REPRESENTATIONS OF THE SUPER JORDAN PLANE

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Abstract. It is shown that the finite-dimensional simple representations of the super Jordan plane $\mathcal{B}$ are one-dimensional. The indecomposable representations of dimension 2 and 3 of $\mathcal{B}$ are classified. Two families of indecomposable representations of $\mathcal{B}$ of arbitrary dimension are presented.

1. Introduction

Nichols algebras are graded connected algebras with a comultiplication in a braided sense. In particular, the Jordan plane and the super Jordan plane are two Nichols algebras that play an important role in the classification of pointed Hopf algebras with finite Gelfand-Kirillov dimension [AAH1, AAH2].

The Jordan plane was first defined in [G] and considered in many papers, e.g. [AS], see also the references in [AAH2, I]. Its representation theory was studied in [I].

The purpose of this note is to begin the study of the representation theory of the super Jordan plane $\mathcal{B}$: we classify the simple finite-dimensional $\mathcal{B}$-modules (all of dimension 1, Theorem 2.6) and the indecomposable $\mathcal{B}$-modules of dimension 2 (Theorem 3.2) and 3 (Theorem 3.11). We also observe that one of the generators of $\mathcal{B}$ has at most two eigenvalues in every indecomposable $\mathcal{B}$-module (Theorem 2.11) and describe two families of indecomposable modules in every dimension.

2. Basic facts

2.1. **Notations and conventions.** Fix an algebraically closed field $\Bbbk$ of characteristic 0; all vector spaces, tensor products, Hom spaces, algebras are over $\Bbbk$. All algebras are associative and all modules are left, unless explicitly stated. Let A be a $\Bbbk$-algebra; then $[\,,\,]$ denotes the Lie bracket given by the commutator. As customary we use indistinctly the languages of modules

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and representations. Denote by ${}_A\mathcal{M}$ the category of finite dimensional A-modules. Given a $\Bbbk$-vector space V, $\mathfrak{gl}(V)$ denotes the Lie algebra of all linear operators on V. The Jacobson radical of an algebra A it will be denoted by Jac A.

2.2. **The Jordan plane.** The *Jordan plane* is the free associative algebra $\mathcal{A}$ in generators y_1 and y_2 subject to the quadratic relation

$$y_1y_2 - y_2y_1 - y_2^2.$$

The algebra $\mathcal{A}$ is a Nichols algebra, $\operatorname{GKdim}\mathcal{A} = 2$ and $\{y_1^a y_2^b : a, b \in \mathbb{N}_0\}$ is a basis of $\mathcal{A}$. By Proposition 3.4 of [I], $\mathcal{A}$ is a Koszul algebra.

2.3. **The super Jordan plane.** Let $x_{21} = x_1x_2 + x_2x_1$ in the free associative algebra in generators x_1 and x_2. Let $\mathcal{B}$ be the algebra generated by x_1 and x_2 with defining relations

$$x_1^2, \tag{2.1}$$

$$x_2x_{21} - x_{21}x_2 - x_1x_{21}. \tag{2.2}$$

The algebra $\mathcal{B}$ (which is graded by $\deg x_1 = \deg x_2 = 1$) was introduced in [AAH1, AAH2] and is called the *super Jordan plane*. Since $\mathcal{B}$ is not a quadratic algebra, it follows that $\mathcal{B}$ is not Koszul; see e. g. § 2.1 of [PP].

Proposition 2.1. [AAH2] *The algebra $\mathcal{B}$ is a Nichols algebra,* $\operatorname{GKdim}\mathcal{B} = 2$ *and $\{x_1^a x_{21}^b x_2^c : a \in \{0, 1\}, b, c \in \mathbb{N}_0\}$ is a basis of $\mathcal{B}$.* $\square$

The following identities are valid in $\mathcal{B}$:

$$x_{21}x_1 = x_1x_{21}, \tag{2.3}$$

$$x_2^2x_1 = x_1x_2^2 + x_1x_2x_1, \tag{2.4}$$

$$x_{21}x_2^2 = (x_2^2 - x_{21})x_{21}. \tag{2.5}$$

Indeed, in presence of (2.1), (2.2) is equivalent to (2.4).

By (2.5) and Proposition 2.1, the subalgebra of the super Jordan plane $\mathcal{B}$ generated by x_2^2 and x_{21}, is isomorphic to the Jordan plane via $y_1 \mapsto x_2^2$ and $y_2 \mapsto x_{21}$.

It is convenient to introduce $s = x_{21}$ and $t = x_2^2$. By (2.5), $st = ts - s^2$ and whence

$$[t, s^n] = ns^{n+1},\ n \geq 1; \quad x_1s = sx_1; \quad x_2t = tx_2; \quad tx_1 = x_1(t + s). \tag{2.6}$$

Lemma 2.2. *Given $b, c \in \mathbb{N}$, we have that*

$$x_{21}^b x_2^c \overset{*}{=} (x_2 - bx_1)x_{21}^b x_2^{c-1}, \qquad x_1x_{21}^b x_2^c \overset{\heartsuit}{=} x_1x_2x_{21}^b x_2^{c-1}.$$

Proof. We prove $*$ by induction. For $b = c = 1$, the relation is valid by (2.2). Suppose that $*$ is valid for $b - 1 > 0$ and $c = 1$. Then

$$\begin{aligned} x_{21}^b x_2 &= x_{21}^{b-1}x_{21}x_2 = x_{21}^{b-1}(x_2x_{21} - x_1x_{21}) \\ &= (x_2 - (b-1)x_1)x_{21}^{b-1}x_{21} - x_{21}^b x_1 = (x_2 - bx_1)x_{21}^b. \end{aligned}$$

Fix $b \in \mathbb{N}$ and assume that the relation is true for $c-1$, with $c > 1$. Thus

$$x_{21}^b x_2^c = (x_{21}^b x_2^{c-1})x_2 = (x_2 - bx_1)x_{21}^b x_2^{c-2} x_2 = (x_2 - bx_1)x_{21}^b x_2^{c-1}.$$

The proof of $\heartsuit$ is similar. $\square$

The next result follows immediately from Proposition 2.1 and Lemma 2.2.

