Survey of braided Hopf algebras

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Abstract. We characterize finite dimensional Yetter–Drinfeld braided Hopf algebras intrinsically and review recent results on braided Hopf algebras from this viewpoint.

The notion of a braided Hopf algebra was introduced by Shahn Majid around 1990 (see his survey [**Mj2**]). Since then, braided Hopf algebras have been studied by many people. Here is a list of recent developments of braided Hopf algebra theory.

Hopf module theorem uniqueness of the integral bijectivity of the antipode	Lyubashenko [**Ly2**],
Frobenius property	Fischman, Montgomery and Schneider [**FMS**],
Nichols–Zoeller theorem	Scharfschwerdt [**Schar**],
the automorphism U corresponding to $\mathcal{S}^2$ first trace formula	Andruskiewitsch and Schneider [**AS**],
second trace formula	Doi [**D3**],
ribbon transformation U^2-formula	Sommerhäuser [**So**],
$\mathcal{S}^4$-formula	Bespalov, Kerler, Lyubashenko and Turaev [**BKLT**].

Most of the known properties of finite dimensional Hopf algebras have been generalized to finite dimensional braided Hopf algebras.

To go ahead, we have to clarify what a braided Hopf algebra is. It should be a natural generalization of usual Hopf algebras over a field. Majid has chosen a categorical way to generalize the Hopf algebra notion. He has considered a braided tensor category as a generalization of a symmetric tensor category and has defined

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a Hopf algebra object in a braided tensor category. Braided Hopf algebras in this sense are well studied by people in Majid school.

The Yetter–Drinfeld category is a very nice example of braided tensor categories. Braided Hopf algebras in this category are algebras and coalgebras in the usual sense so that they can be considered as natural generalization of usual Hopf algebras. It has recently been noticed that finite dimensional Hopf algebra theory can be generalized to finite dimensional Yetter–Drinfeld Hopf algebras. Andruskiewitsch, Doi, Scharfschwerdt, Schneider and Sommerhäuser take this viewpoint. They could be called the Yetter–Drinfeld school. So there are the abstract category school (or Majid school) and the Yetter–Drinfeld school in braided Hopf algebra theory.

In this survey, I take a third way. I define a braided Hopf algebra over a field as an algebraic system without using braided categories. It is an algebra and a coalgebra having a Yang–Baxter operator on it, satisfying some compatibility conditions. The precise definition is given in §5. Our main theorem 5.7 tells that we can characterize finite dimensional Yetter–Drinfeld Hopf algebras completely in this way. To prove this, we require some long preliminaries as developed in the previous 4 sections. Coquasitriangular bialgebras and the FRT construction play essential roles. In §2, we explain Schauenburg's Hopf version of the FRT construction in some detail.

Once one characterizes Yetter–Drinfeld Hopf algebras, one can reformulate all known results on Yetter–Drinfeld Hopf algebras without using Yetter–Drinfeld categories. However, some remark is required concerning distinction of categorical and non-categorical objects (§6). In the final section 7, we survey recent main results on braided Hopf algebras from this viewpoint.

1. CQT bialgebras and the FRT construction

The notion of a coquasitriangular (CQT) bialgebra was first mentioned by Majid [**Mj1**] and then formulated and studied by Larson–Towber [**LT**], Hayashi [**H**] and Schauenburg [**Schau**] around 1991. If R is a Yang–Baxter operator on a finite dimensional vector space V, there is a CQT bialgebra $A(R)$ such that V is a right $A(R)$ comodule and that R is induced from the CQT structure. We review this construction briefly in order to apply it to our characterization of Yetter–Drinfeld braided Hopf algebras.

Throughout the paper, we work over a fixed field $\mathbf{k}$. A CQT bialgebra means a pair (A, σ) where A is a bialgebra and $\sigma : A \times A \to \mathbf{k}$ a (convolution) invertible bilinear form satisfying:

$$\sum \sigma(x_{(1)}, y_{(1)}) x_{(2)} y_{(2)} = \sum \sigma(x_{(2)}, y_{(2)}) y_{(1)} x_{(1)}, \tag{1.1.1}$$

$$\sigma(xy, z) = \sum \sigma(x, z_{(1)}) \sigma(y, z_{(2)}), \tag{1.1.2}$$

$$\sigma(x, yz) = \sum \sigma(x_{(2)}, y) \sigma(x_{(1)}, z) \tag{1.1.3}$$

for $x, y, z \in A$. We use the usual sigma notation.

If (A, σ) is a CQT bialgebra, the category of right A comodules $\mathcal{M}^A$ is a braided tensor category. If V and W are right A comodules, $V \otimes W$ is a right A comodule by the codiagonal coaction

$$V \otimes W \to V \otimes W \otimes A, \quad v \otimes w \mapsto \sum v_{(0)} \otimes w_{(0)} \otimes v_{(1)} w_{(1)}.$$

The CQT structure σ induces the following braiding

$$\tau_{V,W} : V \otimes W \xrightarrow{\sim} W \otimes V, \quad v \otimes w \mapsto \sum \sigma(v_{(1)}, w_{(1)}) w_{(0)} \otimes v_{(0)}. \tag{1.2.1}$$

Refer to [**JS**], [**K**], [**Mo**], for generalities on braided tensor categories.

With the above notation, note that $V \otimes W$ is a right $A \otimes A$ comodule, hence a left $(A \otimes A)^*$ module. The braiding (1.2.1) is simply the action of σ composed with the trivial flip:

$$\tau_{V,W} : V \otimes W \xrightarrow[\sim]{\sigma \rightharpoonup} V \otimes W \xrightarrow[\sim]{\text{flip}} W \otimes V. \tag{1.2.2}$$

This observation will be useful later.

Let V be a finite dimensional vector space with basis $\{e_i\}$. Let $\{e_i^*\}$ be the dual basis for V^*. We put $C_V = V^* \otimes V$. This has the following coalgebra structure.

$$\Delta(f \otimes v) = \sum_i (f \otimes e_i) \otimes (e_i^* \otimes v), \tag{1.3.1}$$

$$\varepsilon(f \otimes v) = f(v) \tag{1.3.2}$$

and V has the following right C_V comodule structure.

$$V \to V \otimes C_V, \quad v \mapsto \sum_i e_i \otimes (e_i^* \otimes v) \tag{1.3.3}$$

where $f \in V^*$, $v \in V$. C_V is called the <u>co-endomorphism</u> coalgebra for V. If C is a coalgebra, there is a $1-1$ correspondence between a coalgebra map $\phi : C_V \to C$ and a right coaction $\rho : V \to V \otimes C$.

Let V and W be finite dimensional vector spaces and $f : V \to W$ a linear map. Let $I(f)$ be the image of

$$W^* \otimes V \xrightarrow{(f^* \otimes \mathrm{id},\, -\,\mathrm{id} \otimes f)} C_V \oplus C_W. \tag{1.4.1}$$

It is the smallest coideal of $C_V \oplus C_W$ such that f becomes a comodule map for $(C_V \oplus C_W)/I(f)$, and called the <u>co-centralizer</u> of f [**T1**, 3.3]. Assume V and W are right comodules for a coalgebra C. Let $\phi_V : C_V \to C$ and $\phi_W : C_W \to C$ be the corresponding coalgebra maps. The image

$$(\phi_V \oplus \phi_W)(I(f)) \tag{1.4.2}$$

is also denoted by the same symbol $I(f)$. It is the smallest coideal of C such that f becomes a $C/I(f)$ comodule map. If $\sigma \in C^*$, then f commutes with the action of σ iff σ vanishes on $I(f)$.

Let V be a finite dimensional vector space. A linear automorphism $R : V \otimes V \xrightarrow{\sim} V \otimes V$ is called a Yang–Baxter operator if it satisfies

$$(R \otimes \mathrm{id})(\mathrm{id} \otimes R)(R \otimes \mathrm{id}) = (\mathrm{id} \otimes R)(R \otimes \mathrm{id})(\mathrm{id} \otimes R) \tag{1.5.1}$$

on $V \otimes V \otimes V$. The pair (V, R) will be called a <u>YB space</u>. The tensor algebra $T(C_V)$ has a canonical bialgebra structure so that $V \otimes V$ becomes a right comodule for it. We put

$$A(R) = T(C_V)/(I(R)) \tag{1.5.2}$$

the quotient bialgebra by the ideal generated by the co-centralizer $I(R)$. This construction is due to Faddeev–Reshetikhin–Takhtajan [**FRT**].

