Uniform functions constants of the motion and their first order perturbation

Domingo Prato and Victor H. Hamity
Fa.M.A.F., Universidad Nacional de Córdoba
Ciudad Universitaria
Córdoba 5000, Argentina

August 29, 2004

The main purpose of this work is to present some uniform functions constant of the motion, either than the well known quantities arising from spacetime symmetries. These constants are usually associated with intrinsic characteristic of the trajectories of a particle in a central potential field. We treat two cases. The first one is the Lenz vector which sometimes is found in the literature[1, 2]; the other case is associated with the isotropic harmonic oscillator, of relative importance in some simple models of classical molecular interaction. The first example is applied to describe the perturbation of the trajectories in the Rutherford scattering and the precession of the Keplerian orbit of a planet. In the other case the conserved quantity is a symmetric tensor. We find the eigenvectors and eigenvalues of that tensor while at the same time we obtain the solution to the problem of calculating the rotation rate of the orbits in first order of a perturbation parameter in the potential energy, by performing a simple coordinate transformation in the Cartesian plane. We think that the present work addresses to many aspects of Mechanics with a didactical interest in other Physics or Mathematics courses.

1 Introduction

It is well known that an isolated mechanical system has a set of conserved quantities associated with its invariance property under the Galilean group. In the Newtonian formulation of the equations of motion this property is characterized by the absence of a total external force and an external net torque acting on the system; it is also required that the internal force between a pair of particles depends only on the distance between the particles[1] These conserved quantities are the well known linear momentum and angular momentum, both as conserved vectors, and the total energy[2] considered a

[1] For the conservation of the angular momentum it is enough that the internal force between a pair of particles be along the same line of action.

[2] The sum of the internal potential energy plus the kinetic energy corresponding to the motion of the system as a whole.

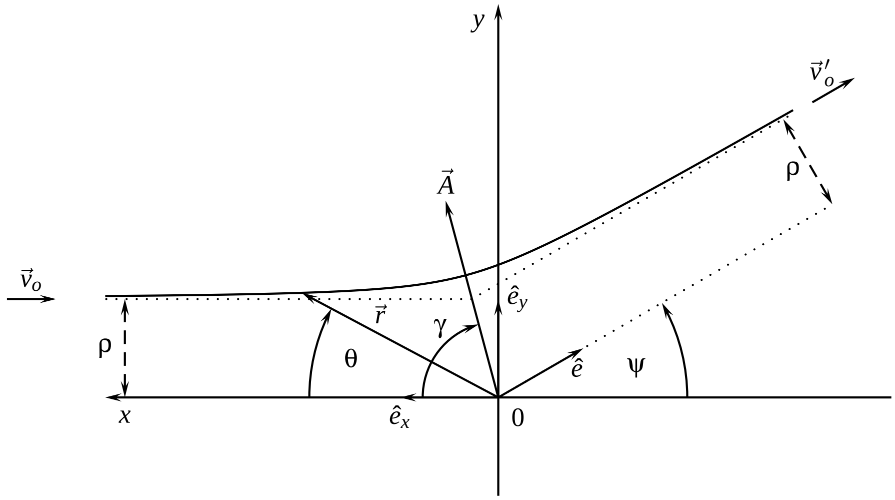

Figure 1: Trajectory and coordinate system for a positive particle scattered by an atomic nucleus.

scalar quantity.

Curiously, these are not all the conserved quantities for a given particular system. It may well happen that we have other conserved quantities (which may be scalars, vectors or, in general, tensors), even for systems which are not isolated. In all cases the conserved quantities are uniform functions of the state of the system (positions and velocities)[1]. Some known examples are the Lenz vector for a particle in Coulomb or Kepler central potential and the "position-velocity" tensor for the spatial isotropic oscillator[2, 3, 4]. The existence of these conserved quantities is useful to find the equation of the trajectory in a simple and quite straightforward manner and to study the changes in those trajectories due to a small perturbation in the original potential energy of the system.

In this work we shall consider the cases of central potentials of the forms $1/r$ and r^2 and their perturbations of the type r^n for some particular values of n.

