DOUBLE CATEGORIES AND QUANTUM GROUPOIDS

NICOLÁS ANDRUSKIEWITSCH AND SONIA NATALE

To Susan Montgomery, on her 60th. birthday

Abstract. We give the construction of a class of weak Hopf algebras (actually, face algebras as defined by Hayashi) associated to a matched pair of groupoids and certain cocycle data. This generalizes a now well-known construction for Hopf algebras, first studied by G. I. Kac in the sixties. Our approach is based on the notion of double groupoids, as introduced by Ehresmann.

Introduction

An *exact factorization* of a group Σ is a pair of subgroups G, F such that the multiplication map induces a bijection $m : F \times G \to \Sigma$. Given an exact factorization of a group Σ, there are a right action $\triangleleft : G \times F \to G$ and a left action $\triangleright : G \times F \to F$ defined by $sx = (s{\triangleright}x)(s{\triangleleft}x)$, for all $s \in G$, $x \in F$. These actions satisfy the compatibility conditions

$$s{\triangleright}xy = (s{\triangleright}x)((s{\triangleleft}x){\triangleright}y), \tag{0.1}$$

$$st{\triangleleft}x = (s{\triangleleft}(t{\triangleright}x))(t{\triangleleft}x), \tag{0.2}$$

for all $s, t \in G$, $x, y \in F$. It follows that $s{\triangleright}1 = 1$ and $1{\triangleleft}x = 1$, for all $s \in G$, $x \in F$. Such a data of groups and compatible actions is called a *matched pair* of groups. Conversely, given a matched pair of groups F, G, one can find a group Σ together with an exact factorization $\Sigma = FG$.

Let $\Bbbk$ be a field. In the early eighties, Takeuchi achieved a construction which, starting from a matched pair F, G of finite groups, gives a (in general not commutative and not cocommutative) Hopf algebra $H := \Bbbk^G \# \Bbbk F$ [T1]. This Hopf algebra fits into an exact sequence

$$1 \longrightarrow \Bbbk^G \xrightarrow{\iota} \Bbbk^G \# \Bbbk F \xrightarrow{\pi} \Bbbk F \longrightarrow 1;$$

$\Bbbk^G \# \Bbbk F$ is called a bismash product; it is semisimple and cosemisimple if the characteristic of $\Bbbk$ is relatively prime to the order of Σ. The same construction was also presented by Majid [Mj1]. A more general instance of this construction can be done by adjoining a certain cohomological data associated to the matched pair: namely a pair of 2-cocycles $\sigma : F \times F \to (\Bbbk^G)^\times$ and $\tau : G \times G \to (\Bbbk^F)^\times$ satisfying appropriate compatibility conditions. In this way, all Hopf algebras H which fit into an exact sequence $1 \longrightarrow \Bbbk^G \xrightarrow{\iota} H \xrightarrow{\pi} \Bbbk F \longrightarrow 1$ are obtained. The compatibility conditions have an elegant description in terms of the total complex associated to a double complex that combines the group cohomologies of G, F and Σ. It turns out that this more general construction, and the cohomology theory behind it, had already been discovered by G. I. Kac [K]. Explicit computations can be done with the help of the so-called Kac exact sequence *loc. cit.* The study of this cohomology theory has been later pursued by Masuoka; see the paper [M] and references therein for details on this topic.

Date: October 7, 2004.

1991 *Mathematics Subject Classification.* 16W30, 20L05, 18D05, 18B40.

Partially supported by ANPCyT, CONICET, Agencia Córdoba Ciencia, Secyt (UNC) and Fundación Antorchas.

Quantum groups appart, this construction gave rise to one of the first genuine examples of non-commutative non-cocommutative Hopf algebras. More recently, it was shown that the Hopf algebras $\mathbb{C}^G\#\mathbb{C}F$ are exactly those having a positive basis [LYZ1].

In our previous paper [AN], we discussed braided Hopf algebras R which fit into an exact sequence $1 \longrightarrow \Bbbk^G \xrightarrow{\iota} R \xrightarrow{\pi} \Bbbk F \longrightarrow 1$. The present paper was inspired by a comment of the referee of [AN], pointing out a pictorial description of the standard basis of $\Bbbk^G\#\Bbbk F$, which gives a more compact form to the constructions. It turns out that this pictorial description, also present in [Mj2, T2, DVVV], can be stated in the language of double categories. These have been introduced by Ehresmann [E]. A double category can be defined as a $\mathcal{C}$-structured category, where $\mathcal{C}$ is the category of small categories and functors; that is, as a category object in the category of small categories.

Roughly, a small double category consists of a set of 'boxes' $\mathcal{B} = \{A, B, \dots\}$, each box having colored edges (and vertices)

$$A = l \begin{array}{c} t \\ \boxed{} \\ b \end{array} r;$$

boxes can be 'horizontally' and 'vertically' composed, both compositions subject to a natural interchange law. The description given in the Appendix of [AN] fits exactly into this framework: here the categories of vertical and horizontal compositions correspond to the transformation groupoids attached to the actions $\triangleleft : G \times F \to G$ and $\triangleright : G \times F \to F$, respectively. In this example the vertical edges of boxes are colored by elements of F, the horizontal edges are colored by elements of G, and it has the particularity that every box is uniquely determined by a pair of adjacent edges. Moreover, in this case there is only one coloring for the 'vertices' of boxes.

It is then natural to ask: what are the double categories that give rise to Hopf algebras in this fashion? First, we shall consider *double groupoids*– double categories where both the horizontal and vertical compositions are invertible– to have antipodes. Now, because of the positive basis Theorem in [LYZ1], we know that the answer should be the double groupoids coming from matched pairs of groups as above. Still, we can ask: what are the double groupoids that give rise to *weak Hopf algebras*?

The notion of weak Hopf algebras or quantum groupoids was recently introduced in [BNSz, BSz] as a non-commutative version of groupoids. A relevant feature is that they give rise to tensor categories. A weak Hopf algebra has an algebra and a coalgebra structure; the comultiplication is multiplicative but it does not preserve the unit. A particular but very important class of weak Hopf algebras was introduced and studied previously by T. Hayashi, see [H] and references therein.

Given a finite double groupoid, we endow the vector space with basis the set of boxes with the groupoid algebra structure of the vertical groupoid, and with the groupoid coalgebra structure of the horizontal groupoid. We found a sufficient condition to get a quantum groupoid; this is condition (2) in Proposition 2.2. It turns out that double groupoids satisfying this condition are equivalent to the *vacant* double groupoids considered by Mackenzie [Ma]: every box be determined by any pair of adjacent edges. Also, vacant double groupoids are in bijective correspondence with matched pairs of groupoids [Ma].

Our main result is that every vacant double groupoid gives rise to a weak Hopf algebra in the way described above; this weak Hopf algebra is semisimple if $\Bbbk$ has characteristic 0. The corresponding construction is also done by adjoining compatible 2-cocycle data. These 2-cocycle data is part of a first quadrant double complex, as in the group case; there is as well a "Kac exact sequence for groupoids". We point out that these weak Hopf algebras have commutative source and target subalgebras, so they are face algebras in the sense of Hayashi [H]. Also, they are involutory. Our construction may alternatively be presented without double groupoids, using just exact factorizations of groupoids. We feel however that the language of double categories is not merely accidental; it also appears again in recent work of Kerler and Lyubashenko on topological quantum field theory [KL].

The paper is organized as follows. The first section is devoted to double categories and double groupoids. For the convenience of the reader not used to the language of double categories, we include many details and proofs. In the second section we discuss vacant double groupoids. We describe vacant double groupoids in group-theoretical sense in Theorem 2.16. The third section contains the construction of semisimple quantum groupoids and a presentation of the Kac exact sequence for groupoids.

The results of this paper were reported by the first author at the University of Clermont-Ferrand, June 2003; at this occasion, he was told by S. Baaj that a topological version of the Kac exact sequence for vacant double groupoids– with set of vertices of cardinal one– has been independiently studied by G. Skandalis, S. Vaes and himself; see [BSV]. The first author also reported this work at the XV Coloquio Latinoamericano de Álgebra, Cococyoc, México, July 2003; at this occasion, G. Böhm, R. Coquereaux and K. Szlachányi pointed out the ressemblance between double groupoids and Ocneanu cells. He also reported this construction at the "Colloque Algèbres de Hopf et invariants topologiques", Luminy, March 2004.

After release of the first version of this paper, positive quasitriangular R-matrices for the weak Hopf algebras introduced here were constructed in [AA], generalizing results from [LYZ2]. See also connections with the quiver-theoretical quantum Yang-Baxter equation and face models in [A]. Explicit examples of matchedpairs of groupoids and some calculations of extensions are presented in [AM].

Acknowledgements. We thank the referee of [AN] for the key remark in her/his report. The research of the second author was done during a postdoctoral stay at the Département de mathématiques et applications of the École Normale Supérieure, Paris; she thanks Marc Rosso for the kind hospitality. Part of the work of the first author was done during a visit to the University of Reims (February - May 2003); he is very grateful to J. Alev for his generous invitation.

1. Double categories and double groupoids

1.1. **Definition of double categories.**

Let $\mathcal{C}$ be a category with pullbacks. Recall that a category object in $\mathcal{C}$ (or a category internal to $\mathcal{C}$) is a collection $(A, O, s, t, \mathrm{id}, m)$, where A ("arrows") and O ("objects") are objects in $\mathcal{C}$; $s, e : A \to O$ ("source" and "end = target", respectively), $\mathrm{id} : O \to A$ ("identities") and $m : A\, {}_e\!\times_s A \to A$ ("composition") are arrows in $\mathcal{C}$; subject to the usual associativity and identity axioms. Similarly, a groupoid object in $\mathcal{C}$ is a category object in $\mathcal{C}$ with all "arrows" invertible, which amounts to the existence of a map $\mathcal{S} : A \to A$ with suitable properties.

Notation. Along this paper, in the case where f, g are composable arrows in a category, their composition $m(f, g)$ will be indicated by juxtaposition: $m(f, g) = fg$ (and not gf).

Definition 1.1. A (small) *double category* $\mathcal{T}$ consists of the following data:

- Four non-empty sets: $\mathcal{B}$ (boxes), $\mathcal{H}$ (horizontal edges), $\mathcal{V}$ (vertical edges) and $\mathcal{P}$ (points);
- eight boundary functions: $t, b : \mathcal{B} \to \mathcal{H}$; $r, l : \mathcal{B} \to \mathcal{V}$; $r, l : \mathcal{H} \to \mathcal{P}$; $t, b : \mathcal{V} \to \mathcal{P}$;
- four identity functions: $\mathrm{id} : \mathcal{P} \to \mathcal{H}$; $\mathrm{id} : \mathcal{P} \to \mathcal{V}$; $\mathbf{id} : \mathcal{H} \to \mathcal{B}$; $\mathbf{id} : \mathcal{V} \to \mathcal{B}$;
- four composition functions, all denoted by m:

$$\mathcal{B}\, {}_b\!\times_t \mathcal{B} \to \mathcal{B} \quad \text{(vertical composition)}, \quad \mathcal{B}\, {}_r\!\times_l \mathcal{B} \to \mathcal{B} \quad \text{(horizontal composition)},$$

$$\mathcal{H}\, {}_r\!\times_l \mathcal{H} \to \mathcal{H}, \quad \mathcal{V}\, {}_b\!\times_t \mathcal{V} \to \mathcal{V};$$

such that the following axioms are satisfied.

Axiom 0. $(\mathcal{B}, \mathcal{H}, t, b, \mathbf{id}, m)$, $(\mathcal{B}, \mathcal{V}, l, r, \mathbf{id}, m)$, $(\mathcal{H}, \mathcal{P}, l, r, \mathrm{id}, m)$, $(\mathcal{V}, \mathcal{P}, t, b, \mathrm{id}, m)$ are categories.

Axiom 1. Four identities between possible functions from $\mathcal{B}$ to $\mathcal{P}$, namely

$$tr = rt, \quad tl = lt, \quad bl = lb, \quad br = rb.$$

This axiom allows to depict graphically $A \in \mathcal{B}$ as a box

$$A = l \ \underset{b}{\overset{t}{\boxed{}}} \ r$$

where $t(A) = t$, $b(A) = b$, $r(A) = r$, $l(A) = l$, and the four vertices of the square representing A are $tl(A)$, $tr(A)$, $bl(A)$, $br(A)$. Of course, t, b, r and l mean, respectively, 'top', 'bottom', 'right' and 'left'. Most of the remaining axioms will be stated in this pictorial representation.

Warning. A box $A \in \mathcal{B}$ is, in general, *not* determined by its four boundaries t, b, r, l.

We shall write $A|B$ if $r(A) = l(B)$ (so that A and B are horizontally composable), and $\dfrac{A}{B}$ if $b(A) = t(B)$ (so that A and B are vertically composable).

The notation AB (respectively, $\genfrac{}{}{0pt}{}{A}{B}$) will indicate the horizontal (respectively, vertical) compositions, whenever A and B are composable in the appropriate sense.

Axiom 2. *Consistency of boundary compositions.* Let $A = l \ \underset{b}{\overset{t}{\boxed{}}} \ r$ and $B = s \ \underset{c}{\overset{u}{\boxed{}}} \ m$ in $\mathcal{B}$.

$$\text{If} \quad A|B, \quad \text{then} \quad AB = l \ \underset{bc}{\overset{tu}{\boxed{}}} \ m, \tag{1.1}$$

$$\text{If} \quad \frac{A}{B}, \quad \text{then} \quad \genfrac{}{}{0pt}{}{A}{B} = ls \ \underset{c}{\overset{t}{\boxed{}}} \ rm. \tag{1.2}$$

The notation $\begin{array}{c|c} A & B \\ \hline C & D \end{array}$ means that all possible horizontal and vertical products are allowed; in view of Axiom 2, this implies that $\dfrac{AB}{CD}$, $\begin{array}{c|c} A & B \\ C & D \end{array}$.

Axiom 3. *Interchange law between horizontal and vertical compositions.* If $\begin{array}{c|c} A & B \\ \hline C & D \end{array}$, then

$$\genfrac{}{}{0pt}{}{AB}{CD} := \genfrac{}{}{0pt}{}{\{AB\}}{\{CD\}} = \begin{Bmatrix} A \\ C \end{Bmatrix} \begin{Bmatrix} B \\ D \end{Bmatrix}. \tag{1.3}$$

A consequence of this Axiom is that, given $r \times s$ boxes A_{ij} with horizontal and vertical compositions allowed as in the following arrangement

$$\begin{array}{c|c|c|c} A_{11} & A_{12} & \dots & A_{1s} \\ \hline A_{21} & A_{22} & \dots & A_{2s} \\ \hline \dots & \dots & \dots & \dots \\ \hline A_{r1} & A_{r2} & \dots & A_{rs} \end{array},$$

then the composition

$$\begin{matrix} A_{11} & A_{12} & \dots & A_{1s} \\ A_{21} & A_{22} & \dots & A_{2s} \\ \dots & \dots & \dots & \dots \\ A_{r1} & A_{r2} & \dots & A_{rs} \end{matrix}$$

is well defined and can be computed associating in any possible way.

Axiom 4. *Horizontal and vertical identities.* The identity functions $\mathbf{id} : \mathcal{H} \to \mathcal{B}$ (vertical identity), $\mathbf{id} : \mathcal{V} \to \mathcal{B}$ (horizontal identity) satisfy

$$\mathbf{id}(g) = \begin{matrix} & g & \\ \mathrm{id}\ l(g) & \boxed{} & \mathrm{id}\ r(g) \\ & g & \end{matrix}, \quad g \in \mathcal{H}; \qquad \mathbf{id}(x) = \begin{matrix} & \mathrm{id}\ t(x) & \\ x & \boxed{} & x \\ & \mathrm{id}\ b(x) & \end{matrix}, \quad x \in \mathcal{V}.$$

Note that, in principle, there is an ambiguity when using the notation id P for an element $P \in \mathcal{P}$; however, this ambiguity disappears in the pictorial representation. When necessary, the notation $\mathrm{id}_{\mathcal{H}} : \mathcal{P} \to \mathcal{H}$ and $\mathrm{id}_{\mathcal{V}} : \mathcal{P} \to \mathcal{V}$ for the corresponding identity maps will be used.

Axiom 5. *Horizontal and vertical identities of the identities of the points.* If $P \in \mathcal{P}$, then $\mathbf{id}\,\mathrm{id}_{\mathcal{H}}\,P = \mathbf{id}\,\mathrm{id}_{\mathcal{V}}\,P$; this box will be denoted Θ_P.

Axiom 6. *Compatibility of the identity with composition of arrows.* If $g, h \in \mathcal{V}$, $x, y \in \mathcal{H}$ are composable, then $\begin{Bmatrix} \mathbf{id}\, g \\ \mathbf{id}\, h \end{Bmatrix} = \mathbf{id}\, gh$, $\{\mathbf{id}\, x\ \mathbf{id}\, y\} = \mathbf{id}\, xy$.

The following lemma is well-known, see for example [BS].