1.6. Theorem. [**LT, H, Schau**] *There is a unique CQT structure σ on $A(R)$ such that the YB operator R equals the braiding $\tau_{V,V}$ (1.2.1) arising from σ.*

Let (A, σ) be a CQT bialgebra, let U, V, W be finite dimensional right A comodules, and let $f : V \to W$ be a linear map.

1.7. Lemma. 1. *The folowing are equivalent.*
(a) $(f \otimes \mathrm{id})\tau_{U,V} = \tau_{U,W}(\mathrm{id} \otimes f)$,
(b) $\sigma(\mathrm{Im}(\phi_U), I(f)) = 0$,
(c) $\sigma^{-1}(\mathrm{Im}(\phi_U), I(f)) = 0$.

2. *The folowing are equivalent.*
(a) $(\mathrm{id} \otimes f)\tau_{V,U} = \tau_{W,U}(f \otimes \mathrm{id})$,
(b) $\sigma(I(f), \mathrm{Im}(\phi_U)) = 0$,
(c) $\sigma^{-1}(I(f), \mathrm{Im}(\phi_U)) = 0$.

Here, $\phi_U : C_U \to A$ denotes the coalgebra map corresponding to the right coaction, and the co-centralizer $I(f)$ is taken inside A. σ^{-1} denotes the convolution inverse of σ.

Proof. 1. By the observation (1.2.2), (1a) means

$$\mathrm{id} \otimes f : U \otimes V \to U \otimes W$$

commutes with the action of σ. This is equivalent to saying that σ vanishes on $I(\mathrm{id} \otimes f) = C_U \otimes I(f)$, yielding (1a)$\Leftrightarrow$(1b). Since σ is convolution invertible, $\mathrm{id} \otimes f$ commutes with the action of σ iff it does with the action of σ^{-1} yielding (1a)$\Leftrightarrow$(1c).

2. This is proved similarly.

□

We say the linear map f <u>commutes with the braiding with</u> U if the identities (1a) and (2a) above hold. Let $\{U_\alpha\}$ be a family of finite dimensional right A comodules such that the images of $\phi_{U_\alpha} : C_{U_\alpha} \to A$ generate A as an algebra. Let $(I(f))$ be the ideal of A generated by the cocentralizer $I(f)$.

1.8. Proposition. *If $f : V \to W$ commutes with the braidings with U_α for all α, then the CQT structure σ on A factors through $A/(I(f))$. Thus $A/(I(f))$ is a quotient CQT bialgebra of A.*

Proof. By using (1.1.2) and (1.1.3), one deduces from 1.7 (1b), (1c) and (2b), (2c) that σ and σ^{-1} vanish on $A \times (I(f))$ and $(I(f)) \times A$, yielding the assertion. □

1.9. Corollary. *Let R be a Yang–Baxter operator on a finite dimensional vector space U, let V and W be finite dimensional right $A(R)$ comodules and let $f : V \to W$ be a linear map. If f commutes with the braiding with U, then $A(R)/(I(f))$ is a quotient CQT bialgebra of $A(R)$.*

2. CQT Hopf algebras and rigid Yang–Baxter operators

A bilinear form σ on a coalgebra C is called skew-invertible if there is a bilinear form σ^s (called the skew-inverse of σ) on C such that

$$\sum \sigma(x_{(1)}, y_{(2)})\sigma^s(x_{(2)}, y_{(1)}) = \varepsilon(x)\varepsilon(y) = \sum \sigma(x_{(2)}, y_{(1)})\sigma^s(x_{(1)}, y_{(2)}) \tag{2.1}$$

for $x, y \in C$.

We begin with a result of Doi [**D1**, Theorem 1.5]:

2.2. Theorem. *Let (A, σ) be a CQT Hopf algebra with antipode $\mathcal{S}$.*

1. *The inverse* σ^{-1} *is given by*

$$\sigma^{-1}(x,y)=\sigma(\mathcal{S}(x),y),\quad x,y\in A.$$

2. σ *is skew-invertible with skew-inverse*

$$\sigma^{s}(x,y)=\sigma(x,\mathcal{S}(y)),\quad x,y\in A.$$

3. σ^{-1} *is skew-invertible with skew-inverse*

$$(\sigma^{-1})^{s}(x,y)=\sigma^{-1}(\mathcal{S}(x),y)=\sigma(\mathcal{S}^2(x),y),\quad x,y\in A.$$

4. *If we put*

$$\lambda(x)=\sum\sigma^{s}(x_{(1)},x_{(2)}),\ \lambda'(x)=\sum(\sigma^{-1})^{s}(x_{(1)},x_{(2)}),\quad x\in A,$$

then λ *and* λ' *are inverses of each other in* A^*.

5. *We have*

$$\mathcal{S}^2(x)=\sum\lambda'(x_{(1)})x_{(2)}\lambda(x_{(3)}),\quad x\in A.$$

This yields the following consequence which will be used in §4.

2.3. Lemma. *Let* (A,σ) *be a CQT Hopf algebra. If* $I\subset A$ *is a bi-ideal such that* $\sigma(A,I)=\sigma(I,A)=\sigma^{-1}(A,I)=\sigma^{-1}(I,A)=0$, *then* $I+\mathcal{S}(I)$ *is a Hopf ideal of* A *and* $A/(I+\mathcal{S}(I))$ *is a quotient CQT Hopf algebra of* A.

This means that every quotient CQT bialgebra has a quotient CQT Hopf algebra.

Proof. Note that $\mathcal{S}$ is bijective and we have

$$\sigma(x,y)=\sigma(\mathcal{S}(x),\mathcal{S}(y)),\quad \sigma^{-1}(x,y)=\sigma^{-1}(\mathcal{S}(x),\mathcal{S}(y))$$

for $x,y\in A$. It follows that $\mathcal{S}(I)$ satisfies the same condition as I. By the same reason, we have

$$\sigma^{s}(A,I)=\sigma^{s}(I,A)=(\sigma^{-1})^{s}(A,I)=(\sigma^{-1})^{s}(I,A)=0.$$

In particular, λ and λ' vanish on I. Hence I is $\mathcal{S}^2$-stable and $I+\mathcal{S}(I)$ becomes a Hopf ideal. The rest is obvious. $\square$

Schauenburg [**Schau**, 3.2.9] has obtained the Hopf algebra version of the FRT construction Theorem 1.6. To review this result (and also for our later applications), we introduce some diagrammatic notations.

Let V be a finite dimensional vector space. Let

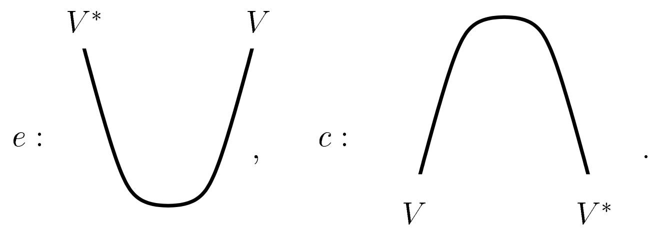
$$e:V^*\otimes V\to\mathbf{k}\quad(\text{resp. } c:\mathbf{k}\to V\otimes V^*)\tag{2.4.1}$$

be the evaluation (resp. coevaluation) map, i.e.,

$$e(f\otimes v)=f(v)\quad(\text{resp. } c(1)=\sum_i e_i\otimes e_i^*)\tag{2.4.2}$$

where $f \in V^*$, $v \in V$, and $\{e_i\}$, $\{e_i^*\}$ are bases of V and V^* dual to each other. These maps will be illustrated by:

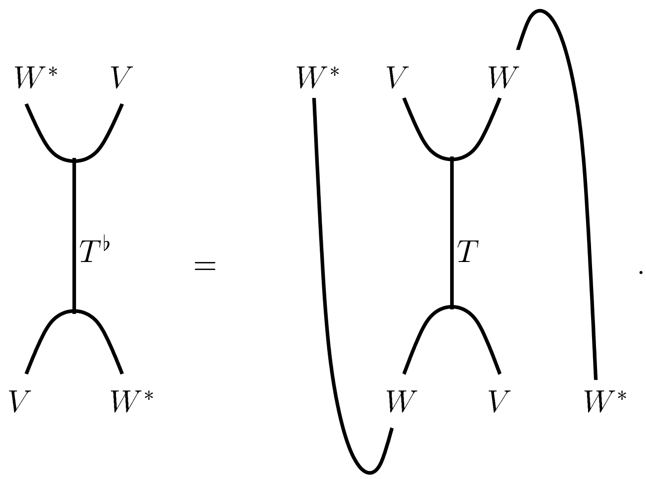
(2.4.3)



All maps go from top to bottom.