Proposition 2.3. *The set $\{1, x_1, x_2, x_1x_2\}$ generates $\mathcal{B}$ as a right $\mathcal{A}$-module.* $\square$

2.4. **Simple modules.** Let (V, ρ) be a finite-dimensional representation of $\mathcal{B}$; set $X_1 = \rho(x_1)$, $X_2 = \rho(x_2)$, $S = \rho(s)$ and $T = \rho(t)$ and

$$V_0 = \ker X_1.$$

Then V_0 is always $\neq 0$ and it is stable under S and T by (2.6). In fact, let $E_{12}(n) \in \mathfrak{gl}(\mathbb{k}^n)$ (or E_{12} if n is clearly from de context) the matrix whose the entry 1×2 is equal to 1 and all other entries are equal to 0. Then the Jordan form of X_1 consists of r blocks like $E_{12}(2)$ and s blocks of size 1 filled by 0. Hence $\dim V = 2r + s$; $r = 0 \iff V = V_0$.

Lemma 2.4. *Assume the previous notations. Then:*

(i) *S and T have a simultaneous eigenvector in V_0.*
(ii) *$W = X_2V_0 \cap V_0$ is a submodule of V.*
(iii) *$U = X_2V_0 + V_0$ is a submodule of V.*

Proof. (i): The subspace of $\mathfrak{gl}(V)$ generated by T and S^n, $n \in \mathbb{N}_0$, is a solvable Lie subalgebra by (2.6); then Lie Theorem applies.

(ii): Clearly $X_1W \subseteq X_1V_0 = \{0\} \subseteq W$. It remains to show that $X_2W \subseteq W$. In fact, let $w \in W$, this is, $w \in V_0$ and $w = X_2v$ for some $v \in V_0$. Clearly $X_2w \in X_2V_0$. Moreover,

$$X_1(X_2w) = X_1X_2^2v \overset{(2.4)}{=} (X_2^2X_1 - X_1X_2X_1)v = 0 \implies X_2w \in W.$$

(iii): Since $\mathcal{B} \cdot V_0 \subseteq X_2V_0$ and $\mathcal{B} \cdot (X_2V_0) \subseteq V_0$, the claim follows. $\square$

Lemma 2.5. *If $V \in {}_{\mathcal{B}}\mathcal{M}$ is simple, then $V = V_0$.*

Proof. Assume that $V \neq V_0$. By Lemma 2.4 we have that $W = X_2V_0 \cap V_0 = 0$ and $V = X_2V_0 + V_0$, so that $V = X_2V_0 \oplus V_0$. By Lemma 2.4 (i), there exists a simultaneous eigenvector $v \in V_0$ of S and T, i.e. there exist $\alpha, \tau \in \mathbb{k}$ such that $Sv = \alpha v$, $Tv = \tau v$.

Now $M = \operatorname{span}\{v, X_2v\} \neq 0$ is a $\mathcal{B}$-submodule of V and by simplicity of V, $M = V$. By our assumption, $X_2v \notin V_0$; hence $\Lambda = \{v, X_2v\}$ is a basis of V. Note that $[X_1]_\Lambda = \begin{pmatrix} 0 & \alpha \\ 0 & 0 \end{pmatrix}$ and $[X_2]_\Lambda = \begin{pmatrix} 0 & \tau \\ 1 & 0 \end{pmatrix}$. The relation $X_2^2X_1 = X_1X_2^2 + X_1X_2X_1$ is satisfied if and only if $\tau\alpha = \alpha\tau + \alpha^2$. Therefore $\alpha = 0$ and $V = V_0$, a contradiction. $\square$

Let $A \in \operatorname{End}(\mathbb{k}^n)$. Denote by $\mathbb{k}_A^n$ the $\mathcal{B}$-module defined by $X_1 = 0$ and $X_2 = A$. Every $\mathcal{B}$-module V with $V = V_0$ is isomorphic to $\mathbb{k}_A^n$ for some A. If $B \in \operatorname{End}(\mathbb{k}^m)$, then $\mathbb{k}_A^n \simeq \mathbb{k}_B^m$ iff $n = m$ and A and B are similar matrices.

Theorem 2.6. *Every simple $\mathcal{B}$-module is isomorphic to $\Bbbk_a^1$ for a unique $a \in \Bbbk$.*

Proof. This follows from Lemma 2.5 and the preceding considerations. $\square$

Corollary 2.7. *Let $\rho : \mathcal{B} \to \operatorname{End} V$ a finite dimensional representation of $\mathcal{B}$ and $B = \rho(\mathcal{B})$. Then there exists an integer s such that $B/\operatorname{Jac} B \simeq \Bbbk^s$ and $\operatorname{Jac} B = \{x \in B : x \text{ is nilpotent}\}$.*

Proof. Since $B/\operatorname{Jac} B$ is semisimple and $\Bbbk$ is algebraically closed, there are positive integers $n_1, \dots, n_s$ such that $B/\operatorname{Jac} B = M_{n_1}(\Bbbk) \times \cdots \times M_{n_s}(\Bbbk)$. The composition

$$\mathcal{B} \xrightarrow{\rho} B \xrightarrow{\pi} B/\operatorname{Jac} B \xrightarrow{\pi_j} M_{n_j}(\Bbbk)$$

is a finite dimensional simple representation of $\mathcal{B}$. Hence, by Theorem (2.6), $n_1 = \cdots = n_s = 1$. Thus, $B/\operatorname{Jac} B \simeq \Bbbk^s$. Let $x \in B$ a nilpotent element. Then $\pi(x)$ is a nilpotent element of $B/\operatorname{Jac} B$. Since $B/\operatorname{Jac} B$ is commutative, we obtain that $\pi(x) \in \operatorname{Jac}(B/\operatorname{Jac} B) = \{0\}$. Hence, $x \in \operatorname{Jac} B$. On the other hand, B finite dimensional implies that $\operatorname{Jac} B$ is a nilpotent ideal. Consequently, $\operatorname{Jac} B = \{x \in B : x \text{ is nilpotent}\}$. $\square$

We also remark:

Proposition 2.8. *If V is an indecomposable $\mathcal{B}$-module with $V = V_0$, then there exist $n \in \mathbb{N}$ and $\lambda \in \Bbbk$ such that V is isomorphic to $\Bbbk_A^n$ where A is the Jordan block of size n with eigenvalue λ.* $\square$

If A is the Jordan block of size n with eigenvalue λ, then denote $\mathcal{A}_\lambda = \Bbbk_A^n$.

