Two-spin subsystem entanglement in spin 1/2 rings with long range interactions

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Abstract

We consider the two-spin subsystem entanglement for eigenstates of the Hamiltonian

$$H = \sum_{1 \leq j < k \leq N} \left(\frac{1}{r_{j,k}} \right)^{\alpha} \sigma_j \cdot \sigma_k$$

for a ring of N spins 1/2 with asssociated spin vector operator $(\hbar/2)\sigma_j$ for the j-th spin. Here $r_{j,k}$ is the chord-distance betwen sites j and k.

The case $\alpha = 2$ corresponds to the solvable Haldane-Shastry model whose spectrum has very high degeneracies not present for $\alpha \neq 2$. Two spin subsystem entanglement shows high sensistivity and distinguishes $\alpha = 2$ from $\alpha \neq 2$. There is no entanglement beyond nearest neighbors for all eigenstates when $\alpha = 2$. Whereas for $\alpha \neq 2$ one has selective entanglement at any distance for eigenstates of sufficiently high energy in a certain interval of α which depends on the energy.

The ground state (which is a singlet only for even N) does not have entanglement beyond nearest neighbors, and the nearest neighbor entanglement is virtually independent of the range of the interaction controlled by α.

PACS numbers: 03.65.Ud,03.67.-a

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I. INTRODUCTION

Since the first studies [1–3] of entanglement in the ground-state of interacting spin 1/2 systems, a considerable amount of work has been devoted to analyze this feature. The Hamiltonians most studied have been those with anti-ferromagnetic nearest neighbor interactions (XX, XY, XYZ, XXZ etc.) between the spins in the presence of an external magnetic field (usually constant) which is the order parameter for a quantum phase transition (i.e., non-analyticity in the ground- state energy; [4]). The fascinating and intricate connections and relations between (mainly two-site subsystem) entanglement and the quantum phase transition have been systematically studied after [5, 6]. At the other extreme, namely models where each spin interacts identically with all the others, there are studies of the Lipkin-Meshkov-Glick model ([7]) which is of the XY type in an external field, and of an equivalent of the BCS model ([8]) which is of the XX type in an external field. Ref. [9] presents a study of entanglement for nearest and next-nearest neighbors for the ground state of a Heisenberg chain with nearest and competing ferromagnetic next-nearest neighbor interactions. A recent review of the subject is [10]. Most of the studies we are aware of restrict to the entanglement analysis of the ground state. Ref. [11] however, presents the nearest neighbor entanglement analysis for nearest neighbor XXX and XY Heisenberg models (with zero field) dealing with all excited states. This reference is the work we are aware of which is most related to the point of view and results we report here.

Here, we are not concerned with the quantum phase transition aspects but concentrate on the dependence of entanglement on the range of the spin pair-interactions. From this point-of-view, one of the basic facts emerging from the studies mentioned above, is that two-site subsystem entanglement in the ground-state is short-ranged in models with finite-range interactions; e.g., in models with nearest neighbor interactions, it vanishes if the sites are not nearest, or next-nearest neighbors. Extrapolating, if in a spin 1/2 chain every spin interacts only with its two neighbors at a distance k, then two-site entanglement in the ground-state will show up for sites at distance k, and basically only for this distance if there are no further contributions to the energy. This is not a general claim we are making; it has just been checked in a number of simple models. We outline a rough argument to sustain this in the concluding remarks. As another motivating example, consider the Hamiltonian for N spins 1/2 ($\sigma_j = (\sigma_j^x, \sigma_j^y, \sigma_j^z)$ is the vector operator formed with the three Pauli operators

associated to the j-th spin)

$$\sum_{1\leq j<k\leq N} (\sigma_j \cdot \sigma_k) + a\ \sigma_1 \cdot \sigma_p\ ,$$

with a real and $p = 2, 3, \cdots, N$, for which the interaction constant for the spin-pair $(1, p)$ is $1 + a$, while for every other pair it is 1. The ground-state shows entanglement for the pair $(1, p)$ only for $0 < a$, and no entanglement for all other pairs independently of the value of a.