Let V and W be finite dimensional vector spaces and let $T : V \otimes W \to W \otimes V$ be a linear map. Lyubashenko [**Ly1**] has introduced the linear map $T^\flat : W^* \otimes V \to V \otimes W^*$ defined as follows

(2.5)



Let (V, R) be a finite dimensional YB space. We say that (V, R) is rigid if $R^\flat : V^* \otimes V \to V \otimes V^*$ is an isomorphism. If this is the case, $(R^{-1})^\flat : V^* \otimes V \to V \otimes V^*$ is also an isomorphism [**LS**, 0.2].

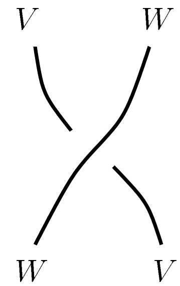
The following is the Hopf algebra version of Theorem 1.6.

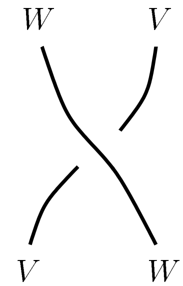
2.6. Theorem ([**Schau**, 3.2.9]). *Let (V, R) be a rigid YB space. There is a CQT Hopf algebra $(H(R), \sigma)$ such that V is a right $H(R)$ comodule and that R coincides with the braiding $\tau_{V,V}$ arising from σ.*

We explain briefly how to construct $H(R)$.

An object of a braided tensor category is called rigid (or finite [**T2**]) if it has a left dual object. From the axiom of braided categories, a rigid object has a right dual object, too. If (A, σ) is a CQT Hopf algebra, a right A comodule V is rigid iff V is finite dimensional [**T2**, 2.11]. In fact, if V is finite dimensional, V^* is a left A comodule. If we turn it into a right A comodule via the antipode $\mathcal{S}$, this gives a left dual of V.

The braiding and its inverse in a braided tensor category will be illustrated as follows:

(2.7) $\tau_{V,W}$: 

, $\tau_{V,W}{}^{-1}$: 

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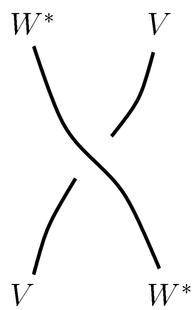
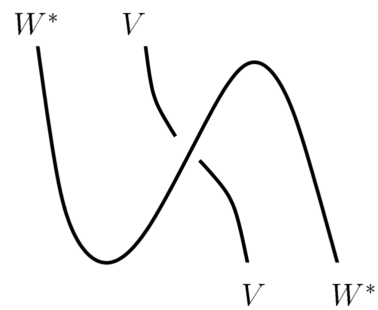
2.8. Proposition. *Let (A,σ) be a CQT Hopf algebra and let V, W be finite dimensional A comodules. Let V^*, W^* be the left duals of V, W. We have*

1. $\tau_{V,W^*} = (\tau^\flat_{V,W})^{-1}$,
2. $\tau_{V^*,W} = (\tau_{V,W}{}^{-1})^\flat$,
3. $\tau_{V^*,W^*} = (\tau_{V,W})^* = (\tau_{V,W})^{\flat\flat}$.

This follows easily from the following diagrams.

(2.8.1) 

= 

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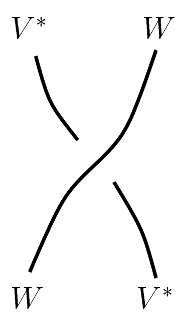
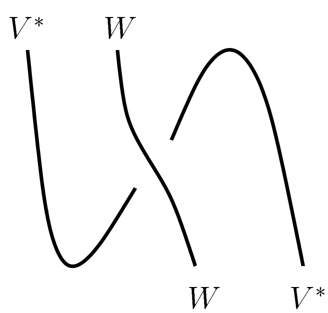
(2.8.2) 

= 

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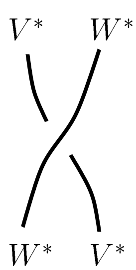
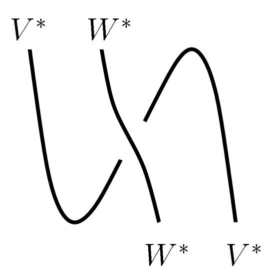
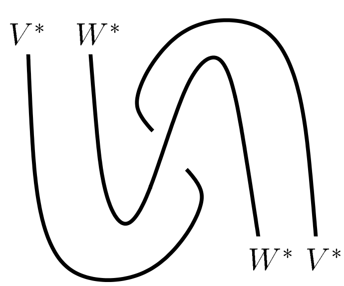
(2.8.3) 

= 

= 

(cf. [**T3**, (2.6.1)-(2.6.3)]).

In particular, this means $(V, \tau_{V,V})$ is a rigid YB space.

Conversely, let (V, R) be a rigid YB space. We put

$$H(R) = T(C_V \oplus C_{V^*})/(I(R), I(e), I(c))$$

the quotient bialgebra of the tensor algebra of $C_V \oplus C_{V^*}$ by the ideal generated by the co-centralizers for

$$R : V \otimes V \to V \otimes V, \quad e : V^* \otimes V \to \mathbf{k}, \quad c : \mathbf{k} \to V \otimes V^*.$$

Note that $V \otimes V$, $V^* \otimes V$, $V \otimes V^*$ and $\mathbf{k}$ are all right comodules for $T(C_V \oplus C_{V^*})$.

Theorem 2.6 is stated explicitly as follows:

2.9. Theorem. 1. *There is a unique CQT structure σ on $H(R)$ such that $R = \tau_{V,V}$ the braiding arising from σ.*
2. *$H(R)$ is a Hopf algebra.*

Proof. 1 Let us write $R = R_{V,V}$. Define linear isomorphisms

$$R_{V,V^*} : V \otimes V^* \xrightarrow{\sim} V^* \otimes V,$$
$$R_{V^*,V} : V^* \otimes V \xrightarrow{\sim} V \otimes V^*,$$
$$R_{V^*,V^*} : V^* \otimes V^* \xrightarrow{\sim} V^* \otimes V^*$$

similarly as 1–3 of 2.8 by replacing τ with R. These maps form a family of Yang–Baxter operators [**Schau**, 3.1.1, 3.1.9]. If we put

$$A(\tilde{R}) = T(C_V \oplus C_{V^*})/(I(R_{V,V}), I(R_{V,V^*}), I(R_{V^*,V}), I(R_{V^*,V^*}))$$

these braidings are induced from a uniquely determined CQT structure σ on $A(\tilde{R})$. This is a multi-vector-space version of Theorem 1.6 (see [**Schau**, 3.2.9]). Since R_{V,V^*}, $R_{V^*,V}$ and R_{V^*,V^*} are constructed from R, e, c, we have

$$I(R_{V,V^*}),\ I(R_{V^*,V}),\ I(R_{V^*,V^*}) \subset (I(R), I(e), I(c))$$

hence $H(R)$ is a quotient bialgebra of $A(\tilde{R})$. It is easy to see that e and c commute with the braidings with V and V^*. Since $A(\tilde{R})$ is generated by the images of C_V and C_{V^*}, applying Proposition 1.8, it follows that $H(R)$ is a quotient CQT bialgebra of $A(\tilde{R})$. This proves 1.