2 The Lenz vector

Let us consider a particle of mass m in a central field with potential energy

$$U(r) = \alpha/r \quad , \tag{1}$$

where α is a positive or negative constant. The vector expression (Lenz vector)

$$\vec{A} = \vec{v} \wedge \vec{J} + \alpha\vec{r}/r \quad , \tag{2}$$

is a uniform function of $(\vec{r},\vec{v})$ constant of the motion[1]. In eq.(2) $\vec{J}$ is the constant angular momentum of the particle, with respect to the center of force, and $\vec{v}$ is the velocity of the particle.

Since $\vec{A}$ is a constant vector it may be computed at any point on the trajectory of the particle. The trajectory may be an ellipse, a parabola or a hyperbola, depending on the value of α and the energy of the particle. The orbits are symmetric respect to the pericenter where $\vec{r}$ and $\vec{v}$ are mutually orthogonal. A simple calculation shows that $\vec{A}$ is along the line joining the center of symmetry with the pericenter. A first application of the Lens vector it is to obtain the equation of the trajectory by taking the scalar product of eq.(2) with $\vec{r}$. For instance, if we are interested in the scattering of positive particles by an atomic nucleus (Rutherford scattering) we take $\alpha > 0$. In this case the energy can only be positive or null and the motion is infinite. The trajectory is a hyperbola ($E > 0$), or a parabola ($E = 0$). Let us consider the case of a hyperbola. The initial data may be chosen as the initial velocity at infinity, $\vec{v}_0$, and the impact parameter ρ [see figure (1)]. Let $(\hat{e}_x, \hat{e}_y, \hat{e}_z)$ be a basis of Cartesian orthonormal vectors; then

$$\begin{aligned} \vec{J} &= mv_0\rho\,\hat{e}_z \ , \ \vec{v}_0 = -v_0\,\hat{e}_x \ , && (3) \\ \vec{A} &= mv_0^2\rho\,\hat{e}_y + \alpha\hat{e}_x \ . && (4) \end{aligned}$$

The particle is scattered in the direction $\hat{e}$ at an angle ψ with respect to the incoming direction. The vector $\vec{A}$ can also be expressed in terms of the outgoing parameters:

$$\vec{A} = mv_0^2\rho\,\hat{e}\wedge\hat{e}_z + \alpha\hat{e} \ . \qquad (5)$$

Taking into account that

$$\hat{e} = -cos\,\psi\,\hat{e}_x + sen\,\psi\,\hat{e}_y \ , \qquad (6)$$

we obtain

$$\begin{aligned} \alpha &= mv_0^2\rho\,sen\,\psi - \alpha\,cos\,\psi \ , && (7) \\ mv_0^2\rho &= mv_0^2\rho\,cos\,\psi + \alpha\,sen\,\psi \ . && (8) \end{aligned}$$

From eqs.(7, 8) we have

$$tg\frac{\psi}{2} = \frac{\alpha}{mv_0^2\rho} \ , \qquad (9)$$

which is the usual expression for the scattering angle ψ in terms of the initial data v_0 and ρ.

The equation of the orbit in polar coordinates (r, θ) is obtained by taking the scalar product of eq.(2) with $\vec{r} = r(cos\,\theta\hat{e}_x + sen\,\theta\hat{e}_y)$:

$$\begin{aligned} \vec{A}\cdot\vec{r} &= (J^2/m) + \alpha r \ \Rightarrow && (10) \\ \frac{1}{r} &= \frac{m}{J^2}[\alpha(cos\,\theta - 1) + mv_0^2\rho\,sen\,\theta] \ . && (11) \end{aligned}$$

2.1 The perturbed Rutherford scattering

Let the potential energy be of the form

$$U(r) = \frac{\alpha}{r} + \frac{\beta}{nr^n} \ \Rightarrow \ \vec{f} = \frac{\alpha\vec{r}}{r^3} + \frac{\beta\vec{r}}{r^{n+2}} \ , \qquad (12)$$

