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A characterization of quantum groups

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I. Main result. $\Bbbk$ algebraically closed field.

Theorem. [ARS] H a **pointed** Hopf algebra with $\Gamma := G(H)$ abelian fin. gen. and **generic infinitesimal braiding**. TFAE:

1. H is a Γ-**reductive** domain with Gelfand-Kirillov dim $< \infty$.

2. The group Γ is free abelian of finite rank, and there exists a **reduced generic datum of finite Cartan type** $\mathcal{D}_{red}$ for Γ such that $H \simeq \mathcal{U}(\mathcal{D}_{red})$ as Hopf algebras.

If H satisfies (2), then H is **reductive** iff $[\Gamma : \Gamma^{\mathbf{2}}]$ is finite.

II. Invariants of a Hopf algebra H.

- $G(H) = \{x \in H - 0 : \Delta(x) = x \otimes x\}$, group of grouplikes.

- The coradical $H_0 =$ sum of all simple subcoalgebras of H.

- The *coradical filtration* is $H_n = \wedge^{n+1} H_0$.

Then: $(H_n)_{n \geq 0}$ is a coalgebra filtration and $\cup_{n \geq 0} H_n = H$.

Furthermore, if H_0 Hopf subalgebra, then $(H_n)_{n \geq 0}$ is an algebra filtration and $\operatorname{gr} H$ is a (graded) Hopf algebra.

H **pointed** iff every simple comodule has dim 1 iff $\Bbbk G(H) \simeq H_0$.

${}^H_H\mathcal{YD}$ = braided tensor category of Yetter-Drinfeld mod. over H.

- $\operatorname{gr} H \simeq R\#H_0$, $R = \oplus_{n\in\mathbb{N}} R(n)$ is a graded braided Hopf algebra.

- $R(0) \simeq \Bbbk$, $R(1) = P(R) =: V$ **infinitesimal braiding** of H.

Fact: subalgebra generated by $V \simeq \mathfrak{B}(V)$ Nichols algebra of V.

Hypothesis: $H_0 = \Bbbk\Gamma$, Γ abelian group.

If $V \in {}^H_H\mathcal{YD}$ and $\dim V < \infty$, then $V = \oplus_{g\in\Gamma,\chi\in\widehat{\Gamma}} V_g^{(\chi)}$.

- V of diagonal type when $V = \oplus_{g\in\Gamma,\chi\in\widehat{\Gamma}} V_g^{\chi} \rightsquigarrow (q_{ij})_{1\le i,j\le\theta}$

generic infinitesimal braiding $\equiv q_{ii} \notin \mathbb{G}_\infty, 1 \le i \le \theta$.

III. Reductivity.

An algebra A **reductive** if all fin.-dim. A-modules are semisimple.

$B \subset A$ a subalgebra. A is B**-reductive** if all finite-dimensional left A-modules which are semisimple over B are semisimple.

H pointed Hopf algebra, $\mathit{\Gamma} := G(H)$.
H $\mathit{\Gamma}$**-reductive** $\equiv$ H $\Bbbk\mathit{\Gamma}$-reductive

IV. Constructions of pointed Hopf algebras.

• Γ is a free abelian group of finite rank s. $\mathbb{I} = \{1, \dots, \theta\}$.

• $(a_{ij}) \in \mathbb{Z}^{\theta \times \theta}$ is a Cartan matrix of finite type; $(d_1, \dots, d_\theta)$ diagonal matrix such that $d_i a_{ij} = d_j a_{ji}$, $d_{ii} > 0$ minimal.

• $\mathcal{X}$ = set of connected components of the Cartan matrix (a_{ij}); if $i, j \in \mathbb{I}$, then $i \sim j \iff i, j \in$ same connected component.

• $(q_I)_{I \in \mathcal{X}}$ is a family of elements in $\Bbbk$ which are not roots of 1.

• $g_1, \dots, g_\theta \in \Gamma$, $\chi_1, \dots, \chi_\theta \in \widehat{\Gamma}$, satisfy

$$\langle \chi_i, g_i \rangle = q_I^{d_i}, \quad \langle \chi_j, g_i \rangle \langle \chi_i, g_j \rangle = q_I^{d_i a_{ij}}, \quad \forall i, j \in \mathbb{I}, \quad i \in I.$$

Two vertices i and j *are linkable* if $i \not\sim j$, $g_i g_j \neq 1$ and $\chi_i \chi_j = \varepsilon$.

A *linking datum* for

$$\Gamma, \quad (a_{ij}), \quad (q_I)_{I \in \mathcal{X}}, \quad g_1, \dots, g_\theta \quad \text{and } \chi_1, \dots, \chi_\theta$$

is a collection $(\lambda_{ij})_{1 \leq i < j \leq \theta, i \not\sim j}$ of elements in $\{0, 1\}$ such that λ_{ij} is arbitrary if i and j are linkable but 0 otherwise. Given a linking datum, we say that two vertices i and j *are linked* if $\lambda_{ij} \neq 0$. The collection

$$\mathcal{D} = \mathcal{D}((a_{ij}), (q_I), (g_i), (\chi_i), (\lambda_{ij})),$$

where (λ_{ij}) is a linking datum, will be called a **generic datum of finite Cartan type** for Γ.