2.5. **Indecomposable modules.** Throughout this subsection, V, X_1, X_2, T and S are as in § 2.4. When V is indecomposable, we will prove that T has a unique eigenvalue. In order to do this, the following relations are useful.

Lemma 2.9. *Let $\lambda \in \Bbbk$, $z := t - \lambda \operatorname{id} \in \mathcal{B}$ and $n \in \mathbb{N}$. Then*

$$z^n x_1 \overset{\clubsuit}{=} x_1 \sum_{j=0}^{n} \frac{n!}{(n-j)!} s^j z^{n-j}, \qquad z^n x_1 x_2 \overset{\diamondsuit}{=} x_1 x_2 \sum_{j=0}^{n} \frac{n!}{(n-j)!} s^j z^{n-j}.$$

Proof. We prove $\clubsuit$ by induction on n; the proof of $\diamondsuit$ is similar. We will use that $x_1 z s^n = x_1 x_{21}^n z + n x_1 s^{n+1}$, which can be verified easily. Note that

$$z x_1 = x_1 z + x_1 x_2 x_1 = x_1 z + x_1 s = x_1(z + s),$$

and whence the formula is true for $n = 1$. Denote $\zeta_{n,j} := \frac{n!}{(n-j)!}$, $0 \le j \le n$. Consider $n > 1$ and assume that the formula is true for $n-1$. Then

$$z^n x_1 = (zx_1) \sum_{j=0}^{n-1} \zeta_{n-1,j} s^j z^{n-1-j} \overset{(2.2)}{=} (x_1 z + x_1 s) \sum_{j=0}^{n-1} \zeta_{n-1,j} s^j z^{n-1-j}$$
$$= \sum_{j=0}^{n-1} \zeta_{n-1,j} (x_1 z s^j) z^{n-1-j} + \sum_{j=0}^{n-1} \zeta_{n-1,j} x_1 s^{j+1} z^{n-1-j} = x_1 \sum_{j=0}^{n} \zeta_{n,j} s^j z^{n-j}.$$

□

Let λ be an eigenvalue of T. Denote by V_λ^T the generalized eigenspace of V associated to λ, i. e. $V_\lambda^T := \cup_{j\ge 0} \ker{(T - \lambda\,\mathrm{id})^j}$

Lemma 2.10. *V_λ^T is a $\mathcal{B}$-submodule of V, for all eigenvalue λ of T.*

Proof. Clearly $V_\lambda^T = \ker{(T-\lambda\,\mathrm{id})^r} = \ker{\left(X_2^2 - \lambda\,\mathrm{id}\right)^r}$, where r is the maximal size of λ-blocks in the Jordan normal form of T. Thus V_λ^T is stable by X_2. It remains to show that it stable by X_1. By Lemma 2.9, if $u \in V_\lambda^T$ then

$$(T - \lambda\,\mathrm{id})^n X_1 u = X_1 \sum_{j=0}^{n} \zeta_{j,n} S^j (T - \lambda\,\mathrm{id})^{n-j} u.$$

By Lemma 2.1 of [I], S is nilpotent. Taking n big enough, it follows that $(T - \lambda\,\mathrm{id})^n X_1 u = 0$ and whence $X_1 u \in V_\lambda^T$. □

Now Lemma 2.10 implies the next result.

Theorem 2.11. *Let $\lambda_1, \dots, \lambda_t$ be the different eigenvalues of T. Then V decomposes into the direct sum of the $\mathcal{B}$-submodules $V_{\lambda_i}^T$.*

In particular, if V is indecomposable then T has a unique eigenvalue. Hence either X_2 has a unique eigenvalue or else the eigenvalues of X_2 are λ and $-\lambda$, with $\lambda \in \Bbbk^\times$. □

Given $\lambda \in \Bbbk$, denote by ${}_{\mathcal{B}}\mathcal{M}_\lambda$ the full subcategory of ${}_{\mathcal{B}}\mathcal{M}$ whose objects are the $\mathcal{B}$-modules V such that $V = \ker{(T - \lambda\,\mathrm{id})^m}$, for some $m \in \mathbb{N}_0$. With this notation, the next result follows immediately from Theorem 2.11.

Corollary 2.12. ${}_{\mathcal{B}}\mathcal{M} \simeq \prod_{\lambda\in\Bbbk} {}_{\mathcal{B}}\mathcal{M}_\lambda$. □

The next result will be useful in § 3.

Lemma 2.13. *Let $\Lambda = \{v_1, \cdots, v_n\}$ be a basis of V such that $[X_1]_\Lambda = E_{12}$ and W a one-dimensional $\mathcal{B}$-submodule of V. Then:*

(i) *If L is a complement (as a $\mathcal{B}$-module) of W in V then $L \cap V_0 = \langle v_1 \rangle$.*
(ii) *$W = \langle v_1 \rangle$ does not have a complement (as a $\mathcal{B}$-submodule) in V.*

Proof. (i): Assume that $W = \langle w \rangle$ and $\{u_1, u_2, \cdots, u_{n-1}\}$ is a basis of L. Since $W_0 = W \cap V_0 \neq 0$, it follows that $W \subset V_0$. Using that v_2 is a linear combination of $w, u_1, u_2, \cdots, u_{n-1}$ we see that $v_1 = X_1 v_2 \in L$; hence $v_1 \in V_0 \cap L$.