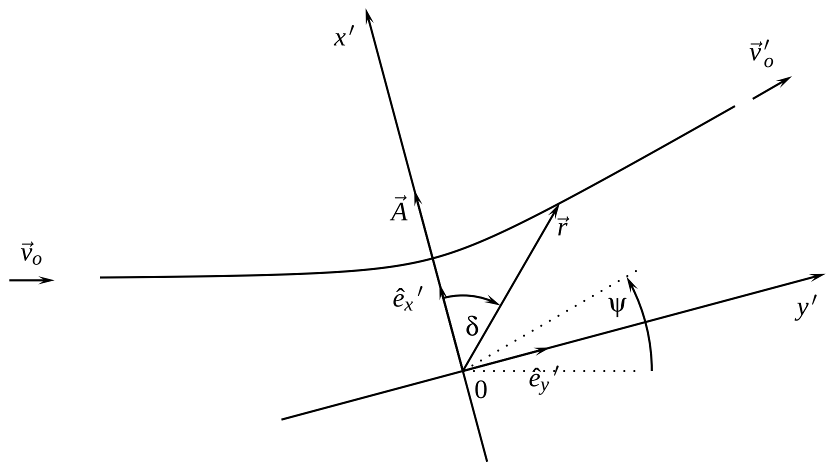

Figure 2: New system of coordinates adapted to the symmetry of the trajectory.

while at the same time we keep expression (2) for the Lenz vector; $\vec{f}$ is the central force acting on the particle. Then

$$\frac{d\vec{A}}{dt} = \frac{\beta}{mr^{n+2}}\vec{r}\wedge\vec{J} \;\Rightarrow\; \frac{d\vec{A}}{d\theta} = \frac{\beta}{Jr^n}\vec{r}\wedge\vec{J}\ . \tag{13}$$

In writing eq.(13) we have used that $(d\theta/dt) = J/mr^2$. We wish to compute the total change of $\vec{A}$, in first order in the parameter β. To this end, we may integrate eq.(13) along the unperturbed trajectory for θ varying from 0 to Θ, where $\Theta = \pi - \psi$. Let us recall that the Lenz vector corresponding to the unperturbed trajectory is a constant vector pointing to the pericenter in the direction $\gamma = \Theta/2$ (see fig. 1). According to eq.(13), the elementary change in $\vec{A}$ due to the perturbation is in the plane of the trajectory and it is perpendicular to $\vec{r}$ at each point. Because of the symmetry of the trajectory with respect to the point of minimal distance to the center, the total change in $\vec{A}$ is only different of zero in the direction perpendicular to the line joining the center of force and the pericenter. Then, to simplify the calculation it is convenient to choose a new system of orthogonal coordinates (x', y') in the plane of the orbit with x' in the direction of the pericenter [see figure (2)]. Then,

$$\vec{r} = rcos\delta\,\hat{e}_x' + rsin\delta\,\hat{e}_y'\ . \tag{14}$$

The only component of $\vec{r}$, in eq.(14), that contribute to the integration of eq.(13) is the x' component. Thus,

$$\vec{A}(\Theta) - \vec{A}(0) = -\hat{e}_y{}'\beta\int_0^{\Theta}\frac{cos\delta}{r^{n-1}}\,d\theta\ , \tag{15}$$

where $\delta = \theta - \Theta/2$. The integration is along the unperturbed trajectory given by eq.(11). To show a definite result let us consider $n = 2$. The $cos\delta$ is obtained taking the scalar product $(\vec{A}\cdot\vec{r})$ in the (x', y') coordinate system and using that $A = (\alpha^2 + m^2 v_0^4 \rho^2)^{1/2}$. Finally,

$$
\begin{aligned}
\Delta\vec{A} &:= \vec{A}(\Theta) - \vec{A}(0) \\
&= -\hat{e}'_y \int_0^\gamma d\theta \frac{2m\beta(\alpha cos\theta + m v_0^2 \rho sen\theta)[\alpha(cos\theta - 1) + m v_0^2 \rho sin\theta]}{J^2(\alpha^2 + m^2 v_0^4 \rho^2)^{1/2}} \ . \qquad (16)
\end{aligned}
$$

The integration is straightforward and the result is notably simplified if we take into account that

$$
cos\gamma = \frac{\alpha}{A} \ \ ; \ \ sen\gamma = \frac{m v_0^2 \rho}{A} \ .
$$

