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PLANCHEREL IDENTITY FOR SEMISIMPLE HOPF ALGEBRAS

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§0. Introduction.

Let k be an algebraically closed field and A a finite Hopf algebra over k. We consider A^* as a left A-module by transposing the left regular action of A and composing with the antipode. Given a nonzero right integral in A^*, one can define a *Fourier transform* on A, which results, thanks to a basic Theorem of Larson and Sweedler [LS], an isomorphism of left A-modules from A onto A^*.

In case A is semisimple, cosemisimple and involutory, we obtain, by means of the calculation of the integral in two different ways and a convenient interpretation of the Plancherel identity for the Fourier transform of A, two formulas relating the algebra and coalgebra structures in A. These formulas, involving the matrix coefficients of A^*, generalize well known identities for finite groups [Se, 6.2, Prop. 11, and Ex. 1]. Because of the analogy with classical harmonic analysis, we call these formulas "inversion formula" and "Plancherel identity", see respectively (2.15) and (2.18).

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In 1975, Kaplansky formulated a series of conjectures on Hopf algebras. One of them states that the dimensions of the irreducible modules over a semisimple Hopf algebra A divide the dimension of A, which would generalize Frobenius Theorem for finite groups. We derive from the inversion formula an expression for the quotient of the dimension of A over the dimension of an irreducible comodule in terms of the matrix coefficients of A and A^*; see (3.3). As an application, when k is a field of characteristic zero, we use this formula to recover a result of Larson [L] that states Frobenius Theorem for semisimple Hopf algebras with an order. We then deduce that a Hopf algebra of order pq which has an order is a group algebra. Here $p < q$ are prime numbers, $q \not\equiv 1 \mod p$.

Fourier transforms for Hopf algebras were first considered by Kac and Paljutkin [KP, §5] in 1966, in the context of finite dimensional Hopf C^*-algebras. For general finite dimensional Hopf algebras, they are implicit in the above mentioned work of Larson and Sweedler. The natural context for them is in fact that of Frobenius algebras. We remark, as well, that Fourier transforms for Hopf algebras have been treated recently in several papers (e.g. [LM], [LM2], [KM], [Kp], [PW]).

Conventions. We shall work over an algebraically closed field k. The notation for Hopf algebras is standard: Δ, $\mathcal{S}$, ϵ, denote respectively the comultiplication, the antipode, the counit; we use Sweedler notation but dropping the summation symbol. Given a Hopf algebra A, we shall regard A as an A^*-bimodule via the transposes of right and left multiplication, that is

$$x \rightharpoonup h = h_{(1)}\langle x, h_{(2)}\rangle, \quad h \leftharpoonup x = \langle x, h_{(1)}\rangle h_{(2)}; \qquad h \in A, x \in A^*.$$

We shall also consider the actions of A^* on A given by

$$h \leftharpoonup x = \mathcal{S}x \rightharpoonup h, \qquad x \rightharpoonup h = h \leftharpoonup \mathcal{S}x.$$

Assume that A is finite dimensional. By abuse of notation we shall denote by the same symbols the analogous actions of A on A^*. Unless explicitly stated A is considered as (left or right) A-module by means of the (left or right) multiplication.

Our references for the theory of Hopf algebras are [Sw], [Mo], [Sch].

§1. Fourier transform on finite Hopf algebras.

Let A be a finite Hopf algebra and let us fix a non-zero right integral $\int \in A^*$; that is $\int x = \langle x, 1\rangle \int$, $\forall x \in A^*$.

Consider the structure of left A-Hopf module in A^* provided by the action $\rightharpoonup$ and the coaction $\rho : A^* \to A \otimes A^*$, defined as follows:

$$(1.1) \qquad \rho(\alpha) = \alpha_{(-1)} \otimes \alpha_{(0)} \quad \text{iff} \quad \langle \alpha, a_{(1)}\rangle\, a_{(2)} = \alpha_{(-1)}\, \langle \alpha_{(0)}, a\rangle.$$

$\forall a \in A$. See [Mo, Lemma 2.1.4].

Definition (1.2). The *Fourier Transform* of A (associated to $\int$) is the map $\mathbb{F} : A \to A^*$, defined by

$$\mathbb{F}(a) = a \rightharpoonup \textstyle\int, \quad a \in A.$$

Thus if we consider A as a left A-Hopf module in the natural way, $\mathbb{F}$ is an isomorphism of left A-Hopf modules from A onto A^* [LS].

Its inverse $\mathbb{G} : A^* \to A$, is given by

$$(1.3) \qquad \mathbb{G}(\alpha) = \Lambda \leftharpoonup \alpha,$$

where $0 \neq \Lambda \in A$ is a left integral such that $\langle \int, \Lambda\rangle = 1$, or equivalently $\Lambda \rightharpoonup \int = \epsilon$. Indeed, $\int$ is a Frobenius homomorphism with dual bases $(\Lambda_{(1)}, \mathcal{S}(\Lambda_{(2)}))$ (see [Sch, Thm. 3.6]); that is

i) $\Lambda_{(1)}\ \langle \int, \mathcal{S}(\Lambda_{(2)})a\rangle = a$, $\forall a \in A$,

ii) $\langle \int, a\Lambda_{(1)}\rangle\ \mathcal{S}(\Lambda_{(2)}) = a$, $\forall a \in A$.