Lemma 1.2. *A double category is a category object in the category of small categories.*

Proof. Let $\mathcal{T}$ be a category object in the category of small categories. Thus $\mathcal{T} = (\mathcal{A}, \mathcal{O}, t, b, \mathrm{id}, m)$, where $\mathcal{A}$ and $\mathcal{O}$ are small categories, $t, b : \mathcal{A} \to \mathcal{O}$, $\mathrm{id} : \mathcal{O} \to \mathcal{A}$ and $m : \mathcal{A}\,{}_b\!\times_t \mathcal{A} \to \mathcal{A}$ are functors subject to associativity and identity axioms.

Write $\mathcal{A} = (\mathcal{B}, \mathcal{V}, l, r, \mathbf{id}, m)$ and $\mathcal{O} = (\mathcal{H}, \mathcal{P}, l, r, \mathrm{id}, m)$. The functors t, b, id and m correspond, respectively, to maps

$$t, b : \mathcal{B} \rightrightarrows \mathcal{H}, \quad \mathrm{id} : \mathcal{H} \to \mathcal{B}, \quad m : \mathcal{B}\,{}_b\!\times_t \mathcal{B} \to \mathcal{B},$$

and

$$t, b : \mathcal{V} \rightrightarrows \mathcal{P}, \quad \mathrm{id} : \mathcal{P} \to \mathcal{V}, \quad m : \mathcal{V}\,{}_b\!\times_t \mathcal{V} \to \mathcal{V}.$$

The associativity and identity constraints relating the functors t, b, id and m, correspond to the fact that $(\mathcal{B}, \mathcal{H}, t, b, \mathbf{id}, m)$ and $(\mathcal{V}, \mathcal{P}, t, b, \mathrm{id}, m)$ are categories. In what follows we shall see that the functoriality of t, b, id and m, corresponds to the axioms 1–6.

The functoriality of t and b amount to the identities

$$tl = lt, \quad tr = rt, \quad bl = lb, \quad br = rb; \tag{1.4}$$

$$tm = m(t \times t), \quad bm = m(b \times b); \tag{1.5}$$

$$t\,\mathrm{id} = \mathrm{id}\,t; \quad b\,\mathrm{id} = \mathrm{id}\,b. \tag{1.6}$$

One sees that (1.4) corresponds to Axiom 1, (1.5) corresponds to (1.1) in Axiom 2, and (1.6) corresponds to the right hand side identity in Axiom 4.

The functoriality of m amounts to

$$m(l\,{}_b\times_t\,l) = lm, \quad m(r\,{}_b\times_t\,r) = rm; \tag{1.7}$$

$$m(m\,{}_b\times_t\,m) = m(m \times m); \tag{1.8}$$

$$m(\mathrm{id}\,{}_b\times_t\,\mathrm{id}) = \mathrm{id}\,m. \tag{1.9}$$

Among these, (1.7) corresponds to identity (1.2) in Axiom 2, (1.8) corresponds to Axiom 3, and (1.9) corresponds to the left hand side identity in Axiom 6.

The functoriality of id amounts to

$$\mathrm{id}\,l = l\,\mathrm{id}, \quad \mathrm{id}\,r = r\,\mathrm{id}; \tag{1.10}$$

$$\mathrm{id}\,m = m(\mathrm{id} \times \mathrm{id}); \tag{1.11}$$

$$\mathrm{id}\,\mathrm{id} = \mathrm{id}\,\mathrm{id}\,. \tag{1.12}$$

Whence, (1.10) corresponds to the left hand side identity in Axiom 4, (1.11) corresponds to the right hand side identity in Axiom 6, and (1.12) corresponds to Axiom 5. The constructions and arguments are reversible, and this finishes the proof of the lemma. $\square$

It is customary to represent a double category in the form of four related categories

$$\begin{array}{ccc} \mathcal{B} & \rightrightarrows & \mathcal{H} \\ \downdownarrows & & \downdownarrows \\ \mathcal{V} & \rightrightarrows & \mathcal{P} \end{array}$$

subject to the above axioms. So that the vertical arrows

$$\begin{array}{cc} \mathcal{B} & \mathcal{H} \\ \downdownarrows, & \downdownarrows, \\ \mathcal{V} & \mathcal{P} \end{array}$$

correspond to the categories $\mathcal{A}$ and $\mathcal{O}$ of 'arrows' and 'objects', respectively, while the horizontal arrows

$$\mathcal{B} \rightrightarrows \mathcal{H}, \quad \mathcal{V} \rightrightarrows \mathcal{P},$$

correspond to the functors $t, b : \mathcal{A} \rightrightarrows \mathcal{O}$.

Remark 1.3. The *transpose* of a double category $\mathcal{T}$ is the double category $\mathcal{T}^t$ with the same boxes and points as $\mathcal{T}$ but interchanging the rôles of the horizontal and vertical categories. A box $B \in \mathcal{B}$ is denoted B^t when regarded in $\mathcal{T}^t$. This remark allows to deduce some "horizontal" statements from "vertical" ones (or vice versa), by passing to the transpose double category.

1.2. **Examples.**

A 2-category is just a double category where all 'vertical arrows' are identities; *i.e.*, where every element of $\mathcal{V}$ is an identity. In this case the elements of $\mathcal{H}$ are the *morphisms* of the 2-category, while the elements of $\mathcal{B}$ are the *2-cells*.

Let $\mathcal{C}$ be a small category. Then the class of all square diagrams in $\mathcal{C}$, that is all the diagrams

$$\begin{array}{ccc} X & \xrightarrow{x} & Y \\ {\scriptstyle f}\Big\downarrow & & \Big\downarrow{\scriptstyle g} \\ Z & \xrightarrow{y} & W \end{array}$$

is a double category (without assuming commutativity of the diagram!).

More examples arise considering all commutative diagrams, or all diagrams whose vertical, respectively horizontal, arrows live in a fixed subcategory. For instance, given a group Σ and two subgroups F and G, one can consider Σ as a category with only one object, and then the double category whose vertical, respectively horizontal, arrows live in F, respectively in G.

A particular case of the preceding remark is the following example, which shows a connection between double categories and some constructions in Hopf algebra theory.

Example 1.4. Let $\triangleleft : G \times F \to G$, $\triangleright : G \times F \to F$, be a matched pair of finite groups. The notation explained in [Mj2] (see also [DVVV], [T2], [AN, Appendix]) coincides with the pictorial representation of the double category defined in what follows.

Take as $\mathcal{P}$ a set with a single element: $\mathcal{P} := \{*\}$. Put also $\mathcal{B} := G \times F$, $\mathcal{H} := G$ and $\mathcal{V} := F$. We have a double category

$$\begin{array}{ccc} G \times F & \rightrightarrows & G \\ \downdownarrows & & \downdownarrows \\ F & \rightrightarrows & \mathcal{P}, \end{array}$$

defined as follows:

$F \rightrightarrows \mathcal{P}$ and $G \rightrightarrows \mathcal{P}$ are the groupoids associated to the group structure on F and G, respectively. The groupoids $G \times F \rightrightarrows G$ and $G \times F \rightrightarrows F$ are the ones corresponding to the actions $\triangleleft$ and $\triangleright$, respectively; see [R, Example 1.2.a]. More precisely, we have:

- $G \times F \rightrightarrows G$ is the category whose objects are the elements of G, the arrows are the elements of $G \times F$ and the source and target maps are defined, respectively, by

$$t := \triangleleft : G \times F \to G, \qquad b := p_1 : G \times F \to G.$$

  The composition $m : (G \times F)\,{}_t\!\times_b (G \times F) \to G \times F$ and the identity map $\mathrm{id} : G \to G \times F$, are determined, respectively, by

$$(g, x).(h, y) := (g, xy), \qquad \mathrm{id}(g) = (g, 1),$$

  for all $g, h \in G$, $x, y \in F$, such that $g \triangleleft x = h$. The inverse map is defined as $(g, x)^{-1} = (g \triangleleft x, x^{-1})$.

- $G \times F \rightrightarrows F$ is the category whose objects are the elements of F, the arrows are the elements of $G \times F$ and the structure maps are defined by

$$r := \triangleright : G \times F \to F, \qquad l := p_2 : G \times F \to F,$$

and composition $m : (G \times F)\, {}_t\!\times_b (G \times F) \to G \times F$ and identity $\mathrm{id} : G \to G \times F$, are determined by

$$(g, x).(h, y) := (hg, x), \qquad \mathrm{id}(x) = (1, x),$$

for all $g, h \in G$, $x, y \in F$, such that $g \triangleright x = y$. The inverse map is given in this case by $(g, x)^{-1} = (g^{-1}, g \triangleright x)$.

1.3. **Basic properties.**

In this section we shall prove some basic properties of double categories and introduce some terminology that will be of use in later sections.

For an element $A \in \mathcal{B}$, we shall use the notation A^h (respectively, A^v) for the horizontal (respectively, vertical) inverse of A, provided they exist; these are defined respectively by the relations

$$AA^h = \mathbf{id}\, l(A), \quad A^h A = \mathbf{id}\, r(A), \qquad \genfrac{}{}{0pt}{}{A}{A^v} = \mathbf{id}\, t(A), \quad \genfrac{}{}{0pt}{}{A^v}{A} = \mathbf{id}\, b(A).$$

Lemma 1.5. *Let $\mathcal{T}$ be a double category and let $A \in \mathcal{B}$. Suppose that $A = l \underset{b}{\overset{t}{\square}} r$ is invertible with respect to horizontal composition. Then $t = t(A), b = b(A) \in \mathcal{H}$ are invertible and we have*

$$A^h = r \underset{b^{-1}}{\overset{t^{-1}}{\square}} l\,.$$

Similarly, if A is invertible with respect to vertical composition, then $l = l(A), r = r(A) \in \mathcal{V}$ are invertible and we have

$$A^v = l^{-1} \underset{t}{\overset{b}{\square}} r^{-1}.$$

Proof. It follows from (1.1), (1.2) and Axiom 4. $\square$

Remark 1.6. Suppose that $g \in \mathcal{V}$ is invertible. It follows from axioms 4, 5 and 6, that $\mathbf{id}\, g$ is vertically invertible and $(\mathbf{id}\, g)^v = \mathbf{id}\, g^{-1}$. Also, if $x \in \mathcal{H}$ is invertible, then $\mathbf{id}\, x$ is horizontally invertible and $(\mathbf{id}\, x)^h = \mathbf{id}\, x^{-1}$.

Lemma 1.7. *Let $\mathcal{T}$ be a double category and let $X, R \in \mathcal{B}$.*

(i) Suppose that $\dfrac{X}{R}$, and that X and R are horizontally invertible. Then $\genfrac{}{}{0pt}{}{X}{R}$ is horizontally invertible, $\dfrac{X^h}{R^h}$ and $\genfrac{}{}{0pt}{}{X^h}{R^h} = \left\{ \genfrac{}{}{0pt}{}{X}{R} \right\}^h$.

(ii) Suppose that $X|R$, and that X and R are vertically invertible. Then XR is vertically invertible, $X^v|R^v$ and $X^vR^v = \{XR\}^v$.

Proof. We prove part (i), part (ii) being entirely similar; it can also be deduced from part (i) by going to the transpose double category. It is clear that $\dfrac{X^h}{R^h}$. On the other hand, using the interchange law, we compute

$$\begin{Bmatrix} X^h \\ R^h \end{Bmatrix} \begin{Bmatrix} X \\ R \end{Bmatrix} = \begin{Bmatrix} X^h X \\ R^h R \end{Bmatrix} = \begin{Bmatrix} \mathbf{id}\, r(X) \\ \mathbf{id}\, r(R) \end{Bmatrix} = \mathbf{id}\, r \begin{Bmatrix} X \\ R \end{Bmatrix},$$

by axioms 6 and 2. A similar computation shows that $\begin{Bmatrix} X \\ R \end{Bmatrix} \begin{Bmatrix} X^h \\ R^h \end{Bmatrix} = \mathbf{id}\, l \begin{Bmatrix} X \\ R \end{Bmatrix}$. This proves the lemma. □

Lemma 1.8. *Let $A \in \mathcal{B}$ such that A is horizontally and vertically invertible. Assume in addition that A^h is vertically invertible and A^v is horizontally invertible. Then $(A^h)^v = (A^v)^h$.*

We shall use the notation $A^{-1} := (A^h)^v = (A^v)^h$; thus $A^{-1} = r^{-1} \underset{t^{-1}}{\overset{b^{-1}}{\square}} l^{-1}$.

Proof. We have $\begin{array}{c|c} (A^v)^h & A^v \\ \hline A^h & A \end{array}$ and also $\begin{array}{c|c} (A^h)^v & A^v \\ \hline A^h & A \end{array}$. Using the axioms, we compute

$$\begin{Bmatrix} (A^v)^h & A^v \\ A^h & A \end{Bmatrix} = \begin{matrix} \{(A^v)^h A^v\} \\ \{A^h A\} \end{matrix} = \begin{matrix} \mathbf{id}\, r(A)^{-1} \\ \mathbf{id}\, r(A) \end{matrix} = \mathbf{id}\, (r(A)^{-1} r(A)) = \Theta_{br(A)}.$$

On the other hand, we have

$$\begin{Bmatrix} (A^h)^v & A^v \\ A^h & A \end{Bmatrix} = \begin{Bmatrix} (A^h)^v \\ A^h \end{Bmatrix} \begin{Bmatrix} A^v \\ A \end{Bmatrix} = \mathbf{id} b(A)^{-1} \mathbf{id} b(A) = \mathbf{id}(b(A)^{-1} b(A)) = \Theta_{rb(A)}.$$

Hence we get $\begin{Bmatrix} (A^v)^h & A^v \\ A^h & A \end{Bmatrix} = \begin{Bmatrix} (A^h)^v & A^v \\ A^h & A \end{Bmatrix}$. The horizontal cancellation of $\begin{Bmatrix} A^v \\ A \end{Bmatrix}$, which is licit after Remark 1.6, and the vertical cancellation of A^h imply the desired identity. □

Remark 1.9. As pointed out by the referee, the basic properties in Subsection 1.3 can also be seen as consequences of Lemma 1.2. For instance, Lemma 1.5 becomes immediate using the fact that r and l are functors $(\mathcal{B} \rightrightarrows \mathcal{H}) \underset{l}{\overset{r}{\rightrightarrows}} (\mathcal{V} \rightrightarrows \mathcal{P})$, and similarly for t and b. Also, Remark 1.6 is a consequence of the fact that $\mathbf{id} : \mathcal{H} \to \mathcal{B}$ and $\mathbf{id} : \mathcal{V} \to \mathcal{B}$ are functors. The proofs of Lemmas 1.7 and 1.8 can also be done using functoriality of the composition map.

1.4. Double groupoids.

A *double groupoid* [E, BS] is a double category such that all the four component categories are groupoids. Note that a double groupoid is the same thing as a groupoid object in the full subcategory **Grpd** of **Cat** whose objects are groupoids. Lemma 1.5 implies that a double category

$$\begin{matrix} \mathcal{B} & \rightrightarrows & \mathcal{H} \\ \downdownarrows & & \downdownarrows \\ \mathcal{V} & \rightrightarrows & \mathcal{P} \end{matrix}$$

is a double groupoid if and only if the component categories $\mathcal{B} \rightrightarrows \mathcal{H}$ and $\mathcal{B} \rightrightarrows \mathcal{V}$ are groupoids.

The transpose of a double groupoid is a double groupoid.

We next include some helpful technical results on general double groupoids.

Lemma 1.10. *Let $\mathcal{T}$ be a double groupoid, and let $A, B, C \in \mathcal{B}$. The following are equivalent:*

(i) $A|B|C$ and $ABC = \mathbf{id}\, t(ABC)$;

(ii) there exist $U, V \in \mathcal{B}$ such that

$$\begin{array}{c|c|c} A & U & \\ \hline & V & C \end{array}, \qquad \begin{matrix} U \\ V \end{matrix} = B, \qquad AU = \mathbf{id}\, t(AB), \qquad VC = \mathbf{id}\, b(BC);$$

(iii) there exist $U', V' \in \mathcal{B}$ such that

$$\begin{array}{c|c|c} & U' & C \\ \hline A & V' & \end{array}, \qquad \begin{matrix} U' \\ V' \end{matrix} = B, \qquad AV' = \mathbf{id}\, b(AB), \qquad U'C = \mathbf{id}\, t(BC).$$

Moreover, in (ii) and (iii) the elements U, V, U', V' are uniquely determined by A, B, C, and we have

$$\begin{array}{c|c|c} A & U & \mathbf{id}\, t(C) \\ \hline \mathbf{id}\, b(A) & V & C \end{array}, \qquad \textit{respectively} \qquad \begin{array}{c|c|c} \mathbf{id}\, t(A) & U' & C \\ \hline A & V' & \mathbf{id}\, b(C) \end{array}.$$

Proof. Let U, V as in (ii). The uniqueness of U and V follows from cancellation properties in a double groupoid. Since $VC = \mathbf{id}\, t(VC)$, $l(V) = \mathrm{id}\; lt(VC)$, and on the other hand $r(\mathbf{id}\, b(A)) = \mathrm{id}\; rb(A)$. Now,

$$lt(VC) = lt(V) = lb(U) = bl(U) = br(A) = rb(A),$$

since $A|U$. This shows that $\mathbf{id}\, b(A)|V$. Similarly, using that $AU = \mathbf{id}\, t(AU)$, we get $r(U) = \mathrm{id}\; rt(AU)$, and

$$rt(AU) = rb(AU) = rb(U) = rt(V) = tr(V) = tl(C) = lt(C).$$

Since $l(\mathbf{id}\, t(C)) = \mathrm{id}\; lt(C)$, we get $U|\mathbf{id}\, t(C)$. Thus we have $\begin{array}{c|c|c} A & U & \mathbf{id}\, t(C) \\ \hline \mathbf{id}\, b(A) & V & C \end{array}$, as claimed. The corresponding facts for U' and V' in (iii) are similarly established.