(ii): It follows at once from (i). □

3. Indecomposable representations of dimension 2 and 3

3.1. **Dimension 2.** In this subsection we describe all 2-dimensional indecomposable representations of $\mathcal{B}$. Fix (V, ρ) a 2-dimensional representation of $\mathcal{B}$.

Lemma 3.1. *If $V \neq V_0$ then V is indecomposable.*

Proof. Suppose that V is decomposable, i.e. there are non-trivial submodules U and W such that $V = U \oplus W$. Then $V_0 = U_0 \oplus W_0 = U \oplus W = V$. $\square$

Define representations of $\mathcal{B}$ on the vector space $\Bbbk^2$ given by $X_1 = E_{12}$ and the following action of x_2:

◇ $X_2 = \begin{pmatrix} a & b \\ 0 & a \end{pmatrix}$, $a, b \in \Bbbk$. This is denoted by $\mathcal{U}_{a,b}$.

◇ $X_2 = \begin{pmatrix} a & 0 \\ 0 & -a \end{pmatrix}$, $a \in \Bbbk^{\times}$. This is denoted by $\mathcal{V}_a$.

It is easy to check that these are indecomposable modules pairwise non-isomorphic.

Theorem 3.2. *Every 2-dimensional indecomposable representation of $\mathcal{B}$ is isomorphic either to $\mathcal{U}_{a,b}$, or to $\mathcal{V}_a$, or to $\Bbbk^2_\lambda$ for unique $a, b, \lambda \in \Bbbk$.*

This confirms Theorem 2.11.

Proof. If $V = V_0$, then Proposition 2.8 applies. Assume that $V_0 \neq 0$; then there exists a basis $\Lambda = \{v_1, v_2\}$ of V such that $[X_1]_\Lambda = E_{12}$. Let $[X_2]_\Lambda = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. Then (2.4) is satisfied if and only if

$$c(a + d) = 0 \qquad \text{and} \qquad d^2 + c = a^2.$$

Suppose that $c \neq 0$. Then by the first equation it follows that $d = -a$. Replacing in the second equation we have that $c = 0$, which is a contradiction. Therefore $c = 0$ and consequently $d = a$ or $d = -a$.

If $d = a$ then $V \simeq \mathcal{U}_{a,b}$. Assume that $d = -a \neq 0$ and take $w_1 = v_1$ and $w_2 = \frac{-b}{2a} v_1 + v_2$. Then $\Omega = \{w_1, w_2\}$ is a basis of V such that $[X_1]_\Omega = E_{12}$ and $[X_2]_\Omega = \begin{pmatrix} a & 0 \\ 0 & -a \end{pmatrix}$. Thus $V \simeq \mathcal{V}_a$. $\square$

Corollary 3.3. *If $\operatorname{Ext}^1(\Bbbk^1_a, \Bbbk^1_b) \neq 0$ then $a = \pm b$.* $\square$

3.2. **Dimension 3.** Let V be a $\mathcal{B}$-module of dimension 3 such that $V \neq V_0$. Throughout this subsection, $\Lambda = \{v_1, v_2, v_3\}$ denotes a basis of V such that $[X_1]_\Lambda = E_{12}$. We define four families of representations of $\mathcal{B}$ on the vector space V determined by the following action of $[X_2]_\Lambda$, for all $a, b, c, d, e \in \Bbbk$:

$$\Theta_1 : \begin{pmatrix} a & b & c \\ 0 & d & e \\ 0 & \frac{a^2-d^2}{e} & -d \end{pmatrix}, \ e \in \Bbbk^\times; \qquad \Theta_2 : \begin{pmatrix} a & b & c \\ 0 & a & 0 \\ 0 & d & e \end{pmatrix};$$

$$\Theta_3 : \begin{pmatrix} a & b & \frac{c^2-a^2}{d} \\ 0 & c & 0 \\ d & e & -a \end{pmatrix}, \ d \in \Bbbk^\times; \qquad \Theta_4 : \begin{pmatrix} a & b & c \\ 0 & -a & 0 \\ 0 & d & e \end{pmatrix}, \ a \in \Bbbk^\times.$$

Lemma 3.4. *The families Θ_1, Θ_2, Θ_3 and Θ_4 contain all 3-dimensional representations of $\mathcal{B}$, up to isomorphism.*

Proof. Let $[X_2]_\Lambda = \begin{pmatrix} \alpha & \beta & \gamma \\ \delta & \epsilon & \zeta \\ \eta & \theta & \iota \end{pmatrix}$. Then (2.4) is valid if and only if

$$\begin{cases} \delta(\alpha+\epsilon) = -\zeta\eta \\ \zeta(\epsilon+\iota) = -\gamma\delta \\ \eta(\alpha+\iota) = -\delta\theta \\ \epsilon^2 - \alpha^2 = -\delta + \gamma\eta - \zeta\theta \end{cases} \tag{3.1}$$

Claim: If the system (3.1) has solution then $\delta = 0$.
Assume that $\delta \neq 0$. If $\zeta = 0$ then $\gamma = 0$ and $\epsilon = -\alpha$. Thus, $\delta = 0$ which is a contradiction. If $\zeta \neq 0$ then

$$\gamma = \frac{-\zeta(\epsilon+\iota)}{\delta}, \quad \eta = \frac{-\delta(\alpha+\epsilon)}{\zeta} \quad \text{and} \quad \theta = \frac{(\alpha+\epsilon)(\alpha+\iota)}{\zeta}.$$