On the positive side, one of the basic properties of quantum entanglement, namely monogamy [12, 13], conspires against the build-up of entanglement of a spin with too many neighbors.

Thus, one of the question we posed ourselves was: will we obtain two-site entanglement in the ground state at large distances if the interaction is long ranged? The other question we posed ourselves is: what about two-site entanglement in excited states?

In this communication we get some answers to both questions, in as much as we present some results on entanglement of 2-spin subsystems for eigenstates of a Hamiltonian where the spins are subject to a long-range interaction inversely proportional to a power of their distance and the external magnetic field vanishes. Our study is not restricted to the ground-state(s), but includes the whole spectrum. In particular, we are interested in which eigenstates show two-site entanglement for long distances, i.e. beyond nearest or next-nearest neighbors. Besides, we are interested in the dependence on the range of the interaction, of the distance for which it is possible to obtain two-site entanglement.

The model is presented in the next section, together with some information on its spectral properties. In section III, we deal with the pair- or two-site entanglement. We give the relation between the concurrence of the eigenstates reduced to the two-site subsystem, and the two-site spin-correlation functions; and present our results emphasizing their qualitative aspects. In the concluding section IV, we highlight those features observed which we consider interesting and provide an argument for the short range of pair-entanglement in the ground state, which proceeds via power law decay of spin correlations.

II. THE MODEL

The model studied is the simplest possible with a very regular dependence on the distance; it is obtained by multiplying the Heisenberg pair interaction $\sigma_j \cdot \sigma_k$ with a factor inversely proportional to a power of the distance between the sites. It is thus an XXX Heisenberg model incorporating a power law decay of the interaction strength. Instead of considering a linear chain with periodic boundary conditions, we consider a ring. Specifically we consider N spins 1/2 and the Heisenberg Hamiltonian is given by

$$H_N(\alpha) = \sum_{1 \le j < k \le N} \frac{\sigma_j \cdot \sigma_k}{(r_{j,k})^\alpha} \ , \quad \alpha > 0 \ ;$$

where

$$r_{j,k} = \frac{\sin(\pi |j-k|/N)}{\sin(\pi/N)} \ , \quad j,k = 1,2,\cdots,N \ ,$$

is proportional to the dimensionless distance (length of the chord) between vertexes j and k in a regular, flat, N-gon whose vertices are numbered consecutively. The constant in the definition is chosen so that the nearest neighbor distance $r_{j,j+1}$ is one. For given N, the number of distinct distances is $[N/2]$ –the largest integer not above $N/2$. We observe that $H_N(0)$ is the isotropic Heisenberg model where each spin interacts identically with every other spin. For $N \ge 2$ fixed, the limit $\alpha \to \infty$ corresponds to the nearest-neighbor antiferromagnetic XXX model studied in [11]. Thus α provides a smooth control of the spatial decay of the interaction as one varies the distance between sites, and also an interpolation parameter between two well known models. We often drop the index N and parameter α in $H_N(\alpha)$, when these are irrelevant.

The Hamiltonian H is "anti-ferromagnetic", while $-H \neq H$ is "ferromagnetic". If N is even then the least energy eigenvalue is non-degenerate, whereas it is degenerate for uneven N.

For $\alpha = 2$ this is the Haldane-Shastry model [14, 15] which is explicitely solvable [16, 17], giving us an effective control on numerical results; this was one of the basic reasons for considering this model (and a ring, not a chain). For $\alpha = 2$, the least eigen-energy is fourfold degenerate for uneven N, and the largest energy eigenvalue (ground-state energy of the ferromagnetic version) is always degenerate with multiplicity $N+1$.