Now, condition i) is equivalent to $\mathbb{F} \circ \mathbb{G} = \mathrm{id}_{A^*}$, and condition ii) is equivalent to $\mathbb{G} \circ \mathbb{F} = \mathrm{id}_A$.

Remark. If we thought of the Fourier Transform only as an isomorphism between the left regular action of A and the transpose of the right regular action on A^*, then we would just get the concept of a Frobenius algebra. The feature of the Hopf algebra case is that the inverse map of a Fourier Transform is also a kind of Fourier Transform of the dual Hopf algebra.

We have nondegenerate bilinear forms on A and A^*, denoted respectively $[\ ,\ ]$ and $[\ ,\ ]_d$ defined as follows:

$$(1.4) \qquad [a, b] = \langle \mathbb{F}(a), b\rangle, \quad \text{and} \quad [\alpha, \beta]_d = \langle \alpha, \mathbb{G}(\beta)\rangle,$$

for all $a, b \in A$, $\alpha, \beta \in A^*$. The following Theorem is immediate from the definitions.

Theorem (1.5) (Plancherel Identity). *Let $a, b \in A$, then $[a, b] = [\mathbb{F}(a), \mathbb{F}(b)]_d$.*

§2. Inversion formula and Plancherel identity for semisimple Hopf algebras.

In this section we interpret the Plancherel identity (1.5) in the case of a semisimple (hence finite, see [Sw., Ex. 1-4, pp. 107]) Hopf algebra A under the aditional hypotheses that A is also cosemisimple and involutory. Along the way we obtain an "inversion formula" relating the algebra and coalgebra structures in A. The keys are two different expressions for the integral on

A, one involving its coalgebra structure and the other its algebra structure. Namely, on one side it can be computed as the projection on the isotypical component of trivial type in the decomposition of A as left A^*-module (see Proposition (2.9) below), and on the other it is, up to a scalar, the character of the left regular representation of A, which in particular implies that the bilinear forms defined in (1.4) are also symmetric.

We will assume that A is a Hopf algebra that satisfies the following three conditions.

i) A is semisimple.

ii) A is cosemisimple.

iii) $\mathcal{S}^2 = \mathrm{id}$.

For instance, if the characteristic of the base field k is zero, then the three conditions are equivalent (see [LR] and [LR2]). Conditions i) and ii) imply that the characteristic of k does not divide the dimension of A [LR2, Th. 2]; also, if we suppose that $\dim A \neq 0$, then condition iii) implies i) and ii).

Denote by $B := A^*$ the dual Hopf algebra of A and let $\hat{A}$ and $\hat{B}$ be, respectively, the sets of isomorphy classes of irreducible representations of A and B. Conditions i) and ii) give, respectively, by the Wedderburn Theorem, an isomorphism of algebras

$$A \simeq \prod_{\rho \in \hat{A}} \mathrm{End}\, V_\rho, \tag{2.1}$$

and an isomorphism of coalgebras

$$A \simeq \bigoplus_{\mu \in \hat{B}} (\mathrm{End}\, W_\mu)^*, \tag{2.2}$$

where V_ρ is the irreducible A-module affording the representation ρ and similarly W_μ. The isomorphism in (2.1) is given by $a \mapsto (\rho(a))_\rho$. On the other hand, if $\mu \in \hat{B}$, $\mu : B \to \mathrm{End}\, W_\mu$ is a surjective algebra map, hence its transpose, ${}^t\mu : (\mathrm{End}\, W_\mu)^* \to B^* = A$, is an injective coalgebra map. The isomorphism in (2.2) is obtained by identifying $\sum_{\mu \in \hat{B}} T_\mu \in \oplus_{\mu \in \hat{B}} (\mathrm{End}\, W_\mu)^*$ with $\sum_{\mu \in \hat{B}} {}^t\mu(T_\mu) \in A$.

From now on, we will assume these identifications.

The isomorphisms (2.1) and (2.2) give two expresions for an element $a \in A$:

$$a = \sum_{\rho \in \hat{A}} a_\rho, \quad a_\rho \in \mathrm{End}\, V_\rho, \tag{2.3}$$

which corresponds to the decomposition of A into simple algebras, and

$$a = \sum_{\mu \in \hat{B}} a_\mu, \quad a_\mu \in (\mathrm{End}\, W_\mu)^*, \tag{2.4}$$

which corresponds to the decomposition of A into simple coalgebras. The a_ρ's may be obtained from the a_μ's by $a_\rho = \sum_{\mu \in \hat{B}} \rho(a_\mu)$. The inversion formula expresses, conversely, the a_μ's in terms of the a_ρ's, relating the algebra and coalgebra structures of A.

As A and B are unimodular, we need not distinguish between right and left integrals. In fact, if we denote by $\mathcal{X}_A \in B$ the character of the left regular representation of A, and pick the integral $\int \in B$ such that $\langle \int, 1 \rangle = 1$ (such integral exists thanks to the semisimplicity of B), then $\mathcal{X}_A = (\dim A) \int$, see [LR2]. That is, as $\dim A \neq 0$,

$$\int = \frac{1}{\dim A} \sum_{\rho \in \hat{A}} m_\rho \mathcal{X}_\rho, \tag{2.5}$$

where $\mathcal{X}_\rho$ denotes the character of the irreducible representation $\rho \in \hat{A}$ and m_ρ is the dimension of V_ρ.