The change in the scattering angle due to the perturbation in the potential becomes:

$$
\delta(\Theta/2)) \simeq \frac{|\Delta\vec{A}|}{A} = \frac{\beta m}{2J^2}(\Theta - sen\Theta) \ . \qquad (17)
$$

Writing $\beta = m\tilde{\beta}$; i.e., considering the perturbing potential energy per unit mass, and introducing the angular momentum per unit mass $H := J/m$, eq.(17) can be expressed as

$$
\delta(\Theta/2)) \simeq \frac{\tilde{\beta}}{2H^2}(\Theta - sen\Theta) \ , \qquad (18)
$$

We see from eq.(18) that $\delta(\Theta/2))$ actually does not depend on the mass of the particle as it is in general in the case of a force field which is proportional to the gradient of a potential energy. This remark will result useful in some applications below.

2.2 The perturbed Kepler orbits

An interesting and somehow important case is to study the shift of the Keplerian orbit of a planet in the solar system due to a possible widened by a massive bulge around the equator of the Sun[3]. Its external gravitational potential energy per unit mass, $\tilde{U}(r,\nu)$, will depend on the distance to the center of the Sun and the azimuthal angle ν measured from the polar axis. Hence, if we developed the potential $\tilde{U}(r,\nu)$ in spherical harmonics, the leading two terms are of the form

$$
\tilde{U}(r,\nu) = -\frac{\tilde{\alpha}}{r} + D\frac{3cos^2\nu - 1}{r^3} + O\left(\frac{1}{r^4}\right) \ . \qquad (19)
$$

The factor D depends, among other quantities, on the deformation of the sphere represented by the extra mass along the equator. We are usually interested in the orbit of a planet in the equatorial plane where $\nu = \pi/2$, then the potential $\tilde{U}(r,\nu)$ acts on it as it has the purely radial dependence

$$
\tilde{U}(r) = -\frac{\tilde{\alpha}}{r} + \frac{\tilde{\beta}}{r^3} \ \ ; \ \ \tilde{\alpha} > 0 \ . \qquad (20)
$$

[3]This change in the sphericity of the Sun will show as a significant quadrupolar term in its external gravitational potential.

The potential energy of a planet of mass m is $U(r) = m\tilde{U}(r)$, which of the form (12) with $n = 3$.

In what follows we shall study the perihelium shift per revolution considering a small perturbation of the Keplerian potential energy of the form β/r^n, in the cases $n = 2, 3$.

Case $n = 2$.

Let

$$U(r) = -\frac{\alpha}{r} + \frac{\beta}{r^2} \quad . \tag{21}$$

Then, using the expression $\vec{A} = \vec{v} \wedge \vec{J} - \alpha\vec{r}/r$, we have

$$\frac{d\vec{A}}{dt} = \frac{2\beta}{m r^3}\,\hat{e}_r \wedge \vec{J} \quad \Rightarrow \quad \frac{d\vec{A}}{d\theta} = \frac{2\beta}{r}\,\hat{e}_r \wedge \hat{e}_z \quad , \tag{22}$$

where we have used that $\vec{J} = m r^2\,\dot{\theta}\,\hat{e}_z$ in a Cartesian orthogonal system with $(\hat{e}_x, \hat{e}_y)$ in the equatorial plane. The unperturbed orbit can be written as

$$\frac{p}{r} = 1 + e\,cos\,\theta \ , \tag{23}$$

where $e = A/\alpha$; $p = J^2/m\alpha$. To obtain eq.(23) we have considered that $\vec{A}$ is constant and it points to the perihelium. If the (perturbed) potential energy is given by eq.(21) the vector $\vec{A}$ will point again to the perihelium after one revolution but it will be shifted by an angle $\Delta\theta = |\Delta\vec{A}|/A$. To calculate $\Delta\vec{A}$, in first order of β, we integrate (22) along the unperturbed orbit from $\theta = 0$ to $\theta = 2\pi$:

$$\begin{aligned} \Delta\vec{A} := \vec{A}(2\pi) - \vec{A}(0) \quad &= \quad -\frac{2\,\beta}{p}\hat{e}_z \wedge \int_0^{2\pi} \hat{e}_r(1 + e\,cos\,\theta)\,d\theta \\ &= \quad -\frac{2\,\beta m\pi A}{J^2}\,\hat{e}_y \ . \end{aligned} \tag{24}$$

Then,

$$\Delta\theta = -\frac{2\,\tilde{\beta}\pi}{H^2} \ , \tag{25}$$

where we have introduced again the constants per unit mass $\tilde{\beta}$ and H.