We shall show that (i) $\iff$ (ii). The proof of the equivalence of (i) and (iii) is similar and left to the reader.

(i) $\Longrightarrow$ (ii). Let $x = t(A)t(B) \in \mathcal{H}$. Note that

$$r(A^h) = l(A) = \mathrm{id}\; lt(AB) = \mathrm{id}\; l(x) = l(\mathbf{id}\, x);$$

so that $A^h|\mathbf{id}\, x$. Define $U := A^h \mathbf{id}\, x$ and $V := \begin{matrix} U^v \\ B \end{matrix}$. To see that V is well defined we compute

$$b(U^v) = t(U) = t(A^h)x = t(A)^{-1}t(A)t(B) = t(B),$$

and thus $\dfrac{U^v}{B}$. By definition, we have that $A|U$, $\dfrac{U}{V}$, $AU = \mathbf{id}\, t(AU)$ and $\begin{smallmatrix}U\\V\end{smallmatrix} = B$. Also,

$$r(V) = r(U^v)r(B) = r((A^h)^v)r(\mathbf{id}\, x)r(B) = l(A)^{-1}\,\mathrm{id}\ r(x)r(B) = l(C),$$

the last equality since $B|C$ and $ABC = \mathbf{id}\, t(ABC)$. Thus $V|C$.

We now observe that

$$r(U) = \mathrm{id}\ rt(B) = \mathrm{id}\ tr(B) = \mathrm{id}\ tl(C) = \mathrm{id}\ lt(C),$$
$$l(V) = l(U^v)l(B) = l((A^h\mathbf{id}x)^v)l(B) = l(A^{-1}\mathbf{id}x)l(B) = r(A)^{-1}l(B) = \mathrm{id}\ br(A) = \mathrm{id}\ rb(A);$$

hence $U|\mathbf{id}\, t(C)$ and $\mathbf{id}\, b(A)|V$. Thus $\begin{array}{c|c|c} A & U & \mathbf{id}\, t(C) \\ \hline \mathbf{id}\, b(A) & V & C \end{array}$. On the other hand $\mathbf{id}\, t(ABC) = ABC = \left\{\begin{smallmatrix} A & U & \mathbf{id}\, t(C) \\ \mathbf{id}\, b(A) & V & C \end{smallmatrix}\right\}$. This implies that $VC = \mathbf{id}\, t(VC)$, and part (ii) follows.

(ii) $\implies$ (i). Since $VC = \mathbf{id}\, t(VC)$, $l(V) = \mathrm{id}\ lt(VC)$. Also, since $B = \begin{smallmatrix}U\\V\end{smallmatrix}$, and $A|U$, we have $l(B) = l(U)l(V) = l(U) = r(A)$; therefore $A|B$. Similarly, the assumptions $C|V$ and $AU = \mathbf{id}\, t(AU)$ imply that $r(B) = r(U)r(V) = r(V) = l(C)$; hence $B|C$.

Finally, the interchange law gives $ABC = \left\{\begin{smallmatrix} A & U & \mathbf{id}\, t(C) \\ \mathbf{id}\, b(A) & V & C \end{smallmatrix}\right\} = \mathbf{id}\, t(ABC)$, and (i) follows. □

The following lemma is dual to Lemma 1.10. Its proof is left to the reader.

Lemma 1.11. *Let $\mathcal{T}$ be a double groupoid, and let $A, B, C \in \mathcal{B}$. The following are equivalent:*

(i) $\begin{array}{c} A \\ \hline B \\ \hline C \end{array}$ *and* $\begin{smallmatrix}A\\B\\C\end{smallmatrix} = \mathbf{id}\, l\begin{pmatrix}A\\B\\C\end{pmatrix}$;

(ii) there exist $U, V \in \mathcal{B}$ such that

$$\begin{array}{c|c} A & \\ \hline U & V \\ \hline & C \end{array}, \qquad UV = B, \qquad \begin{smallmatrix}A\\U\end{smallmatrix} = \mathbf{id}\, l\begin{pmatrix}A\\U\end{pmatrix}, \qquad \begin{smallmatrix}V\\C\end{smallmatrix} = \mathbf{id}\, l\begin{pmatrix}V\\C\end{pmatrix};$$

(iii) there exist $U', V' \in \mathcal{B}$ such that

$$\begin{array}{c|c} & A \\ \hline U' & V' \\ \hline C & \end{array}, \qquad U'V' = B, \qquad \begin{smallmatrix}A\\V'\end{smallmatrix} = \mathbf{id}\, l\begin{pmatrix}A\\V'\end{pmatrix}, \qquad \begin{smallmatrix}U'\\C\end{smallmatrix} = \mathbf{id}\, l\begin{pmatrix}U'\\C\end{pmatrix}.$$

The elements U, V, U', V' in (ii) and (iii) are uniquely determined by A, B, C, and we have

$$\begin{array}{c|c} A & \mathbf{id}\, r(A) \\ \hline U & V \\ \hline \mathbf{id}\, l(C) & C \end{array}, \quad \textit{respectively} \quad \begin{array}{c|c} \mathbf{id}\, l(A) & A \\ \hline U' & V' \\ \hline C & \mathbf{id}\, r(C) \end{array}.$$

□

The following result is needed in the proof of Theorem 3.1.

Lemma 1.12. *The following properties hold in a double groupoid $\mathcal{T}$.*

(a) Let $A, X, Y, Z \in \mathcal{B}$ such that

$$\begin{array}{c|c|c} & X^{-1} & \\ \hline X & Y & Z \\ \hline & Z^{-1} & \end{array} \quad . \tag{1.13}$$

Then the following conditions are equivalent:

$$XYZ = A. \tag{1.14}$$

$$\begin{Bmatrix} X^{-1} \\ Y \\ Z^{-1} \end{Bmatrix} = A^{-1}. \tag{1.15}$$

(b) The collection $X = A = Z$, $Y = A^h$ satisfies (1.13)*,* (1.14) *and* (1.15)*.*

Proof. Part (b) being straightforward, we prove (a). Suppose that (1.14) holds. By assumption we have $b(X)b(Y)b(Z) = b(A)$, and $b(Y) = t(Z^{-1}) = b(Z)^{-1}$. Therefore $b(X) = b(A) = t(A^v)$. Similarly, $t(Z) = t(A) = b(A^v)$. Also, $r(X^{-1}) = l(X)^{-1} = l(A)^{-1} = l(A^v)$, and $r(A^v) = r(A)^{-1} = r(Z)^{-1} = l(Z^{-1})$. This implies that

$$\begin{array}{c|c|c} X^v & X^{-1} & A^v \\ \hline X & Y & Z \\ \hline A^v & Z^{-1} & Z^v \end{array} \quad . \tag{1.16}$$

We compute in two different ways, using the interchange law:

$$\begin{aligned} \begin{Bmatrix} X^v & X^{-1} & A^v \\ X & Y & Z \\ A^v & Z^{-1} & Z^v \end{Bmatrix} &= \begin{matrix} \{X^v X^{-1} A^v\} \\ \{XYZ\} \\ \{A^v Z^{-1} Z^v\} \end{matrix} = \begin{Bmatrix} A^v \\ A \\ A^v \end{Bmatrix} = A^v \\ &= \begin{Bmatrix} X^v \\ X \\ A^v \end{Bmatrix} \begin{Bmatrix} X^{-1} \\ Y \\ Z^{-1} \end{Bmatrix} \begin{Bmatrix} A^v \\ Z \\ Z^v \end{Bmatrix} = A^v \begin{Bmatrix} X^{-1} \\ Y \\ Z^{-1} \end{Bmatrix} A^v; \end{aligned}$$

thus $\begin{Bmatrix} X^{-1} \\ Y \\ Z^{-1} \end{Bmatrix} = (A^v)^h = A^{-1}$, as claimed.

Conversely, suppose that (1.15) holds. Then, in the transpose double category $\mathcal{T}^t$,

$$\begin{array}{c|c|c} & X^t & \\ \hline (X^t)^{-1} & Y^t & (Z^t)^{-1} \\ \hline & Z^t & \end{array}, \quad \text{and} \quad (X^t)^{-1} Y^t (Z^t)^{-1} = (A^t)^{-1};$$

thus $\begin{Bmatrix} X^t \\ Y^t \\ Z^t \end{Bmatrix} = A^t$ in $\mathcal{T}^t$ by the preceding; that is, (1.14) holds in $\mathcal{T}$. $\square$

2. Vacant double groupoids

2.1. **Definition and basic properties.**

The notion of vacant double groupoids appears in [Ma, Definition 2.11].

Definition 2.1. Let $\mathcal{T}$ be a double groupoid. We shall say that $\mathcal{T}$ is *vacant* if for any $g \in \mathcal{V}$, $x \in \mathcal{H}$ such that $r(x) = t(g)$, there is exactly one $X \in \mathcal{B}$ such that $X = \begin{matrix} x & \\ \square & g \end{matrix}$.

We give an alternative description of vacant double groupoids that we have found in the course of our research; see condition 2 below. This will be useful in Section 3.

Proposition 2.2. *Let $\mathcal{T}$ be a double groupoid. The following are equivalent.*

1. *$\mathcal{T}$ is vacant.*
2. *For all $R, S, P \in \mathcal{B}$ such that $\frac{R}{S}$ and $P| \left\{ \begin{matrix} R \\ S \end{matrix} \right\}$, there exist unique $X, Y \in \mathcal{B}$ such that $\begin{array}{c|c} X & R \\ \hline Y & S \end{array}$ and $P = \frac{X}{Y}$.*
3. *For any $f \in \mathcal{V}$, $y \in \mathcal{H}$ such that $l(y) = b(f)$, there is exactly one $Z \in \mathcal{B}$ such that $Z = \begin{matrix} f & \square \\ & y \end{matrix}$.*
4. *For all $T, U, Q \in \mathcal{B}$ such that $T|U$ and $\frac{Q}{TU}$, there exist unique $V, Z \in \mathcal{B}$ such that $\begin{array}{c|c} V & Z \\ \hline T & U \end{array}$ and $Q = VZ$.*
5. *For any $f \in \mathcal{V}$, $x \in \mathcal{H}$ such that $l(x) = t(f)$, there is exactly one $Z \in \mathcal{B}$ such that $Z = \begin{matrix} & x \\ f & \square \end{matrix}$.*
6. *For any $g \in \mathcal{V}$, $y \in \mathcal{H}$ such that $r(y) = b(g)$, there is exactly one $Z \in \mathcal{B}$ such that $Z = \begin{matrix} \square & g \\ y & \end{matrix}$.*
7. *For all $A, B, X, Y \in \mathcal{B}$ such that $XY = \frac{A}{B}$, there exist unique $U, V, R, S \in \mathcal{B}$ with*

$$\begin{array}{c|c} U & V \\ \hline R & S \end{array}, \quad UV = A, \quad RS = B, \quad \frac{U}{R} = X, \quad \frac{V}{S} = Y. \tag{2.1}$$

Note that condition (4) says that the transpose double groupoid $\mathcal{T}^t$ is vacant.

Proof. (1) $\implies$ (2). Let $R, S, P \in \mathcal{B}$ such that $\frac{R}{S}$ and $P| \left\{ \begin{matrix} R \\ S \end{matrix} \right\}$. Let $x = t(P)$, $g = l(R)$ and let X be the unique box of the form $\begin{matrix} x & \\ \square & g \end{matrix}$. Set $Y = \frac{X^v}{P}$; then clearly $\begin{array}{c|c} X & R \\ \hline Y & S \end{array}$ and $P = \frac{X}{Y}$. Furthermore, if X', Y' are boxes with these properties then clearly X' should be of the form $\overset{x}{\square}\, g$, hence $X' = X$, by the uniqueness condition in (1); *a fortiori* $Y' = Y$.

(2) $\implies$ (1). Let $g \in \mathcal{V}$, $x \in \mathcal{H}$ such that $r(x) = t(g)$. Put $P = \mathbf{id}\, x$, $R = \mathbf{id}\, g$, $S = R^v = \mathbf{id}\, g^{-1}$; by part (2), there exist unique $X, Y \in \mathcal{B}$ such that $\begin{array}{c|c} X & R \\ \hline Y & S \end{array}$ and $P = \begin{matrix} X \\ Y \end{matrix}$. Clearly, $X = \overset{x}{\square}\, g$.

Let now X' be of the form $\overset{x}{\square}\, g$. Let $Y' := \begin{matrix} (X')^v \\ P \end{matrix}$; then $\begin{array}{c|c} X' & R \\ \hline Y' & S \end{array}$ and $P = \begin{matrix} X' \\ Y' \end{matrix}$. By the uniqueness in part (2), $X = X'$ (and $Y = Y'$).

(3) $\iff$ (4). This follows from the equivalence (1) $\iff$ (2) for $\mathcal{T}^t$.

(1) $\iff$ (3). If Z is of the form $f\, \underset{y}{\square}$, then Z^{-1} is of the form $\overset{y^{-1}}{\square}\, f^{-1}$. This remark implies the desired equivalence.

The proofs of the equivalences (1) $\iff$ (5) and (1) $\iff$ (6) are analogous, using Z^h and Z^v respectively.

(5) $\iff$ (7). A bijective correspondence between the set of quadruples (U, V, R, S) satisfying (2.1) and the set $\mathfrak{C} := \left\{ U \in \mathcal{B} : U = l(A) \overset{t(X)}{\square} \right\}$, is established by assigning to each $U \in \mathfrak{C}$ the quadruple $\left(U, U^h A, \begin{matrix} U^v \\ X \end{matrix}, \begin{matrix} U^{-1} A^v \\ Y \end{matrix} \right)$. The definition of $\mathfrak{C}$ guarantees that this map is well defined. Moreover, this defines a bijection, whose inverse is determined by the law $(U, V, R, S) \mapsto U$. $\square$

Example 2.3. (i) The double category attached to a matched pair of finite groups as in in Example 1.4 is a vacant double groupoid.

(ii) Let $\mathcal{G}$ be any finite groupoid and consider the double category $\mathcal{T}$ of commuting square diagrams with vertical arrows in $\mathcal{G}$ but with horizontal arrows only identities. Then $\mathcal{T}$ is a vacant double groupoid.

We now complete the information in Lemma 1.12.

Lemma 2.4. *Let $\mathcal{T}$ be a vacant double groupoid and let $A \in \mathcal{B}$. There exists exactly one collection $X, Y, Z \in \mathcal{B}$ such that* (1.13), (1.14) *and* (1.15) *hold, namely $X = Z = A$, $Y = A^h$.*

Proof. By Lemma 1.12, the collection $X = Z = A$, $Y = A^h$ satisfies (1.13), (1.14) and (1.15). On the other hand, suppose that $X, Y, Z \in \mathcal{B}$ is any collection satisfying (1.13), (1.14) and (1.15). Then $l(X) = l(A)$ and $b(X)^{-1} = t(X^{-1}) = t(A^{-1}) = b(A)^{-1}$, hence $X = A$. Similarly, $Z = A$ and then necessarily $Y = A^h$. $\square$

We list several technical facts about vacant double groupoids that are needed later in this paper. The straightforward proof is left to the reader.