The spectrum for $\alpha = 2$ is highly degenerate with comparatively few eigenvalues and very

different from the spectrum for $\alpha \neq 2$ and same N. For example for $N = 8$, where the Hilbert space has dimension $2^8 = 256$, H has 5 eigenvalues for $\alpha = 0$, 19 for $\alpha = 2$ and 45 for $0 < \alpha \neq 2$, except for a discrete and finite set of values of α, where a crossing or two reduces the number of distinct eigenvalues by 1 or 2 (see figure 1); the nearest-neighbor XXX model corresponding to $\alpha = \infty$ has 40 eigenvalues. These qualitative features do not depend on N.

Due to the absence of computable necessary and sufficent conditions for multipartite entanglement in mixed states, we study only the entanglement of the possible pairs of spins, that is two-site entanglement. It suffices to consider the pairs $(1, j)$ for $j = 2, \cdots, [N/2] + 1$; corresponding to the possible distances. Given a state ρ of system of N spins, $\rho_{j,k}$ denotes the reduction of ρ to the subsystem with components (i.e., sites) j and k. The entanglement of this reduced state is detected and quantified by its concurrence, [18].

Our analysis proceeds as follows. One has

$$\sigma_j \cdot \sigma_k = 2\Pi_{j,k} - \mathbf{1} \ ,$$

where $\Pi_{j,k}$ is the transposition interchanging the j-th and k-th components of the product basis vectors

$$|\epsilon_1, \epsilon_2, \cdots, \epsilon_N\rangle \ , \quad \epsilon_n = \pm \ , \quad n = 1, 2, \cdots, N \ , \tag{1}$$

where $\sigma^z|\pm\rangle = \pm\,|\pm\rangle$. That is,

$$\Pi_{j,k}|\epsilon_1, \epsilon_2, \cdots, \epsilon_j, \cdots, \epsilon_k, \cdots, \epsilon_N\rangle = |\epsilon_1, \epsilon_2, \cdots, \epsilon_k, \cdots, \epsilon_j, \cdots, \epsilon_N\rangle \ .$$

To simplify the structure of the corresponding matrix in the above basis, we consider the operator (Hamiltonian)

$$\widetilde{H}_N(\alpha) = \frac{1}{2} \sum_{1 \leq j < k \leq N} \left(\frac{1}{r_{j,k}}\right)^{\alpha} (\Pi_{j,k} - \mathbf{1})$$

which differs from $H_N(\alpha)$ by an additive, (N, α)-dependent constant (see [11]). Thus $H_N(\alpha)$ and $\widetilde{H}_N(\alpha)$ have the same number of distinct eigenvalues with the same multiplicities and the same spectral orthoprojectors. Figure 1 shows the spectrum of $\widetilde{H}$ for $N = 8$ as a function of α up to $\alpha = 5$. We find that beyond α about 7.29 there are no crossings, but eigenvalue curves do approach each other asymptotically leading to 40 eigen-energies

in the nearest-neighbor model ($\alpha = \infty$). The largest eigen-energy of $\widetilde{H}_N(\alpha)$ is zero, it is $(N+1)$-fold degenerate for every $\alpha \geq 0$, and the corresponding spectral orthoprojector is independent of α [19]. In fact an orthonormal basis of this eigen-space is easily described; is consists of $N+1$ vectors each of which is the normalized sum of the $\begin{pmatrix} N \\ s \end{pmatrix}$ vectors of the form (1) where exactly s of the ϵ_n's are $+$; and $s = 0, 1, \cdots, N$.

III. ENTANGLEMENT

Denote the spectrum with *spec*; suppose

$$H = \sum_{E \in spec(H)} E P_E \ , \ \ P_E P_{E'} = \delta_{E,E'} P_E \ , \ \ \sum_{E \in spec(H)} P_E = \mathbf{1} \ ,$$

is the spectral decomposition of the Hamiltonian H. Then $tr(P_E)$ is the multiplicity (degeneracy) of the eigen-energy E.