Case $n = 3$.

Let now

$$U(r) = -\frac{\alpha}{r} + \frac{\gamma}{r^3} \quad . \tag{26}$$

By a similar calculation to the previous case we obtain the result

$$\Delta\theta = -\frac{6\tilde{\gamma}\pi\tilde{\alpha}}{H^4} \ , \tag{27}$$

where $\gamma = \tilde{\gamma} m$, $\alpha = \tilde{\alpha} m$ and $J = Hm$. This case corresponds to a flattening of the solar sphere into an ellipsoid (or. equivalently, to a surplus mass located at the ring around the equator of the Sun, as we have discussed). Astronomers observe the shift in the perihelium motion after many revolutions of the planet. In particular, we may express the perihelium shift per century by the formula

$$P := \frac{\Delta\theta}{T} = -\frac{6\pi\tilde{\gamma}\tilde{\alpha}}{H^4 T} \quad , \tag{28}$$

where T is the period of revolution expressed in units of centuries[4]. Since $r^2(d\theta/dt) = H$, we have that, approximately, $2\pi R^2 = HT$; where R is the mean distance planet-Sun. Then, we can write eq.(28) as

$$P = -\frac{6\pi\tilde{\gamma}\tilde{\alpha}T^3}{(2\pi)^4 R^8} \quad . \tag{29}$$

Finally, if we use Kepler's third law: $(T^2/R^3) = C$, where C is a constant of the same value for all the planets[5], we have

$$P = KR^{-7/2} \quad , \tag{30}$$

where K is a constant that resumes all the previous constants and it is the same for all the planets. We notice that the perihelium shift is greater for the planets closer to the Sun. The measured value[5] of P for Mercury is $P_M = (43,11 \pm 0,45)$ *arc seconds per century*. Since the value of $\tilde{\gamma}$ for the Sun is quite uncertain, we may obtain the value of K from the value P_M and R_M for Mercury and compute the values of P from eq.(30). The best fit to the observational data[6] is given by a straight line in the form

$$lnP = a\,lnR + b \quad , \tag{31}$$

with slope $a = -2,30 \pm 0,26$. The theoretical value according to eq.(30) is $a = -3,5$, which is outside the range of uncertainty of the observational data and shows that the model is not at all satisfactory. To overcome this difficulty was one of the results contained in the three classical tests of General Relativity which predicts[6] a slope $a = -2,5$ for the residual perihelium advance of the planets observed by the astronomers[6].

3 The isotropic harmonic oscillator

By an isotropic harmonic oscillator we mean a particle of mass m in a central field of force with a potential energy given by

$$U(r) = \frac{1}{2}kr^2 \quad . \tag{32}$$

[4] For instance, if the planet is the Earth, then $T = 10^{-2}$ centuries.

[5] Actually, C depends on the mass of the planet trough the ratio *mass of the planet/reduce mass of the system planet-Sun*. The differences were not observable by Kepler and they are not significant for the present discussion.

[6] The residual perihelium advance of the planets is that which may not be associated with any other cause that excludes the Sun.

Clearly, the equation of motion are separable in Cartesian coordinates. The motion of the particle is in a plane that we choose as the (x,y) one. Thus the functions of motion, with an adequate choice of the origen of t and the orientation of the coordinate system, can be written as

$$\begin{aligned} x(t) &= A\,cos\,\omega t \quad , \\ y(t) &= B\,sen\,\omega t \quad , \end{aligned} \tag{33}$$

where $\omega^2 = k/m = 1/\alpha$. The trajectory corresponding to eq.(33) is an ellipse with its center at the origen and axis along the direction of the coordinate axis:

$$\frac{x^2}{A^2} + \frac{y^2}{B^2} = 1 \quad . \tag{34}$$

It is straightforward to show that the tensor

$$F_{ij} = x_i x_j + \alpha v_i v_j \quad , \tag{35}$$

where $x_i = (x,y)$; $v_i = (v_x.v_y)$; $\alpha\omega^2 = 1$, is a constant of motion; i.e.,

$$\frac{dF_{ij}}{dt} = 0 \quad , \tag{36}$$

when the equations of motion are satisfied. From eqs.(33) and (35) the components of F are

$$F_{ij} = \begin{pmatrix} A^2 & 0 \\ 0 & B^2 \end{pmatrix} \quad . \tag{37}$$

The eigenvalues of F_{ij} are A^2 and B^2.; the corresponding eigenvectors are along the direction of the axis of symmetry of the ellipse (34).

3.1 Orbit perturbation by changing the potential in first order

We wish to study now how the trajectory changes when we perturb the potential energy by adding to it a term of the form, for instance,

$$\delta U(r) = \frac{\beta}{r^4} \quad , \tag{38}$$

where $\beta \ll (A^2 + B^2)^3 k$. It is clear that the tensor (35) will not longer be constant. We have:

$$\frac{dF_{ij}}{dt} = \frac{4\alpha\beta}{mr^6}(x_i v_j + x_j v_i) \quad . \tag{39}$$

After a lapse of one period the tensor F will no longer be diagonal as in eq.(37); all its components will change by first order terms keeping its symmetry property. However, it is possible to perform a rotation of the coordinate system by an angle ψ to have F in a diagonal form again, with the new axis of symmetry of the ellipse along the axis of the rotated coordinate system [see figure (3)]. To obtain the angle ψ it is enough to calculate along the unperturbed trajectory, in the original coordinate system, the

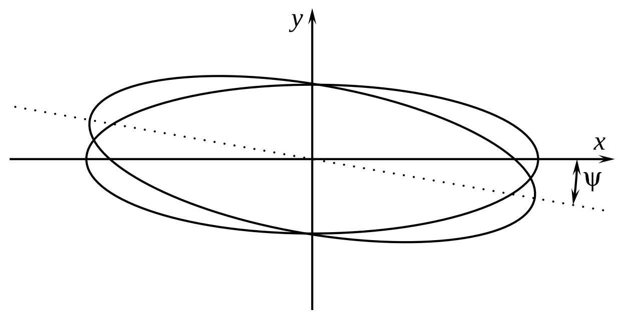

Figure 3: Two elliptical orbits that show the rotation angle ψ of the symmetry axis after a revolution.

expression of F after a revolution and find its eigenvectors which are, precisely, along the direction of the symmetry axis of the rotated trajectory. Then, the angle ψ is given by the expression

$$tag\,\psi = \frac{\mathcal{X}_y}{\mathcal{X}_x} \quad , \tag{40}$$

where $\mathcal{X}_x$ y $\mathcal{X}_y$ are the coordinate components of the new eigenvector that form an angle ψ with the x-axis. To have the tensor F after a period we integrate eq.(39) on the unperturbed orbit for ωt from 0 to 2π. Thus,

$$F_{ij}(2\pi) - F_{ij}(0) = \frac{4\alpha\beta}{\omega m}\int_0^{2\pi} \frac{(x_i v_j + x_j v_i)}{r^6} d(\omega t) \quad . \tag{41}$$

The initial value $F_{ij}(0)$ is the tensor (37). In general $F_{ij}(2\pi)$ will have the form

$$F_{ij}(2\pi) = F_{ij}(0) + O_{ij} \quad , \tag{42}$$

where O_{ij} is of order β. Similarly, the eigenvectors and eigenvalues of $F_{ij}(2\pi)$ can be calculated up to the first order in β. We would obtain $\mathcal{X}_y = O(\beta)$; $\mathcal{X}_x = 1 + O(\beta)$ and the eigenvalues $\lambda_1 = A^2 + O(\beta)$; $\lambda_2 = B^2 + O(\beta)$. By a simple calculation it is possible to show that the components of the eigenvector $\mathcal{X}$ can be chosen as

$$\mathcal{X}_x = 1 \quad ; \quad \mathcal{X}_y = \frac{F_{12}(2\pi)}{\lambda_1 - F_{22}(2\pi)} \quad . \tag{43}$$