Proof of 2 will be given in Appendix. $\square$

3. The Yetter–Drinfeld category

So far, we have worked with CQT bialgebras and Hopf algebras. But many authors prefer to work with Yetter–Drinfeld categories. We recall the definition. Let L be a Hopf algebra with bijective antipode. Let V be a right L module and right L comodule. It is called a Yetter–Drinfeld module if we have

$$\sum v_{(0)} \cdot h_{(1)} \otimes v_{(1)} h_{(2)} = \sum (v \cdot h_{(2)})_{(0)} \otimes h_{(1)} (v \cdot h_{(2)})_{(1)} \tag{3.1}$$

for $h \in L$, $v \in V$ (see [**Mo**, 10.6.10]). If V and W are right L Yetter–Drinfeld modules, then so is $V \otimes W$ with the diagonal action and coaction. The category of right L Yetter–Drinfeld modules $\mathcal{YD}_L^L$ is a braided tensor category with braiding

$$\tau_{V,W} : V \otimes W \xrightarrow{\sim} W \otimes V, \quad v \otimes w \mapsto \sum w_{(0)} \otimes v \cdot w_{(1)}. \tag{3.2}$$

Let (A, σ) be a CQT Hopf algebra. If V is a right A comodule, it is a right A module with action

$$v \cdot a = \sum \sigma(v_{(1)}, a) v_{(0)}, \quad v \in V, \ a \in A. \tag{3.3}$$

The Yetter–Drinfeld condition (3.1) is satisfied. Thus V becomes a right A Yetter–Drinfeld module. If V and W are right A comodules, the right A action on $V \otimes W$ coincides with the diagonal action. It is easy to see that the braiding $\tau_{V,W}$ of (1.2.1) coincides with (3.2). We note that the antipode of A is bijective (2.2). We conclude that we have a canonical functor

$$\mathcal{M}^A \to \mathcal{YD}^A_A, \quad V \mapsto V \tag{3.4}$$

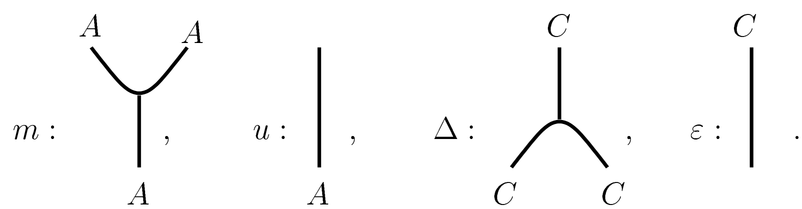
which preserves the tensor product, the unit object and the braiding (cf. [**Mo**, 10.6.14]).

4. Bialgebras and Hopf algebras in braided tensor categories

We review the notions of bialgebras and Hopf algebras in a braided tensor category following [**Mj2, T2**] as well as their basic properties.

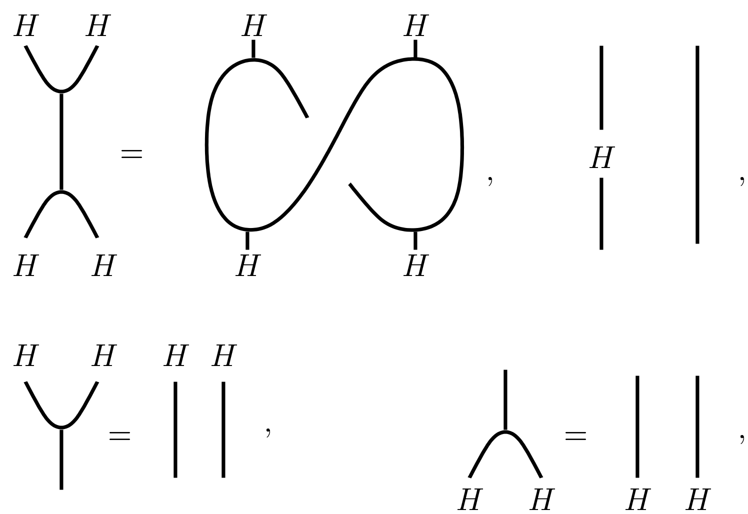
Let $\mathcal{M}$ be a braided tensor category with unit object I. An algebra (resp. coalgebra) in $\mathcal{M}$ means a triple (A, m, u) (resp. (C, Δ, ε)) where $A \in \mathcal{M}$ (resp. $C \in \mathcal{M}$), $m : A \otimes A \to A$, $u : I \to A$ (resp. $\Delta : C \to C \otimes C$, $\varepsilon : C \to I$) satisfying the (co)associativity and the (co)unit condition. The structures are illustrated by symbols:

(4.1)



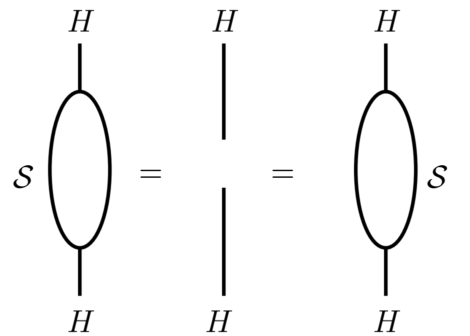
A bialgebra in $\mathcal{M}$ means a 5-tuple $(H, m, u, \Delta, \varepsilon)$ where $H \in \mathcal{M}$, (H, m, u) is an algebra and (H, Δ, ε) a coalgebra satisfying the following compatibility condition:

(4.2)

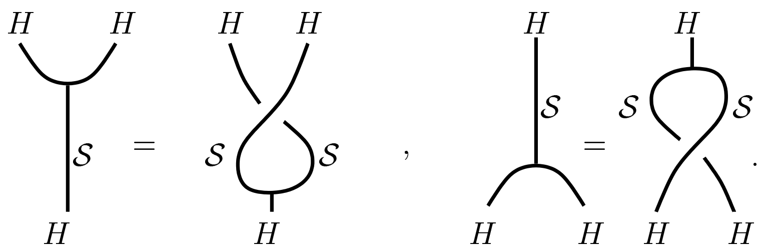


i.e., $\Delta \circ m = (m \otimes m)(\mathrm{id} \otimes \tau \otimes \mathrm{id})(\Delta \otimes \Delta)$, $\varepsilon \circ u = \mathrm{id}$, $\varepsilon \circ m = \varepsilon \otimes \varepsilon$, $\Delta \circ u = u \otimes u$. We use the notation (2.7).

If H is a bialgebra, the set of $\mathcal{M}$ maps $H \to H$ becomes a monoid under the convolution product. If the identity is convolution invertible, the inverse $\mathcal{S}$ is called the antipode and H a Hopf algebra in this case. Thus the antipode $\mathcal{S}$ is characterized by the diagram

(4.3) 

It is known [**Mj2**, Lemma 2.3] that $\mathcal{S}$ is a braided anti-bialgebra map in the following sense:

(4.4) 

5. Braided bialgebras and Hopf algebras

In this section we define braided bialgebras and Hopf algebras over $\mathbf{k}$ without using braided tensor categories. We discuss relations with bialgebras and Hopf algebras in some known braided categories.

5.1. Definition. A braided $\mathbf{k}$ bialgebra means a 6-tuple $(H, m, u, \Delta, \varepsilon, R)$ where H is a $\mathbf{k}$ vector space and the following are $\mathbf{k}$ linear maps

$$m : H \otimes H \to H,\ u : \mathbf{k} \to H,\ \Delta : H \to H \otimes H,\ \varepsilon : H \to \mathbf{k},\ R : H \otimes H \xrightarrow{\sim} H \otimes H$$

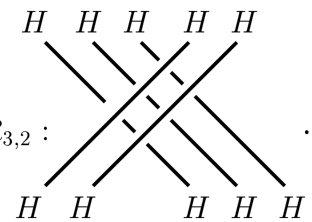
satisfying the following conditions:

1. (H, m, u) is a $\mathbf{k}$ algebra,
2. (H, Δ, ε) is a $\mathbf{k}$ coalgebra,
3. R is a Yang–Baxter operator on H,
4. The structures m, u, Δ and ε commute with R,
5. $\varepsilon : H \to \mathbf{k}$ is an algebra map and $u : \mathbf{k} \to H$ is a coalgebra map,
6. we have
$$\Delta \circ m = (m \otimes m)(\mathrm{id} \otimes R \otimes \mathrm{id})(\Delta \otimes \Delta).$$

Although we do not work in a braided category, we will use the previous diagrammatic notations. We use (4.1) to denote m, u, Δ and ε for H, and (2.7) to denote R and R^{-1}. The braiding R induces braidings

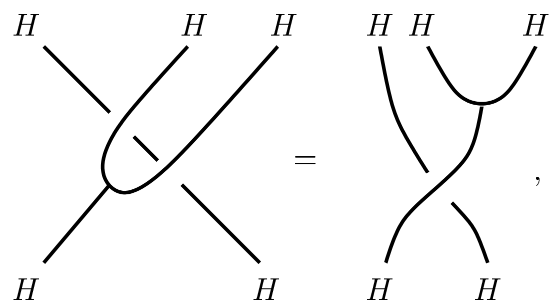
$$R_{m,n} : H^{\otimes m} \otimes H^{\otimes n} \xrightarrow{\sim} H^{\otimes n} \otimes H^{\otimes m}$$

in a canonical way. For instance, we have

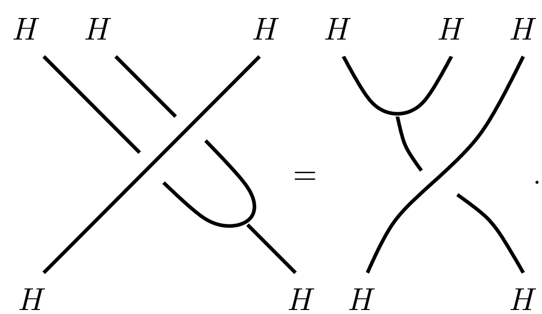
$R_{3,2}$: 

.

