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Integrated Analysis of Experimental Data

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Integrated Analysis of Experimental Data

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Abstract

A challenging important need is to extract from experimental data the correct values for the characteristic parameters while, at the same time minimizing their uncertainties. However, when we have at hand experimental data yielded from different experiments sharing the same characteristic parameters, it is difficult to find a procedure to take full advantage of the whole data package by combining the various sets of experimental data into a single fit, and consequently to determine unique values for such parameters and their uncertainties. In this work we develop a procedure allowing the integration of the various data sets into a grand set which in turn is analyzed into a fitting procedure producing unique values for the characteristic parameters in a single shot. This procedure is successfully applied, employing DataFit, a commercially available software, in two cases, providing full evidence and support about the reliability of the proposed procedure. Additionally, a comparison of the values of the uncertainties affecting the characteristic parameters with those obtained by standard methods is presented.

I – Introduction

In the realm of experimental sciences to carry on experiments and to gather data in order, for example, to verify a proposed model to describe a system or, by means of a reliable model, to determine the values for the characteristic parameters of such a system, is a demanding activity requiring time, supplies, equipment, manpower, etc. For these reasons the analysis of the collected information must be optimized obtaining the best values for the characteristic parameters and minimizing the uncertainties affecting them.

In this work we develop a procedure that combines various measurement sets obtained in different experiments in a unique integrated analysis allowing the determination of the best values for the characteristic parameters.

These sets of measurements may have been produced in two well differentiated situations. a) The same experiment is repeated, in the same system and in identical experimental conditions, a number of times. b) Different experiments sharing characteristic parameters and that may be repeated in identical experimental conditions a convenient number of times.

It is necessary that the measurements be produced under repeatability conditions, which are: (i) To employ the same measurement procedure; (ii) The observer/operator is the same; (iii) The same instruments are used; (iv) The instruments are used in the same conditions; (v) The measurements are carried out in the same place; and (vi) The measurements are carried out in a short period of time.

In Section II the procedure is described, and in Section III application examples are given and the highlights of the procedure are pointed out.

II – Procedure

Let us consider that experiments are performed on a system which may be described by a function such as that given by the equation

$$y(x) = f(P_1, P_2, \dots, V_1, V_2, \dots, x) \tag{1}$$

where:

— x identifies the independent variable in our experiment, for example time, temperature, etc.

— y is the dependent variable, namely position, specific heat, NQR (Nuclear Quadrupole Resonance) frequency, magnetization, etc.

— P_1, P_2, etc. are characteristic parameters of the system being studied whose values are wanted to be determined and which may not be obtained by other means.

— V_1, V_2, etc. are working parameters which neither depend on nor affect the characteristic parameters, but instead may be varied in order to produce distinct sets of measurements. These parameters are known and may be determined by other methods.

Furthermore, in order to simplify let us consider the case where we have two characteristic parameters P_1 and P_2 as well as two working parameters V_1 and V_2 and that x stands for time t. Generalization is straightforward.

III – A – Repeated Single Experiment

In this case it is analyzed the results of an experiment that may be repeated a number of times n and, in this way, we end up with n data sets of the form given by

$$\begin{array}{lll} y_1(t_a) = f(P_1, P_2, V_{11}, V_{21}, t_a) & 1 \le a \le m_a & 0 \le t_a < T_1 \\ y_2(t_b) = f(P_1, P_2, V_{12}, V_{22}, t_b) & 1 \le b \le m_b & 0 \le t_b < T_2 \\ & \vdots & \\ y_i(t_l) = f(P_1, P_2, V_{1i}, V_{2i}, t_l) & 1 \le a \le m_a & 0 \le t_l < T_i \\ & \vdots & \\ y_n(t_p) = f(P_1, P_2, V_{1n}, V_{2n}, t_p) & 1 \le p \le m_p & 0 \le t_p < T_n \end{array} \tag{2}$$

If each set is analyzed on its own we will end up obtaining n pairs of values (P_{1i}, P_{2i}), $1 \le i \le n$, for the characteristic parameters. Also, it has been allowed for each set to have different number of data points (m_a, m_b, … , m_p) as well as spanning different time intervals (T_1, T_2, … , T_n).

Since the characteristic parameters have a unique value for the particular system, it turns out more attractive and effective from a physical stand point to ask ourselves: How can we manage the integration of all of the data from the n sets given in Eqs. 2 in a single analysis producing unique values for the characteristic parameters P_1 and P_2?

To this end what must be done is to construct an array of the n data sets into a grand data set where, for physical reasons, the characteristic parameters P_1 and P_2 possess the same value since the system itself has not been modified and the measurements have been carried out in exactly under the same experimental conditions. Simultaneously, it is necessary to keep track of each individual set in order to allow the determination of the other experimental parameters such as, for example, V_1, V_2 and the time intervals t under which the different experiments are run.

We propose to build such a grand data set through the ordered union of the n sets appropriately identifying them by means of the independent variable t.

Among the commercial software, DataFit[1] was chosen since it combines simplicity, easiness and is particularly useful to the objective in this work.

