq 1$ and $\chi_i^{N_i} = 1$, but 0 otherwise;
2) a scalar $\lambda_{ij} \in k$ for each i, j, $1 \le i < j \le \theta$; it is is arbitrary if $g_ig_j \neq 1$ and $\chi_i\chi_j = 1$, but 0 otherwise.

Remark 5.3. If $\lambda_{ij} \neq 0$ and $\lambda_{ih} \neq 0$, then $\chi_i\chi_j = 1$ and $\chi_i\chi_h = 1$; hence $\chi_j = \chi_h$. If in addition the order N_i of $q_i := \chi_i(g_i)$ is odd, then $j = h$ [2].

Definition 5.4. *Let Γ be a finite abelian group, $\mathcal{R}$ a quantum linear space and $\mathcal{D}$ a compatible datum. Keep the notation above. Let $\mathcal{A}(\Gamma, \mathcal{R}, \mathcal{D})$ be the algebra presented by generators h_ℓ, $1 \le \ell \le \nu$, and a_i, $1 \le i \le \theta$ with defining relations*

$$h_\ell^{M_\ell} = 1, \ 1 \le \ell \le \nu; \tag{15}$$
$$h_\ell h_t = h_t h_\ell, \ 1 \le t < \ell \le \nu; \tag{16}$$
$$a_i h_\ell = \chi_i^{-1}(y_\ell) h_\ell a_i, \ 1 \le \ell \le \nu, \ 1 \le i \le \theta; \tag{17}$$
$$a_i^{N_i} = \mu_i \left(1 - g_i^{N_i}\right), \ 1 \le i \le \theta; \tag{18}$$
$$a_j a_i = \chi_i(g_j) a_i a_j + \lambda_{ij} \left(1 - g_i g_j\right), \ 1 \le i < j \le \theta. \tag{19}$$

There exists a unique Hopf algebra structure on $\mathcal{A}(\Gamma, \mathcal{R}, \mathcal{D})$ such that

$$\Delta(h_\ell) = h_\ell \otimes h_\ell, \qquad \Delta(a_i) = a_i \otimes 1 + g_i \otimes a_i, \quad 1 \le \ell \le \nu, \quad 1 \le i \le \theta. \tag{20}$$

The set of monomials $h_1^{r_1} \dots h_\nu^{r_\nu} a_1^{s_1} \dots a_\theta^{r_\theta}$, $0 \le r_\ell < M_\ell$, $0 \le s_i < N_i$, $1 \le \ell \le \nu$, $1 \le i \le \theta$, is a basis of $\mathcal{A}(\Gamma, \mathcal{R}, \mathcal{D})$. See [2].

Let $\widetilde{\sigma} : \Gamma \times \Gamma \to \mathbf{k}^\times$ be a normalized group cocycle and let $\psi : \mathcal{A}(\Gamma, \mathcal{R}, \mathcal{D}) \to \mathbf{k}[\Gamma]$ be the coalgebra map given by

$$\psi(h_1^{r_1} \dots h_\nu^{r_\nu} a_1^{s_1} \dots a_\theta^{r_\theta}) = h_1^{r_1} \dots h_\nu^{r_\nu} \delta_{s_1,0} \dots \delta_{r_\theta,0};$$

then $\sigma : \mathcal{A}(\Gamma,\mathcal{R},\mathcal{D})\otimes\mathcal{A}(\Gamma,\mathcal{R},\mathcal{D}) \to k$, $\sigma = \widetilde{\sigma}(\psi\otimes\psi)$ is a normalized cocycle. Here we extended linearly $\widetilde{\sigma}$ to $\mathbf{k}[\Gamma]\times\mathbf{k}[\Gamma]\to\mathbf{k}$. It is harmless to write σ instead of $\widetilde{\sigma}$ in what follows.

Let $\Psi_{i,n-1} = \prod_{1\le h\le n}\sigma(g_i, g_i^h)$. Let ω_i^{-1} be a fixed N_i-th root of ψ_{i,N_i}.