(4) is interpreted in terms of these braidings (see below Lemma 1.7). For example, "m commutes with R" means we have



,



.

(5) and (6) are illustrated by diagrams (4.2).

Let H be a finite dimensional braided bialgebra with braiding R. By Theorem 1.6, H is a right comodule for the CQT bialgebra $A(R)$ and R is induced from the CQT structure. By Corollary 1.9, $A = A(R)/(I(m), I(u), I(\Delta), I(\varepsilon))$ is a quotient CQT bialgebra of A. By construction, m, u, Δ and ε are A comodule maps. It is obvious to see that $(H, m, u, \Delta, \varepsilon)$ becomes a braided bialgebra in the braided tensor category $\mathcal{M}^A$. This proves:

5.2. Proposition. 1. *Let A be a CQT bialgebra. If $(H, m, u, \Delta, \varepsilon)$ is a bialgebra in $\mathcal{M}^A$, then $(H, m, u, \Delta, \varepsilon, \tau_{H,H})$ is a braided* **k** *bialgebra, where $\tau_{H,H}$ means the braiding in $\mathcal{M}^A$.*
2. *If $(H, m, u, \Delta, \varepsilon, R)$ is a finite dimensional braided* **k** *bialgebra, there is a CQT bialgebra A such that*

(a) *H is a right A comodule,*
(b) $R = \tau_{H,H}$,
(c) $(H, m, u, \Delta, \varepsilon)$ *is a bialgebra in* $\mathcal{M}^A$.

5.3. Definition. A braided bialgebra H is called rigid if it is finite dimensional and the braiding R is rigid (see below (2.5)).

If H is a rigid braided bialgebra, it is a right comodule for the CQT Hopf algebra $H(R)$ (Theorem 2.9). Condition (4) of 5.1 implies that m, u, Δ and ε commute with the braiding with H^*, too. More generally, assume (A, σ) is a CQT Hopf algebra in Lemma 1.7. One can show easily if f commutes with the braiding with U, then it also commutes with the braiding with U^* (use (2.8)). Applying Proposition 1.8, we see $H(R)/(I(m), I(u), I(\Delta), I(\varepsilon))$ is a quotient CQT bialgebra. By Lemma 2.3, this admits a quotient CQT Hopf algebra. Thus we have:

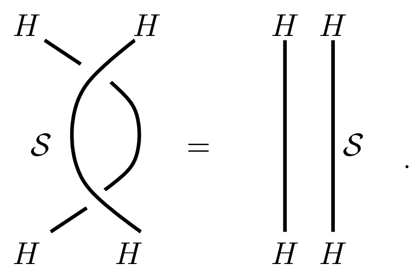
5.4. Proposition. 1. *Let A be a CQT Hopf algebra. If H is a finite dimensional bialgebra in $\mathcal{M}^A$, then it is a rigid braided bialgebra.*
2. *Conversely, if H is a rigid braided bialgebra, it can be realized as a bialgebra in $\mathcal{M}^A$ for some CQT Hopf algebra A.*

If H is a braided bialgebra, it is an algebra and a coalgebra. Hence $\mathrm{End}_{\mathbf{k}}(H)$ has the structure of an algebra with the convolution product. If the identity is invertible, the inverse $\mathcal{S}$ is called the antipode. A braided bialgebra with antipode is called a braided Hopf algebra.

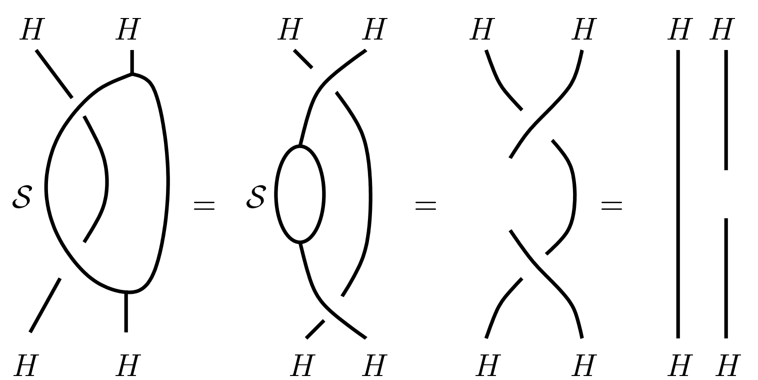
5.5. Proposition. *Let H be a braided Hopf algebra with braiding R and antipode $\mathcal{S}$. Then $\mathcal{S}$ commutes with R.*

Proof. The antipode $\mathcal{S}$ satisfies condition (4.3). We claim we have

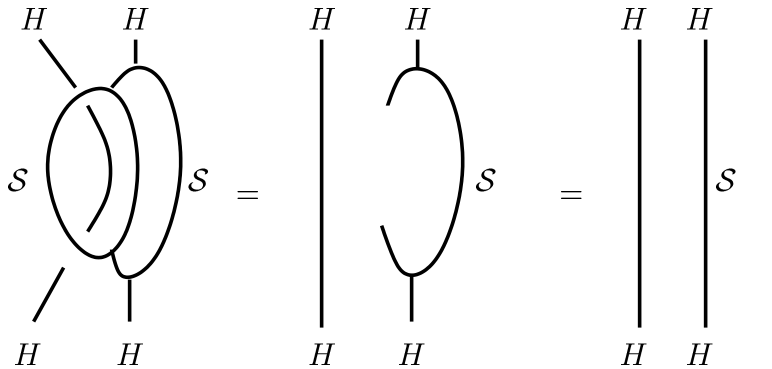
(5.5.1)



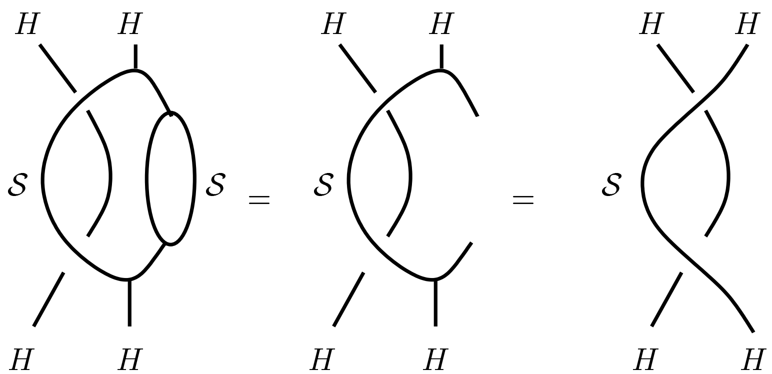
In fact, since m, u, Δ and ε commute with R, we have



Hence we have

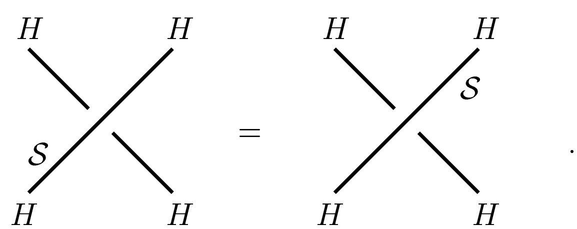


By the (co)associativity, the left-hand side equals



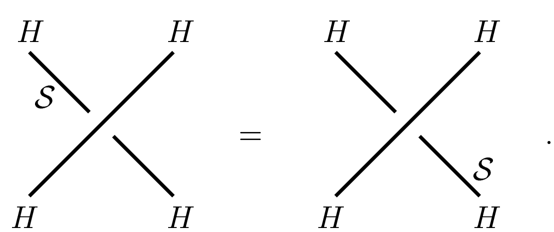
This yields (5.5.1). It implies immediately

(5.5.2)



Similarly we have

(5.5.3)



Hence $\mathcal{S}$ commutes with R. $\square$

Using this proposition, we have the following Hopf version of 5.2 and 5.4. The proof is completely similar.