From the last equation of (3.1), $\delta = 0$ which is again a contradiction.
Assume $\delta = 0$. Thus $\zeta\eta = 0$. If $\zeta \neq 0$ then $\eta = 0$, $\iota = -\epsilon$ and $\theta = \frac{\alpha^2-\epsilon^2}{\zeta}$. Hence V belongs to the family Θ_1. When $\zeta = 0$ and $\eta \neq 0$, it follows that $\iota = -\alpha$ and $\gamma = \frac{\epsilon^2-\alpha^2}{\eta}$. Thus, V belongs to the family Θ_3. If $\zeta = 0$ and $\eta = 0$ then $\epsilon = |\alpha|$. In this case, V belongs to the families Θ_2 or Θ_4. □

Remark 3.5. Let L a $\mathcal{B}$-submodule of V of dimension 2 such that $L \cap V_0$ is one-dimensional. Fix $\overline{L} := L/(L \cap V_0) = \langle \overline{u} \rangle$. Since $u \notin V_0$, we can suppose that $u = \alpha v_1 + v_2 + \gamma v_3 \in L$, with $\alpha, \gamma \in \Bbbk$.

Proposition 3.6. *Let V be a $\mathcal{B}$-module. Then:*

(i) *the representations in the family Θ_1 are always indecomposable;*
(ii) *a representation in the family Θ_2 is indecomposable if and only if $c \neq 0$ and $e = a$ or $d \neq 0$ and $e = a$;*
(iii) *the representations in the family Θ_3 are always indecomposable;*
(iv) *a representation in the family Θ_4 is indecomposable if and only if $c \neq 0$ and $e = a$ or $d \neq 0$ and $e = -a$.*

Proof. (i): The unique one-dimensional $\mathcal{B}$-submodule of V is $\langle v_1 \rangle$ which does not have complement by Lemma 2.13 (ii).

(ii): Let V be a representation of $\mathcal{B}$ of the type Θ_2. Suppose that $W = \langle w \rangle$ is a one-dimensional $\mathcal{B}$-submodule of V. Since $W \subset V_0$, see § 2.4, $w = \alpha v_1 + \beta v_3$, with $\alpha, \beta \in \Bbbk$. Note that $X_2 w = \gamma w$, $\gamma \in \Bbbk$, if and only if $\beta(\gamma - e) = 0$ and $\alpha(\gamma - a) = \beta c$. Consequently, the one-dimensional $\mathcal{B}$-submodules of V are:

◇ $\langle v_1 \rangle$, $\langle v_3 \rangle$, $c = 0$, $e \neq a$,
◇ $\langle \alpha v_1 + \beta v_3 \rangle$, $c = 0$, $e = a$,
◇ $\langle v_1 \rangle$, $\langle v_1 + \frac{e-a}{c} v_3 \rangle$, $c \neq 0$, $e \neq a$
◇ $\langle v_1 \rangle$, $c \neq 0$, $e = a$.

Assume $e \neq a$. If $c \neq 0$, $V = \langle v_1 + \frac{e-a}{c} v_3 \rangle \oplus \langle v_1, v_1 + v_2 + \frac{d}{a-e} v_3 \rangle$. If $c = 0$, $V = \langle v_3 \rangle \oplus \langle v_1, v_1 + v_2 + \frac{d}{a-e} v_3 \rangle$. If $c = d = 0$, $V = \langle v_1, v_2 \rangle \oplus \langle v_3 \rangle$. Hence, V is decomposable.
Conversely, suppose $e = a$ and $c \neq 0$. Then the unique one-dimensional $\mathcal{B}$-submodule of V is $\langle v_1 \rangle$ which does not have complement. Suppose that $e = a$ and $d \neq 0$. Assume that W is a one-dimensional $\mathcal{B}$-submodule of V which admits a complement $L = \langle u_1, u_2 \rangle$. Then by Lemma 2.13 (ii), $v_1 \in L \cap V_0$. By Remark 3.5, $\overline{L} = \langle \overline{u} \rangle$ where $u = \alpha v_1 + v_2 + \beta v_3$. Thus $X_2 \overline{u} = \gamma \overline{u}$, $\gamma \in \Bbbk$, if and only if $\gamma = a$ and $\beta(a - e) = d$. Since $d \neq 0$ and $e = a$ then W does not have complement in V.

(iii): Suppose that $W = \langle w \rangle$ is a one-dimensional $\mathcal{B}$-submodule of V which admits a complement L. Then by Lemma 2.13 (ii), $\langle v_1 \rangle = L \cap V_0$, which is a contradiction because $d \neq 0$.

(iv): Analogous to item (ii). □

3.2.1. *Isomorphism classes in* Θ_1. Assume V in the family Θ_1. We distinguish: for all $a, b, c, d, e \in \Bbbk$

◇ $X_2 = \begin{pmatrix} a & b & c \\ 0 & d & e \\ 0 & \frac{a^2-d^2}{e} & -d \end{pmatrix}$, $e \in \Bbbk^\times$. This is denoted by $\mathcal{Y}_{a,b,c,d,e}$.

◇ $X_2 = \begin{pmatrix} a & b & 0 \\ 0 & a & 1 \\ 0 & 0 & -a \end{pmatrix}$. This is denoted by $\mathcal{U}^{a,b}$.

By Proposition 3.6 (i), these representations are indecomposable. Note that $\mathcal{U}^{a,b} = \mathcal{Y}_{a,b,0,a,1}$.

Proposition 3.7. *Every* 3*-dimensional indecomposable representation* V *of* $\mathcal{B}$ *in* Θ_1 *is isomorphic either to* $\mathcal{U}^{a,b}$*, or to* $\mathcal{Y}_{a,b,c,d,e}$*. Moreover,*

$$\mathcal{Y}_{a,b,c,d,e} \simeq \mathcal{Y}_{a,b',c',d',e'} \text{ if and only if } (a - d')\frac{ce' - c'e}{e'} = e(b' - b) + c(d' - d).$$

In particular, $\mathcal{U}^{a,b} \simeq \mathcal{U}^{a,b'}$ *if and only if* $b = b'$.