By the same reason, any integral Λ in A is a multiple of the character $\mathcal{X}_B$. The condition $\langle \int, \Lambda \rangle = 1$ implies $\langle \varepsilon, \Lambda \rangle = \dim A$, thus

$$\Lambda = \sum_{\mu \in \hat{B}} n_\mu \mathcal{X}_\mu, \tag{2.6}$$

where $n_\mu = \dim W_\mu$, and $\mathcal{X}_\mu$ is the character of $\mu \in \hat{B}$.

Given $\mu \in \hat{B}$, we shall identify $W_\mu \otimes W_\mu^*$ with $\mathrm{End}\, W_\mu \subseteq B$ in the usual way, and $W_\mu^* \otimes W_\mu$ with $(\mathrm{End}\, W_\mu)^* \subseteq A$ via the map ϕ^μ in the top of the following commutative diagram:

$$\begin{array}{ccc} W_\mu^* \otimes W_\mu & \xrightarrow{\phi^\mu} & (\mathrm{End}\, W_\mu)^* \subseteq A \\ \tau \big\downarrow & & \big\uparrow \mathrm{Tr} \\ W_\mu \otimes W_\mu^* & \xrightarrow[\text{canonical}]{} & \mathrm{End}\, W_\mu \subseteq B. \end{array} \tag{2.7}$$

Here τ is the usual twist and Tr is the canonical isomorphism provided by the trace.

Given $v \in W_\mu$, $\alpha \in W_\mu{}^*$, the *matrix coefficient* $\phi^\mu_{\alpha\otimes v} \in A$ is the image under ϕ^μ of $\alpha \otimes v$. Explicitly, $\langle \phi^\mu_{\alpha\otimes v}, b\rangle = \langle \alpha, \mu(b)(v)\rangle$, for $b \in B$. In particular, if $\tau \in \hat{B}$ and $w \otimes \beta \in W_\tau \otimes W_\tau^*$, then

$$\langle \phi^\mu_{\alpha\otimes v}, w \otimes \beta\rangle = \delta_{\mu,\tau}\, \langle \alpha, w\rangle\langle \beta, v\rangle. \tag{2.8}$$

Proposition (2.9). *Let $\mu, \tau \in \hat{B}$ and denote by $\mu^d \in \hat{B}$ the representation of B in $W_\mu{}^*$ induced by the antipode, i.e. $\mu^d(x) = {}^t(\mu(\mathcal{S}x))$, $x \in B$. Then we have:*

a) $\mathcal{S}(\phi^\mu_{\alpha\otimes v}) = \phi^{\mu^d}_{v\otimes\alpha}$, $\forall \alpha \in W_\mu{}^*, v \in W_\mu$.

b) $W_\mu \otimes W_\tau$ contains the trivial representation of B iff $\tau = \mu^d$. In particular $(\operatorname{End} W_\mu)^*.(\operatorname{End} W_\tau)^*$ contains $k1 = (\operatorname{End} W_\varepsilon)^*$ iff $\tau = \mu^d$.

c) $\langle \int, (\operatorname{End} W_\mu)^*\rangle = 0$, except when μ is trivial. Therefore the integral $\int$ can be identified with the projection onto $(\operatorname{End} W_\varepsilon)^* = k1$ with kernel $\oplus_{\mu\neq\varepsilon}(\operatorname{End} W_\mu)^*$.

d) The characteristic of k does not divide the dimension n_μ of the irreducible A-comodule W_μ.

Proof. a). It follows from the definitions.

b). The canonical isomorphism

$$W_\mu \otimes W_{\tau^d} \longrightarrow \operatorname{Hom}(W_\tau, W_\mu)$$

is a B-linear map. The isotypic component of trivial type in $\operatorname{Hom}(W_\tau, W_\mu)$ is exactly the space of B-linear maps. Therefore, if W_μ and W_τ are irreducible, the multiplicity of the trivial representation in $W_\mu \otimes W_{\tau^d}$ is 1 (respectively 0) iff W_μ and W_τ are (resp. are not) isomorphic. This says that $W_\mu \otimes W_\tau$ contains the trivial representation iff $W_\tau \simeq W_{\mu^d}$.

For the last assertion observe that for each $\mu \in \hat{B}$, $(\operatorname{End} W_\mu)^*$ is the isotypical component of type μ in A and that the multiplication map $A \otimes A \to A$ is a morphism of B-modules.

c). In general, if C is any coalgebra such that $1 \notin C$, then $\langle \int, C\rangle = 0$. Indeed, $\langle \int, a_{(1)}\rangle a_{(2)} = \langle \int, a\rangle 1 \in k1 \cap C = 0$, $\forall a \in C$. This proves the first assertion and the second follows from the first.

d). See [L2]. □

Note. Parts a) to c) of Proposition (2.9) hold more generally for cosemisimple Hopf algebras over an algebraically closed field of arbitrary characteristic. See [Sw2]. Part d) is valid for involutory cosemisimple Hopf algebras, see [L2].

Let (v) be a basis of W_μ, and $(\alpha) \subset W_\mu{}^*$ its dual basis; then $\{\phi^\mu_{\alpha\otimes v}\}$, $\alpha \in (\alpha)$, $v \in (v)$, is a basis of $(\operatorname{End} W_\mu)^* \subset A$.