5.6. Proposition. 1. *Let A be a CQT bialgebra (resp. Hopf algebra). If H is a Hopf algebra (resp. finite dimensional Hopf algebra) in $\mathcal{M}^A$, then it is a braided Hopf algebra (resp. rigid braided Hopf algebra).*

2. *Conversely, let H be a finite dimensional (resp. rigid) braided Hopf algebra. Then it can be realized as a Hopf algebra in $\mathcal{M}^A$ for some CQT bialgebra (resp. Hopf algebra) A.*

Let L be a Hopf algebra with bijective antipode. A bialgebra (resp. Hopf algebra) in the Yetter–Drinfeld category $\mathcal{YD}_L^L$ is called a Yetter–Drinfeld bialgebra (resp. Hopf algebra). By means of the functor (3.4), we can characterize rigid Yetter–Drinfeld Hopf algebras as follows:

5.7. Theorem. 1. *A Yetter–Drinfeld bialgebra (resp. Hopf algebra) is a braided bialgebra (resp. Hopf algebra). It is rigid if finite dimensional.*

2. *Conversely, an arbitrary rigid braided bialgebra (resp. Hopf algebra) can be realized as a Yetter–Drinfeld bialgebra (resp. Hopf algebra).*

Obviously, there are many possible realizations in 2.

6. "Categorical" vs. "non-categorical"

The previous Theorem 5.7 characterizes finite dimensional Yetter–Drinfeld Hopf algebras allowing to reformulate all known results on Yetter–Drinfeld Hopf algebras in terms of braided Hopf algebras. However, we should be careful in distinction of categorical objects and non-categorical ones. Assume we work in the Yetter–Drinfeld category $\mathcal{YD}_L^L$. For an object V in $\mathcal{YD}_L^L$, we should distinguish subobjects in the category among all subspaces of V. If V, W are two objects in $\mathcal{YD}_L^L$, we should distinguish morphisms $V \to W$ in $\mathcal{YD}_L^L$ among all linear maps $V \to W$.

Let (V, R) be a finite dimensional YB space. A linear endomorphism $f : V \to V$ is called an endomorphism of (V, R) if

$$R(f \otimes f) = (f \otimes f)R. \tag{6.1}$$

We say it is <u>categorical</u> if f commutes with R, i.e.,

$$R(f \otimes \mathrm{id}) = (\mathrm{id} \otimes f)R, \quad R(\mathrm{id} \otimes f) = (f \otimes \mathrm{id})R. \tag{6.2}$$

Obviously, (6.2) implies (6.1). If f is a categorical endomorphism, there is a CQT bialgebra A such that

1. V is a right A comodule,
2. $R = \tau_{V,V}$,
3. f is a right A comodule map, by the argument above 5.2.

If (V, R) is rigid, we can take a CQT Hopf algebra as A, and this means f can be realized as a Yetter–Drinfeld homomorphism.

Let H be a rigid braided Hopf algebra. An automorphism of H means a linear automorphism f of H preserving all structures $m, u, \Delta, \varepsilon$ and R. (f preserves R meaning (6.1)). Then, f commutes with $\mathcal{S}$ automatically. We say f is a categorical automorphism if (6.2) is satisfied. This is the case iff H can be realized as a Yetter–Drinfeld Hopf algebra in such a way that f is a morphism in the category.

Let (V, R) be a YB space. A YB subspace means a subspace W of V such that

$$R : W \otimes W \xrightarrow{\sim} W \otimes W. \tag{6.3}$$

The restriction R_W is a Yang–Baxter operator on W. We call W a categorical YB subspace if R induces isomorphisms

$$R : V \otimes W \xrightarrow{\sim} W \otimes V \text{ and } R : W \otimes V \xrightarrow{\sim} V \otimes W. \tag{6.4}$$

Obviously (6.4) implies (6.3).

Assume V is finite dimensional. The subspace $N(W) = N^{\perp} \otimes W$ of $C_V = V^* \otimes V$ is a coideal which is smallest among all coideals I such that W is a C_V/I comodule. It is called the co-normalizer of W [**T1**, Prop. 3.3].

Let (A, σ) be a CQT bialgebra and V a finite dimensional right A comodule. It is a YB space with YB operator $\tau_{V,V}$.

6.5. Lemma. *If W is a categorical YB subspace of V, then we have*

$$\begin{aligned}\sigma(\operatorname{Im}\phi_V, N(W)) = \sigma(N(W), \operatorname{Im}\phi_V) &= \sigma^{-1}(\operatorname{Im}\phi_V, N(W)) \\ &= \sigma^{-1}(N(W), \operatorname{Im}\phi_V) = 0,\end{aligned}$$

where $N(W)$ means the image of $N(W)$ by $\phi_V : C_V \to A$.

Proof. This is proved similarly as Lemma 1.7. In fact, (6.4) means $V \otimes W$ and $W \otimes V$ are stable under the action of σ. Since $N(V \otimes W) = C_V \otimes N(W)$ and $N(W \otimes V) = N(W) \otimes C_V$, this means σ and σ^{-1} vanish on $C_V \otimes N(W)$ and $N(W) \otimes C_V$. □

The converse is also true.

If A is generated by $\operatorname{Im}\phi_V$, it follows that $A/(N(W))$ is a quotient CQT bialgebra of A.

6.6. Proposition. *Let (V, R) be a finite dimensional YB space and let W be a categorical YB subspace of V.*

1. *There is a CQT bialgebra (A, σ) such that*
   (a) *V is a right A comodule,*
   (b) *$R = \tau_{V,V}$,*
   (c) *W is an A subcomodule of V.*
2. *If (V, R) is rigid, (A, σ) can be taken to be a CQT Hopf algebra. In particular, (W, R_W) is also rigid.*

Proof. 1. This follows immediately by applying 6.5 to $A(R)$.
2. If (A, σ) is a CQT Hopf algebra in Lemma 6.5, then σ and σ^{-1} vanish on $\operatorname{Im}\phi_{V^*} \times N(W)$ and $N(W) \times \operatorname{Im}\phi_{V^*}$, too. Use 2.8 to see this. The assertion follows by applying this to $H(R)$ and using Lemma 2.3.

□

In particular, we can speak of a categorical braided Hopf subalgebra of a braided Hopf algebra.

7. Survey of braided Hopf algebras

We have characterized finite dimensional Yetter–Drinfeld Hopf algebras in Theorem 5.7. This allows us to reformulate recent results on finite dimensional Yetter–Drinfeld Hopf algebras intrinsically. In fact, it is always possible to prove them in our context. We review some of main results and notions below.

Let H be a rigid (hence finite dimensional) braided Hopf algebra over $\mathbf{k}$.

7.1. Proposition. *The antipode $\mathcal{S}$ is bijective.*

See [**T2**, 4.1 Theorem].

7.2. Proposition. *H is a Frobenius algebra and coalgebra.*

See [**FMS**, 5.8 Cor.], [**D2**, Theorem 3]. By a Frobenius coalgebra, we mean the dual algebra H^* is Frobenius. See [**DT**, 8.3 Example].

7.3. Theorem. *Let K be a categorical braided Hopf subalgebra of H. Then H is a free left and right K module.*

See [**Schar, T4**]. This may be called the braided Nichols–Zoeller theorem [**Mo**, 3.1]. Some effort to generalize to non-categorical braided Hopf subalgebras is found in [**G**].

7.4. Definition. We put

$$I_r(H) = \{t \in H \mid th = \varepsilon(h)t, \ \forall h \in H\},$$
$$I_\ell(H) = \{t \in H \mid ht = \varepsilon(h)t, \ \forall h \in H\}$$

which are called right and left integral spaces. These spaces are 1-dimensional [**Ly2**, Theorem 1.6], [**FMS**, 5.8 Cor], [**T2**, 4.6 Theorem], [**D2**, Theorem 3]. These are categorical YB subspaces of H (see (6.4)), since each of them is presented as the equalizer of two morphisms in a braided category, where H is realized as a Hopf algebra. If we take a non-zero right integral $t \in H$, there is an algebra map $\alpha : H \to \mathbf{k}$ such that

$$ht = \alpha(h)t, \quad h \in N. \tag{7.4.1}$$

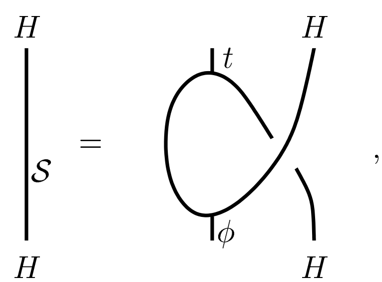
It is called the modular function. Since $I_r(H)$ is categorical, it follows that α is categorical. More strongly, if we realize H as a Hopf algebra in a braided category, then α is a morphism in the category.