Proof. Since $\langle X_2 v_1 \rangle = \operatorname{Im} X_1$, we obtain that a is invariant. Consider the indecomposable representation $\mathcal{Y}_{a,b',c',d',e'}$ of $\mathcal{B}$. If $d' = a$, taking the basis $\{v_1, \frac{c'}{e'} v_1 + v_2, \frac{1}{e'} v_3\}$ we conclude that $\mathcal{Y}_{a,b',c',d',e'} \simeq \mathcal{U}^{a,b'}$.

Note that $\mathcal{Y}_{a,b,c,d,e}$ and $\mathcal{Y}_{a,b',c,d',e'}$ are isomorphic if and only if there exists a basis $\{w_1, w_2, w_3\}$ of V such that $X_1w_1 = X_1w_3 = 0$, $X_1w_2 = w_1$, $X_2w_1 = aw_1$, $X_2w_2 = b'w_1 + d'w_2 + \frac{a^2-d'^2}{e'}w_3$ and $X_2w_3 = c'w_1 + e'w_2 - d'w_3$. Since $\langle v_1\rangle = \operatorname{Im} X_1$ and V_0 has dimension 2, then we can consider $w_1 = v_1$, $w_2 = \lambda_1v_1 + \lambda_2v_2 + \lambda_3v_3$ and $w_3 = \beta_1v_1 + \beta_3v_3$, $\lambda_1, \lambda_2, \lambda_3, \beta_1, \beta_3 \in \Bbbk$. Then, $\mathcal{Y}_{a,b,c,d,e} \simeq \mathcal{Y}_{a,b',c,d',e'}$ if and only if $(a-d')\dfrac{ce'-c'e}{e'} = e(b'-b)+c(d'-d)$. $\square$

3.2.2. *Isomorphism classes in* Θ_2. Consider V in the family Θ_2 and the following distinguish representations: for all $a \in \Bbbk$

- ◇ $X_2 = \begin{pmatrix} a & 0 & 0 \\ 0 & a & 0 \\ 0 & 1 & a \end{pmatrix}$. This is denoted by $\mathcal{R}_a$.
- ◇ $X_2 = \begin{pmatrix} a & 0 & 1 \\ 0 & a & 0 \\ 0 & 0 & a \end{pmatrix}$. This is denoted by $\mathcal{S}_a$.
- ◇ $X_2 = \begin{pmatrix} a & 0 & b \\ 0 & a & 0 \\ 0 & c & a \end{pmatrix}$, $b \in \Bbbk^\times$ or $c \in \Bbbk^\times$. This is denoted by $\mathcal{T}_{a,b,c}$.

By Proposition 3.6 (ii), these are indecomposable representations. Notice that $\mathcal{R}_a = \mathcal{T}_{a,0,1}$ and $\mathcal{S}_a = \mathcal{T}_{a,1,0}$.

Proposition 3.8. *Every* 3*-dimensional indecomposable representation* V *of* $\mathcal{B}$ *in* Θ_2 *is isomorphic either to* $\mathcal{R}_a$*, or to* $\mathcal{S}_a$ *or to* $\mathcal{T}_{a,b,c}$*. Moreover,* $\mathcal{T}_{a,b,c}$ *and* $\mathcal{T}_{a,b',c'}$ *are isomorphic if and only if* $bc = b'c'$*.*

Proof. Let V' the representation of $\mathcal{B}$ given by

$$[X_2]_\Lambda = \begin{pmatrix} a & d' & b' \\ 0 & a & 0 \\ 0 & c' & a \end{pmatrix}.$$

If $b' = 0$, then by Proposition 3.6 (ii) we have that $c' \neq 0$. In this case, taking the basis $\{v_1, v_2, d'v_1 + c'v_3\}$ of V', we conclude that $V' \simeq \mathcal{R}_a$. Similarly, if $c' = 0$ then $b' \neq 0$. Taking the basis $\{v_1, v_2 - \frac{d'}{b'}v_3, \frac{1}{b'}v_3\}$ of V', we obtain that $V' \simeq \mathcal{S}_a$. If $b, b', c, c' \in \Bbbk^\times$, taking the basis $\{v_1, v_2, \frac{d'}{c'}v_1 + v_3\}$, it follows that $V' \simeq \mathcal{T}_{a,b',c'}$.

Finally, notice that $\mathcal{T}_{a,b,c} \simeq \mathcal{T}_{a,b',c'}$ if and only if there exists a basis $\{w_1, w_2, w_3\}$ of $\Bbbk^3$ such that $X_1w_1 = X_1w_3 = 0$, $X_1w_2 = w_1$, $X_2w_1 = aw_1$, $X_2w_2 = aw_2 + cw_3$ and $X_2w_3 = bw_1 + aw_3$. We can assume $w_1 = v_1$, $w_2 = \lambda_1v_1 + \lambda_2v_2 + \lambda_3v_3$ and $w_3 = \beta_1v_1 + \beta_3v_3$, $\lambda_1, \lambda_2, \lambda_3, \beta_1, \beta_3 \in \Bbbk$. Note that $X_2w_2 = w_1$ if and only if $\lambda_2 = 1$. Moreover, $X_2w_2 = aw_2 + cw_3$ and $X_2w_3 = bw_1 + aw_3$ if and only if $bc = b'c'$. $\square$

3.2.3. *Isomorphism classes in* Θ_3. Consider V in the family Θ_3 and the following distinguished representations: for all $a, b, c, d, e \in \Bbbk$

$\diamond$ $X_2 = \begin{pmatrix} a & b & \frac{c^2-a^2}{d} \\ 0 & c & 0 \\ d & e & -a \end{pmatrix}$, $d \in \Bbbk^\times$. This is denoted by $\mathcal{W}_{a,b,c,d,e}$.

$\diamond$ $X_2 = \begin{pmatrix} a & b & 0 \\ 0 & a & 0 \\ 1 & 0 & -a \end{pmatrix}$. This is denoted by $\mathcal{U}_{a,b}$.