Lemma 5.5. *If σ is symmetric, then $\mathcal{A}(\Gamma,\mathcal{R},\mathcal{D})_\sigma$ is isomorphic to $\mathcal{A}(\Gamma,\mathcal{R},\mathcal{D}_\sigma)$, where $\mathcal{D}_\sigma = (\mu_i, \omega_i\omega_j\sigma(g_i,g_j)\lambda_{ij})$.*

Proof. Let $\widehat{h_\ell}$, $\widehat{a_i}$ be the elements h_ℓ, a_i when regarded in $\mathcal{A}(\Gamma,\mathcal{R},\mathcal{D})_\sigma$. We claim that

$$\widehat{h_\ell}\cdot_\sigma\widehat{h_t} = h_\ell h_t; \tag{21}$$

$$\widehat{a_i}\cdot_\sigma\widehat{h_t} = \sigma(g_i,h_t)a_ih_t, \quad \widehat{h_t}\cdot_\sigma\widehat{a_i} = \sigma(h_t,g_i)a_ih_t; \tag{22}$$

$$\widehat{a_i}\cdot_\sigma\widehat{a_j} = \sigma(g_i,g_j)a_ia_j; \tag{23}$$

$$\widehat{a_i}^n = \Psi_{i,n}a_i^n. \tag{24}$$

Indeed, (21), (22) and (23) follow easily from the definition (13) of the twisted multiplication using that $\Delta^2(h) = h\otimes h\otimes h$, $\Delta^2(a_i) = a_i\otimes 1\otimes 1 + g_i\otimes a_i\otimes 1 + g_i\otimes g_i\otimes a_i$. In turn, (24) follows from (13) by induction; the first step is (23) and the inductive step uses the expression

$$\Delta^2(a_i^n) = \sum_{0\le h\le n}\sum_{0\le s\le h}\binom{n}{h}_{q_i}\binom{h}{s}_{q_i} a_i^s g_i^{n-s}\otimes a_i^{h-s}g_i^{n-h}\otimes a_i^{n-h}g_i^{n-h},$$

which follows applying twice the quantum binomial formula.
Let now $\widetilde{h_t}$, $\widetilde{a_i}$ be the generators of $\mathcal{A}(\Gamma,\mathcal{R},\mathcal{D})s$ satisfying the relations $\widetilde{(15)}$, $\cdots$, $\widetilde{(19)}$ analogous to (15), ..., (19) but with respect to the data $\mathcal{D}_\sigma$. We claim that there exists a Hopf algebra isomorphism $\mathcal{F} : \mathcal{A}(\Gamma,\mathcal{R},\mathcal{D})s \to \mathcal{A}(\Gamma,\mathcal{R},\mathcal{D})_\sigma$ such that

$$\mathcal{F}(\widetilde{h_t}) = \widehat{h_t}, \quad \mathcal{F}(\widetilde{a_i}) = \omega_i\widehat{a_i}, \quad 1\le t\le\nu, \quad 1\le i\le\theta.$$

We need only to verify that relations $\widetilde{(15)}$, $\cdots$, $\widetilde{(19)}$ hold. Relations $\widetilde{(15)}$, $\widetilde{(16)}$, follow immediately from (21); $\widetilde{(17)}$ from (22) and $\widetilde{(19)}$ from (23) using the symmetry of σ; $\widetilde{(18)}$ from (24) by the choice of ω_i. $\square$

Lemma 5.6. *Let $\mathcal{D} = (\mu_i, \lambda_{ij})$ be a compatible datum of rank $\theta = 2$. Let $\mathcal{E}$ be the compatible datum (μ_i, η_{ij}) where $\eta_{ij} = 1$ if $\lambda_{ij}\ne 0$ and 0 otherwise. Then $\mathcal{A}(\Gamma,\mathcal{R},\mathcal{D})$ is quasi-isomorphic to $\mathcal{A}(\Gamma,\mathcal{R},\mathcal{E})$.*