To this end, artificial shifts of the time variables are introduced corresponding to each data set, such as

$$
\begin{aligned}
y_1(t_j) &= f(P_1, P_2, V_{11}, V_{21}, t_j), 1 \le j \le m_1, 0 \le t_j < T_1 \\
y_2(t_k) &= f(P_1, P_2, V_{12}, V_{22}, t_k), 1 \le k \le m_2, T_1 \le t_k < T_1 + T_2 \\
&\vdots \\
y_i(t_l) &= f(P_1, P_2, V_{1i}, V_{2i}, t_l), 1 \le l \le m_i, T_1 + T_2 \ldots + T_{i-1} \le t_l < T_1 + T_2 \ldots + T_i \\
&\vdots \\
y_n(t_m) &= f(P_1, P_2, V_{1n}, V_{2n}, t_m), 1 \le m \le m_n, T_1 + T_2 \ldots + T_{n-1} \le t_m < T_1 + T_2 \ldots + T_n
\end{aligned}
\tag{3}
$$

where, in general, having performed the ith repetition of the experiment, m_i indicates the number of (t_l, $y_i(t_l)$) data pairs collected in the ith set while time t_l spans the interval $[T_1+ T_2+\ldots +T_{i-1} , T_1+ T_2+\ldots +T_{i-1} +T_i)$.
Therefore, with the proposed union of these n sets a unique grand set is obtained, which consists of $M = \sum_{i=1}^{n} m_i$ data pairs (y_k, t_k) with $1 \le k \le M$ and $t_k \, \varepsilon \, [0, T)$ with $T = \sum_{i=1}^{n} T_i$. The ordering of the (y_k, t_k) pairs is the one given in Eqs. 3.

Thus we have prepared the data in an appropriate way to carry on the integrated analysis of the n measurement sets by means of DataFit, hence determining single values for parameters P_1 y P_2 optimizing the fitting of the n measurements sets simultaneously.

III – B – Repeated Different Experiments

In this case we consider the analysis of different experiments realized on the same system that may be repeated. For the sake of clarity, let us consider two different experiments performed on the same system, sharing characteristic parameters, which may be repeated as in the previous case.

Let the functions g and h describe the two experiments under analysis. Let us also assume that, following the procedure described in the previous point, the times have been adequately shifted in order to carry on the proposed analysis

$$
\begin{aligned}
&u_1(t_a) = g(P_1, P_2, V_{11}, V_{21}, t_a), \quad 1 \le a < k_{u1}, \qquad 0 \le t_a \le T_{u1} \\
&u_2(t_b) = g(P_1, P_2, V_{12}, V_{22}, t_b), \quad 1 \le b < k_{u2}, \qquad T_{u1} \le t_b \le T_{u1} + T_{u2} \\
&\qquad \vdots \\
&u_i(t_c) = g(P_1, P_2, V_{1i}, V_{2i}, t_c), \quad 1 \le c < k_{ui}, \qquad T_{u1} + \cdots T_{u(i-1)} \le t_c \le T_{u1} + \cdots + T_{ui} \\
&\qquad \vdots \\
&u_n(t_d) = g(P_1, P_2, V_{1n}, V_{2n}, t_d), \quad 1 \le d < k_{un}, \qquad T_{u1} + \cdots T_{u(n-1)} \le t_d \le T_{u1} + \cdots + T_{un}
\end{aligned}
$$

$$T_u = \sum_{i=1}^{n} T_{ui} \tag{4}$$

$$
\begin{aligned}
&v_1(t_e) = h(P_1, P_2, V_{11}, V_{21}, t_e), \quad 1 \le e < k_{v1}, \qquad T_u \le t_e < T_u + T_{v1} \\
&v_2(t_f) = h(P_1, P_2, V_{12}, V_{22}, t_f), \quad 1 \le f < k_{v2}, \qquad T_u + T_{v1} \le t_f < T_u + \cdots + T_{v2} \\
&\qquad \vdots \\
&v_j(t_k) = h(P_1, P_2, V_{1j}, V_{2j}, t_k), \quad 1 \le k < k_{vj}, \qquad T_u + T_{v1} \cdots + T_{v(k-1)} \le t_k < T_u + \cdots + T_{vk} \\
&\qquad \vdots \\
&v_m(t_l) = h(P_1, P_2, V_{1m}, V_{2m}, t_l), \quad 1 \le l < k_{vm}, \qquad T_u + T_{v1} \cdots T_{v(m-1)} \le t_l \le T_u + \cdots + T_{vm}
\end{aligned}
$$

$$T_v = \sum_{j=1}^{m} T_{vj}$$

where time t in set $1a$ takes k_{u1} values and spans the interval $[0, T_{u1})$, in set $2b$ takes k_{u2} values and spans the interval $[T_{u1}, T_{u1}+T_{u2})$, etc., and k_{ui} and k_{vi} indicate the number of (t_i, u_i) and (t_j, y_j) data pairs collected while performing the *ith* and the *jth* repetition of the experiments described by g and h, respectively.

The analysis follows similarly as that developed in the previous point. To proceed it is necessary to perform the union of these $n+m$ data sets in order to build the grand set, which consists of $N+M$, $N = \sum_{i=1}^{n} k_{ui}$ and $M = \sum_{j=1}^{m} k_{vj}$, data pairs $(u_i(t_c),\ t_c)$ with $1 \le k_{ui} \le N$, $t_c \in (0, T_u = \sum_{i=1}^{n} T_{ui})$ and $t_v \in \left(T_u, T_u + T_v = T_u + \sum_{j=1}^{m} T_{vj}\right)$. The order of the ordered pairs is that given in Eqs. 4. It may be mentioned that there are other equivalent ways of building the grand set and the choice as to what is the preferred way to build it up to our decision.

In this way the data has been properly prepared to carry on the integrated fit of the $n+m$ sets of measurements by means of DataFit in a unique analysis thus determining unique values for the parameters P_1 and P_2 optimizing the fitting of all of the data simultaneously.

IV – Employing DataFit

In the “Model Editor” screen is where DataFit defines the function Y to be used to fit the data, allowing this definition to take place through the definition of a series of partial functions Fi with $1 \le i \le 9$.