By Proposition 3.6 (iii), these representations are indecomposable. Observe that $\mathcal{U}_{a,b} = \mathcal{W}_{a,b,a,1,0}$.

Proposition 3.9. *Every* 3*-dimensional indecomposable representation* V *of* $\mathcal{B}$ *in* Θ_3 *is isomorphic either to* $\mathcal{U}_{a,b}$*, or to* $\mathcal{W}_{a,b,c,d,e}$*. Moreover,*

$$\mathcal{W}_{a,b,c,d,e} \simeq \mathcal{W}_{a',b',c,d',e'} \text{ if and only if } \frac{ae-bd-ce}{d} = \frac{a'e'-b'd'-ce'}{d'}.$$

In particular, $\mathcal{U}_{a,b} \simeq \mathcal{U}_{a,b'}$ *iff* $b = b'$.

Proof. Since the characteristic polynomial of X_2 is $(t-c)^2(t+c)$, c is an invariant. Let the indecomposable representation $\mathcal{W}_{a',b',c,d',e'}$ of $\mathcal{B}$. If $c = a$, taking the basis $\{v_1, -\frac{e}{d}v_1 + v_2, dv_3\}$ we conclude that $\mathcal{W}_{a',b',c,d',e'} \simeq \mathcal{U}_{a',b'}$. Note that $\mathcal{W}_{a,b,c,d,e} \simeq \mathcal{W}_{a',b',c,d',e'}$ if and only if there is a basis $\{w_1, w_2, w_3\}$ of $\Bbbk^3$ such that $X_1w_1 = X_1w_3 = 0$, $X_1w_2 = w_1$, $X_2w_1 = aw_1$, $x_2w_2 = aw_2+cw_3$ and $X_2w_3 = bw_1+aw_3$. We can assume $w_1 = v_1$, $w_2 = \lambda_1v_1+\lambda_2v_2+\lambda_3v_3$ and $w_3 = \beta_1v_1 + \beta_3v_3$, where $\lambda_1, \lambda_2, \lambda_3, \beta_1, \beta_3 \in \Bbbk$. However $X_2w_1 = a'w_1 + d'w_3$ if and only if $\beta_1 = \frac{a-a'}{d'}$ e $\beta_3 = \frac{d}{d'}$. With this choose of β_1 and β_3 we have that $X_2w_3 = \frac{c^2-a'^2}{d'}w_1 - a'w_3$. Finally, $X_2w_2 = b_1w_1 + cw_2 + e'w_3$ if and only if $\frac{ae-bd-ce}{d} = \frac{a'e'-b'd'-ce'}{d'}$. □

3.2.4. *Isomorphism classes in* Θ_4. Consider V in Θ_4 and the following distinguish representations: for all $a \in \Bbbk^\times$

$\diamond$ $X_2 = \begin{pmatrix} a & 0 & 1 \\ 0 & -a & 0 \\ 0 & 0 & a \end{pmatrix}$. This is denoted by $\mathcal{V}^a$.

$\diamond$ $X_2 = \begin{pmatrix} a & 0 & 0 \\ 0 & -a & 0 \\ 0 & 1 & -a \end{pmatrix}$. This is denoted by $\mathcal{V}_a$.

By Proposition 3.6 (iv), these are indecomposable representations pairwise non-isomorphic.

Proposition 3.10. *Every* 3*-dimensional indecomposable representation* V *of* $\mathcal{B}$ *in* Θ_4 *is isomorphic either to* $\mathcal{V}^a$ *or to* $\mathcal{V}_a$ *for unique* $a \in \Bbbk^\times$*.*

Proof. Let V' be a 3-dimensional indecomposable representation of $\mathcal{B}$ such that

$$[X_2]_\Lambda = \begin{pmatrix} a & b & c \\ 0 & -a & 0 \\ 0 & d & e \end{pmatrix}, \ a \in \Bbbk^\times.$$

Since V' is indecomposable, by Proposition 3.6 (iv) we have that $c \neq 0$ and $e = a$ or $d \neq 0$ and $e = -a$. If $c \neq 0$ and $e = a$, taking the basis $\{v_1, \frac{cd-2ab}{4a^2}v_1 + v_2 - \frac{d}{2a}v_3, v_1 + \frac{1}{c}v_3\}$ of V', we obtain $V' \simeq \mathcal{V}^a$. If $d \neq 0$ and $e = -a$, taking the basis $\{v_1, -\frac{2ab+cd}{4a^2}v_1 + v_2, -\frac{dc}{2a}v_1 + dv_3\}$ of V', it follows that $V' \simeq \mathcal{V}_a$. □

3.2.5. *Classification of indecomposable 3-dimensional $\mathcal{B}$-modules.*

Theorem 3.11. *Every 3-dimensional indecomposable $\mathcal{B}$-module is isomorphic either to $\Bbbk^3_\lambda$ for a unique λ, or else to a representation in one of the families Θ_j, $j = 1, 2, 3, 4$, with the constraints described in Proposition 3.6. The isomorphism classes are described in Propositions 3.7, 3.8, 3.9 and 3.10.* □

Again, this agrees with Theorem 2.11.

Remark 3.12. It is straightforward to verify that two 3-dimensional indecomposable representations of $\mathcal{B}$ that belong to different families Θ_i, $i = 1, 2, 3, 4$, are not isomorphic.

4. Families of indecomposable $\mathcal{B}$-modules

Throughout this section (V, ρ) is an n-dimensional representation of $\mathcal{B}$, $\Lambda = \{v_1, \dots, v_n\}$ is a basis of V, $X_1 = \rho(x_1)$, $X_2 = \rho(x_2)$ and $[X_1]_\Lambda = E_{12}$.