Let $a \in A$, $a = \sum_{\mu\in\hat{B}} a_\mu$, with $a_\mu \in (\operatorname{End} W_\mu)^*$. Each a_μ has an expression of the form

$$a_\mu = \sum_{\alpha, v} a^\mu_{\alpha,v}\phi^\mu_{\alpha\otimes v}, \quad \text{for some } a^\mu_{\alpha,v} \in k. \tag{2.10}$$

We want to compute $\mathbb{F}(\phi^\mu_{\alpha\otimes v})$, for $\mu \in \hat{B}$, and $\alpha \in W_\mu{}^*$, $v \in W_\mu$.

Proposition (2.11). $\mathbb{F}(\phi^\mu_{\alpha\otimes v}) = \dfrac{1}{n_\mu} v \otimes \alpha$.

Proof. It is enough to show that the restriction of $\mathbb{G}$ to $W_\mu \otimes W_\mu^*$ satisfies

$$\mathbb{G}(v \otimes \alpha) = n_\mu \phi^\mu_{\alpha\otimes v};$$

since $n_\mu \neq 0$ by (2.9) d), we can then apply $\mathbb{F}$ to both sides of the equality and get the Proposition.

Now, $\mathbb{G}(v \otimes \alpha) = \Lambda \leftharpoonup v \otimes \alpha = \sum_{\tau\in\hat{B}} n_\tau(\mathcal{X}_\tau \leftharpoonup v \otimes \alpha)$. Let $b \in B$ and write $b = \sum_{\sigma\in\hat{B}} b_\sigma$, with $b_\sigma \in \operatorname{End} W_\sigma$. Then

$$\langle \mathcal{X}_\tau \leftharpoonup v \otimes \alpha, b\rangle = \sum_{\sigma\in\hat{B}} \langle \mathcal{X}_\tau, (v \otimes \alpha)b_\sigma\rangle = \delta_{\tau,\mu} \operatorname{Tr}((v \otimes \alpha)b_\mu) = \delta_{\tau,\mu}\langle \phi^\mu_{\alpha\otimes v}, b_\mu\rangle.$$

So, $\mathcal{X}_\tau \leftharpoonup v \otimes \alpha$ equals $\phi^\mu_{\alpha\otimes v}$ if $\tau = \mu$, and 0 otherwise. □

Corollary (2.12). *For $\alpha \in W_\mu{}^*$, $\beta \in W_\tau{}^*$, $v \in W_\mu$, $u \in W_\tau$*

$$\langle \textstyle\int, \mathcal{S}(\phi^\mu_{\alpha\otimes v})\phi^\tau_{\beta\otimes u}\rangle = \delta_{\mu,\tau}\dfrac{1}{n_\mu}\langle \alpha, u\rangle\langle \beta, v\rangle.$$

Proof. By (2.11) and (2.8)

$$\langle \textstyle\int, \mathcal{S}(\phi^\mu_{\alpha\otimes v})\phi^\tau_{\beta\otimes u}\rangle = \langle \mathbb{F}(\phi^\mu_{\alpha\otimes v}), \phi^\tau_{\beta\otimes u}\rangle = \delta_{\mu,\tau}\dfrac{1}{n_\mu}\langle \alpha, u\rangle\langle \beta, v\rangle. \quad \square$$

Note. Corollary (2.12) follows from [DK, Prop. 3.4 and 3.5], using that $\mathcal{S}^2 = \mathrm{id}$. The result [DK, Prop. 3.5] (for arbitrary cosemisimple Hopf algebras) is equivalent to the following [A, (1.8)]

$$(\int, ab) = \mathrm{Tr}({}^t S T \mathcal{M}), \tag{2.13}$$

if $a \in A$ is identified with $S \in \mathrm{End}\, W_{\mu^d}$ and $b \in A$ is identified with $T \in \mathrm{End}\, W_\mu$. Here $\mathcal{M} \in \mathrm{End}\, W_\mu$ is a B-isomorphism between W_μ and $W_{\mu^{dd}}$ such that $\mathrm{Tr}(\mathcal{M}) = 1$. The existence of such $\mathcal{M}$ follows from [L2]. It is important here the following result due to Larson [L2, Thm. 3.3]: The antipode of a cosemisimple Hopf algebra is bijective.

The key for the following Proposition is (2.13); part i) is contained in [A] and part ii) was motivated by a question of H.-J. Schneider. This result, however, will not be used in the rest of the article.

Proposition 2.14. *Let A be a (possibly infinite dimensional) cosemisimple Hopf algebra. Let $(a|b) := (\int, ab)$ be the Killing form of A.*

i) $(\ |\ )$ is symmetric iff $\mathcal{S}^2 = \mathrm{id}$.

ii) $(a|b) = (\mathcal{S}^2(b)|a)$ for all $a, b \in A$ iff $\mathcal{S}^4 = \mathrm{id}$.

Proof. i). Let $a \in A$ corresponding to $S \in \mathrm{End}\, W_{\mu^d}$ and $b \in A$ corresponding to $T \in \mathrm{End}\, W_\mu$. By (2.13), $(a|b) = \mathrm{Tr}({}^t S T \mathcal{M})$. Let us compute $(b|a)$. By the definition of $\mathcal{M}$,

$$({}^t\mathcal{M})^{-1} \mu^d(\mathcal{S}^2 x) = \mu^d(x)({}^t\mathcal{M})^{-1}, \qquad \text{for all } x \in B.$$