For the sake of simplicity let us consider the case in which the one experiment has been repeated three times. Generalization is straightforward. This situation is shown in Ec. 5

$$\begin{aligned} y_{1i} &= f(P_1, P_2, V_{11}, V_{21}, t_i) \quad 1 \le i < m_1 \quad 0 \le t_i < T_1 \\ y_{2j} &= f\left(P_1, P_2, V_{12}, V_{22}, t_j\right) \quad 1 \le j < m_2 \quad T_1 \le t_j < T_1 + T_2 \\ y_{3k} &= f(P_1, P_2, V_{13}, V_{23}, t_k) \quad 1 \le k < m_3 \quad T_1 + T_2 \le t_k \le T_1 + T_2 + T_3 \end{aligned} \tag{5}$$

The way to proceed with DataFit is shown in Appendix A.1 (Section VII).

Running DataFit will allow determining the two shared characteristic parameters P_1 and P_2 as well as the six non-shared experimental parameters V_{11}, V_{21}, V_{12}, V_{22}, V_{13} and V_{23}, with their corresponding standard deviations.

In the following section examples are given.

V – Examples

Two examples will be developed in detail: (a) The decay of the transverse magnetization in the Carr-Purcell-Meiboom-Gill (CPMG)[2] sequence, Nuclear Magnetic Resonance (NMR) experiment; and (b) The flywheel experiment, a basic experiment in Mechanics[3].

The CPMG corresponds to the repeated single experiment case treated in Section III-A, while the flywheel corresponds to the repeated different experiments case treated in Section III-B.

NMR-CPMG Decay Signal

The decay of the transversal component of the magnetization in an NMR experiment may be determined by means of the CPMG pulse sequence. This signal may, in general, be well described by an exponential decay; its amplitude is related to the amount of substance and to the gain of the analizer while the characteristic time constant is a property of the compound producing the signal. In the present case the NMR measurements have been carried out employing a Minispec[4] instrument using the ^1H in a brine prepared with a concentration of 50 ppm of ClNa. The signals amplitude obtained are well described by an exponential decay as that given in Eq. 7

$$S_i(t) = A_i\, exp\left(-\frac{t}{\tau 2_i}\right) + S_i\, t + O_i \qquad 0 \le t \le T_i \qquad 1 \le i \le n \tag{6}$$

The experiment is repeated n times in exactly the same experimental conditions while the time t sweeps the interval $0 \le t \le T$, and the data is gathered at times evenly spaced by about 6 ms. In the present case the experiment has been repeated eight times. Due to the experimental conditions it is expected that the characteristic parameters A_i and $\tau 2_i$ should have the same value in all eight measurements. The same situation holds for S_i and O_i which are added in order to account for the presence of base line drifts produced by the NMR spectrometer during the experiments.
Employing T = 12000 *ms* the DataFit expressions are shown in Appendix A.2 (Section VIII). The results obtained by means of the integrated fit are given in Table 1 and depicted in Fig. 1.

Table 1: Values of the characteristic parameters in Eqs. a.2

Parameter	Value
A (*AU**)	15.634 ± 0.004

$\tau 2$ (*ms*)	2444 ± 1

* *AU* stands for Arbitrary Units.

The uncertainties –standard deviations– in the fitted parameters using DataFit are evaluated following the Guide ISO/IEC 98-3:2008.[5]

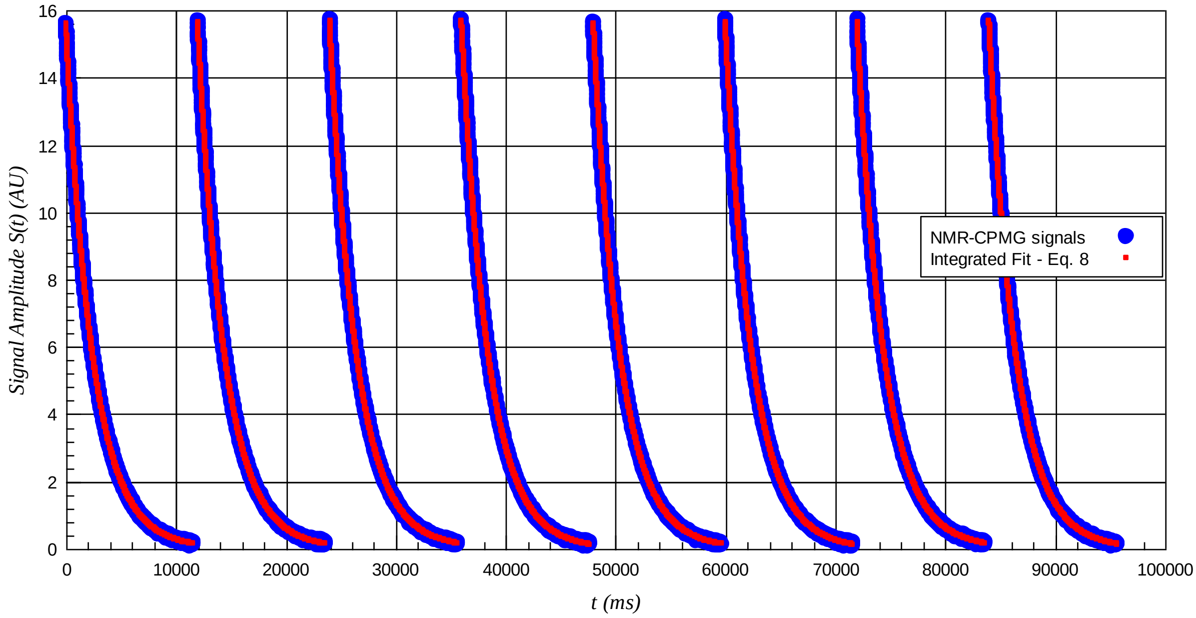

Figure 1: The eight NMR–CPMG brine signal amplitudes (blue) are depicted along the integrated fit (red) achieved by means of Eq. 6. As may be seen the description of the experimental data is excellent.