4.1. **The family $\mathcal{U}_a$.** Let $a \in \Bbbk$. Consider the following action of X_2 on V:

$$[X_2]_\Lambda = \begin{pmatrix} a & 0 & 0 & 0 & \dots & 0 & 0 \\ 0 & a & 0 & 0 & \dots & 0 & 0 \\ 0 & 1 & a & 0 & \dots & 0 & 0 \\ 0 & 0 & 1 & a & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 1 & a & 0 \\ 0 & 0 & 0 & \dots & 0 & 1 & a \end{pmatrix}.$$

Clearly V with this action is a $\mathcal{B}$-module which will be denoted by $\mathcal{U}_a$.

Lemma 4.1. *Let $\mathcal{W}$ be a proper $\mathcal{B}$-submodule of $\mathcal{U}_a$. Then:*

(i) $v_2 \notin \mathcal{W}$;

(ii) *If $v = \sum_{i=1}^n \lambda_i v_i \in \mathcal{W}$ then $\lambda_2 = 0$.*

Proof. (i): Suppose $v_2 \in \mathcal{W}$. Then $v_1 = X_1 v_2 \in \mathcal{W}$ and $X_2 v_2 = av_2 + v_3 \in \mathcal{W}$. Hence $v_3 \in \mathcal{W}$. Again, $X_2 v_3 = av_3 + v_4 \in \mathcal{W}$ and consequently $v_4 \in \mathcal{W}$. With this procedure, we obtain that $\Lambda \subset \mathcal{W}$. Thus, $\mathcal{W} = \mathcal{U}_a$ and we have a contradiction.

(ii): Assume $\lambda_2 \neq 0$ and fix $w_1 = \lambda_2^{-1} v$. Thus $w_1 = \alpha_1 v_1 + v_2 + \ldots + \alpha_n v_n$, where $\alpha_i = \lambda_2^{-1}\lambda_i$, for all $1 \le i \le n$. Consider the following elements of V:

$$w_j := v_{j+1} + \alpha_3 v_{j+2} + \ldots + \alpha_{n-j+1} v_n, \quad \text{for all } 2 \le j \le n-2.$$

By a straightforward calculation, we obtain that $X_2 w_j = a w_j + w_{j+1}$, for all $1 \leq j \leq n-2$. Thus, $w_1, \dots, w_{n-2} \in \mathcal{W}$ and $X_2 w_{n-2} = a w_{n-2} + v_n$. Therefore, $v_n \in \mathcal{W}$. But $w_{n-2} = v_{n-1} + \alpha_3 v_n$ and whence $v_{n-1} \in \mathcal{W}$. By this procedure, it follows that $v_3, \dots, v_n \in \mathcal{W}$. From $v_1 = X_1 w_1 \in \mathcal{W}$, it follows that $v_2 \in \mathcal{W}$ which contradicts (i). □

Theorem 4.2. *$\mathcal{U}_a$ is an indecomposable $\mathcal{B}$-module, for all $n \geq 2$.*

Proof. Suppose $\mathcal{U}_a$ decomposable. Let $\mathcal{W}$, $\widetilde{\mathcal{W}}$ be nontrivial $\mathcal{B}$-submodules of $\mathcal{U}_a$ such that $\mathcal{U}_a = \mathcal{W} \oplus \widetilde{\mathcal{W}}$. Consider $\{w_1, \dots, w_r\}$ and $\{w_{r+1}, \dots, w_n\}$ basis of $\mathcal{W}$ and $\widetilde{\mathcal{W}}$ respectively. By Lemma 4.1, $w_i = \lambda_{i1} v_1 + \lambda_{i3} v_3 + \dots + \lambda_{in} v_n$, for all $1 \leq i \leq n$. Since $v_2 \in \mathcal{U}_a$, there exist $\alpha_1, \dots, \alpha_n \in \Bbbk$ such that $v_2 = \alpha_1 w_1 + \dots + \alpha_n w_n$, a contradiction. □

4.2. **The family $\mathcal{V}_a$.** Let $a \in \Bbbk^\times$. Consider the following action of X_2 on V:

$$[X_2]_\Lambda = \begin{pmatrix} a & 0 & 0 & 0 & \dots & 0 & 0 \\ 0 & -a & 0 & 0 & \dots & 0 & 0 \\ 0 & 1 & -a & 0 & \dots & 0 & 0 \\ 0 & 0 & 1 & -a & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 1 & -a & 0 \\ 0 & 0 & 0 & \dots & 0 & 1 & -a \end{pmatrix}.$$

Notice that V is a $\mathcal{B}$-module which will be denoted by $\mathcal{V}_a$. Since $a \neq 0$, $\mathcal{U}_a$ and $\mathcal{V}_a$ are not isomorphic.

Theorem 4.3. *$\mathcal{V}_a$ is an indecomposable $\mathcal{B}$-module, for all $n \geq 2$.*

Proof. Let $\mathcal{W}$ a proper $\mathcal{B}$-submodule of $\mathcal{V}_a$. As in Lemma 4.1 (i), we can show that $v_2 \notin \mathcal{W}$. Let $v \in \mathcal{W}$ such that $v = \sum_{i=1}^n \lambda_i v_i$. Assume that $\lambda_2 \neq 0$ and consider $u := \lambda_2^{-1} v \in \mathcal{W}$. Then $v_1 = X_1 u \in \mathcal{W}$. Take $w_1 := u - \lambda_2^{-1}\lambda_1 v_1$ and note that $w_1 = \alpha_2 v_2 + \dots + \alpha_n v_n$, where $\alpha_i = \lambda_2^{-1}\lambda_i$, for all $2 \leq i \leq n$. Considering the following elements of V

$$w_j := v_{j+1} + \alpha_3 v_{j+2} + \dots + \alpha_{n-j+1} v_n, \quad \text{for all } 2 \leq j \leq n-2,$$

it follows $X_2 w_j = -a w_j + w_{j+1}$, for all $1 \leq j \leq n-2$. As in Lemma 4.1, this implies that $v_2 \in \mathcal{W}$ which is a contradiction. Thus, the result follows as in Theorem 4.2. □

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