Proposition 2.5. *Let $\mathcal{T}$ be a vacant double groupoid, $C \in \mathcal{B}$.*

(i) *If a horizontal (resp. vertical) side of C is an identity, then C is a vertical (resp. horizontal) identity.*

(ii) *The set of pairs of boxes (A, B) such that* $\begin{array}{c|c} A & B \\ \hline C & \end{array}$, $AB = \operatorname{id} t(AB)$ *and* $\begin{matrix} A \\ C \end{matrix} = \operatorname{id} l \begin{pmatrix} A \\ C \end{pmatrix}$ *is either*

$$\begin{cases} \emptyset, & \textit{if } C \textit{ is not a horizontal identity,} \\ \{(\Theta_P, \operatorname{id} x) :\ x \in \mathcal{H},\ l(x) = P\}, & \textit{if } C = \operatorname{id} g,\ P = t(g). \end{cases}$$

(iii) *The set of pairs of boxes (A, B) such that* $\begin{array}{c|c} & C \\ \hline A & B \end{array}$, $AB = \operatorname{id} b(AB)$ *and* $\begin{matrix} C \\ B \end{matrix} = \operatorname{id} r \begin{pmatrix} C \\ B \end{pmatrix}$ *is either*

$$\begin{cases} \emptyset, & \textit{if } C \textit{ is not a horizontal identity,} \\ \{(\operatorname{id} h, \Theta_Q) :\ h \in \mathcal{V},\ r(h) = Q\}, & \textit{if } C = \operatorname{id} g,\ Q = b(g). \end{cases}$$

(iv) *The set of pairs of boxes (A, B) such that* $\begin{array}{c|c} A & B \\ \hline B^{-1} & \end{array}$, $AB = C$ *is either*

$$\begin{cases} \emptyset, & \textit{if } C \textit{ is not a horizontal identity,} \\ \left\{\left(\begin{matrix} & z \\ g & \square \end{matrix}\ ,\ \begin{matrix} z^{-1} & \\ \square & g \end{matrix} \right) :\ z \in \mathcal{H},\ l(z) = t(g)\right\}, & \textit{if } C = \operatorname{id} g. \end{cases}$$

(v) *The set of pairs of boxes (A, B) such that* $\begin{array}{c|c} & A^{-1} \\ \hline A & B \end{array}$, $AB = C$ *is either*

$$\begin{cases} \emptyset, & \textit{if } C \textit{ is not a horizontal identity,} \\ \left\{\left(\begin{matrix} g & \square \\ & w^{-1} \end{matrix}\ ,\ \begin{matrix} \square & g \\ w & \end{matrix} \right) :\ w \in \mathcal{H},\ r(w) = b(g)\right\}, & \textit{if } C = \operatorname{id} g. \end{cases}$$

□

2.2. Matched pairs of groupoids.

We shall now give a characterization of vacant double groupoids in terms of matched pairs of groupoids. This characterization is due to Mackenzie [Ma].

Definition 2.6. Let $\mathcal{G}$ be a groupoid with base $\mathcal{P}$ and source and target maps $s, e : \mathcal{G} \rightrightarrows \mathcal{P}$. Let also $p : \mathcal{E} \to \mathcal{P}$ be a map. A *left action* of $\mathcal{G}$ on p is a map $\triangleright : \mathcal{G}\,{}_e\!\times_p \mathcal{E} \to \mathcal{E}$ such that

(2.2) $p(g\triangleright x) = s(g)$,

(2.3) $g\triangleright(h\triangleright x) = gh\triangleright x$,

(2.4) $\mathbf{id}\, p(x) \triangleright x = x$,

for all $g, h \in \mathcal{G}$, $x \in \mathcal{E}$ composable in the appropiate sense.

Hence, if $\mathcal{E}_b := p^{-1}(b)$, then the action of $g \in \mathcal{G}$ is an isomorphism $g\triangleright\_\_ : \mathcal{E}_{t(g)} \to \mathcal{E}_{s(g)}$. This somewhat unpleasant notation is originated by our choice of juxtaposition to denote composition.

Given actions of $\mathcal{G}$ on $p : \mathcal{E} \to \mathcal{P}$ and $p' : \mathcal{E}' \to \mathcal{P}$, a map $\phi : \mathcal{E} \to \mathcal{E}'$ is said to *intertwine* the actions if $p = \phi p'$ and $\phi(g\triangleright x) = g\triangleright\phi(x)$, for all $g \in \mathcal{G}$, $x \in \mathcal{E}$ such that $e(g) = p(x)$.

An action is *trivial* if there exists a set X such that $\mathcal{E} = \mathcal{P} \times X$, p is the first projection and $g\triangleright(e(g), x) = (s(g), x)$ for all $x \in X$, $g \in \mathcal{G}$.

Similarly, a *right action* of $\mathcal{G}$ on p is a map $\triangleleft : \mathcal{E}\,{}_p\!\times_s \mathcal{G} \to \mathcal{E}$ such that

(2.5) $p(x\triangleleft g) = e(g)$,

(2.6) $(x\triangleleft g)\triangleleft h = x\triangleleft gh$,

(2.7) $x\triangleleft \mathbf{id}\, p(x) = x$,

for all $g, h \in \mathcal{G}$, $x \in \mathcal{E}$ composable in the appropiate sense.

It is convenient to set the following notation: a *wide subgroupoid* of a groupoid $\mathcal{D}$ is a groupoid $\mathcal{V}$ provided with a functor $F : \mathcal{V} \to \mathcal{D}$ which is the identity on the objects, and induces inclusions on the hom sets. In other words, it has the same base, and (perhaps) less arrows.

The next two definitions generalize corresponding notions for finite groups, *cf.* [Ma, Definition 2.14].

Definition 2.7. A *matched pair of groupoids* is a pair of groupoids $t, b : \mathcal{V} \rightrightarrows \mathcal{P}$, $l, r : \mathcal{H} \rightrightarrows \mathcal{P}$, on the same base $\mathcal{P}$, endowed with a left action $\triangleright : \mathcal{H}\,{}_r\!\times_t \mathcal{V} \to \mathcal{V}$ of $\mathcal{H}$ on $t : \mathcal{V} \to \mathcal{P}$, and a right action $\triangleleft : \mathcal{H}\,{}_r\!\times_t \mathcal{V} \to \mathcal{H}$ of $\mathcal{V}$ on $r : \mathcal{H} \to \mathcal{P}$, satisfying

(2.8) $b(x\triangleright g) = l(x\triangleleft g)$,

(2.9) $x\triangleright fg = (x\triangleright f)((x\triangleleft f)\triangleright g)$,

(2.10) $xy\triangleleft g = (x\triangleleft(y\triangleright g))(y\triangleleft g)$,

for all $f, g \in \mathcal{V}$, $x, y \in \mathcal{H}$ such that the compositions are possible.

We claim that

$$x\triangleright\mathbf{id}\, r(x) = \mathbf{id}\, l(x), \qquad \text{for all } x \in \mathcal{H}, \tag{2.11}$$

$$\mathbf{id}\, t(g)\triangleleft g = \mathbf{id}\, b(g), \qquad \text{for all } g \in \mathcal{V}. \tag{2.12}$$

Indeed, $x\triangleright\mathbf{id}\, r(x) = x\triangleright(\mathbf{id}\, r(x)\mathbf{id}\, r(x)) = (x\triangleright\mathbf{id}\, r(x))\,((x\triangleleft\mathbf{id}\, r(x))\triangleright\mathbf{id}\, r(x)) = (x\triangleright\mathbf{id}\, r(x))\,(x\triangleright\mathbf{id}\, r(x))$, by (2.9) and (2.7). Since $t(x\triangleright\mathbf{id}\, r(x)) = l(x)$ by (2.2), (2.11) follows. Similarly (2.12) follows from (2.10), (2.4) and (2.5).

Definition 2.8. Let $\mathcal{D} \rightrightarrows \mathcal{P}$ be a groupoid. An *exact factorization* of $\mathcal{D}$ is a pair of wide subgroupoids $\mathcal{V}$, $\mathcal{H}$, such that for any $\alpha \in \mathcal{D}$, there exist unique $f \in \mathcal{V}$, $y \in \mathcal{H}$, such that $\alpha = fy$; that is, if the multiplication map $\mathcal{V}\,{}_b\!\times_l \mathcal{H} \to \mathcal{D}$ is a bijection.

Proposition 2.9. [Ma, Theorems 2.10 and 2.15] *The following notions are equivalent.*

(1) *Matched pairs of groupoids.*
(2) *Groupoids with an exact factorization.*
(3) *Vacant double groupoids.*

If $\mathcal{V}$, $\mathcal{H}$ is a matched pair of groupoids, the groupoid arising in (2) will be denoted $\mathcal{D} = \mathcal{V} \bowtie \mathcal{H}$ and called the *diagonal* groupoid.

Proof. (1) $\implies$ (2). Let $\triangleright : \mathcal{H}\,{}_r\!\times_t \mathcal{V} \to \mathcal{V}$, $\triangleleft : \mathcal{H}\,{}_r\!\times_t \mathcal{V} \to \mathcal{H}$ be a matched pair of groupoids on the same base $\mathcal{P}$. Let $\mathcal{D}$ be the groupoid on the base $\mathcal{P}$, with arrows $\mathcal{V}\,{}_b\!\times_l \mathcal{H}$, source and target maps $\alpha, \beta : \mathcal{V}\,{}_b\!\times_l \mathcal{H} \rightrightarrows \mathcal{P}$ given by $\alpha(f, y) = t(f)$, $\beta(f, y) = r(y)$, and composition defined by the rule: $(f, y)(h, z) = (f(y \triangleright h), (y \triangleleft h)z)$. We shall denote the arrow corresponding to $(f, y) \in \mathcal{V}\,{}_b\!\times_l \mathcal{H}$ by $\begin{array}{c|c} f & \\ \hline & y \end{array}$. A straightforward verification shows that $\mathcal{D}$ is indeed a well-defined groupoid. We identify $\mathcal{H}$, resp. $\mathcal{V}$, with the arrows of the form $\begin{array}{c|c} \mathbf{id}\, l(y) & \\ \hline & y \end{array}$, resp. $\begin{array}{c|c} f & \\ \hline & \mathbf{id}\, b(f) \end{array}$. Then the pair $\mathcal{V}, \mathcal{H}$ is an exact factorization of $\mathcal{D}$.

(2) $\implies$ (1). Let $\mathcal{D}$ be a groupoid and let $\mathcal{V}, \mathcal{H}$ be an exact factorization of $\mathcal{D}$. Define $\triangleright : \mathcal{H}\,{}_r\!\times_t \mathcal{V} \to \mathcal{V}$, and $\triangleleft : \mathcal{H}\,{}_r\!\times_t \mathcal{V} \to \mathcal{H}$, by the formulas $xg = (x \triangleright g)(x \triangleleft g)$, $(x, g) \in \mathcal{H}\,{}_r\!\times_t \mathcal{V}$. The uniqueness of the factorization implies that $\triangleright$, $\triangleleft$ are well-defined. It is not difficult to see that these make $\mathcal{V}$, $\mathcal{H}$ into a matched pair of groupoids.

(3) $\implies$ (1). Let $\mathcal{T}$ be a vacant double groupoid. Given $g \in \mathcal{V}$, $x \in \mathcal{H}$ such that $r(x) = t(g)$, we set $x \triangleleft g := b(X)$, $x \triangleright g := l(X)$ where $X \in \mathcal{B}$ is the unique box such that $X = \begin{array}{cc} x & \\ \boxed{} & g \end{array}$. That is, $X = \begin{array}{ccc} & x & \\ x \triangleright g & \boxed{} & g \\ & x \triangleleft g & \end{array}$. The uniqueness gives at once that $\mathcal{V}$, $\mathcal{H}$, together with $\triangleleft$, $\triangleright$, is a matched pair of groupoids.

(1) $\implies$ (3). Let $\mathcal{V}$, $\mathcal{H}$ be groupoids with the same set of objects $\mathcal{P}$, endowed with functions $\triangleright$, $\triangleleft$. Let $\mathcal{B} := \mathcal{H}\,{}_r\!\times_t \mathcal{V}$; we denote $X = (x, g) \in \mathcal{H}\,{}_r\!\times_t \mathcal{V}$ by $X = \begin{array}{ccc} & x & \\ x \triangleright g & \boxed{} & g \\ & x \triangleleft g & \end{array}$. We leave to the reader the verification that this gives rise to a vacant double groupoid. $\square$

Remark 2.10. Let $\mathcal{V}$, $\mathcal{H}$, be an exact factorization of a groupoid $\mathcal{D}$ and let $\mathcal{T}$ be the corresponding vacant double groupoid. Then $\mathcal{H}$, $\mathcal{V}$, is also an exact factorization of $\mathcal{D}$; the corresponding vacant double groupoid is $\mathcal{T}^t$.

2.3. **Structure of vacant double groupoids.**

2.3.1. *Structure of groupoids.* We first briefly recall the well-known structure of groupoids. Let $\mathcal{G} \rightrightarrows \mathcal{P}$ be a groupoid. We shall denote by $\mathcal{G}(x, y)$ the set of arrows from x to y; the set $\mathcal{G}(x, x)$ will be denoted by $\mathcal{G}(x)$.

There are two basic examples of groupoids:

- A group G, considered as the set of arrows of a category with a single object.
- An equivalence relation R on a set $\mathcal{P}$; s and e are respectively the first and the second projection, and the composition is given by $(x, y)(y, v) = (x, v)$. We shall denote by $\mathcal{P}^2$ the equivalence relation where all the elements of $\mathcal{P}$ are related; $\mathcal{P}^2$ is called the *coarse* groupoid on $\mathcal{P}$.

If $\mathcal{G} \rightrightarrows \mathcal{P}$ and $\mathcal{G}' \rightrightarrows \mathcal{P}'$ are two groupoids, then two basic operations are:

- The disjoint union $\mathcal{G} \coprod \mathcal{G}'$, a groupoid on the base $\mathcal{P} \coprod \mathcal{P}'$.
- The direct product $\mathcal{G} \times \mathcal{G}'$, a groupoid on the base $\mathcal{P} \times \mathcal{P}'$.

The structure of any groupoid can be described with the help of these basic examples and operations. Namely, let $\mathcal{G} \rightrightarrows \mathcal{P}$ be any groupoid and define an equivalence relation on $\mathcal{P}$ by $x \sim y$ iff $\mathcal{G}(x, y) \neq \emptyset$. We say that $\mathcal{G}$ is *connected* if $x \sim y$ for all $x, y \in \mathcal{P}$. The opposite case is a *group bundle*: this is a groupoid such that $x \sim y$ implies $x = y$. The trivial group bundle is $\mathcal{P} \rightrightarrows \mathcal{P}$, $s = e = \mathrm{id}$.

- If $\mathcal{G}$ is connected, then $\mathcal{G} \simeq \mathcal{G}(x) \times \mathcal{P}^2$, where x is any element of $\mathcal{P}$.

Indeed, choose an arrow $\tau_y \in \mathcal{G}(x, y)$ and define $F : \mathcal{G} \to \mathcal{G}(x) \times \mathcal{P}^2$, $F(\alpha) = (\tau_z^{-1}\alpha\tau_y, (y, z))$ if $y = s(\alpha)$, $z = t(\alpha)$. Then F is an isomorphism of groupoids.

- In general, let P be an equivalence class of $\sim$ and let $\mathcal{G}_P$ be corresponding groupoid on the base P; so that $\mathcal{G}_P(x, y) = \mathcal{G}(x, y)$, for all $x, y \in P$. Then $\mathcal{G} \simeq \coprod_{P \in \mathcal{P}/\sim} \mathcal{G}_P$.

These remarks provide the general structure of a groupoid.

2.3.2. *Structure of wide subgroupoids.* We now give a description of a wide subgroupoid of a connected groupoid in group-theoretical terms. We fix a finite non-empty set $\mathcal{P}$, a point $O \in \mathcal{P}$, and a finite group $D =: D(O)$. Let $\mathcal{D} = D(O) \times \mathcal{P}^2$ be the corresponding connected groupoid.

Lemma 2.11. *There is a bijective correspondence between the following data:*

(1) *Wide subgroupoids $\mathcal{H}$ of $\mathcal{D}$;*
(2) *collections $(\sim_H, (H_P)_{P \in \mathcal{P}}, (\overline{d_{PQ}})_{P \sim_H Q})$, where*
  - *$\sim_H$ is an equivalence relation on $\mathcal{P}$,*
  - *H_P is a subgroup of D, $P \in \mathcal{P}$,*
  - *$\overline{d_{PQ}}$ is an element of $H_P \backslash D / H_Q$ such that for any representative d_{PQ}, the following hold:*

$$d_{PQ} H_Q = H_P d_{PQ}, \quad \text{if} \quad P \sim_H Q, \tag{2.13}$$

$$d_{PQ} d_{QR} \in H_P d_{PR}, \quad \text{if} \quad P \sim_H Q \sim_H R, \tag{2.14}$$

$$d_{PP} \in H_P, \quad P \in \mathcal{P}. \tag{2.15}$$

Proof. For each $P \in \mathcal{P}$, fix $\tau_P \in \mathcal{D}(O,P)$. The correspondence is not natural; it depends on the choice of the family $(\tau_P)_{P\in\mathcal{P}}$.

(1) $\implies$ (2). The equivalence relation $\sim_H$ is defined by $P \sim_H Q$ iff $\mathcal{H}(P,Q) \neq \emptyset$. Given $P \in \mathcal{P}$, the subgroup H_P is defined by $H_P := \tau_P \mathcal{H}(P) \tau_P^{-1}$. If $P \sim_H Q$, choose $h_{PQ} \in \mathcal{H}(P,Q)$ and set $d_{PQ} := \tau_P h_{PQ} \tau_Q^{-1}$. Then

$$H_Q = \tau_Q \mathcal{H}(Q) \tau_Q^{-1} = \tau_Q h_{PQ}^{-1} \mathcal{H}(P) h_{PQ} \tau_Q^{-1} = d_{PQ}^{-1} \tau_P \mathcal{H}(P) \tau_P^{-1} d_{PQ} = d_{PQ}^{-1} H_P d_{PQ}.$$

This proves that condition (2.13) is satisfied. Conditions (2.14) and (2.15) are similarly verified.