In order to show the improvement achieved by means of the proposed integrated analysis it is convenient to compare these results with those obtained treating the eight signals individually. The characteristic parameters obtained for the eight signals are given in Table 2.

Table 2: Values obtained for the characteristic parameters along with their standard deviations are given for each of the eight signals analyzed with DataFit using the fitting function given in Eq. 6.

Signal	A (*AU*)	$\tau 2$ (*ms*)
1	15.6065 ± 0.0095	2440.8 ± 2.5
2	15.6520 ± 0.0091	2447.6 ± 2.4
3	15.6092 ± 0.0089	2435.1 ± 2.4
4	15.7360 ± 0.0092	2460.1 ± 2.4
5	15.6152 ± 0.0089	2450.5 ± 2.4
6	15.6118 ± 0.0090	2434.6 ± 2.4
7	15.5633 ± 0.0087	2434.9 ± 2.3
8	15.6783 ± 0.0091	2448.0 ± 2.4

As may be seen in Table 2 the characteristic parameters, *A* and $\tau 2$, have close but different values. Two questions arise: (a) How to determine the values to be used? and (b) What are their standard deviations?

A mean value ($\overline{\tau 2}$, and $\bar{A}$) and its standard deviation ($\sigma_{\overline{\tau 2}}$, and $\sigma_{\bar{A}}$) may be evaluated in various ways. Three of them are as follows: (a) applying the basic Gauss theory for random errors[6]; (b) applying the procedure defined in the GUM (Guide to the expression of Uncertainty in

Measurement ISO/IEC 98-3:2008)[5]; and (c) applying the methodology developed in this integrated analysis and employing DataFit.

The results of these three ways for τ2 are shown in Eqs. 7.

Results obtained following Gauss theory for random errors

$$\overline{\tau 2} = \frac{1}{8}\sum_{\mathrm{i}=1}^{8} \tau 2_{\mathrm{i}} = 2443.95\ ms$$

$$\sigma_{\overline{\tau 2}} = \sqrt{\frac{\sum_{\mathrm{i}=1}^{8}(\tau 2_{\mathrm{i}} - \overline{\tau 2})^2}{8-1}} = 9.2\ ms \qquad \Delta\overline{\tau 2} = \frac{\sigma_{\overline{\tau 2}}}{\sqrt{8}} = 3.3\ ms \qquad (7-a)$$

$$\tau 2 = \overline{\tau 2} \pm \Delta\overline{\tau 2} \cong 2444 \pm 3\ ms$$

Results obtained following GUM

$$\boldsymbol{u}_{\tau 2} = \frac{\boldsymbol{u}_{\tau 2_{\boldsymbol{i}}}}{\sqrt{8}} \cong \frac{2.4}{\sqrt{8}} = 0.85\ ms$$

$$\tau 2 = (2443.95 \pm 0.85)\ ms \cong (2444 \pm 1)\ ms \qquad (7-b)$$

According to GUM $\boldsymbol{u}_{\tau 2}$ is the uncertainty in the parameter τ2.

Results obtained following this work

$$\tau 2 = \overline{\tau 2} \pm \Delta\overline{\tau 2} = (2443.94 \pm 0.88)\ ms \cong (2444 \pm 1)\ ms \qquad (7-\mathrm{c})$$

Similarly the results obtained for *A* are shown in Eqs. 10.

Results obtained following Gauss theory for random errors

$$\bar{A} = \frac{1}{8}\sum_{\mathrm{i}=1}^{8} A_{\mathrm{i}} = 15.634\ AU$$

$$\sigma_{\bar{A}} = \sqrt{\frac{\sum_{\mathrm{i}=1}^{8}(A_{\mathrm{i}} - \bar{A})^2}{8-1}} = 0.053\ \mathrm{AU} \qquad \Delta\bar{A} = \frac{\sigma_{\bar{A}}}{\sqrt{8}} = 0.19\ AU \qquad (8-a)$$

$$A = \bar{A} \pm \Delta\bar{A} \cong (15.634 \pm 0.019)\ AU \cong (15.63 \pm 0.02)\ AU$$

Results obtained following GUM

$$\boldsymbol{u}_{A} = \frac{\boldsymbol{u}_{A_i}}{\sqrt{8}} \cong \frac{0.0091}{\sqrt{8}} = 0.0032\ AU$$

$$A = (15.6340 \pm 0.0032)\ AU \cong (15.634 \pm 0.003)\ AU \qquad (8-b)$$

According to GUM $\boldsymbol{u}_{\mathrm{A}}$ is the uncertainty in the parameter *A*.

Results obtained following this work

$$A = (15.6339 \pm 0.0037)\ AU \cong (15.634 \pm 0.004)\ AU \qquad (8 - c)$$

The following conclusions are in order:

1) The mean values $\overline{\tau 2}$ and $\bar{A}$ agree with those determined by means the integrated analysis, thus supporting the reliability of the procedure developed in this work.
2) The GUM standard deviations agree with those determined through the integrated analysis, thus bringing additional support to the procedure developed in this work.
3) The close agreement in the results given by Eqs. 7b with 7c and 8b with 8c indicates that the evaluation of uncertainties by DataFit closely follows the GUM.
4) It is generally observed a greater uncertainty in applying the Gauss standard theory, giving a clear indication of the difficulty of implementing it with a certain type of experimental data. A more detailed study (e.g. as suggested in the book by P.R. Bevington and D.K. Robinson[8]) can lead to decrease the difference between the uncertainties obtained by different methods.
5) The integrated procedure allows saving time and effort since, in the case under analysis, the situation was solved with just one fit instead of the eight required by other means. These savings increase with the number of acquired signals.
6) What has been pointed out for $\tau 2$ and *A* is also valid for the other parameters (*O* and *S* in Eq. 6).
7) The integrated analysis procedure provides support to the expectation that the various measurements have been carried on the same initial experimental conditions, thus producing repeatability.
8) The integrated analysis methodology allows determining in an elegant form and in just one attempt unique values for the characteristic parameters (*A* and $\tau 2$) as well for others that may not be so relevant (*O* and *S*).