Proof. By Lemma 5.5, it is enough to find a symmetric cocycle $\sigma : \Gamma\times\Gamma\to\mathbf{k}^\times$ such that $\omega_1\omega_2\sigma(g_1,g_2)\lambda_{12} = 1$ whenever $\lambda_{12}\ne 0$. Assume that σ is the coboundary of a map $f : \Gamma\to\mathbf{k}^\times$ satisfying $f(1) = 1$. Then $\sigma(g_1,g_2) = f(g_1)f(g_2)f(g_1g_2)^{-1}$ and

$$\begin{aligned}\Psi_{1,n} &= \sigma(g_1,g_1)\sigma(g_1,g_1^2)\dots\sigma(g_1,g_1^{n-1})\\ &= f(g_1)f(g_1)f(g_1^2)^{-1}f(g_1)f(g_1^2)f(g_1^3)^{-1}\dots f(g_1)f(g_1^{n-1})f(g_1^n)^{-1}\\ &= f(g_1)^nf(g_1^n)^{-1}.\end{aligned}$$

We can choose $\omega_1 = f(g_1)^{-1}\sqrt[N_1]{f(g_1^{N_1})}$ for some N_1-th root of $f(g_1^{N_1})$ and similarly for ω_2. So that we are reduced to find a function f such that $f(1) = 1$ and

$$\lambda_{12}^{-1} = \sqrt[N_1]{f(g_1^{N_1})}\sqrt[N_2]{f(g_2^{N_2})}f(g_1g_2)^{-1}.$$

Since $\lambda_{12} \neq 0$, $g_1g_2 \neq 1$ and such f exists. □

Proposition 5.7. *There are only a finite number of quasi-isomorphism types of pointed Hopf algebras of order p^4.*

Proof. This follows at once from Lemma 5.6 and the classification results in the previous Sections of the paper. □

References

[1] N. Andruskiewitsch and H.-J. Schneider, Hopf Algebras of order p^2 and braided Hopf algebras of order p, *J. Algebra* **199** (1998), 430-454.

[2] N. Andruskiewitsch and H.-J. Schneider, Lifting of Quantum Linear Spaces and Pointed Hopf Algebras of order p^3, *J. Algebra* **209** (1998), 659-691.

[3] N. Andruskiewitsch and H.-J. Schneider, Finite Quantum Groups and Cartan Matrices, *Adv. Math.*, to appear.

[4] M. Beattie, S. Dăscălescu, and L. Grünenfelder, On the number of types of finite-dimensional Hopf algebras, *Inventiones Math.* **136** (1999), 1-7.

[5] S. Caenepeel and S. Dăscălescu, Pointed Hopf algebras of dimension p^3, *J. Algebra* **209** (1998), 622-634.

[6] S. Caenepeel, S. Dăscălescu, and Ş. Raianu, Classifying Pointed Hopf algebras of dimension 16, *Comm. Algebra*, to appear.

[7] S. Gelaki, On pointed Hopf algebras and Kaplansky's tenth conjecture, *J. Algebra* **209** (1998), 635-657.

[8] M. Graña, Pointed Hopf algebras of dimension 32, preprint.

[9] A. Masuoka, Self dual Hopf algebras of dimension p^3 obtained by extension *J. Algebra* **178** (1995), 791-806.

[10] S. Montgomery, Hopf algebras and their actions on rings, American Mathematical Society, Providence, 1993.

[11] W.D. Nichols, Bialgebras of type one, *Comm. Algebra* **6** (1978), 1521-1552.

[12] D. Ştefan and F. Van Oystaeyen, Hochschild cohomology and coradical filtration of pointed coalgebras. Applications, *J. Algebra* **210** (1998), 535-556.

[13] M. E. Sweedler, Hopf algebras, Benjamin, New York, 1969.

[14] R. Williams, Finite dimensional Hopf algebras, Ph. D. Thesis, Florida State Univ., 1988.

FAMAF. UNC. (5000), C. Universitaria, Córdoba, Argentina,
e-mail andrus@mate.uncor.edu

Mathematisches Seminar der Universität München, Theresienstrasse 39, D-8033 München, Germany,
e-mail hansschrz.mathematik.uni-muenchen.de