Similarly, we can define right and left integral spaces in H^*, $I_r(H^*)$ and $I_\ell(H^*)$ which are 1-dimensional. If we take a non-zero right integral $\phi \in H^*$, there is a group-like element $a \in H$ such that

$$\sum h_{(1)}\phi(h_{(2)}) = \phi(h)a, \quad h \in H. \tag{7.4.2}$$

It is called the modular element. Dually to the above, we see $\mathrm{Ker}(\phi)$ is a categorical YB subspace of H and $a : \mathbf{k} \to H$ is a categorical coalgebra map. The pair can be chosen so that $\phi(t) = 1$ [**D2**, Theorem 3]. Then we have the following $\mathcal{S} - \phi - t$ relation:

(7.4.3)

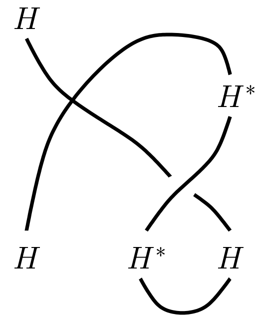


i.e., $\mathcal{S}(h) = \sum(\phi \otimes \mathrm{id})(t_{(1)} \otimes R(t_{(2)} \otimes h))$, $h \in H$.

This relation is the starting point of the study of braided biFrobenius algebras [**DT**, §7].

7.5. Definition. We define a linear map $\theta : H \to H$ called the ribbon map as follows

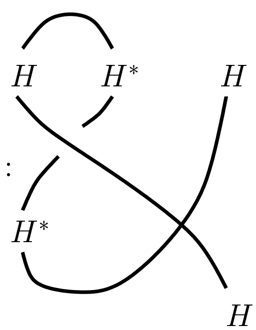
(7.5.1) θ :



where means the trivial flip.

It is invertible with inverse

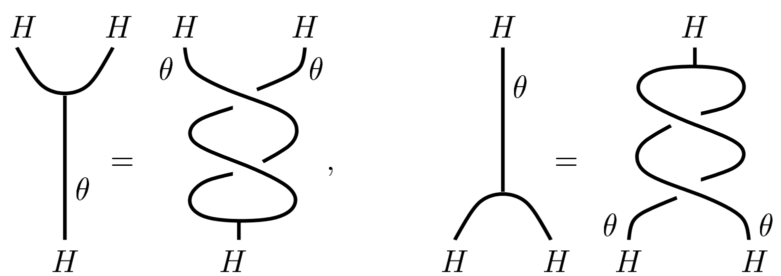
(7.5.2) θ^{-1} :



.

This was introduced by Sommerhäuser [**So**, 3.5]. It is called the ribbon map, since we have

(7.5.3)



In general, θ is a non-categorical YB automorphism of (H, R).

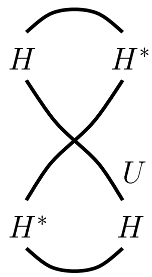
7.6. Proposition. *Put $U = \mathcal{S}^2\theta = \theta\mathcal{S}^2$. It is a (non-categorical) automorphism of the braided Hopf algebra H.*

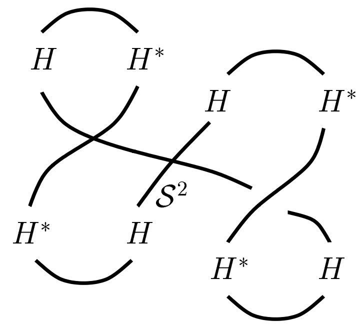
This follows from (7.5.3) and 4.4. The automorphism U is due to [**D3**, (25)]. This was introduced in [**AS**] with notation $\mathcal{J}$.

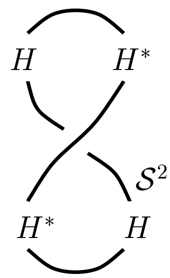
7.7. Theorem (First trace formula). *Choose right integrals $\phi \in H^*$ and $t \in H$ in such a way that $\phi(t) = 1$. Then we have*

$$\mathrm{tr}(U) = \phi(1)\varepsilon(t).$$

See [**AS**, Theorem 7.3], [**D3**, 2.2]. As noted in [**AS**, ibid.], the trace $\mathrm{tr}(U)$ is interpreted as the braided trace $\mathrm{tr}_{\wp}(\mathcal{S}^2)$ as follows:

$\mathrm{tr}(U) \quad =$ 

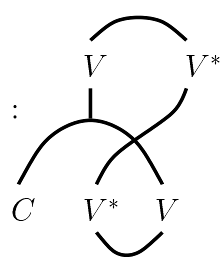
$=$ 

$=$ 

$= \quad \mathrm{tr}_{\wp}(\mathcal{S}^2).$

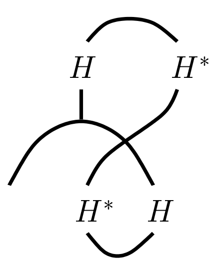
Doi [**D3**] has established the second trace formula for a finite dimensional Yetter–Drinfeld Hopf algebra (with some involutory condition). We review his result below.

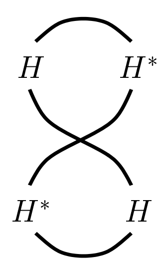
In general, if V is a finite dimensional left C comodule for a coalgebra C, its Larson character $\chi(V) \in C$ [**L**, p. 355] is defined as follows:

(7.8.1) $\chi(V):$ 

.

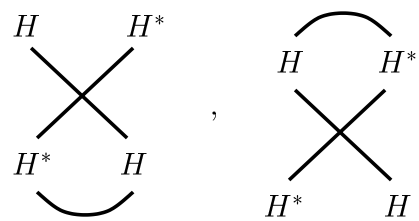
Doi puts $y = \chi(H)$ for the regular comodule structure. It is a cocommutative element and we have $U(y) = y$. The following diagram shows that $\varepsilon(y) = \dim H$.

(7.8.2) $\varepsilon(y) \; =$ 

$=$ 

$= \; \mathrm{tr}(\mathrm{id}).$

Assume that H is realized as a Hopf algebra in $\mathcal{YD}_L^L$ for an involutory Hopf algebra L. Now "L is involutory" means "left dual=right dual" in $\mathcal{YD}_L^L$. Hence the maps

(7.8.3) 

are morphisms in $\mathcal{YD}_L^L$ (i.e., categorical).

7.8. Theorem (Second trace formula). *If H is a finite dimensional Hopf algebra in $\mathcal{YD}_L^L$ for an involutory Hopf algebra L, then*

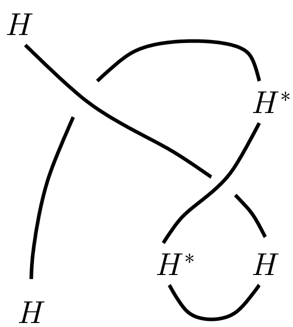
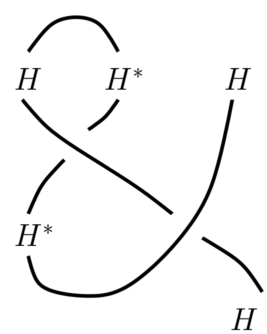
$$(\dim H)\,\mathrm{tr}(U|_{Hy}) = \phi(1)\varepsilon(t)$$

where $\phi \in H^*$ and $t \in H$ are right integrals with $\phi(t) = 1$.