Flywheel

This is a standard experiment carried out in the Mechanics General Physics Laboratory, whose experimental setup is shown in Fig. 2.

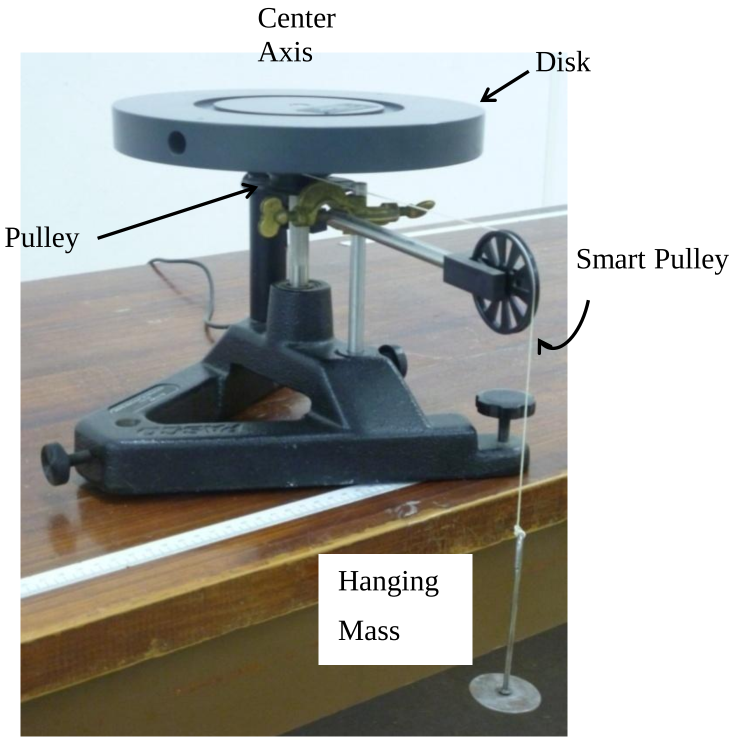

Figure 2: The main parts of the flywheel experimental setup are shown (ring is shown in Fig. 3).

Two main concepts are worked out: (a) The moment of inertia of a rotating flywheel (disk and ring); and (b) The flywheel kinetic energy losses by friction, mainly at the ball bearings.

The main parts and other considerations are:

- The flywheel and the spool, having radius r, are fixed to the axis that is able to rotate.
- The axis sits on the supports by low friction ball bearings.
- The total moment of inertia I of the system includes the flywheel, the axis, the spool and the ball bearings.
- The moment of inertia of the flywheel may easily be changed.
- A mass m hangs from a string which is wound about the spool. The string is no extensible, and its mass and diameter are negligible compared to corresponding quantities of the system.
- The mass may easily be changed.
- The falling of m is what puts the system into motion. Since the string has a limited length, after a while the mass drops from the spool and the system keeps in motion freely being acted only by the friction on the ball bearings. This friction will finally bring the system to rest.
- The experiment has two well differentiated stages: (i) The falling of m, starting with the system initially at rest, puts the system under motion transferring energy until m drops from the spool; and (ii) The rotating system, due to the friction at the ball bearings, dissipates its kinetic energy until, eventually, comes to rest.

- The available equipment[7] allows the acquisition of the angle $\theta(t)$ rotated by the flywheel as a function of time t.
- The friction may be assumed as a dissipative couple τ_F applied to the system at the ball bearings.
- This dissipative couple may be written as $\tau_F = -I\alpha$, where α may be thought as a negative acceleration applied to the system, reducing its kinetic energy until total rest (eventually) is achieved.
- The characteristic parameters of the system that we are interested in are I and α.

With these elements we may write down the equations of motion and their solutions

$$\frac{d^2\theta}{dt^2} = \ddot{\theta} = \frac{mgr - I\alpha}{I + mr^2}$$

$$m \neq 0 \quad \Rightarrow \quad \theta_L(t) = \frac{\ddot{\theta}_L}{2}t^2 + \dot{\theta}_{L0}t + \theta_{L0}\ , \ \ddot{\theta}_L = \frac{mgr - I\alpha}{I + mr^2} \tag{9}$$

$$m = 0 \quad \Rightarrow \quad \theta_F(t) = \frac{\ddot{\theta}_F}{2}t^2 + \dot{\theta}_{F0}t + \theta_{F0}\ , \ \ddot{\theta}_F = -\alpha$$

Here we have used $\theta, \dot{\theta}, \ddot{\theta}$ to denote the angles rotated by the disc and its first and second derivatives respectively as a function of time t. Also, mg is the weight of the loaded mass, r is the pulley radius, $I\alpha$ is the opposing torque produced by the viscose force, and I is the flywhell moment of inertia.
In equation (9) L and F stand for "loaded" ($m \neq 0$) and "free" ($m = 0$), respectively$\dot{\theta}_{L0}$, $\dot{\theta}_{F0}$, θ_{L0}, and θ_{F0}, are angular velocities and angles required to satisfy initial conditions of the system and containing information of no interest.

Typically, the standard procedure to determine I and α is as follows: (a) To carry on a number of experiments for $\theta_F(t)$ from which a similar number of values for α are gathered, and from these values to obtain a representative value which may be given, for example, by the mean average; and (b) To carry on a number of experiments for $\theta_L(t)$ (repeating the experiment and/or varying, for example m and/or r) and, employing α previously determined, obtain a similar number of values for I given by

$$I = \frac{mr\left(g - \ddot{\theta}_L r\right)}{\ddot{\theta}_L + \alpha} \tag{10}$$

and from these obtain a representative value, as that given, for example, by the mean average. Furthermore there is not a unique procedure, thus providing different values for the characteristic parameters. But, clearly the characteristic parameters, on physical grounds, have unique and well defined values. This assertion is viewed more consistent with the spirit of the integrated analysis.