Hence $\frac{1}{\mathrm{Tr}\,\mathcal{M}^{-1}}({}^t\mathcal{M})^{-1}$ plays the rôle of $\mathcal{M}$ for μ^d. Now, as b corresponds to $T \in \mathrm{End}\, W_\mu$, then it corresponds to $\mathcal{M}^{-1} T \mathcal{M} \in \mathrm{End}\, W_{\mu^{dd}}$. Applying (2.13) to μ^d we get

$$\begin{aligned}(b|a) &= \frac{1}{\mathrm{Tr}\,\mathcal{M}^{-1}} \mathrm{Tr}({}^t(\mathcal{M}^{-1} T \mathcal{M}) S ({}^t\mathcal{M})^{-1}) \\ &= \frac{1}{\mathrm{Tr}\,\mathcal{M}^{-1}} \mathrm{Tr}({}^t T ({}^t\mathcal{M})^{-1} S) \\ &= \frac{1}{\mathrm{Tr}\,\mathcal{M}^{-1}} \mathrm{Tr}({}^t S \mathcal{M}^{-1} T).\end{aligned}$$

Thus $(a|b) = (b|a)$ for all such a, b iff $\mathrm{Tr}({}^t S T \mathcal{M}) = \frac{1}{\mathrm{Tr}\,\mathcal{M}^{-1}} \mathrm{Tr}({}^t S \mathcal{M}^{-1} T)$, iff $\mathcal{M} = (\dim W_\mu)^{-1} \mathrm{id}_{W_\mu}$. As $\mu \in \hat{B}$ is arbitrary, this is equivalent to $\mathcal{S}^2 = \mathrm{id}$. Indeed $\mathcal{S}^2(b)$ corresponds to $\mathcal{M} T \mathcal{M}^{-1} \in \mathrm{End}\, W_\mu$.

ii). By the preceding, $\mathcal{S}^4(b)$ corresponds to $\mathcal{M}^2 T \mathcal{M}^{-2} \in \mathrm{End}\, W_\mu$. On the other hand, $\mathcal{S}^2(b)$ corresponds to $T \in \mathrm{End}\, W_{\mu^{dd}}$, hence

$$(\mathcal{S}^2(b)|a) = \frac{1}{\mathrm{Tr}\,\mathcal{M}^{-1}} \mathrm{Tr}({}^t T \mathcal{S}^t(\mathcal{M}^{-1})).$$

Thus $(a|b) = (\mathcal{S}^2(b)|a)$ iff $\mathcal{M}^2 = \frac{1}{\mathrm{Tr}\,\mathcal{M}^{-1}} \mathrm{id}_{W_\mu}$, iff $\mathcal{S}^4 = \mathrm{id}$. □

We can now prove our first main result.

Theorem (2.15) (Inversion Formula). *Keeping the notation above, the coefficients $a^\mu_{\alpha,v}$ in (2.10) are given by the formula*

$$a^\mu_{\alpha,v} = \frac{n_\mu}{\dim A} \sum_{\rho \in \hat{A}} m_\rho \mathcal{X}_\rho \left(\phi^{\mu^d}_{\alpha^* \otimes v^*} a_\rho\right).$$

Here we indicate by $\alpha^* \in (v)$, the element defined by $\langle \alpha^*, \beta \rangle = \delta_{\alpha,\beta}$, $\beta \in (\alpha)$, and analogously for $v^* \in (\alpha)$.

Proof. We have, for all $\alpha \in (\alpha)$ and $v \in (v)$,

$$a^\mu_{\alpha,v} = n_\mu \langle \mathbb{F}(\phi^\mu_{v^* \otimes \alpha^*}), a \rangle, \tag{2.16}$$

which follows from Proposition (2.11).

On the other hand, if we express the integral in terms of the character of A, we obtain

$$\begin{aligned} n_\mu \langle \mathbb{F}(\phi^\mu_{v^* \otimes \alpha^*}), a \rangle &= n_\mu \langle \int, \phi^{\mu^d}_{\alpha^* \otimes v^*} a \rangle = \\ &\frac{n_\mu}{\dim A} \sum_{\rho \in \hat{A}} m_\rho \mathcal{X}_\rho \left(\phi^{\mu^d}_{\alpha^* \otimes v^*} a\right) = \frac{n_\mu}{\dim A} \sum_{\rho \in \hat{A}} m_\rho \mathcal{X}_\rho \left(\phi^{\mu^d}_{\alpha^* \otimes v^*} a_\rho\right), \end{aligned}$$

which gives the claimed identity in view of (2.16). □

We next want to interpret Plancherel identity (1.5).

Lemma (2.17). *Let $\mu \in \hat{B}$, and let $\alpha, \beta \in W_\mu{}^*$, $u, v \in W_\mu$. Then*

$$[v \otimes \alpha, u \otimes \beta]_d = n_\mu \langle \alpha, u \rangle \langle \beta, v \rangle.$$

Proof. By definition and (2.8)

$$[v \otimes \alpha, u \otimes \beta]_d = \langle v \otimes \alpha, \mathbb{G}(u \otimes \beta) \rangle = \langle v \otimes \alpha, n_\mu \phi^\mu_{\beta \otimes u} \rangle = n_\mu \langle \alpha, u \rangle \langle \beta, v \rangle. \quad □$$

Let a and b in A. Keep the notation in (2.10) for a and b.