If we choose another element $\widetilde{h_{PQ}} \in \mathcal{H}(P,Q)$ then $\widetilde{d_{PQ}} := \tau_P \widetilde{h_{PQ}} \tau_Q^{-1}$ has the same class in $H_P \backslash D / H_Q$ as d_{PQ}. Clearly, if (2.13), (2.14), (2.15) are true for some choice of representatives of $\overline{d_{PQ}}$ then they are true for any choice. This finishes the proof of the first implication.

(2) $\implies$ (1). Define a wide subgroupoid $\mathcal{H}$ of $\mathcal{D}$ as follows: if $P, Q \in \mathcal{P}$, then

$$\mathcal{D}(P,Q) \supseteq \mathcal{H}(P,Q) := \begin{cases} \emptyset, & \text{if } \quad P \not\sim_H Q; \\ \tau_P^{-1} H_P d_{PQ} \tau_Q, & \text{if } \quad P \sim_H Q. \end{cases}$$

We have to check that $\mathcal{H}$ is stable under composition, inverses and identities. First, if $P \sim_H Q \sim_H R$ then

$$\begin{aligned} \mathcal{H}(P,Q)\mathcal{H}(Q,R) &= (\tau_P^{-1} H_P d_{PQ} \tau_Q)(\tau_Q^{-1} H_Q d_{QR} \tau_R) \\ &= \tau_P^{-1} H_P d_{PQ} d_{QR} \tau_R \\ &= \tau_P^{-1} H_P d_{PR} \tau_R, \end{aligned}$$

where the first equality is by definition, the second by (2.13) and the third by (2.14). Next, if $P \sim_H Q$, then

$$\mathcal{H}(P,Q)^{-1} = \tau_Q^{-1} d_{PQ}^{-1} H_P \tau_P = \tau_Q^{-1} H_Q d_{PQ}^{-1} \tau_P = \tau_Q^{-1} H_Q d_{QP} \tau_P = \mathcal{H}(Q,P),$$

using several times (2.13), (2.14) and (2.15). Similarly, $\mathrm{id}_P \in \mathcal{H}(P,P)$ by (2.15). The second implication is proved. $\square$

2.3.3. *Double equivalence relations.* Let $\mathcal{P}$ be a finite non-empty set. Let $\sim_H$, $\sim_V$ be two equivalence relations on $\mathcal{P}$. Let $\mathcal{V} \rightrightarrows \mathcal{P}$ be the groupoid defined by the relation $\sim_V$, let $\begin{matrix}\mathcal{H}\\ \downdownarrows \\ \mathcal{P}\end{matrix}$ be the groupoid defined by the relation $\sim_H$, and let

$$\mathcal{B} = \left\{ \begin{pmatrix} P & Q \\ R & S \end{pmatrix} \in \mathcal{P}^{2\times 2} : P \sim_H Q,\ P \sim_V R,\ R \sim_H S,\ Q \sim_V S \right\}.$$

Let $\sim_D$ be the relation defined as follows: $P \sim_D Q$ if there exists $R \in \mathcal{P}$ such that $P \sim_H R$, $R \sim_V Q$. We shall sometimes denote this as $\begin{matrix} P & — & R \\ & & | \\ & & Q \end{matrix}$.

Lemma 2.12. *(a). The maps* $\mathcal{B} \rightrightarrows \mathcal{H}$, $\begin{matrix}\mathcal{B}\\ \downdownarrows \\ \mathcal{V}\end{matrix}$ *given by* $\begin{pmatrix}P & Q\\ R & S\end{pmatrix} \rightrightarrows \begin{matrix}\begin{pmatrix}P & Q\end{pmatrix}\\ \begin{pmatrix}R & S\end{pmatrix}\end{matrix}$, $\begin{pmatrix}P & Q\\ R & S\end{pmatrix} \rightrightarrows \begin{pmatrix}P\\ R\end{pmatrix}$, $\begin{pmatrix}Q\\ S\end{pmatrix}$, *with evident composition, define a double groupoid* $\begin{matrix}\mathcal{B} & \rightrightarrows & \mathcal{H}\\ \downdownarrows & & \downdownarrows\\ \mathcal{V} & \rightrightarrows & \mathcal{P}\end{matrix}$*; it will be called a* double equivalence relation.

(b). The relation $\sim_D$ *is an equivalence relation if and only if it is symmetric.*

(c). The double equivalence relation is vacant if and only if $\sim_D$ *is an equivalence relation and the following condition holds:*

(2.16) *If* $R, S \in \mathcal{P}$, $R \sim_H S$, $R \sim_V S$, *then* $R = S$.

(d). Let $\begin{matrix}\mathcal{B} & \rightrightarrows & \mathcal{H}\\ \downdownarrows & & \downdownarrows\\ \mathcal{V} & \rightrightarrows & \mathcal{P}\end{matrix}$ *be any* vacant *double groupoid and let* $\sim_H$, $\sim_V$ *be the equivalence relations on* $\mathcal{P}$ *defined by* $\mathcal{H}$, $\mathcal{V}$ *respectively. Then* $\sim_D$ *is an equivalence relation on* $\mathcal{P}$.

Proof. Part (a) is left to the reader.

(b). The relation $\sim_D$ is clearly reflexive. Assume that $\sim_D$ is symmetric. Let $P, Q, T \in \mathcal{P}$ such that $P \sim_D Q$, $Q \sim_D T$. Then there exist $R, S \in \mathcal{P}$ such that $\begin{matrix}P & — & R\\ & & |\\ & & Q\end{matrix}$, $\begin{matrix}Q & — & S\\ & & |\\ & & T\end{matrix}$. But then $\begin{matrix}S & — & Q\\ & & |\\ & & R\end{matrix}$, *i. e.* $S \sim_D R$. By symmetry, there exists $V \in \mathcal{P}$ such that $\begin{matrix}S & — & Q\\ | & & |\\ V & — & R\end{matrix}$. Hence $\begin{matrix}P & — & V\\ & & |\\ & & T\end{matrix}$, *i. e.* $P \sim_D T$.

(c). Assume that the double equivalence relation is vacant. If $P \sim_D Q$, there exists $R \in \mathcal{P}$ such that $\begin{matrix}P & — & R\\ & & |\\ & & Q\end{matrix}$. By vacancy, there exists $V \in \mathcal{P}$ such that $\begin{pmatrix}P & R\\ V & Q\end{pmatrix} \in \mathcal{B}$, that is, we have $\begin{matrix}P & — & R\\ | & & |\\ V & — & Q\end{matrix}$, and thus $Q \sim_D P$. It follows that $\sim_D$ is symmetric and hence an equivalence relation, by (b). Now let $R, S \in \mathcal{P}$ be such that $\begin{matrix}R & — & S\\ & & |\\ & & R\end{matrix}$. Then both $\begin{matrix}R & — & S\\ | & & |\\ R & — & R\end{matrix}$ and $\begin{matrix}R & — & S\\ | & & |\\ S & — & R\end{matrix}$ belong to $\mathcal{B}$, and by vacancy $R = S$. Thus (2.16) holds.

Conversely, assume that $\sim_D$ is an equivalence relation and (2.16) holds. Then any $\begin{matrix}P & — & R\\ & & |\\ & & Q\end{matrix}$ can be extended to a box $\begin{matrix}P & — & R\\ | & & |\\ V & — & Q\end{matrix}$ in $\mathcal{B}$, since $\sim_D$ is an equivalence relation. If also $\begin{matrix}P & — & R\\ | & & |\\ U & — & Q\end{matrix}$ then clearly $\begin{matrix}U & — & V\\ & & |\\ & & U\end{matrix}$ and hence $U = V$ by (2.16).

(d). By vacancy, the relation $\sim_D$ is symmetric, then apply (b). $\square$

Example 2.13. Let Σ be a group, let F, G be subgroups of Σ, acting on Σ respectively on the left and on the right by multiplication. Let $\sim_H$, $\sim_V$ be the equivalence relations on Σ defined by these actions. Then the corresponding $\sim_D$ is an equivalence relation, where the associated partition of Σ is that given by the double cosets FqG, $q \in \Sigma$. Moreover, (2.16) holds if and only if $G \cap gFg^{-1} = 1$ for $g \in \Sigma$. (For instance, if the orders of F and G are relatively prime).

Definition 2.14. We shall say that a double equivalence relation is *connected* if the associated relation $\sim_D$ is so; that is, if any two elements of $\mathcal{P}$ are connected by $\sim_D$.

Let r, s be natural numbers. Let $\mathbb{X}_{rs}$ be the double equivalence relation on the set $\{1, \dots, r\} \times \{1, \dots, s\}$ and with side relations

$$(i, j) \sim_H (k, l) \iff i = k, \qquad (i, j) \sim_V (k, l) \iff j = l.$$

Clearly, $\mathbb{X}_{rs}$ is a *connected* vacant equivalence relation.

Proposition 2.15. *Any finite double equivalence relation which is vacant and connected is isomorphic to* $\mathbb{X}_{rs}$.

Proof. Let $Y_1, \dots, Y_r$ be the classes of $\sim_H$ in $\mathcal{P}$, and assume that $Y_1 = \{1, \dots, s\}$. We shall define a bijection $\phi_i : Y_1 \to Y_i$, $2 \le i \le s$ and shall prove that for $j \in Y_1$ and $k \in Y_i$, $j \sim_V k$ if and only if $k = \phi_i(j)$. So, let us fix i and set $\phi = \phi_i$. Fix $a \in Y_i$. If $j \in Y_1$, by connectedness and (2.16), there exists a unique k such that $\begin{matrix} j & \\ | & \\ k & — a \end{matrix}$. Set $\phi_i(j) = k$. We claim that ϕ_i is bijective.

Indeed, assume that $\phi_i(j) = k = \phi_i(h)$. Then $\begin{matrix} j & — j \\ | & \\ k & \end{matrix}$ and $\begin{matrix} h & — j \\ | & \\ k & \end{matrix}$; thus $j = h$ and ϕ is injective. Also, let $k \in Y_i$. Then $\begin{matrix} 1 & \\ | & \\ \phi(1) & — k \end{matrix}$, hence there exists a unique $j \in Y_1$ such that $\begin{matrix} 1 & — & j \\ | & & | \\ \phi(1) & — & k \end{matrix}$; thus $\phi(j) = k$ and ϕ is surjective. $\square$

2.3.4. *Structure of vacant double groupoids.* Let $\mathcal{T}_1 = \begin{matrix} \mathcal{B}_1 & \rightrightarrows & \mathcal{H}_1 \\ \downdownarrows & & \downdownarrows \\ \mathcal{V}_1 & \rightrightarrows & \mathcal{P}_1 \end{matrix}$ and $\mathcal{T}_2 = \begin{matrix} \mathcal{B}_2 & \rightrightarrows & \mathcal{H}_2 \\ \downdownarrows & & \downdownarrows \\ \mathcal{V}_2 & \rightrightarrows & \mathcal{P}_2 \end{matrix}$ be double groupoids. Then we can define double groupoids

$$\mathcal{T}_1 \coprod \mathcal{T}_2 = \begin{matrix} \mathcal{B}_1 \coprod \mathcal{B}_2 & \rightrightarrows & \mathcal{H}_1 \coprod \mathcal{H}_2 \\ \downdownarrows & & \downdownarrows \\ \mathcal{V}_1 \coprod \mathcal{V}_2 & \rightrightarrows & \mathcal{P}_1 \coprod \mathcal{P}_2 \end{matrix}, \qquad \mathcal{T}_1 \times \mathcal{T}_2 = \begin{matrix} \mathcal{B}_1 \times \mathcal{B}_2 & \rightrightarrows & \mathcal{H}_1 \times \mathcal{H}_2 \\ \downdownarrows & & \downdownarrows \\ \mathcal{V}_1 \times \mathcal{V}_2 & \rightrightarrows & \mathcal{P}_1 \times \mathcal{P}_2 \end{matrix},$$

and similarly for families of double groupoids. If $\mathcal{T}_i$, $i \in I$, is a family of *vacant* double groupoids then $\coprod_{i \in I} \mathcal{T}_i$ and $\times_{i \in I} \mathcal{T}_i$ are also vacant.

Now, let $\mathcal{T}$ be a vacant double groupoid on the base $\mathcal{P}$, let $\mathcal{D}$ be the corresponding diagonal groupoid and let $\sim_D$ be the associated equivalence relation. Then $\mathcal{T} = \coprod_{P \in \mathcal{P}/\sim_D} \mathcal{T}_P$ where $\mathcal{T}_P$ is the "full" double groupoid with base the class P.

Let us say that a vacant double groupoid $\mathcal{T}$ is *connected* if the diagonal relation $\sim_D$ is transitive. The preceding remark shows that it is enough to consider connected vacant double groupoids.

As in subsection 2.3.2 above, we fix a finite non-empty set $\mathcal{P}$, a point $O \in \mathcal{P}$, and a finite group D, and set $\mathcal{D} = D(O) \times \mathcal{P}^2$.

Theorem 2.16. *Let $\mathcal{H}$, $\mathcal{V}$ be wide subgroupoids of $\mathcal{D}$ associated to data $(\sim_H, (H_P)_{P\in\mathcal{P}}, (\overline{d_{PQ}})_{P\sim_H Q})$ and $(\sim_V, (V_P)_{P\in\mathcal{P}}, (\overline{e_{PQ}})_{P\sim_H Q})$, respectively, as in Lemma 2.11.*

The following are equivalent:

1. *$\mathcal{D} = \mathcal{V}\mathcal{H}$ is an exact factorization.*
2. *The following conditions hold:*
   (a) *For all $P, Q \in \mathcal{P}$, one has*

$$D = \coprod_{R\in\mathcal{P}: P\sim_H R, R\sim_V Q} V_P\, e_{PR} d_{RQ}\, H_Q. \tag{2.17}$$

   (b) *For all $P \in \mathcal{P}$, $V_P \cap H_P = e$.*

Note that (a) implies that $\sim_D$ is an equivalence relation on $\mathcal{P}$, *cf.* subsection 2.3.3 above; this agrees with Lemma 2.12 (d).

Proof. (1) $\implies$ (2).We show (a). Since $\mathcal{T}$ is vacant, we have

$$\begin{aligned} D = \tau_P \mathcal{D}(P,Q)\tau_Q^{-1} &= \tau_P \left(\coprod_{R\in\mathcal{P}: P\sim_V R, R\sim_H Q} \mathcal{V}(P,R)\mathcal{H}(R,Q) \right) \tau_Q^{-1} \\ &= \coprod_{R\in\mathcal{P}: P\sim_V R, R\sim_H Q} V_P e_{PR} \tau_R \tau_R^{-1} H_R d_{RQ} = \coprod_{R\in\mathcal{P}: P\sim_V R, R\sim_H Q} V_P e_{PR} d_{RQ} H_Q, \end{aligned}$$

as claimed. We show (b). Let $g \in V_P \cap H_P$. Then $\tau_P g \tau_P^{-1} \in \mathcal{V}(P) \cap \mathcal{H}(P)$; but $\mathcal{V}(P) \cap \mathcal{H}(P) = \mathrm{id}_P$ since $\mathcal{T}$ is vacant. Thus $g = e$.

(2) $\implies$ (1). Let $x \in \mathcal{H}$, $g \in \mathcal{V}$ such that $S := r(x) = t(g)$, and set $P = l(x)$, $Q = b(g)$. That is, we have $\begin{array}{c|} \overset{x}{\overline{\quad}} \\ \end{array}_{\,g}$. Now $\gamma := xg \in \mathcal{D}(P,Q) = \coprod_{R\in\mathcal{P}: P\sim_V R, R\sim_H Q} \mathcal{V}(P,R)\mathcal{H}(R,Q)$ by assumption (a). Thus there exist $R \in \mathcal{P}$ (unique!), $f \in \mathcal{V}(P,R)$ and $y \in \mathcal{H}(R,Q)$ such that $\gamma = fy$, in other words $f\,\underset{y}{\overset{x}{\square}}\,g \in \mathcal{B}$. Moreover, assume that also $h\,\underset{w}{\overset{x}{\square}}\,g \in \mathcal{B}$; note $f \in \mathcal{V}(P,R)$ and $y \in \mathcal{H}(R,Q)$. Then $z := h^{-1}f = wy^{-1} \in \mathcal{V}(R) \cap \mathcal{H}(R)$; by hypothesis (b), $z = e$. This implies that $\mathcal{T} = \mathcal{V}\mathcal{H}$ is an exact factorization. $\square$

3. Weak Hopf algebras arising from a vacant double groupoid

Let F, G be a matched pair as in Example 1.4. As explained in many places, see *e. g.* [AN, 5.3], a bicrossed product Hopf algebra $\Bbbk^{G\tau}\#_\sigma \Bbbk F$ admits a convenient realization in the vector space with basis $\mathcal{B}$. In this section we shall discuss a generalization of this construction.