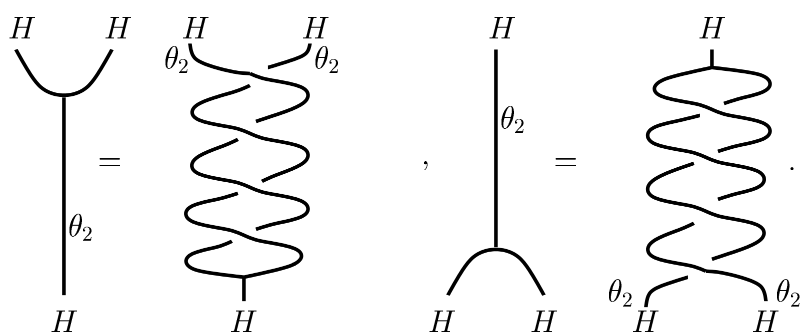
This is [**D3**, 2.6 Theorem].

It is possible to formulate this theorem without using Yetter–Drinfeld categories. It is enough to assume the maps of (7.8.3) are categorical.

Bespalov, Kerler, Lyubashenko and Turaev [**BKLT**] have established Radford's $\mathcal{S}^4$ formula for braided Hopf algebras. They have worked in a braided tensor category. We reformulate their main results in our context. Let H be a rigid braided Hopf algebra. Some analogue of θ is defined as follows:

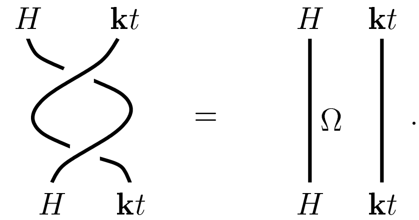
θ_2 :  , θ_2^{-1} :  .

This is called the double ribbon map, since it satisfies:



The map θ_2 is categorical in contrast to θ. It follows that the composite $\mathcal{S}^4\theta_2 = \theta_2\mathcal{S}^4$ is a categorical automorphism of H.

Take right integrals $\phi \in H^*$ and $t \in H$ as before. The space $\mathbf{k}t$ is a categorical YB subspace of H. Hence there is a linear automorphism Ω of H such that



One sees Ω is a categorical automorphism of H. Let $a \in H$ and $\alpha \in H^*$ be the modular element and function, respectively. Then we have

7.9. Theorem ([**BKLT**, Theorem 3.6]).

$$\mathcal{S}^4\theta_2 = (a \cdot a^{-1})(\alpha^{-1} \rightharpoonup \ \leftharpoonup \alpha)\Omega.$$

Here, $a \cdot a^{-1}$, $\alpha^{-1} \rightharpoonup \ \leftharpoonup \alpha$ and Ω are all categorical automorphisms of H commuting with one another. Generalization of this formula to braided bF algebras is found in [**DT**, 16.3 Theorem].

Appendix. Proof of Theorem 2.9 (2)

We begin with the following observation. Let A be a bialgebra and let V be a finite dimensional right A comodule with corresponding coalgebra map $\phi_V : C_V \to A$.

A.1. Proposition. *V has a left dual in the tensor category $\mathcal{M}^A$ iff ϕ_V is convolution invertible.*

Proof. The 'if' part. V^* is a left C_V comodule. The inverse $\phi_V^{-1} : C_V \to A$ is an opposite coalgebra map. If we view V^* as a right A comodule through ϕ_V^{-1}, this gives a left dual of V.

The 'only if' part. Assume V^* has a right A comodule structure $f \mapsto \sum f_{(0)} \otimes f_{(1)}$ such that $e : V^* \otimes V \to \mathbf{k}$ and $c : \mathbf{k} \to V \otimes V^*$ are both A comodule maps. Let $\{e_i\}$ and $\{e_i^*\}$ be bases of V and V^* dual with each other. The fact that $c : 1 \mapsto \sum_i e_i \otimes e_i^*$ is an A comodule map means

(A.1.1) $$\sum_i e_{i,(0)} \otimes e^*_{i,(0)} \otimes e_{i,(1)}e^*_{i,(1)} = \sum_i e_i \otimes e_i^* \otimes 1.$$

Define $\psi : C_V = V^* \otimes V \to A$ by putting

(A.1.2) $$\psi(f \otimes v) = \sum f_{(0)}(v)f_{(1)}, \ f \in V^*, \ v \in V.$$

Recall that $\phi_V : C_V \to A$ is defined by

(A.1.3) $$\phi_V(f \otimes v) = \sum f(v_{(0)})v_{(1)}.$$

We claim $\phi_V * \psi = u\varepsilon$. In fact,

$$\begin{aligned}(\phi_V * \psi)(f \otimes v) &= \sum_i \phi_V(f \otimes e_i)\psi(e_i^* \otimes v) \\ &= \sum_i f(e_{i,(0)})e_{i,(1)}e^*_{i,(0)}(v)e^*_{i,(1)} \qquad \text{(by (A.1.1))} \\ &= \sum_i f(e_i)e_i^*(v) = f(v) = \varepsilon(f \otimes v).\end{aligned}$$

Similarly we have $\psi * \phi_V = u\varepsilon$. Hence ϕ_V is convolution invertible. □

Put $\bar{A} = A^{\mathrm{op}\cdot\mathrm{cop}}$. Call a right A comodule V rigid if it is finite dimensional and has a left dual in $\mathcal{M}^A$. Then V has a right $\bar{A}$ comodule structure through coalgebra map $\phi_V^{-1} : C_V \to \bar{A}$.

A.2. Proposition. *Let V and W be right A comodules.*

1. *If V and W are rigid, then $V \otimes W$ is rigid and its $\bar{A}$ comodule structure is the tensor product of the $\bar{A}$ comodules V and W.*
2. *Assume V and W are rigid. If $f : V \to W$ is an A comodule map, then it is an $\bar{A}$ comodule map.*

Proof. 1. If we identify $C_{V\otimes W} = C_V \otimes C_W$, then $\phi_{V\otimes W}$ factors as:

$$\phi_{V\otimes W} : C_V \otimes C_W \xrightarrow{\phi_V\otimes\phi_W} A \otimes A \xrightarrow{m} A.$$

If ϕ_V and ϕ_W are convolution invertible, then $\phi_{V\otimes W}$ is invertible with inverse

$$\phi^{-1}_{V\otimes W} : C_V \otimes C_W \xrightarrow{\phi_V^{-1}\otimes\phi_W^{-1}} A \otimes A \xrightarrow{m^{op}} A.$$

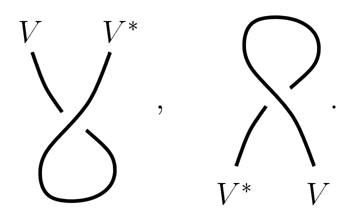
This proves the assertion.

2. In general, assume V and W are finite dimensional right comodules for a coalgebra C, $f : V \to W$ a linear map and $\phi : C \to A$ a convolution invertible linear map with an algebra A. It is easy to see if ϕ vanishes on $I(f)$, then ϕ^{-1} also vanishes on $I(f)$. It is enough to apply this to $C = C_V \oplus C_W$, $\phi = (\phi_V, \phi_W)$.

□

Proof of Theorem 2.9, 2.

Put $A = H(R)$ which is a CQT bialgebra by 2.9, 1. V and V^* are finite dimensional A comodules. Since $e : V^* \otimes V \to \mathbf{k}$ and $c : \mathbf{k} \to V \otimes V^*$ are A comodule maps, V^* is a left dual of V in $\mathcal{M}^A$. The braidings $R_{V^*,V}$ and R_{V,V^*} are A comodule maps by the proof of 2.9, 1. Hence the following pairings are A comodule maps:



By using them, we may view V as a left dual of V^* in $\mathcal{M}^A$. Thus V and V^* are rigid A comodules. They have $\bar{A}$ comodule structures. By Proposition A.2, we see $R : V \otimes V \to V \otimes V$, $e : V^* \otimes V \to \mathbf{k}$ and $c : \mathbf{k} \to V \otimes V^*$ are all $\bar{A}$ comodule maps. By construction, the coalgebra maps $C_V \to \bar{A}$, $C_{V^*} \to \bar{A}$ (corresponding to the $\bar{A}$ comodule structures) induce a bialgebra map $\mathcal{S} : H(R) = A \to \bar{A}$. One has $\mathrm{id} * \mathcal{S} = \mathcal{S} * \mathrm{id} = u\varepsilon$ on $C_V \oplus C_{V^*}$, since $\mathcal{S}$ coincides with ϕ_V^{-1} on C_V and $\phi_{V^*}^{-1}$ on C_{V^*}. Since $C_V \oplus C_{V^*}$ generates A, it follows that $\mathcal{S}$ is the antipode of $H(R)$. $\square$

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