Theorem (2.18) (Explicit Plancherel Identity).

$$\sum_{\rho\in\hat{A}} m_\rho \mathcal{X}_\rho(ab) = \dim A \sum_{\mu\in\hat{B}} \frac{1}{n_\mu} \sum_{\beta,u} a^{\mu}_{u^*,\beta^*} b^{\mu^d}_{u,\beta}.$$

Proof. By (2.5)

$$\sum_{\rho\in\hat{A}} m_\rho \mathcal{X}_\rho(ab) = \dim A \ \langle f, ab\rangle = \dim A \ \langle f, \sum_{\mu,\tau\in\hat{B}} a_\mu b_\tau\rangle.$$

This equals, by Proposition (2.9) b) and c),

$$\dim A \sum_{\mu\in\hat{B}} \langle f, a_\mu b_{\mu^d}\rangle = \dim A \sum_{\mu\in\hat{B}} \langle f, \mathcal{S}^2(a_\mu) b_{\mu^d}\rangle = \dim A \sum_{\mu\in\hat{B}} \langle \mathbb{F}\mathcal{S}(a_\mu), b_{\mu^d}\rangle = \dim A \sum_{\mu\in\hat{B}} [\mathcal{S}(a_\mu), b_{\mu^d}].$$

The first equality because $\mathcal{S}^2 = \mathrm{id}$. Now we use the Plancherel identity (1.5) to obtain

$$\sum_{\rho\in\hat{A}} m_\rho \mathcal{X}_\rho(ab) = \dim A \sum_{\mu\in\hat{B}} [\mathbb{F}(\mathcal{S}a_\mu), \mathbb{F}(b_{\mu^d})]_d. \tag{2.19}$$

Let now $\mu \in \hat{B}$, and let us compute $[\mathbb{F}(\mathcal{S}a_\mu), \mathbb{F}(b_{\mu^d})]_d$. By Proposition (2.9) a), $\mathcal{S}(a_\mu) = \sum_{\alpha,v} a^{\mu}_{\alpha,v} \phi^{\mu^d}_{v\otimes\alpha}$. So that, using (2.11), we may write

$$\mathbb{F}(\mathcal{S}a_\mu) = \frac{1}{n_\mu} \sum_{\alpha,v} a^{\mu}_{\alpha,v}\ \alpha\otimes v, \quad \text{and} \quad \mathbb{F}(b_{\mu^d}) = \frac{1}{n_\mu} \sum_{\beta,u} b^{\mu^d}_{u,\beta}\ \beta\otimes u.$$

Thus,

$$[\mathbb{F}(\mathcal{S}a_\mu), \mathbb{F}(b_{\mu^d})]_d = \frac{1}{{n_\mu}^2} \sum_{\alpha,\beta,u,v} a^{\mu}_{\alpha,v} b^{\mu^d}_{u,\beta}\ [\alpha\otimes v, \beta\otimes u]_d = \frac{1}{{n_\mu}^2} \sum_{\alpha,\beta,u,v} a^{\mu}_{\alpha,v} b^{\mu^d}_{u,\beta}\ n_\mu \langle \alpha, u\rangle\langle \beta, v\rangle,$$

the last equality thanks to Lemma (2.17) applied to μ^d. Now, as $(u, v, \dots)$ and $(\alpha, \beta, \dots)$ are dual bases, we have

$$[\mathbb{F}(\mathcal{S}a_\mu), \mathbb{F}(b_{\mu^d})]_d = \frac{1}{n_\mu} \sum_{\beta,u} a^{\mu}_{u^*,\beta^*} b^{\mu^d}_{u,\beta}. \tag{2.20}$$

Putting together (2.19) and (2.20), we get the claim. □

Example. Let G be a finite group, $A = kG$ its group algebra, $\hat{G}$, the set of isomorphy classes of irreducible representations of G. In this setting the inversion formula and Plancherel identity give, respectively, for $u = \sum_{s\in G} u_s s = \sum_{\rho\in\hat{G}} u_\rho$

$$u_s = \frac{1}{|G|} \sum_{\rho\in\hat{G}} m_\rho \mathcal{X}_\rho(s^{-1}u_\rho), \quad \text{and} \quad \sum_{\rho\in\hat{G}} m_\rho \mathcal{X}_\rho(uv) = |G| \sum_{s\in G} v_s u_{s^{-1}}.$$

See [Se, 6.2, Prop. 11 and Ex. 1].

§3. On the dimensions of the irreducible modules.

As an application of the above results, and in particular of the inversion formula (2.15), we will prove a formula for the quotient $\dfrac{\dim A}{n_\mu}$.

We remark that it was conjectured by Kaplansky in 1975 that the dimensions of the irreducible modules divide the dimension of A. In the case of finite groups this is true, and is known as Frobenius Theorem [Se, 6.5, Cor. 2].

Let $\rho \in \hat{A}$. Fix a basis $(v_i)_{i=1}^{m_\rho}$ of V_ρ and let $(v_i^*)_{i=1}^{m_\rho}$ be the dual basis in V_ρ^*. Consider the matrix coefficients $\rho_{ij} := \psi^{\rho}_{v_i^*\otimes v_j} \in B$; that is,

$$\langle \rho_{ij}, a\rangle = \langle \psi^{\rho}_{v_i^*\otimes v_j}, a\rangle = \langle v_i^*, \rho(a)(v_j)\rangle,$$

$\forall a \in A$, $1 \le i, j \le m_\rho$.

The following properties are easily checked.

$$\Delta(\rho_{ij}) = \sum_{k=1}^{m_\rho} \rho_{ik} \otimes \rho_{kj}, \quad \epsilon(\rho_{ij}) = \delta_{ij}. \tag{3.1}$$

Also,

$$\mathcal{X}_\rho = \sum_{i=1}^{m_\rho} \rho_{ii}, \tag{3.2}$$

where $\mathcal{X}_\rho$ denotes the character of V_ρ.