At this point, we must recall Theorem 2 of [20] which says that any subspace of dimension at least 2 of the four-dimensional Hilbert space of two q-bits contains at least one product vector. Thus if the eigen-energy E of a two spin 1/2 system is degenerate, then there is a separable eigenvector to E. Extensions of this result to sufficiently high dimensional subspaces of N (> 2) q-bits are possible but this is not the point of this paper. To analyze entanglement in the case of degenerate eigenvalues, we must consider the (uniform) eigen-state

$$\rho(E) := P_E / tr(P_E) \ , \ \ E \in spec(H) \ .$$

Notice that this state is obtained by mixing with equal weights (namely $1/tr(P_E)$) any (pairwise orthogonal) pure eigen-states whose corresponding vectors constitute an orthonormal basis of the eigenspace of the eigenvalue E of H. This is the very same eigen-state notion as that used in [11].

For the Haldane-Shastry model ($\alpha = 2$), we use the known eigenvalues and degeneracies [17], and determine the orthoprojectors by: $P_E = \prod_{E' \neq E} \frac{H - E'}{E - E'}$. For $\alpha \neq 2$ we determine the

eigen-energies and corresponding spectral projections numerically.

Due to the particular symmetry of H or $\widetilde{H}$ which commutes with $\sum_{j=1}^{N}\sigma_j^z$ and with $\prod_j \sigma_j^z$ ([11, 19, 21]) the reduced density operators (in the basis (1) of product eigenvectors of σ_j^z, $j = 1, 2, \cdots, N$) for any pair (j,k) of spins have the same structure namely

$$\begin{pmatrix} a & 0 & 0 & 0 \\ 0 & b & c & 0 \\ 0 & c & b & 0 \\ 0 & 0 & 0 & a \end{pmatrix} ,$$

where $a, b \geq 0$, with $a + b = 1/2$; and c is real with $|c| \leq b$. The concurrence of this 2-spin state is $\max\{0, 2\max\{a, b + |c|\} - 1\} = \max\{2(|c| - a), 0\}$. But the isotropic nature of H ($SU(2)$ invariance), implies (as observed in [11] for the n-n model)

$$2c = tr(\rho_{j,k}(E)\sigma_j^x\sigma_k^x) = tr(\rho_{j,k}(E)\sigma_j^y\sigma_k^y) = tr(\rho_{j,k}(E)\sigma_j^z\sigma_k^z) = 2(a - b) \; . \tag{2}$$

From this we conclude that $b = (1 - 2c)/4$, $a = (1 + 2c)/4$, and $-1/2 \leq c \leq 1/6$. The concurrence is

$$\begin{cases} 0 & , \quad \text{if } -1/6 \leq c \leq 1/6 \\ -3c - 1/2 & , \quad \text{if } -1/2 \leq c < -1/6 \end{cases} . \tag{3}$$

Thus, for a given pair (j,k), there will be entanglement for this pair in the state $\rho(E)$ if and only if the correlation function $tr(\rho(E)\sigma_j^\omega\sigma_k^\omega)$ ($\omega = x, y, z$) which is independent of ω, is in the interval $[-1, -1/3)$.

The above structure of $\rho_{j,k}$ also implies that the reduced density matrix for the j-th site is simply and always just $(1/2)I$ which is the maximally mixed state for a spin 1/2, and the expectation value of σ_j is the zero vector. This remarkable feature of the model is independent of N, $\alpha > 0$ and the eigen-energy considered. As a consequence, the Meyer-Wallach measure [25] (which, when the state is pure, is a true measure of entanglement and not just a measure of degree of mixture) given by $2 - (2/N)\sum_{j=1}^{N} tr(\rho_j^2)$ and often misused as an indicator of multipartite entanglement, is identically equal to 1 (and thus of no use whatsoever) for every eigen-state $\rho(E)$ independently of N, $\alpha > 0$ and the eigen-energy considered, even for the maximal energy where the eigenstate is separable.