3.1. Weak Hopf algebras (quantum groupoids).

Recall [BNSz, BSz] that a *weak bialgebra* structure on a vector space H over a field $\Bbbk$ consists of an associative algebra structure $(H, m, 1)$, a coassociative coalgebra structure (H, Δ, ε), such that the following are satisfied:

$$\Delta(ab) = \Delta(a)\Delta(b), \qquad \forall a, b \in H. \tag{3.1}$$

$$\Delta^{(2)}(1) = (\Delta(1) \otimes 1)(1 \otimes \Delta(1)) = (1 \otimes \Delta(1))(\Delta(1) \otimes 1). \tag{3.2}$$

$$\varepsilon(abc) = \varepsilon(ab_1)\varepsilon(b_2c) = \varepsilon(ab_2)\varepsilon(b_1c), \qquad \forall a, b, c \in H. \tag{3.3}$$

A weak bialgebra H is called a *weak Hopf algebra* or a *quantum groupoid* if there exists a linear map $\mathcal{S} : H \to H$ satisfying

$$m(\mathrm{id} \otimes \mathcal{S})\Delta(h) = (\varepsilon \otimes \mathrm{id})(\Delta(1)(h \otimes 1)) =: \varepsilon_t(h), \tag{3.4}$$

$$m(\mathcal{S} \otimes \mathrm{id})\Delta(h) = (\mathrm{id} \otimes \varepsilon)((1 \otimes h)\Delta(1)) =: \varepsilon_s(h), \tag{3.5}$$

$$m^{(2)}(\mathcal{S} \otimes \mathrm{id} \otimes \mathcal{S})\Delta^{(2)} = \mathcal{S}, \tag{3.6}$$

for all $h \in H$. The maps ε_s, ε_t are respectively called the source and target maps; their images are respectively called the source and target subalgebras. See [NV] for a survey on quantum groupoids. It is known that a weak Hopf algebra is a true Hopf algebra if and only if $\Delta(1) = 1 \otimes 1$.

3.2. Weak Hopf algebras arising from vacant double groupoids.

Let $\mathcal{T}$ be a *finite* double groupoid, that is, $\mathcal{B}$, $\mathcal{V}$, $\mathcal{H}$ and $\mathcal{P}$ are finite sets.

Let $\Bbbk$ be a field (*) and let $\Bbbk\mathcal{T}$ denote the $\Bbbk$-vector space with basis $\mathcal{B}$ together with the following structures.

Algebra structure. Consider the groupoid algebra structure on $\Bbbk\mathcal{T}$ corresponding to the groupoid $\mathcal{B} \rightrightarrows \mathcal{H}$. Thus the multiplication in $\Bbbk\mathcal{T}$ is given by

$$A.B = \begin{cases} \dfrac{A}{B}, & \text{if } \dfrac{A}{B}, \\ 0, & \text{otherwise,} \end{cases}$$

for all $A, B \in \mathcal{B}$. This multiplication is associative and has a unit $\mathbf{1} := \sum_{x \in \mathcal{H}} \mathbf{id}\, x$. We shall also consider, for any $P \in \mathcal{P}$, the elements

$$_P\mathbf{1} = \sum_{x \in \mathcal{H}, l(x) = P} \mathbf{id}\, x, \qquad \mathbf{1}_P = \sum_{x \in \mathcal{H}, r(x) = P} \mathbf{id}\, x.$$

Clearly, $_P\mathbf{1}\ _Q\mathbf{1} = \delta_{P,Q}\ _P\mathbf{1}$, $\mathbf{1}_P\mathbf{1}_Q = \delta_{P,Q}\mathbf{1}_P$, for all $P, Q \in \mathcal{P}$. Hence the subalgebras $\Bbbk\mathcal{T}_s$, respectively $\Bbbk\mathcal{T}_t$, generated by $\mathbf{1}_P$, $P \in \mathcal{P}$, respectively by $_P\mathbf{1}$, $P \in \mathcal{P}$, are commutative of dimension $|\mathcal{P}|$.

Coalgebra structure. Dually, we consider the coalgebra structure on $\Bbbk\mathcal{T}$ dual to the algebra structure of the groupoid algebra corresponding to the groupoid $\mathcal{B} \rightrightarrows \mathcal{V}$. This means that the comultiplication of $\Bbbk\mathcal{T}$ is determined by

$$\Delta(A) = \sum B \otimes C, \qquad A \in \mathcal{B},$$

where the sum runs over all B, C with $B|C$ and $A = BC$. This comultiplication is coassociative and has counit $\varepsilon : \Bbbk\mathcal{T} \to \Bbbk$ given by

$$\varepsilon(A) = \begin{cases} 1, & \text{if } A = \mathbf{id}\, l(A), \\ 0, & \text{otherwise.} \end{cases}$$

(*) Most of the constructions in this section are valid over an arbitrary commutative ring.

Theorem 3.1. *$\Bbbk\mathcal{T}$ is a quantum groupoid if and only if $\mathcal{T}$ is a vacant double groupoid. If this is the case, the antipode is defined by $\mathcal{S}(A) = A^{-1}$, $\forall A \in \mathcal{B}$; and the source and target subalgebras are respectively $\Bbbk\mathcal{T}_s$, $\Bbbk\mathcal{T}_t$.*

Proof. Let $A, B \in \mathcal{B}$. It follows from the definitions that

$$\Delta(A).\Delta(B) = \sum \genfrac{}{}{0pt}{}{U}{R} \otimes \genfrac{}{}{0pt}{}{V}{S},$$

where the sum runs over all elements $U, V, R, S \in \mathcal{B}$, such that $\begin{array}{c|c} U & V \\ \hline R & S \end{array}$, $UV = A$ and $RS = B$. It is thus clear that $\Delta(A).\Delta(B) = 0 = \Delta(A.B)$, if A and B are not vertically composable. So assume that $\dfrac{A}{B}$. Then

$$\Delta(A.B) = \sum_{XY = \genfrac{}{}{0pt}{}{A}{B}} X \otimes Y.$$

Since $(X \otimes Y)_{X,Y \in \mathcal{B}}$ is a basis of $\Bbbk\mathcal{T} \otimes \Bbbk\mathcal{T}$, we see that $\Delta(A).\Delta(B) = \Delta(A.B)$ if and only if

$$1 = \#\left\{ \begin{pmatrix} U & V \\ R & S \end{pmatrix} \in \mathcal{B}^4 : \begin{array}{c|c} U & V \\ \hline R & S \end{array}, \quad UV = A, \quad RS = B, \quad \genfrac{}{}{0pt}{}{U}{R} = X, \quad \genfrac{}{}{0pt}{}{V}{S} = Y \right\}$$

for all X, $Y \in \mathcal{B}$ such that $X|Y$, $XY = \genfrac{}{}{0pt}{}{A}{B}$. By Proposition 2.2, we conclude that (3.1) holds if and only if $\mathcal{T}$ is vacant.

Assume for the rest of the proof that $\mathcal{T}$ is vacant. We next prove the relationships (3.2) and (3.3). We have $\Delta(\mathbf{1}) = \sum_{A|B,\, AB = \mathbf{id}\, t(AB)} A \otimes B$. But if $AB = \mathbf{id}\, t(AB)$ then the right side of B is an identity, hence $B = \mathbf{id}\, t(B)$ and the same for A. Thus

$$\Delta(\mathbf{1}) = \sum_{x,y \in \mathcal{H}: x|y} \mathbf{id}\, x \otimes \mathbf{id}\, y.$$

Therefore, $\Delta^{(2)}(\mathbf{1}) = \sum_{x,y,z \in \mathcal{H}: x|y|z} \mathbf{id}\, x \otimes \mathbf{id}\, y \otimes \mathbf{id}\, z = (\Delta(\mathbf{1}) \otimes \mathbf{1})(\mathbf{1} \otimes \Delta(\mathbf{1})) = (\mathbf{1} \otimes \Delta(\mathbf{1}))(\Delta(\mathbf{1}) \otimes \mathbf{1})$. This establishes (3.2). The proof of (3.3) is similar.

We next consider the axioms of the antipode. We first treat (3.4) and (3.5). Using Proposition 2.5 (ii), we see that

$$\varepsilon_t(C) = \begin{cases} 0 & \text{if } C \text{ is not a horizontal identity,} \\ {}_P\mathbf{1}, & \text{if } C = \operatorname{id} g,\ P = t(g), \end{cases} \qquad \text{for all } C \in \mathcal{B}.$$

This coincides with the left hand side of (3.4), by Proposition 2.5 (iv). Similarly, using Proposition 2.5 (iii), we see that

$$\varepsilon_s(C) = \begin{cases} 0 & \text{if } C \text{ is not a horizontal identity,} \\ \mathbf{1}_P, & \text{if } C = \operatorname{id} g,\ P = b(g), \end{cases} \qquad \text{for all } C \in \mathcal{B}.$$

This coincides with the left hand side of (3.4), by Proposition 2.5 (v). Finally, the relation (3.6) is equivalent to the identity

$$\sum_{X,Y,Z} \begin{Bmatrix} X^{-1} \\ Y \\ Z^{-1} \end{Bmatrix} = A^{-1},$$

for all $A \in \mathcal{B}$, where the sum runs over all X, Y and Z in $\mathcal{B}$ such that $\begin{array}{c|c|c} & X^{-1} & \\ \hline X & Y & Z \\ \hline & Z^{-1} & \end{array}$ and $XYZ = A$.

By Lemmas 1.12 and 2.4, the left hand side equals A^{-1}. $\square$

By construction, $\Bbbk\mathcal{T}$ is the groupoid algebra of the vertical groupoid $\mathcal{B} \rightrightarrows \mathcal{H}$. Using the description in 2.3.1, this groupoid is isomorphic to $\coprod_{H\in\mathcal{H}/\sim_h} \mathcal{B}_H$, where $\sim_h$ is the equivalence relation in $\mathcal{H}$ defined by $\mathcal{B} \rightrightarrows \mathcal{H}$ and $\mathcal{B}_H \rightrightarrows H$ is the connected groupoid $\mathcal{B}(x) \times H^2$ on the class H, $x \in H$. Therefore, there is an isomorphism of algebras

$$\Bbbk\mathcal{T} \simeq \oplus_{H\in\mathcal{H}/\sim_h} \Bbbk\mathcal{B}(x) \otimes M_{n(H)}(\Bbbk), \tag{3.7}$$

where $\Bbbk\mathcal{B}(x)$ is the group algebra, and $n(H) = |H|$. Similarly, we have an isomorphism of coalgebras

$$\Bbbk\mathcal{T} \simeq \oplus_{V\in\mathcal{V}/\sim_v} \Bbbk\mathcal{B}(g)^* \otimes M_{m(V)}(\Bbbk)^*, \tag{3.8}$$

where $\sim_v$ is the equivalence relation defined by the horizontal groupoid $\mathcal{B} \rightrightarrows \mathcal{V}$, $\Bbbk\mathcal{B}(g)$ is the group algebra of $\mathcal{B}(g)$, $g \in V$, and $m(V) = |V|$.

Example 3.2. Suppose that $\Bbbk\mathcal{T}$ is simple as an algebra. Then
(1) $\mathcal{B} \rightrightarrows \mathcal{H}$ is the coarse groupoid on $\mathcal{H}$;
(2) $\mathcal{H} \rightrightarrows \mathcal{P}$ is a trivial group bundle;
(3) $\mathcal{V} \rightrightarrows \mathcal{P}$ is the coarse groupoid on $\mathcal{P}$;
(4) $\mathcal{B} \rightrightarrows \mathcal{V}$ is a trivial group bundle.

This means that as a weak Hopf algebra $\Bbbk\mathcal{T}$ is the groupoid algebra of the vertical groupoid $\mathcal{B} \rightrightarrows \mathcal{H}$, and in particular it is cocommutative.

Proof. By (3.7), $\Bbbk\mathcal{T}$ is simple iff $|\mathcal{H}/\sim_h| = 1$ and $\mathcal{B}(x)$ is trivial for any $x \in \mathcal{H}$. Hence (1). If $x : P \to Q$ is in $\mathcal{H}$, then there is a box B connecting it to $\operatorname{id} P$; that is, there exists $g \in \mathcal{V}$ such that $B = x{\triangleright}g \begin{array}{c} \operatorname{id} \\ \boxed{} \\ x \end{array} g$ with $x = \operatorname{id} P{\triangleleft}g = \operatorname{id} Q$ by (2.12). Hence (2). If $P, Q \in \mathcal{P}$, there is a unique box connecting $\operatorname{id} P$ and $\operatorname{id} Q$; the vertical sides connect P and Q, hence (3). Finally, let $B := \begin{array}{c} x \\ \boxed{} \\ \end{array} g$ be any box in $\mathcal{B}$. Then $x = \operatorname{id}_P$ by (2), and $B = \mathbf{id}g$, by vacancy; hence (4). $\square$

The proofs of the following statements are straightforward and are left to the reader.

Proposition 3.3. *Let $\mathcal{T}_1$, $\mathcal{T}_2$ be finite vacant double groupoids. Then there are isomorphisms of quantum groupoids* $\Bbbk(\mathcal{T}_1 \coprod \mathcal{T}_2) \simeq \Bbbk\mathcal{T}_1 \times \Bbbk\mathcal{T}_2$, $\Bbbk(\mathcal{T}_1 \times \mathcal{T}_2) \simeq \Bbbk\mathcal{T}_1 \otimes \Bbbk\mathcal{T}_2$. $\square$

Proposition 3.4. *Let $\mathcal{T}$ be a finite vacant double groupoid and assume that $\Bbbk = \mathbb{C}$. Then $\mathbb{C}\mathcal{T}$ is a C^* quantum groupoid* [BNSz], *with the involution uniquely defined by $A^* = A^v$, $A \in \mathcal{B}$.* $\square$

3.3. **Extensions with cocycles.**

We begin by recalling the following definition, which fits into the general framework of groupoid cohomology, due to Westman [We], see [R].

Definition 3.5. Let $s, e : \mathcal{G} \rightrightarrows \mathcal{P}$ be a groupoid. A *normalized 2-cocycle* on $\mathcal{G}$ with values in $\Bbbk^\times$ is a function $\sigma : \mathcal{G}\,{}_s\!\times_e \mathcal{G} \to \Bbbk^\times$ such that

(3.9) $\sigma(\alpha, \beta)\sigma(\alpha\beta, \gamma) = \sigma(\beta, \gamma)\sigma(\alpha, \beta\gamma)$;

(3.10) $\sigma(\alpha, \mathrm{id}\ e(\alpha)) = \sigma(\mathrm{id}\ s(\alpha), \alpha) = 1$,

for all composable $\alpha, \beta, \gamma \in \mathcal{G}$.

Let now $\mathcal{T}$ be a double groupoid. A *normalized vertical 2-cocycle* is a 2-cocycle on the groupoid $\mathcal{B} \rightrightarrows \mathcal{H}$; similarly, a *normalized horizontal 2-cocycle* is a 2-cocycle on the groupoid $\mathcal{B} \rightrightarrows \mathcal{V}$. Thus, a normalized vertical 2-cocycle is a function σ on the set of all pairs (A, B) with $\dfrac{A}{B}$ with values in $\Bbbk^\times$ such that

(3.11) If $\begin{array}{c} A \\ \hline B \\ \hline C \end{array}$, then $\sigma(A, B)\sigma(\tfrac{A}{B}, C) = \sigma(B, C)\sigma(A, \tfrac{B}{C})$.

(3.12) If A or B is a vertical identity, then $\sigma(A, B) = 1$.

Letting $A = B^v = C$, we deduce that

(3.13) $\sigma(A, A^v) = \sigma(A^v, A)$.

Analogously, a normalized horizontal 2-cocycle is a function τ on the set of all pairs (A, B) with $A|B$, such that

(3.14) If $A|B|C$, then $\tau(A, B)\tau(AB, C) = \tau(B, C)\tau(A, BC)$.

(3.15) If A or B is a horizontal identity, then $\tau(A, B) = 1$.

Letting $A = B^h = C$, we deduce that

(3.16) $\tau(A, A^h) = \tau(A^h, A)$.

Definition 3.6. Let $\mathcal{T}$ be a double groupoid. A *normalized 2-cocycle* on $\mathcal{T}$ with values in $\Bbbk^\times$ is a pair (σ, τ), where σ is a normalized vertical 2-cocycle, τ is normalized horizontal 2-cocycle, and the following property holds:

$$\text{If } \begin{array}{c|c} \mathrm{A} & \mathrm{B} \\ \hline \mathrm{C} & \mathrm{D} \end{array}, \quad \text{then} \quad \sigma(AB, CD)\tau\begin{pmatrix} \frac{A}{C}, \frac{B}{D} \end{pmatrix} = \tau(A, B)\tau(C, D)\sigma(A, C)\sigma(B, D). \tag{3.17}$$

Remark 3.7. (i). Let $x, y \in \mathcal{H}$, $x|y$ so that $\begin{array}{c|c} \mathbf{id}\,x & \mathbf{id}\,y \\ \hline \mathbf{id}\,x & \mathbf{id}\,y \end{array}$. By (3.12) and (3.17), we have

$$\tau(\mathbf{id}\,x, \mathbf{id}\,y) = 1. \tag{3.18}$$

(ii). Let $f, g \in \mathcal{V}$, $f|g$ so that $\begin{array}{c|c} \mathbf{id}\, f & \mathbf{id}\, f \\ \hline \mathbf{id}\, g & \mathbf{id}\, g \end{array}$. By (3.15) and (3.17), we have

$$\sigma(\mathbf{id}\, f, \mathbf{id}\, g) = 1. \tag{3.19}$$

Given a normalized vertical 2-cocycle σ and a normalized horizontal 2-cocycle τ on the double groupoid $\mathcal{T}$, one may consider the σ-*twisted* groupoid algebra structure and, dually, the τ-*twisted* groupoid coalgebra structure on the vector space $\Bbbk\mathcal{T}$ with basis $\mathcal{B}$. The following theorem asserts that, provided that $\mathcal{T}$ is vacant, the compatibility condition (3.17) guarantees that these two structures combine into a weak Hopf algebra structure.