With the notation above, and letting $\rho^d_{ij} := \psi^{\rho^d}_{v_i\otimes v_j^*}$, we have

Theorem (3.3). *Let $\mu \in \hat{B}$, and let $(v) \subseteq W_\mu$, $(\alpha) \subseteq W_\mu{}^*$ be dual bases. If $\alpha \in (\alpha)$ and $v \in (v)$, then*

$$\frac{\dim A}{n_\mu} = \sum_{\rho \in \hat{A}} m_\rho \sum_{i,k=1}^{m_\rho} \langle v^*, \mu(\rho_{ki}^d)(\alpha^*)\rangle \, \langle \alpha, \mu(\rho_{ik})(v)\rangle.$$

Observation. If $v \in W_\mu$ and $\alpha \in W_\mu^*$ satisfy $\langle \alpha, v\rangle = 1$, then (3.3) implies the suggestive formula

$$\frac{\dim A}{n_\mu} = \sum_{\rho \in \hat{A}} m_\rho \sum_{i,k=1}^{m_\rho} \langle \alpha, \mu(\rho_{ki}^d)(v)\rangle \, \langle \alpha, \mu(\rho_{ik})(v)\rangle.$$

Proof. Specializing the inversion formula (2.15) in $a = \phi^\mu_{\alpha\otimes v}$, we get

$$1 = \frac{n_\mu}{\dim A} \sum_{\rho \in A} m_\rho \, \mathcal{X}_\rho \left(\phi^{\mu^d}_{\alpha^*\otimes v^*} \phi^\mu_{\alpha\otimes v}\right).$$

Hence,

$$\frac{\dim A}{n_\mu} = \sum_{\rho \in \hat{A}} m_\rho \, \mathcal{X}_\rho \left(\phi^{\mu^d}_{\alpha^*\otimes v^*} \phi^\mu_{\alpha\otimes v}\right) = \sum_{\rho \in \hat{A}} m_\rho \, \langle \Delta(\mathcal{X}_\rho), \phi^{\mu^d}_{\alpha^*\otimes v^*} \otimes \phi^\mu_{\alpha\otimes v}\rangle =$$

$$\sum_{\rho \in \hat{A}} m_\rho \sum_{i,k=1}^{m_\rho} \langle \rho_{ik}, \phi^{\mu^d}_{\alpha^*\otimes v^*}\rangle \, \langle \rho_{ki}, \phi^\mu_{\alpha\otimes v}\rangle,$$

the last equality by (3.1) and (3.2). Using the definition of the matrix coefficients, and (2.9) a), this equals

$$\sum_{\rho \in \hat{A}} m_\rho \sum_{i,k=1}^{m_\rho} \langle \mu^d(\rho_{ik})(v^*), \alpha^*\rangle \, \langle \alpha, \mu(\rho_{ki})(v)\rangle =$$

$$\sum_{\rho \in \hat{A}} m_\rho \sum_{i,k=1}^{m_\rho} \langle v^*, \mu(S\rho_{ik})(\alpha^*)\rangle \, \langle \alpha, \mu(\rho_{ki})(v)\rangle =$$

$$\sum_{\rho \in \hat{A}} m_\rho \sum_{i,k=1}^{m_\rho} \langle v^*, \mu(\rho_{ki}^d)(\alpha^*)\rangle \, \langle \alpha, \mu(\rho_{ki})(v)\rangle,$$

as claimed. □

§4. Application to Hopf algebras with an order.

From now on we will assume that the characteristic of k is zero.

Let K be an algebraic number field and let R be the ring of algebraic integers in K. We may assume that $K \subseteq k$. An R-order of A is a Hopf algebra A^R over R such that $A \simeq A^R \otimes_R k$ as Hopf algebras over k, and A^R is projective and finitely generated as R-module [L, §1]. As an application of Theorem (3.3) we will give a new proof of a result of Larson [L, Prop.4.2] which states that under the assumption that A has an order, the dimensions of the irreducible modules divide the dimension of A.

Remark. H.-J. Schneider has pointed out to us a third way to get Larson's Theorem via an argument, which involves Casimir elements, that appears in [KMe].

In this section we will assume that A has an R-order. For any ring extension $R \subseteq S \subsetneq k$, we will denote $A^S := A^R \otimes_R S$, so that $A = A^k$.

Lemma (4.1). *Let H be a Hopf algebra over a field L. If $L \subseteq F$ is a field extension of L and $H^F = H \otimes_L F$, then H is semisimple iff H^F is. (Thus the semisimplicity of a Hopf algebra does not depend on the base field).*

Proof. It follows from Maschke Theorem for Hopf algebras. □

If H is any algebra over a field L and F is an extension field of L, for any H-module V we may consider the H^F-module $V^F := V \otimes_L F$. L is called a *splitting field* for H if for every irreducible H-module V, and for every field extension F of L, V^F is an irreducible H^F-module [CR, Def. (29.12)]. In the case of Hopf algebras we have the following Lemma. Compare with [CR, Th. (29.16)].

Lemma (4.2). *Let H be a finite Hopf algebra over an algebraic number field K, then there exists a finite extension field L of K which is a splitting field for H.* □

We now return to the case under consideration.