From eqs. (40), (42) and (43) we can calculate the angle ψ up to order β by

$$\psi \simeq \frac{F_{12}(2\pi)}{A^2 - B^2} \quad . \tag{44}$$

Then, from eq.(41)

$$\begin{aligned} F_{12}(2\pi) &= \frac{4\alpha\beta AB}{m}\int_0^{2\pi} \frac{(cos^2\theta - sen^2\theta)}{(A^2 cos^2\theta + B^2 sen^2\theta)^3}\, d\theta \\ &= -\frac{3\pi\alpha\beta\,(A^4 - B^4)}{m A^4 B^4} \quad , \end{aligned} \tag{45}$$

where we have put $\theta = \omega t$. Finally,

$$\psi \simeq -\frac{3\pi\alpha\beta\,(A^2+B^2)}{mA^4B^4} \quad . \tag{46}$$

The velocity of rotation of the ellipse' axis is

$$\Omega := \frac{\psi\,\omega}{2\pi} = -\frac{3\tilde{\beta}\,(A^2+B^2)}{2\omega A^4 B^4} \quad , \tag{47}$$

where $\tilde{\beta} = \beta/m$ and we have used that $\alpha\omega^2 = 1$. Eq.(47) coincides with the result obtained by Kotkin and Serbo[7] through a different method.

4 Final comments

The main purpose of this work was to bring to the attention of students and teachers the existence of another uniform functions constant of the motion rather than the well known quantities arising from spacetime symmetries. We have treated, in some details, just two cases. The first one is the Lenz vector which sometimes is found in the literature[1, 2]; the other case is associated with the isotropic harmonic oscillator of relative importance in some simple models of classical molecular interaction. The first example was applied to describe the perturbation in the trajectories by the addition in the potential energy of a small term depending only on the radial variable. As an interesting application of the method, we computed the change in the outgoing direction of a scattered particle as predicted in the Rutherford scattering. Another problem that we considered was the precession of the Keplerian orbits of the planets in the solar system in two cases. One of the cases has especial relevance since it may be used with great simplicity to investigate the magnitude and characteristic of the effect on the trajectories of the planets if the Sun shows a massive protuberance along its equator. We revised this aspect of the problem following the discussion presented by Adler et al[6], that ends up with a comparison with the prediction and validity of General Relativity.

On the other hand, we believe that the study of a spherically symmetric perturbation of the potential energy of the isotropic harmonic oscillator is an apparent opportunity to present a case in which the conserved quantity is a tensor. Thus, we have to find the eigenvectors and eigenvalues of that tensor while at the same time we obtain the rate of rotation of the orbits by performing a simple coordinate rotation in the Cartesian plane. We think that the present work addresses to many aspects of Mechanics with a didactical interest in other Physics or Mathematics courses.

References

[1] Landau L.D. and Lifchitz E.M.: "Mechanics", *Course of theoretical Physics, Volume 1*, 3rd. edition, Pergamon Press, New York (1976).

[2] Goldstein H.: "Classical Mechanics", Chaps. 3 and 9, 2nd. edition, Addison-Wesley Pu. Co., Inc. (1980).

[3] An interesting article in which the Lenz vector may emerge as a Noether invariant is: Struckmeir J. and Reidel C., Phys. Rev. E **66**, 066605 (2002).

[4] For the Lenz vector in quantum mechanics see: Burkhart C.E. and Leventhal J.J., Am. J. Phys. **72** (8), 1013 (2004) and the page by Jacob J. Leventhal: www.umsl.edu/~jjl/homepage/chapters/Chap_7.doc.

[5] Duncombe R.L., "Relativity effects for the 3 inner planets", Astron. J. **61**, 174-175 (1956).

[6] Adler R. Bazin M. and Schiffer M.: "Introduction to General Relativity", §6.6, McGraw-Hill Book Co., New York (1965).

[7] Kotkin G.L. and Serbo V.G.: "Problemas de Mecánica Clásica", problem 22-a, MIR Ed., Moscow (1980). Original Edition in Russian in 1977.