Let us apply the above developed procedure in order to determine the moment of inertia of a given rigid body. In the present case this given rigid body is a hollow cylinder (or ring) as that shown in Fig. 3

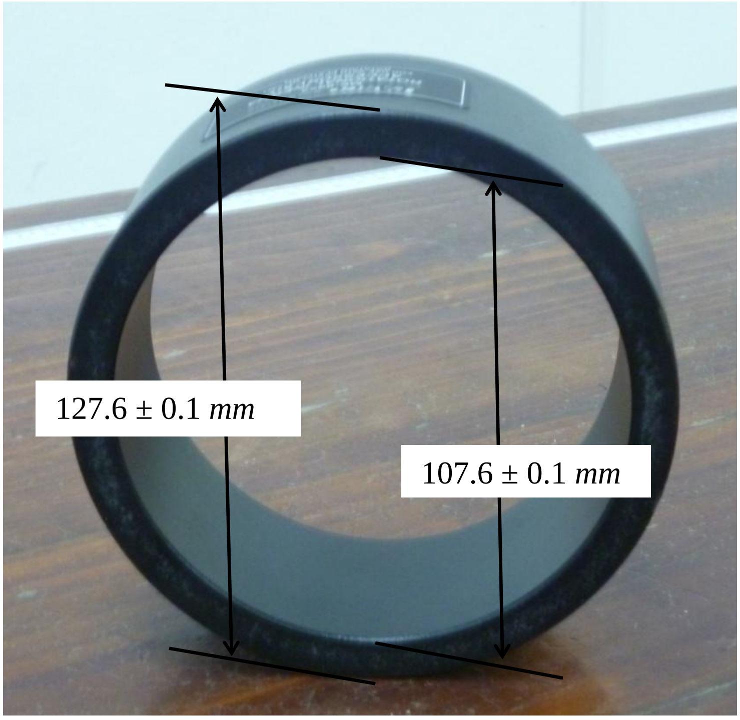

Figure 3: The geometry of the hollow cylinder is sketched. The material what it is made of is homogenous throughout its entire volume and its mass is $M_{hc} = 1.273 \pm 0.001\ kg$. The geometric moment of inertia $I_{hcg} = 5.102 \pm 0.065\ 10^{-3} kg\ m^2$. The uncertainty in I_{hcg} has been evaluated following GUM[5].

The main objective is to compare the value deduced by means of the present integrated procedure with that geometrically obtained. Let us call I_{hcip} and I_{hcg} the moments of inertia of the hollow cylinder (*hc*) obtained by means of this integrated procedure (*ip*) and geometrically determined (*g*). This hollow cylinder may be easily mounted on to a disc rigidly attached to the axis and the spool as shown in Fig. 2. Let I_{disc} be the moment of inertia of the disc, axis, spool and ball bearings.

Thus we have four different experiments: (i) I_{disc} with *m* falling; (ii) I_{disc} free; (iii) $I_{disc} + I_{hcip}$ with *m* falling; and (iv) $I_{disc} + I_{hcip}$ free. In the present case each of these four experiments was repeated three times, therefore the grand set has twelve components. The idea is to apply the integrated procedure developed in this work in order to determine I_{hcip}, I_{disc}, $\alpha_{disc+hc}$ and α_{disc} are the angular accelerations acting on the free system when the hollow cylinder is attached or not to the disc, respectively. Furthermore, $\alpha_{disc+hc}$ and $\alpha_{disc+hc}$ are allowed to possess different values in order to account for the different loads acting on the ball bearings in both cases.

Since the values for *m* and *r* were kept constant throughout the experiments, and since *g* is a constant, Eqs. 9 may be rewritten as

$$\begin{gathered} m = 50.7\ 10^{-3} kg \quad g = 9.81\ ms^{-2} \quad r = 12.7\ 10^{-3} m \\ mgr = 6.3166\ 10^{-3} kgm^2 \quad mr^2 = 8.1774\ 10^{-6} kgm^2 \\ \ddot{\theta}_L = \frac{mgr - I\alpha}{I + mr^2} = \frac{6.3166\ 10^{-3} - I\alpha}{I + 8.1774\ 10^{-6}} \end{gathered} \tag{11}$$

Thus the four cases above mentioned may be written as

$$m \neq 0;\ \ I = I_{disc} + I_{hc}\ \Rightarrow$$

$$\Rightarrow\ \theta_L(t) = \frac{6.3166\ 10^{-3} - (I_{disc} + I_{hc})\alpha_{disc+hc}}{2\ (I_{disc} + I_{hc} + 8.1774\ 10^{-6})} t^2 + \dot{\theta}_{L0,disc+hc} t + \theta_{L0,disc+hc} \qquad (12-a)$$

$$m = 0;\ \ I = I_{disc} + I_{hc}\ \Rightarrow$$

$$\Rightarrow\ \theta_F(t) = -\frac{\alpha_{disc+hc}}{2} t^2 + \dot{\theta}_{F0,disc+hc} t + \theta_{F0,disc+hc} \qquad (12-b)$$

$$m \neq 0;\ \ I = I_{disc}\ \Rightarrow$$

$$\Rightarrow\ \theta_L(t) = \frac{6.3166\ 10^{-3} - I_{disc}\alpha_{disc}}{2\ (I_{disc} + 8.1774\ 10^{-6})} t^2 + \dot{\theta}_{L0,disc} t + \theta_{L0,disc} \qquad (12-c)$$

$$m = 0;\ \ I = I_{disc}\ \Rightarrow$$

$$\Rightarrow\ \theta_F(t) = -\frac{\alpha_{disc}}{2} t^2 + \dot{\theta}_{F0,disc} t + \theta_{F0,disc} \qquad (12-d)$$

where the characteristic parameters to be determined are I_{disc}, I_{hc}, $\alpha_{disc+hc}$ and α_{disc}.