Remark 4.3. It is clear that the dimensions of the irreducible modules are the same in any extension A^F of A^L, where L is a splitting field for A^K. Just observe that $\dim_F V^F = \dim_L V$ for any A^L-module V, and any field extension $L \subseteq F$, and use [CR, Th. (29.13)]. Thus to show that the dimensions of its irreducible modules divide the dimension of A, it is enough to show that the proper occurs for A^L, where L is a splitting field for A^K. Also, by Lemma (4.2), we may suppose that L is an algebraic number field, and apply the above remarks to its algebraic closure in k (which is also a splitting field for A^K).

We will need the following Lemma. See [K, Thm. 1], [Sch, Lemma (4.11)].

Lemma (4.4). *Let L be an algebraic number field, S the ring of integers in L, and $\overline{L}$ its algebraic closure in k. Suppose H is an S-algebra which is finitely generated and projective as an S-module, and such that $H^{\overline{L}}$ is a semisimple algebra over L. Then for any $H^{\overline{L}}$-module V there exists a $\overline{L}$-basis such that the entries of the matrices representing the action of elements of H in this basis are algebraic integers.* □

Remark (4.5). Let A^S be an order of A over the ring S, so that A^S is finitely generated and projective as S-module, then $(A^S)^* := \mathrm{Hom}_S(A^S, S)$ also is. Hence the canonical map $(A^S)^* \otimes_S k \to \mathrm{Hom}_S(A^S, k)$ is a k-linear isomorphism. On the other hand, recall the isomorphism, $\mathrm{Hom}_S(A^S, k) \simeq \mathrm{Hom}_k(A, k)$, also canonical. These give a map $(A^S)^* \otimes_S k \to \mathrm{Hom}_k(A, k) = A^* = B$, which is easily verified to be an isomorphism of Hopf algebras over k. Thus $(A^S)^*$ is an S-order of B.

Theorem (4.6). *The dimensions of the irreducible A-modules divide the dimension of A.*

Proof. By Remark (4.3), it will suffice to prove the claim for $A^{\overline{L}}$, where L is a splitting field of A^K which is an algebraic number field, and $\overline{L}$ denotes the algebraic closure in k of L. As $A^{\overline{L}}$ is a semisimple Hopf algebra over the algebraically closed field $\overline{L}$, we may apply Theorem (3.3).

Denote by S the ring of integers of L, so $A^S = A^R \otimes_R S$ is a finitely generated and projective S-module, and we may apply Lemma (4.4) to the irreducible module V_ρ, $\rho \in \widehat{A^{\overline{L}}}$.

Choose then a basis (v_i) of V_ρ such that the elements of A^S are represented, in this basis, by matrices with entries algebraic integers. Call E the integral extension of S generated by the matrix entries of $\rho(A^S)$. So that the matrix coefficients $\rho_{ij} \in B$ corresponding to the basis (v_i), map A^S into E. Moreover, they map $A^E = E\ A^S$ into E. In other words, we have $\rho_{ij} \in (A^E)^*$, $\forall \rho \in \widehat{A^{\overline{L}}}$.

Fix now $\mu \in \widehat{B^{\overline{L}}}$. By Remark (4.6), and again applying Lemma (4.4) now to the irreducible $B^{\overline{L}}$-module W_μ, $\mu \in \widehat{B^{\overline{L}}}$, we can take the basis (v) such that the entries of the matrices representing elements of $(A^E)^* = B^E$ are algebraic integers. In particular, $\langle \alpha, \mu(\rho_{ik})(v) \rangle$ are algebraic integers for all ρ in $\widehat{A^{\overline{L}}}$. But then, by Theorem (3.3) the rational number $\dfrac{\dim A}{n_\mu}$ is an algebraic integer, hence an integer. Thus the dimensions of the irreducible comodules divide the dimension of A. Interchanging the rôles of A and B, the result follows. □

Semisimple Hopf algebras of order pq. Let p and q be prime numbers, $p < q$, $q \not\equiv 1 \mod p$. Let A be a semisimple Hopf algebra with an order. We will prove that if the dimension of A is pq, then A is isomorphic to $k\mathbb{Z}_{pq}$, hence it is commutative and cocommutative.

Using the isomorphisms (2.1) and (2.2), and Theorem (4.6), we may write

$$pq = \dim A = \sum_{\rho \in \widehat{A}} (m_\rho)^2 = \sum_{\mu \in \widehat{B}} (n_\mu)^2,$$

with $m_\rho | \dim A$, and $n_\mu | \dim A$, $\forall \rho \in \widehat{A}$, $\mu \in \widehat{B}$. Observe also that $(n_\mu)^2$ and $(m_\rho)^2$ are strictly less than pq. Thus, each n_μ and m_ρ must equal either 1 or p.

Call b_1 (respectively b_2), the number of $\mu \in \widehat{B}$ such that $n_\mu = 1$ (respectively $n_\mu = p$). Then $pq = b_1 + b_2p^2$, so that $p | b_1$. But as $b_1 = |G(A)|$, where $G(A)$ is the group of grouplike elements in A, by the Nichols-Zoeller Theorem [NZ], $b_1 | pq$. Hence $b_1 = p$ or pq.

If $b_1 = pq$, then $A = kG(A)$ is a group algebra and the result follows from elementary group theory. But if $b_1 = p$, $pq = p + b_2p^2$, hence $q = 1 + b_2p \equiv 1 \mod p$ against the assumption.

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