Definition. $\mathcal{D}$ a generic datum of finite Cartan type for Γ. $U(\mathcal{D})$ = algebra presented by gens. $a_1, \dots, a_\theta$, $y_1^{\pm 1}, \dots, y_s^{\pm 1}$, rels.

$$y_m^{\pm 1} y_h^{\pm 1} = y_h^{\pm 1} y_m^{\pm 1}, \quad y_m^{\pm 1} y_m^{\mp 1} = 1, \qquad 1 \le m, h \le s,$$
$$y_h a_j = \chi_j(y_h) a_j y_h, \qquad 1 \le h \le s,\ 1 \le j \le \theta,$$
$$(\operatorname{ad} a_i)^{1-a_{ij}} a_j = 0, \qquad 1 \le i \ne j \le \theta, \quad i \sim j,$$
$$a_i a_j - \chi_j(g_i) a_j a_i = \lambda_{ij}(1 - g_i g_j), \qquad 1 \le i < j \le \theta, \quad i \not\sim j.$$

Theorem. [AS] $U(\mathcal{D})$ is a pointed Hopf algebra with

$$\Delta y_h = y_h \otimes y_h, \qquad \Delta a_i = a_i \otimes 1 + g_i \otimes a_i, \qquad 1 \le h \le s,\ 1 \le i \le \theta.$$

$U(\mathcal{D})$ has a PBW-basis given by monomials in the root vectors. The associated graded Hopf algebra $\operatorname{gr} U(\mathcal{D})$ is isomorphic to $\mathfrak{B}(V)\#\Bbbk\Gamma$; $U(\mathcal{D})$ is a domain with finite GK-dim.

Theorem. [AS, AA] H a pointed Hopf algebra with fin. generated abelian $G(H)$, and generic infinitesimal braiding. TFAE:

(a). H is a domain with finite Gelfand-Kirillov dimension.

(b). The group $\Gamma := G(H)$ is free abelian of finite rank, and there exists a generic datum of finite Cartan type $\mathcal{D}$ for Γ such that $H \simeq U(\mathcal{D})$ as Hopf algebras.

V. Reduced data and Levi-type decomposition.

$\mathcal{D}$ a generic datum of finite Cartan type, linking parameter λ.

- $\mathbb{I}^{\mathsf{s}} = \{h \in \mathbb{I} : h \text{ is not linked}\}$; $\mathbb{I}' = \mathbb{I} - \mathbb{I}^{\mathsf{s}}$; $X' = X_{\mathbb{I}'}$;

- $\approx$, the equivalence relation on $\mathbb{I}'$ defined by $\mathcal{D}'$;

- $\lambda'_{ij} = \begin{cases} \lambda_{ij}, & \text{if } i \not\sim j \\ 0, & \text{if } i \sim j, \end{cases}$ for all $i, j \in \mathbb{I}', i \not\approx j$.

Then λ' is a linking parameter for $\mathcal{D}'$.

- $\mathcal{D}' = \mathcal{D}((a_{ij})_{i,j\in\mathbb{I}'}, (q_I)_{I\in X'}, (g_i)_{i\in\mathbb{I}'}, (\chi_i)_{i\in\mathbb{I}'}, (\lambda_{ij})_{i,j\in\mathbb{I}'})$

Theorem. [ARS]

There are Hopf algebra morphisms

$$\Phi : \mathcal{U}(\mathcal{D},\lambda) \to \mathcal{U}(\mathcal{D}',\lambda'), \quad \Psi : \mathcal{U}(\mathcal{D}',\lambda') \to \mathcal{U}(\mathcal{D},\lambda), \quad \Phi\Psi = \mathrm{id}\,.$$

$K = \mathcal{U}(\mathcal{D},\lambda)^{\mathrm{co}\Phi}$ is a braided Hopf algebra in ${}^{\mathcal{U}(\mathcal{D}')}_{\mathcal{U}(\mathcal{D}')}\mathcal{YD}$ and

$$K\#\mathcal{U}(\mathcal{D}',\lambda') \cong \mathcal{U}(\mathcal{D},\lambda)$$

The algebra K is generated by the set

$$S = \{\mathrm{ad}(x_{i_1}\cdots x_{i_n})(x_h) \mid h \in L,\, n \geq 0,\, i_\nu \in \mathbb{I}',\, i_\nu \sim h, 0 \leq \nu \leq n\}.$$

A linking parameter $\mathcal{D}$ is *perfect* if and only if any vertex is linked.

$\mathcal{D}$ a generic datum of finite Cartan type, linking parameter $\lambda \implies \mathcal{U}(\mathcal{D}, \lambda) \twoheadrightarrow \mathcal{U}(\mathcal{D}', \lambda')$ with perfect linking parameter λ'.

$\mathcal{D}_{red}$ = half of $\mathcal{D}$ with perfect linking parameter.

$\mathcal{U}(\mathcal{D}_{red})$ = corresponding Hopf algebra.

Γ^2 = subgroup of Γ generated by $g_i g_j$: i and j linked.

References.

[AA] N. A. and I. Angiono. On Nichols algebras with generic braiding. In "Modules and Comodules". Trends in Mathematics. Brzezinski, T.; Gmez Pardo, J.L.; Shestakov, I.; Smith, P.F. (Eds.), pp. 47-64 (2008).

[ARS] N. A., D. Radford and H.-J. Schneider. Complete reducibility theorems for modules over pointed Hopf algebras. J. Algebra 324 (2010), pp. 2932-2970, Special Issue in honor of Susan Montgomery.

[AS] N. A. and H.-J. Schneider. A characterization of quantum groups . J. Reine Angew. Math. 577, 81–104 (2004).