A time $T = 200\,s$ to separate and identify the twelve sets in the grand set, was found to be appropriate.

The expressions to be used in DataFit to carry on the fitting are shown in Appendix A.3 (Section IX).

As mentioned above each of the experiments was repeated three times. The first three runs are those with $I_{disc} + I_{hcip}$ while the remaining three are those with I_{disc}. The initial part –the short one- of each run is that with the falling *m*, while the final part –the long one– corresponds to the free one. Thus, the gathered experimental data along with the fit achieved by means of Eqs. 12 are depicted in Fig. 4.

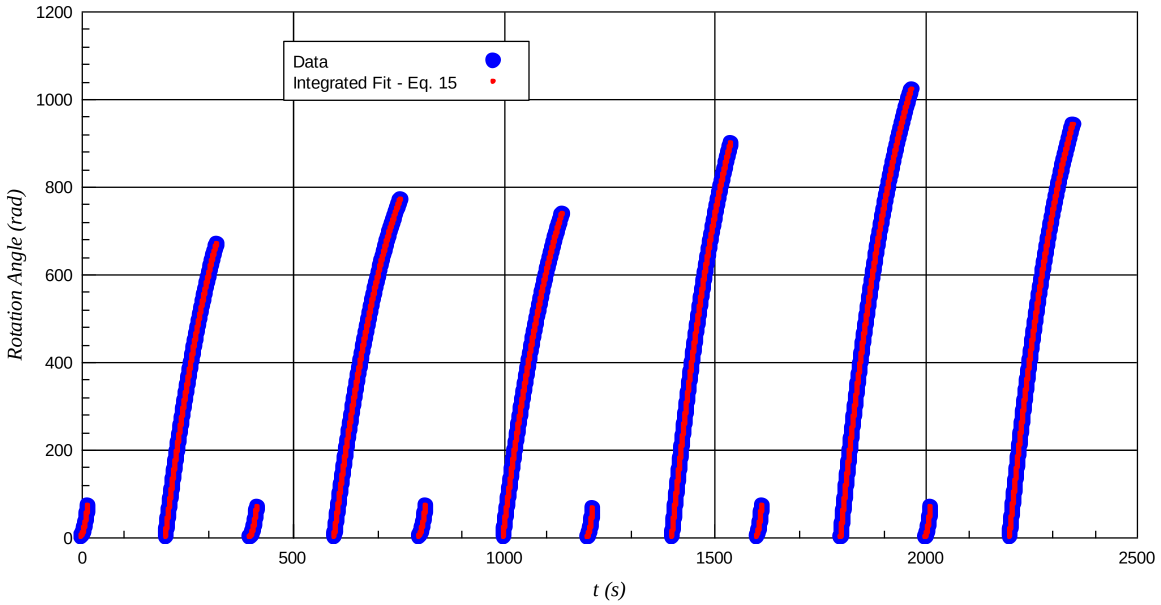

Figure 4: The twelve runs (blue) are depicted along the integrated fit achieved by means of Eqs. 12. As may be seen the description of the experimental data is excellent.

The values obtained for the characteristic parameters are those given in Table 3:

Table 3: Values obtained for the characteristic parameters	
$\alpha_{disc+hc}$	$(3.2314 \pm 0.0028)\ 10^{-2}$ rad/s^2
α_{disc}	$(3.9711 \pm 0.0021)\ 10^{-2}$ rad/s^2
I_{disc}	$(9.286 \pm 0.098)\ 10^{-3}$ kg m^2
I_{hcip}	$(5.110 \pm 0.017)\ 10^{-3}$ kg m^2

Let us consider particularly I_{hc}, i.e. the moment of inertia of the body we are interested in. Analyzing each of the six runs –consistent in the loaded and free components- by means of Eqs. 12 we end up with three independent values for $I_{disc+hc}$ and for I_{disc}.
For the sake of being clear the chosen procedure is as follows. Let us consider one of the three runs with I_{disc} (Eqs. 12-c and 12-d). Fitting the "free" part of the data using Eq. 12-d, α_{disc} is determined. By introducing this value into Eq. 12-c and fitting the "loaded" part of the data, a value is determined for I_{disc}.

With these values for $I_{disc+hc}$ and for I_{disc} nine values for $I_{hc} = I_{disc+hc} - I_{disc}$ are determined whose mean value and standard deviation are $I_{hc} = (5.100 \pm 0.014)\ 10^{-3} kg\ m^2$.

Thus, we end up with three values for the moment of inertia of the hollow cylinder $I_{hcg} = (5.102 \pm 0.065)\ 10^{-3} kg\ m^2$, $I_{hcip} = (5.110 \pm 0.017)\ 10^{-3} kg\ m^2$, and $I_{hc} = (5.100 \pm 0.014)\ 10^{-3} kg\ m^2$. These results lead to the following comments:

1) The value determined for I_{hcip} agrees with the geometric I_{hcg} and run-by-run I_{hc} values, thus supporting the reliability of the procedure developed in this work.
2) The integrated procedure allows saving time and effort since, in the case under analysis, the situation was solved with just one fit instead of the twelve required by other means. This saving increases with the number of acquired signals.