Theorem 3.8. *Let $\mathcal{T}$ be a vacant double groupoid and let (σ, τ) be a normalized 2-cocycle on $\mathcal{T}$ with values in $\Bbbk^\times$.*

(i) Let $\Bbbk^\tau_\sigma\mathcal{T}$ be the vector space with basis $\mathcal{B}$ and multiplication and comultiplication defined, respectively by

- *$A.B = \sigma(A, B)\frac{A}{B}$, if $\frac{A}{B}$, and 0 otherwise.*
- *$\Delta(A) = \sum \tau(B, C) B \otimes C$, where the sum is over all pairs (B, C) with $B|C$ and $A = BC$.*

Then $\Bbbk^\tau_\sigma\mathcal{T}$ is a quantum groupoid with antipode defined by

$$\mathcal{S}(A) = \tau(A, A^h)^{-1}\, \sigma(A^{-1}, A^h)^{-1}\, A^{-1}. \tag{3.20}$$

The source and target subalgebras are, respectively, the subspaces spanned by $(\mathbf{1}_P)_{P\in\mathcal{P}}$ and $({}_P\mathbf{1})_{P\in\mathcal{P}}$; so they are commutative of dimension $|\mathcal{P}|$.

(ii) Let (ν, η) be another normalized 2-cocycle on $\mathcal{T}$ with values in $\Bbbk^\times$. Let $\psi : \mathcal{T} \to \Bbbk^\times$ be a map and let $\Psi : \Bbbk^\tau_\sigma\mathcal{T} \to \Bbbk^\eta_\nu\mathcal{T}$ be the linear map given by $\Psi(B) = \psi(B)B$, $B \in \mathcal{B}$. Then Ψ is an isomorphism of quantum groupoids if and only if

$$\psi\begin{pmatrix} A \\ B \end{pmatrix} \sigma(A, B) = \psi(A)\psi(B)\nu(A, B), \quad \textit{for all } A, B \in \mathcal{B} \textit{ such that } \frac{A}{B}; \tag{3.21}$$

$$\psi(CD)\eta(C, D) = \psi(C)\psi(D)\tau(C, D), \quad \textit{for all } C, D \in \mathcal{B} \textit{ such that } C|D. \tag{3.22}$$

Proof. (i). Straightforward computations show that the multiplication is associative with unit 1 (because of the cocycle and unitary conditions on σ), and that the comultiplication is coassociative with counit ε (because of the cocycle and unitary conditions on τ). By (3.17), Δ is multiplicative. Now $\Delta(\mathbf{1}) = \sum_{x,y\in\mathcal{H}:x|y} \mathbf{id}\, x \otimes \mathbf{id}\, y$ by (3.18); we can conclude the validity of (3.2), using (3.19). The proof of (3.3) is similar. The proof of (3.4) and (3.5) is as in the proof of Theorem 3.1; (3.13) and (3.16) are needed. Note that the source and target maps coincide with those of $\Bbbk\mathcal{T}$. We next prove (3.6). Given $A \in \mathcal{B}$, we compute:

$$\begin{aligned} m^{(2)}(\mathcal{S} \otimes \mathrm{id} \otimes \mathcal{S})\Delta^{(2)}(A) &= m^{(2)}(\mathcal{S} \otimes \mathrm{id} \otimes \mathcal{S}) \left(\sum_{X|Y|Z, XYZ=A} \tau(X, Y)\tau(XY, Z) X \otimes Y \otimes Z \right) \\ &= \sum \tau(X, Y)\tau(XY, Z)\tau(X, X^h)^{-1}\, \sigma(X^{-1}, X^h)^{-1}\tau(Z, Z^h)^{-1}\, \sigma(Z^{-1}, Z^h)^{-1} \\ &\qquad \times \sigma(X^{-1}, Y)\sigma(\begin{smallmatrix} X^{-1} \\ Y \end{smallmatrix}, Z^{-1}) \begin{Bmatrix} X^{-1} \\ Y \\ Z^{-1} \end{Bmatrix} \end{aligned}$$

$$= \tau(A, A^h)\tau(\mathrm{id}_{l(A)}, A)\tau(A, A^h)^{-1}\,\sigma(A^{-1}, A^h)^{-1}\tau(A, A^h)^{-1}\,\sigma(A^{-1}, A^h)^{-1}$$
$$\times\,\sigma(A^{-1}, A^h)\sigma(\mathrm{id}_{b(A)^{-1}}, A^{-1})\,A^{-1}$$
$$= \tau(A, A^h)^{-1}\,\sigma(A^{-1}, A^h)^{-1}\,A^{-1} = \mathcal{S}(A),$$

where the second sum is over all X, Y, Z such that $\begin{array}{c|c|c} & X^{-1} & \\ \hline X & Y & Z \\ \hline & Z^{-1} & \end{array}$ and $XYZ = A$; the third equality is by Lemma 2.4; and the fourth is clear. The proof of (ii) is straightforward. □

Proposition 3.9. *Let $\mathcal{T}$ be a vacant double groupoid and let (σ, τ) be a normalized 2-cocycle on $\mathcal{T}$ with values in $\Bbbk^\times$. Then $\Bbbk^\tau_\sigma\mathcal{T}$ is a Hopf algebra if and only if $\mathcal{T}$ arises from a matched pair of groups.*

Proof. $\Delta(1) = 1 \otimes 1$ iff $\sum_{x,y\in\mathcal{H}:x|y} \mathbf{id}\,x \otimes \mathbf{id}\,y = \sum_{v,w\in\mathcal{H}} \mathbf{id}\,v \otimes \mathbf{id}\,w$. Thus, $\mathbf{id}\,x|\mathbf{id}\,y$ for any $x, y \in \mathcal{H}$. Hence $\#\mathcal{P} = 1$, *i. e.* $\mathcal{T}$ arises from a matched pair of groups. The converse is well-known. □

Proposition 3.10. *Let $\mathcal{T}$ be a vacant double groupoid and let (σ, τ) be a normalized 2-cocycle on $\mathcal{T}$ with values in $\Bbbk^\times$. Then $\Bbbk^\tau_\sigma\mathcal{T}$ is involutory. If char $\Bbbk = 0$, then $\Bbbk^\tau_\sigma\mathcal{T}$ is semisimple and cosemisimple.*

Proof. Let $A \in \mathcal{B}$. We compute

$$\mathcal{S}^2(A) = \tau(A, A^h)^{-1}\,\sigma(A^{-1}, A^h)^{-1}\,\mathcal{S}(A^{-1}) = \tau(A, A^h)^{-1}\,\sigma(A^{-1}, A^h)^{-1}\,\tau(A^{-1}, A^v)^{-1}\,\sigma(A, A^v)^{-1}\,A.$$

Now, $\begin{array}{c|c} A & A^h \\ \hline A^v & A^{-1} \end{array}$ implies that $1 = \tau(A, A^h)\tau(A^v, A^{-1})\sigma(A, A^v)\sigma(A^h, A^{-1})$. We conclude, using (3.13) and (3.16), that $\mathcal{S}^2(A) = A$. The second statement follows then from [N, Corollary 6.5]. □

The category of finite-dimensional representations of a weak Hopf algebra over $\Bbbk$ admits the structure of a $\Bbbk$-linear rigid monoidal category [NV]. Recall from [ENO] the definition of a *multifusion category*: this is a semisimple $\Bbbk$-linear rigid tensor category with finitely many isoclasses of simple objects and finite dimensional hom-spaces. Recall also that a multifusion category is called a *fusion* category if in addition the unit object is *simple*.

Proposition 3.11. *Let $\mathcal{T}$ be a vacant double groupoid and let (σ, τ) be a normalized 2-cocycle on $\mathcal{T}$ with values in $\Bbbk^\times$.*

(i). The unit object of the category $\operatorname{Rep}\Bbbk^\tau_\sigma\mathcal{T}$ is simple if and only if $\mathcal{V} \rightrightarrows \mathcal{P}$ is connected.

(ii). If char $\Bbbk = 0$, then the category $\operatorname{Rep}\Bbbk^\tau_\sigma\mathcal{T}$ of finite dimensional $\Bbbk^\tau_\sigma\mathcal{T}$-modules is a multifusion category. It is fusion if and only if $\mathcal{V} \rightrightarrows \mathcal{P}$ is connected.

Proof. We prove (i). We have already observed that the target subalgebra– which is the unit object of $\operatorname{Rep}\Bbbk^\tau_\sigma\mathcal{T}$ by general reasons– is the span of the elements ${}_P\mathbf{1}$. Let $\sim_V$ be the equivalence relation in $\mathcal{P}$ induced by $\mathcal{V} \rightrightarrows \mathcal{P}$. We claim that the subspaces $\sum_{P\in X} \Bbbk_P\mathbf{1}$, for X an equivalence class of $\sim_V$, are the simple subobjects of $\Bbbk^\tau_\sigma\mathcal{T}_t$. Indeed, for all $A \in \mathcal{B}$, we have

$$A.{}_P\mathbf{1} = \epsilon_t(A\,{}_P\mathbf{1}) = \begin{cases} {}_Q\mathbf{1}, & \text{if } A = \mathrm{id}\,g, \text{ for some } g \in \mathcal{V} \text{ such that } b(g) = P,\ t(g) = Q, \\ 0, & \text{otherwise.} \end{cases}$$

The claim is proved. Now (ii) follows from 3.10, general results on weak Hopf algebras and (i). □

Proposition 3.12. *Let $\mathcal{T}$ be a vacant double groupoid and let (σ, τ) be a normalized 2-cocycle on $\mathcal{T}$ with values in $\Bbbk^{\times}$. Then (τ, σ) is a normalized 2-cocycle on the transpose double groupoid $\mathcal{T}^t$ and the quantum groupoid $\Bbbk_\tau^\sigma \mathcal{T}^t$ is dual to $\Bbbk_\sigma^\tau \mathcal{T}$.*

Proof. The duality is given by the bilinear form $(B|C) = \delta_{B,C^t}$, B, $C \in \mathcal{B}$. □

3.4. **The category** $\operatorname{Rep} \Bbbk\mathcal{T}$**.**

Let $\mathcal{T}$ be a finite vacant double groupoid, and let $\Bbbk\mathcal{T}$ be the quantum groupoid associated to $\mathcal{T}$ as in Theorem 3.1. Consider the category $\mathcal{C} := \operatorname{Rep} \Bbbk\mathcal{T}$ of finite dimensional representations of $\Bbbk\mathcal{T}$. Our aim in this subsection is to sketch a combinatorial description of the category $\operatorname{Rep} \Bbbk\mathcal{T}$ in groupoid-theoretical terms. We shall follow the lines in [NV, 5.1].

Suppose that $s, e : \mathcal{G} \rightrightarrows \mathcal{P}$ is a groupoid. A $\Bbbk$-*linear* $\mathcal{G}$*-bundle*, or $\mathcal{G}$-bundle for short, is a map $p : V \to \mathcal{P}$ together with an action of $\mathcal{G}$ on p, and such that
(i) each fiber V_b ($b \in \mathcal{P}$) is a vector space over $\Bbbk$;
(ii) for all $g \in \mathcal{G}$ the map $g : V_{e(g)} \to V_{s(g)}$ is a linear isomorphism.

So one may think of a $\mathcal{G}$-bundle as a $\mathcal{P}$-graded vector space $V = \oplus_{b \in \mathcal{P}} V_b$ endowed with a linear $\mathcal{G}$-action $g : V_{e(g)} \to V_{s(g)}$, $g \in \mathcal{G}$.

Remark 3.13. The category $\mathcal{G}$-bund of (finite dimensional) $\Bbbk$-linear $\mathcal{G}$-bundles is equivalent to the category of (finite dimensional) representations of the groupoid algebra $\Bbbk\mathcal{G}$. The equivalence is defined as follows: for a $\Bbbk\mathcal{G}$-module V, we let the $\mathcal{P}$-grading on V be given by $V_b = \operatorname{id} b.V$, for all $b \in \mathcal{P}$.

Let now $\mathcal{T}$ be a vacant double groupoid. Let $\mathcal{T}$-bund be the category of $\Bbbk$-linear bundles over the vertical groupoid $\mathcal{B} \rightrightarrows \mathcal{H}$. Thus, the objects of $\mathcal{T}$-bund are $\mathcal{H}$-graded vector spaces endowed with a left action of the vertical groupoid $\mathcal{B} \rightrightarrows \mathcal{H}$ by linear isomorphisms. There is a structure of rigid monoidal category on $\mathcal{T}$-bund:

- *Tensor product.* If V, W are $\mathcal{T}$-bundles then $V \otimes W := \oplus_{z \in \mathcal{H}} (V \otimes W)_z$, where

$$(V \otimes W)_z = \sum_{xy=z} V_x \otimes_{\Bbbk} W_y, \qquad z \in \mathcal{H}.$$

(Note that this difers from $V \otimes_{\Bbbk} W$ by the fact that we are not taking all summands $V_x \otimes_{\Bbbk} W_y$ but only those for which x and y are composable.) The action of $\mathcal{B}$ on $V \otimes W$ is given by Δ.

- *Unit object.* This is the target subalgebra $\Bbbk\mathcal{T}_t = \oplus_{P \in \mathcal{P}} \Bbbk_P \mathbf{1}$, with $\mathcal{H}$-grading defined by

$$(\Bbbk\mathcal{T}_t)_x = \begin{cases} 0, & \text{if } x \text{ is not an identity,} \\ \Bbbk_P \mathbf{1}, & \text{if } x = \operatorname{id} P, \end{cases} \qquad \text{for all } x \in \mathcal{H}.$$

  and $\mathcal{B}$-action $A._P\mathbf{1} = \epsilon_t(A\, _P\mathbf{1})$.

- The *dual* V^* of an object $V = \oplus_{x \in \mathcal{H}} V_x \in \mathcal{C}$ has $\mathcal{H}$-grading $(V^*)_x = (V_{x^{-1}})^*$, $x \in \mathcal{H}$; $\mathcal{B}$-action $A := (A^{-1})^* : (V^*)_{b(A)} \to (V^*)_{t(A)}$, for all $A \in \mathcal{B}$.

With remark 3.13 in mind, we can describe the monoidal structure in $\operatorname{Rep} \Bbbk\mathcal{T}$.

Proposition 3.14. *Let $\mathcal{T}$ be a vacant double groupoid. Assume that $\operatorname{char} \Bbbk = 0$. The category $\mathcal{T}$-bund is a multifusion category over $\Bbbk$ and it is monoidally equivalent to* $\operatorname{Rep} \Bbbk\mathcal{T}$.