3) The integrated analysis procedure provides support to the expectation that the various measurements have been carried on in identical conditions, thus producing repeatability as indicated above in the previous example.
4) The integrated analysis methodology allows determining in an elegant form and in just one attempt unique values for the characteristic parameters $\left(\alpha_{disc+hc}, \alpha_{disc}, I_{disc}\text{ and } I_{hcip}\right)$ as well for others that may not be so relevant and related to the initial conditions.
5) What has been pointed out for I_{hcip} is also valid for the remaining characteristic parameters.

VI – Discussion and Conclusions

In this work we have successfully developed a procedure to integrate various related files and to analyze them in a single fit. The proposed method, if it can be applied, effectively implies a more realistic analysis of data resulting in a better measurement quality. Briefly, the reason for this is that for the "integrated analysis" all the experimental points have the same "status". Not so in other ways in which each data set yields an average value of equal weight, independent of the set to which it belongs. Which causes small data sets have higher relative weights. The problem could be overcome including the notion of "statistical weights", which continues to be a patch in an attempt to get closer to what we have been called the "integrated analysis".

Two examples were analyzed employing DataFit. Various main advantages may be pointed out: (a) A unique value is determined for the common characteristic parameters –which are the ones we are most interested in– in a single shot; (b) The uncertainties determined for these characteristic parameters turn out, as expected, to be generally smaller than those obtained by standard methods; (c) As a byproduct non characteristic parameters such as baseline drifts, initial conditions, etc., are also determined; (d) Data handling is easy; (e) Computer time is shortened; and (f) Analysis of results is shortened and improved.

The method has, perhaps, against the need to know how to program.

VII – Appendix A.1

Let us consider the case in which the one experiment has been repeated three times. Generalization is straightforward. This situation is shown in Ec. 5. The way to proceed with DataFit is as follows:

```
F1=f(P1,P2,V11,V21,x)
F2=f(P1,P2,V12,V22,x-T1)
F3=f(P1,P2,V13,V23,x-(T1+T2))                    (a.1)
Y=if(x<T1,F1,if(x<(T1+T2),F2,F3))
```

Advantage has been taken of the sentence “if” implemented in DataFit. Functions F1, F2, F3 and Y must be written with the proper syntax. Particularly, DataFit assigns “*x*” to the independent variable (that is the reason as to why we have switched *t* by *x*).

VIII – Appendix A.2

For the example of NMR-CPMG Decay Signal, the DataFit expressions are:

F1 = 12000
F2 = A*exp(-(x-0*F1)/τ2)+O+S*(x-0*F1)
F3 = A*exp(-(x-1*F1)/τ2)+O+S*(x-1*F1)
F4 = A*exp(-(x-2*F1)/τ2)+O+S*(x-2*F1)
F5 = A*exp(-(x-3*F1)/τ2)+O+S*(x-3*F1) (a.2)
F6 = A*exp(-(x-4*F1)/τ2)+O+S*(x-4*F1)
F7 = A*exp(-(x-5*F1)/τ2)+O+S*(x-5*F1)
F8 = A*exp(-(x-6*F1)/τ2)+O+S*(x-6*F1)
F9 = A*exp(-(x-7*F1)/τ2)+O+S*(x-7*F1)
Y = if(x<1*F1, F2, if(x<2*F1, F3, if(x<3*F1, F4, if(x<4*F1, F5, if(x<5*F1, F6,if(x<6*F1, F7, if(x<7*F1, F8, F9)))))))

Each of the eight files has about 2000 data points, thus the grand file contains about 16000 points. In each of these eight files the independent variable takes evenly spaced values in the $[0, T = 12000\ ms)$ interval.

IX – Appendix A.3

For the case of flywheel the expressions to be used in DataFit to carry on the fitting are:

F1 = 200
F2 = if(x<1*F1, ((6.3166E-3)-(Idisc+Ihc)*Adischc)/(2*((Idisc+Ihc)+8.1774E-6))*(x-0*F1)^2+VL1*(x-0*F1)+XL1, if(x<2*F1, -Adischc/2*(x-1*F1)^2+VF1*(x-1*F1)+XF1, 0))
F3 = if(x<3*F1, ((6.3166E-3)-(Idisc+Ihc)*Adischc)/(2*((Idisc+Ihc)+8.1774E-6))*(x-2*F1)^2+VL2*(x-2*F1)+XL2, if(x<4*F1, -Adischc/2*(x-3*F1)^2+VF2*(x-3*F1)+XF2, 0))
F4 = if(x<5*F1, ((6.3166E-3)-(Idisc+Ihc)*Adischc)/(2*((Idisc+Ihc)+8.1774E-6))*(x-4*F1)^2+VL3*(x-4*F1)+XL3, if(x<6*F1, -Adischc/2*(x-5*F1)^2+VF3*(x-5*F1)+XF3, 0)) (a.3)
F5 = if(x<7*F1, ((6.3166E-3)-Idisc*Adisc)/(2*(Idisc+8.1774E-6))*(x-6*F1)^2+VL4*(x-6*F1)+XL4, if(x<8*F1, -Adisc/2*(x-7*F1)^2+VF4*(x-7*F1)+XF4, 0))
F6 = if(x<9*F1, ((6.3166E-3)-Idisc*Adisc)/(2*(Idisc+8.1774E-6))*(x-8*F1)^2+VL5*(x-8*F1)+XL5, if(x<10*F1, -Adisc/2*(x-9*F1)^2+VF5*(x-9*F1)+XF5, 0))
F7 = if(x<11*F1, ((6.3166E-3)-Idisc*Adisc)/(2*(Idisc+8.1774E-6))*(x-10*F1)^2+VL6*(x-10*F1)+XL6, if(x<12*F1, -Adisc/2*(x-11*F1)^2+VF6*(x-11*F1)+XF6, 0))
Y = if(x<2*F1, F2, if(x<4*F1, F3, if(x<6*F1, F4, if(x<8*F1, F5, if(x<10*F1, F6, if(x<12*F1, F7, 0))))))

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