Proof. The expressions for the tensor product, unit object and duals are a translation of the formulas in [NV] to the language of $\mathcal{T}$-bundles. For instance, the unit isomorphism $\Bbbk\mathcal{T}_t \otimes V \to V$ is given as follows: for any $z \in \mathcal{H}$, we have $(\Bbbk\mathcal{T}_t \otimes_{\Bbbk} V)_z = \Bbbk_{l(z)}\mathbf{1} \otimes_{\Bbbk} V_z$; the isomorphism $\Bbbk\mathcal{T}_t \otimes V \to V$ is determined by its homogeneous components $(\Bbbk\mathcal{T}_t \otimes_{\Bbbk} V)_z \to V_z$, given by the action of ${}_{l(z)}\mathbf{1}$, which is the identity on V_z. The unit isomorphisms on the right and the evaluation and coevaluation maps for the duals are translated similarly from $\operatorname{Rep} \Bbbk\mathcal{T}$. $\square$

3.5. A Kac exact sequence for matched pairs of groupoids.

We first recall the well-known definition of the groupoid cohomology via standard resolutions [We, R]. Let $s, e : \mathcal{G} \rightrightarrows \mathcal{P}$ be a groupoid. In this subsection, we shall denote by $\mathcal{G}^{(0)} := \mathcal{P}$ the base of $\mathcal{G}$, $\mathcal{G}^{(1)} := \mathcal{G}$ and

$$\mathcal{G}^{(n)} = \{(x_1, \dots, x_n) \in \mathcal{G}^n : x_1|x_2|\dots|x_{n-1}|x_n\}, \qquad n \geq 2.$$

Let M be an abelian group and let

$$C^n(\mathcal{G}, M) = \{f : \mathcal{G}^{(n)} \to M : f(x_1, \dots, x_n) = 0, \text{ if some } x_i \in \mathcal{G}^{(0)}\}.$$

The cohomology groups $H^n(\mathcal{G}, M)$ of $\mathcal{G}$ with coefficients in M are the cohomology groups of the complex

$$\begin{aligned} 0 \longrightarrow C^0(\mathcal{G}, M) \xrightarrow{d^0} C^1(\mathcal{G}, M) \xrightarrow{d^1} C^2(\mathcal{G}, M) \xrightarrow{d^2} \dots \\ \longrightarrow C^n(\mathcal{G}, M) \xrightarrow{d^n} C^{n+1}(\mathcal{G}, M) \longrightarrow \dots \end{aligned} \tag{3.23}$$

where

$$\begin{aligned} d^0 f(x) &= f(e(x)) - f(s(x)), \\ d^n f(x_1, \dots, x_{n+1}) &= f(x_2, \dots, x_{n+1}) + \sum_{1 \leq i \leq n} (-1)^i f(x_1, \dots, x_i x_{i+1}, \dots, x_{n+1}) \\ &\quad + (-1)^{n+1} f(x_1, \dots, x_n). \end{aligned} \tag{3.24}$$

Let now $\mathcal{T}$ be a double groupoid: $\begin{matrix} \mathcal{B} & \rightrightarrows & \mathcal{H} \\ \downdownarrows & & \downdownarrows \\ \mathcal{V} & \rightrightarrows & \mathcal{P} \end{matrix}$. Let

$$\begin{aligned} \mathcal{B}^{(0,0)} &:= \mathcal{P}, \\ \mathcal{B}^{(0,s)} &:= \{(x_1, \dots, x_s) \in \mathcal{H}^s : x_1|x_2 \dots |x_s\} = \mathcal{H}^{(s)}, \quad s > 0, \\ \mathcal{B}^{(r,0)} &:= \{(g_1, \dots, g_r) \in \mathcal{V}^r : g_1|g_2 \dots |g_r\} = \mathcal{V}^{(r)}, \quad r > 0, \\ \mathcal{B}^{(r,s)} &:= \left\{ \begin{pmatrix} A_{11} & A_{12} & \dots & A_{1s} \\ A_{21} & A_{22} & \dots & A_{2s} \\ \dots & \dots & \dots & \dots \\ A_{r1} & A_{r2} & \dots & A_{rs} \end{pmatrix} \in \mathcal{B}^{r \times s} : \begin{array}{c|c|c|c} A_{11} & A_{12} & \dots & A_{1s} \\ \hline A_{21} & A_{22} & \dots & A_{2s} \\ \hline \dots & \dots & \dots & \dots \\ \hline A_{r1} & A_{r2} & \dots & A_{rs} \end{array} \right\}, \quad r, s > 0. \end{aligned}$$

Let M be an abelian group and let $D^{r,s} = D^{r,s}(\mathcal{T}, M)$, $r, s \geq 0$, be defined by

$$D^{r,s} := \left\{ f : \mathcal{B}^{(r,s)} \to M : f \begin{pmatrix} A_{11} & \dots & A_{1s} \\ A_{21} & \dots & A_{2s} \\ \dots & \dots & \dots \\ A_{r1} & \dots & A_{rs} \end{pmatrix} = 0, \text{ if} \begin{cases} \text{either } r > 1, s > 0 \text{ and } A_{ij} \in \mathcal{V}, \\ \text{or } r > 1, s = 0 \text{ and } A_{i0} \in \mathcal{P}, \\ \text{or } r > 0, s > 1 \text{ and } A_{ij} \in \mathcal{H}, \\ \text{or } r = 0, s > 1 \text{ and } A_{0j} \in \mathcal{P}. \end{cases} \right\}.$$

Let $d_H = d_H^{r,s} : D^{r,s} \to D^{r,s+1}$, $d_V = d_V^{r,s} : D^{r,s} \to D^{r+1,s}$ be, respectively, the horizontal and vertical coboundary maps defined as follows:

- If $r = 0$, d_H is as in (3.24);
- if $s = 0$, d_V is as in (3.24);
- if $r = 0$, $s > 0$,

$$d_V^{0,s} f(A_{11}, \dots, A_{1s}) = f(b(A_{11}), \dots, b(A_{1s})) - f(t(A_{11}), \dots, t(A_{1s}));$$

- if $r > 0$, $s = 0$,

$$d_H^{r,0} f \begin{pmatrix} A_{11} \\ \dots \\ A_{r1} \end{pmatrix} = f(r(A_{11}), \dots, r(A_{r1})) - f(l(A_{11}), \dots, l(A_{r1}));$$

- if $r > 0$ and $s > 0$,

$$\begin{aligned} d_V^{r,s} f \begin{pmatrix} A_{11} & \dots & A_{1s} \\ \dots & \dots & \dots \\ A_{r1} & \dots & A_{rs} \\ A_{r+1,1} & \dots & A_{r+1,s} \end{pmatrix} &= f \begin{pmatrix} A_{21} & \dots & A_{2s} \\ \dots & \dots & \dots \\ A_{r1} & \dots & A_{rs} \\ A_{r+1,1} & \dots & A_{r+1,s} \end{pmatrix} \\ &+ \sum_{1 \le i \le r} (-1)^i f \begin{pmatrix} A_{11} & \dots & A_{1s} \\ \dots & \dots & \dots \\ \begin{Bmatrix} A_{i1} \\ A_{i+1,1} \end{Bmatrix} & \dots & \begin{Bmatrix} A_{is} \\ A_{i+1,s} \end{Bmatrix} \\ \dots & \dots & \dots \\ A_{r+1,1} & \dots & A_{r+1,s} \end{pmatrix} \\ &+ (-1)^{r+1} f \begin{pmatrix} A_{11} & \dots & A_{1s} \\ \dots & \dots & \dots \\ A_{r1} & \dots & A_{rs} \end{pmatrix};\end{aligned}$$

$$\begin{aligned} d_H^{r,s} f \begin{pmatrix} A_{11} & \dots & A_{1,s+1} \\ \dots & \dots & \dots \\ A_{r1} & \dots & A_{r,s+1} \end{pmatrix} &= f \begin{pmatrix} A_{12} & \dots & A_{1,s+1} \\ \dots & \dots & \dots \\ A_{r2} & \dots & A_{r,s+1} \end{pmatrix} \\ &+ \sum_{1 \le j \le s} (-1)^j f \begin{pmatrix} A_{11} & \dots & \{A_{1,j} A_{1,j+1}\} & \dots & A_{1,s+1} \\ \dots & \dots & \dots & \dots & \dots \\ A_{r,1} & \dots & \{A_{r,j} A_{1,j+1}\} & \dots & A_{r,s+1} \end{pmatrix} \\ &+ (-1)^{s+1} f \begin{pmatrix} A_{11} & \dots & A_{1s} \\ \dots & \dots & \dots \\ A_{r1} & \dots & A_{rs} \end{pmatrix}.\end{aligned}$$

A straightforward computation shows that the following diagram commutes:

$$\begin{array}{ccc} D^{r+1,s} & \xrightarrow{d_H} & D^{r+1,s+1} \\ \uparrow{\scriptstyle d_V} & & \uparrow{\scriptstyle d_V} \\ D^{r,s} & \xrightarrow{d_H} & D^{r,s+1} \end{array}$$

Thus, there is a double cochain complex

$$D^{\cdot\cdot} = \begin{tikzcd} \vdots & & & \\ D^{2,0} \arrow[u] \arrow[r, "d_H"] & \vdots\cdots & & \\ D^{1,0} \arrow[u, "d_V"] \arrow[r, "d_H"] & D^{1,1} \arrow[u, "-d_V"] \arrow[r, "d_H"] & \vdots\cdots & \\ D^{0,0} \arrow[u, "d_V"] \arrow[r, "d_H"] & D^{0,1} \arrow[u, "-d_V"] \arrow[r, "d_H"] & D^{0,2} \arrow[u] \arrow[r] & \cdots , \end{tikzcd}$$

with the usual "sign trick": the vertical differential is $(-1)^s d_V^{r,s}$. We then remove the edges of this double complex setting $A^{r,s}(\mathcal{T}, M) = A^{r,s} := D^{r+1,s+1}$, $r, s \geq 0$; and denote by $E^{\cdot\cdot}(\mathcal{T}, M) = E^{\cdot\cdot}$ the double complex consisting only of the edges of $D^{\cdot\cdot}$. Compare with [M, pp. 173 ff.].

We are now ready to state a result inspired in the celebrated Kac exact sequence [K, (3.14)]. Let $\mathcal{T}$ be a *vacant* double groupoid and let $\mathcal{D} = \mathcal{V} \bowtie \mathcal{H}$ be corresponding diagonal groupoid, see Prop. 2.9.

Proposition 3.15. *There is an exact sequence*

$$\begin{aligned} &0 \to H^1(\mathcal{D}, M) \to H^1(\mathcal{H}, M) \oplus H^1(\mathcal{V}, M) \to H^0(\operatorname{Tot} A(\mathcal{T}, M)^{\cdot\cdot}, M) \\ &\to H^2(\mathcal{D}, M) \to H^2(\mathcal{H}, M) \oplus H^2(\mathcal{V}, M) \to H^1(\operatorname{Tot} A(\mathcal{T}, M)^{\cdot\cdot}, M) \\ &\to H^3(\mathcal{D}, M) \to H^3(\mathcal{H}, M) \oplus H^3(\mathcal{V}, M). \end{aligned} \tag{3.25}$$

Proof. The short exact sequence of double complexes $0 \to A^{\cdot\cdot} \to D^{\cdot\cdot} \to E^{\cdot\cdot} \to 0$ (with $A^{\cdot\cdot}$ "shifted") induces a long exact sequence in cohomology. It is clear that $H^n(\operatorname{Tot} E^{\cdot\cdot}(\mathcal{T}, M)) = H^n(\mathcal{H}, M) \oplus H^n(\mathcal{V}, M)$, for $n > 0$. We claim that

$$H^n(\operatorname{Tot} D^{\cdot\cdot}(\mathcal{T}, M))) = H^n(\mathcal{D}, M), \quad n > 0. \tag{3.26}$$

Indeed, $H^{\cdot}(\mathcal{D}, M)$ are the cohomology groups of a complex $\operatorname{Hom}_{\Bbbk\mathcal{D}}(F^{\cdot}, M)$, where $F^{\cdot}$ is some free resolution of the trivial $\mathcal{D}$-module. Now, arguing as in [M, Lemma 1.7], we see that $H^{\cdot}(\operatorname{Tot} D^{\cdot\cdot}(\mathcal{T}, M)))$ are also the cohomology groups of a complex $\operatorname{Hom}_{\Bbbk\mathcal{D}}(G^{\cdot}, M)$, where $G^{\cdot}$ is another free resolution of the trivial $\mathcal{D}$-module; this implies (3.26). □

If $M = \Bbbk^\times$, it is natural to denote

$$\operatorname{Aut}(\Bbbk\mathcal{T}) = H^0(\operatorname{Tot} A^{\cdot\cdot}(\mathcal{T}, \Bbbk^\times)), \tag{3.27}$$

$$\operatorname{Opext}(\Bbbk^{\mathcal{V}}, \Bbbk\mathcal{H}) = H^1(\operatorname{Tot} A^{\cdot\cdot}(\mathcal{T}, \Bbbk^\times)), \tag{3.28}$$

by Theorem 3.8 and in view of an extension theory of quantum groupoids yet to be explored. Then (3.25) has in this case the familiar expression

$$\begin{aligned} &0 \to H^1(\mathcal{D}, \Bbbk^\times) \to H^1(\mathcal{H}, \Bbbk^\times) \oplus H^1(\mathcal{V}, \Bbbk^\times) \to \operatorname{Aut}(\Bbbk\mathcal{T}) \\ &\to H^2(\mathcal{D}, \Bbbk^\times) \to H^2(\mathcal{H}, \Bbbk^\times) \oplus H^2(\mathcal{V}, \Bbbk^\times) \to \operatorname{Opext}(\Bbbk\mathcal{T}) \\ &\to H^3(\mathcal{D}, \Bbbk^\times) \to H^3(\mathcal{H}, \Bbbk^\times) \oplus H^3(\mathcal{V}, \Bbbk^\times). \end{aligned} \tag{3.29}$$

3.6. Conclusion.

We have introduced families of quantum groupoids and *a fortiori* of tensor categories. To be sure that these tensor categories are really new, we have to explicitly compute first the $\operatorname{Opext}(\Bbbk\mathcal{T})$ groups, and second to analyze when the corresponding quantum groupoids give rise to equivalent tensor categories. We shall address both questions in subsequent work.

References

[AA] M. Aguiar and N. Andruskiewitsch, *Representations of matched pairs of groupoids and applications to weak Hopf algebras*, Contemp. Math. (to appear), `math.QA/0402118` (2004).

[A] N. Andruskiewitsch, *On the quiver-theoretical quantum Yang-Baxter equation*, `math.QA/0402269` (2004).

[AM] N. Andruskiewitsch and M. Mombelli, *Examples of weak Hopf algebras arising from vacant double groupoids*, `math.QA/0405374` (2004).

[AN] N. Andruskiewitsch and S. Natale, *Braided Hopf algebras arising from matched pairs of groups*, J. Pure Appl. Alg. **182**, 119–149 (2003).

[BSV] S. Baaj, G. Skandalis and S. Vaes, *Measurable Kac cohomology for Bicrossed Products*, preprint `math.OA/0307172`.

[BNSz] G. Böhm, F. Nill and K. Szlachányi, *Weak Hopf algebras I. Integral theory and C^*-structure*, J. Algebra **221**, 385–438 (1999).

[BSz] G. Böhm and K. Szlachányi, *A coassociative C^*-quantum group with nonintegral dimensions*, Lett. in Math. Phys. **35**, 437–456 (1996).

[BS] R. Brown and C. Spencer, *Double groupoids and crossed modules*, Cahiers Topo. et Géo. Diff. **XVII**, 343–364 (1976).

[DVVV] R. Dijkgraaf, C. Vafa, E. Verlinde and H. Verlinde, *The operator algebra of orbifold models*, Commun. Math. Phys. **123**, 485–526 (1989).

[E] C. Ehresmann, *Catégories doubles et catégories structures*, C. R. Acad. Sci. Paris **256**, 1198–1201 (1963).

[ENO] P. Etingof, D. Nikshych and V. Ostrik, *On fusion categories*, preprint math.QA/0203060 (2002).

[H] T. Hayashi, *A brief introduction to face algebras*, in "New trends in Hopf Algebra Theory"; Contemp. Math. **267** (2000), 161–176.

[K] G. Kac, *Extensions of groups to ring groups*, Math. USSR Sbornik **5**, 451–474 (1968).

[KL] T. Kerler and V. Lyubashenko, *Non-Semisimple Topological Quantum Field Theories for 3-Manifolds with Corners*, Lecture Notes in Math. **1765**, Springer-Verlag, Berlin (2001).

[LYZ1] Jiang-Hua Lu, Min Yan and Yong-Chang Zhu, *On Hopf algebras with positive bases*, J. Algebra **237**, 421–445 (2001).

[LYZ2] Jiang-Hua Lu, Min Yan and Yong-Chang Zhu, *Quasi-triangular structures on Hopf algebras with positive bases*, in "New trends in Hopf Algebra Theory"; Contemp. Math. **267**, 339–356 (2000).

[Ma] K. Mackenzie, *Double Lie algebroides and Second-order Geometry, I*, Adv. Math. **94**, 180–239 (1992).

[Mj1] S. Majid, *Physics for algebraists: Non-commutative and non-cocommutative Hopf algebras by a bicrossproduct construction*, J. Algebra **130**, 17–64 (1990).

[Mj2] S. Majid, *Foundations of quantum group theory*, Cambridge Univ. Press, Cambridge (1995).

[M] A. Masuoka, *Hopf algebra extensions and cohomology*, Math. Sci. Res. Inst. Publ. **43**, 167–209 (2002).

[N] D. Nikshych, *On the structure of weak Hopf algebras*, Adv. Math. **170**, 257–286 (2002).

[NV] D. Nikshych and L. Vainerman, *Finite quantum groupoids and their applications*, Math. Sci. Res. Inst. Publ. **43**, 211–262 (2002).

[R] J. Renault, *A groupoid approach to C^*-algebras*, Lecture Notes in Math. **793**, Springer-Verlag, Berlin (1980).

[T1] M. Takeuchi,*Matched pairs of groups and bismash products of Hopf algebras*, Commun. Alg. **9**, 841–882 (1981).

[T2] M. Takeuchi, *Survey on matched pairs of groups. An elementary approach to the ESS-LYZ theory*, preprint (2001); Banach Center Publ., to appear.

[T3] M. Takeuchi, *Survey of braided Hopf algebras*, Contemp. Math. **267**, 301–324 (2000).

[We] J. Westman, *Groupoid theory in algebra, topology and analysis*, preprint (1971); Univ. of California at Irvine.

Facultad de Matemática, Astronomía y Física
Universidad Nacional de Córdoba
CIEM – CONICET
(5000) Ciudad Universitaria, Córdoba, Argentina
E-mail address: andrus@mate.uncor.edu, *URL:* http://www.mate.uncor.edu/andrus
E-mail address: Sonia.Natale@dma.ens.fr, natale@mate.uncor.edu
URL: http://www.mate.uncor.edu/natale