This is not surprising at all, since the N-qubit trace $2^{-N}\mathbf{1} = (\mathbf{1}/2) \otimes (\mathbf{1}/2) \otimes \cdots \otimes (\mathbf{1}/2)$ which is a product-state and hence separable has maximal Meyer-Wallach measure 1. But the model does provide a rather blatant example of a whole zoo of eigenstates with very different entanglement properties which goes alltogether undetected by the Meyer-Wallach measure. Meyer and Wallach of course proved that their measure is an entanglement measure for pure states; they never claimed anything about mixed states.

Another remarkable feature is that $\rho_{j,k}(0)$, the reduced density matrix for sites (j,k) for the eigen-state of maximal energy which was described above and seen to be independent of α, is also independent of the pair (j,k), [19]. It turns out that this maximal energy eigen-state (ground-state of the ferromagnetic model) does not show entanglement at all site distances.

Our calculations were performed for $N = 2, 3, 4, 5, 6, 7, 8$. Up to the degeneracies in the lowest eigen-energy, the qualitative features are independent of N in that range. The graphs shown corrrespond to $N = 8$.

Here is a list of some of our observations for the mentioned values of N (others will follow). Some of these extend or complement results of [11] which correspond to the nearest neighbor XXX model ($\alpha = \infty$):

- The isotropic model $H_N(0)$ shows no two-site entanglement for all distances at every eigen-energy.

- In the ground-state there is exclusively nearest neighbor two-site entanglement for every $\alpha > 0$ ($\alpha > 0$ was sampled rather completely only for $N = 8$); the corresponding concurrence is a slowly varying increasing function of $\alpha > 0$ which is discontinuous at 0 (see figure 6).

- For every $\alpha > 0$ nearest neighbor entanglement appears only in the first few energy levels (i.e., for $N = 8$, the first four or five energy levels). If an excited state presents nearest neighbor entanglement for some $\alpha > 0$ then the corresponding concurrence is below that of the ground-state nearest neighbor concurrence for that value of α.

However, nearest neighbor concurrence is not generally a decreasing function of the energy for fixed $\alpha > 0$ (see figure 2).

- The distinctive feature of the case $\alpha = 2$ with respect to the cases $\alpha \neq 2$ is simple: *For $\alpha = 2$ there is no two-site entanglement beyond nearest neighbors at all eigen-energies. For $0 < \alpha \neq 2$ two-site entanglement for other possible distances appear at some excited eigen-energy level.* In fact, two-site entanglement for all possible distances is present for every α at some excited energy-level except for α in a certain bounded interval which depends on N.

- Except for exceptional values of α one finds excited states where the concurrence for sites further apart than nearest neighbors have greater concurrence than the ground-state nearest neighbor concurrence. For example, for $N = 8$ and almost all $\alpha > 0$ ($\alpha = 1$ is exceptional) one finds some excited state where the concurrence for sites at maximal distance is always above the nearest neighbor concurrence for the ground state.

- In the ferromagnetic model $-H$, the ground-state does not show pair-entanglement at all distances and the same is true for the low energy eigenstates, but the number of the states with this property decreases with α. The first excited states which exhibit two-site entanglement do so for the largest distances. Only high energy eigen-states show nearest-neighbor entanglement. All this is seen in figures 2-5.

The series of four figures 2-5 show the concurrence of $\rho_{j,k}(E)$ for the possible eigen-energies E for $N = 8$ and distinct values of α. In all these figures the dot corresponds to nearest neighbor distance ($|j - k| = 1$), the cross $\times$ to the next-nearest neighbor distance ($|j - k| = 2$), the star $*$ to $|j - k| = 3$ and the empty square $\Box$ to the maximal distance $|j - k| = 4$.

We mention a feature of the nearest neighbor model which is apparent in figure 5 and was observed in [11]. In this model, the eigen-energies are proportional to the spin correlation function of the corresponding eigen-state (Eq. (8) of Ref. [11]). Nearest neighbor concurrence where it is positive, is a linear decreasing function of the energy: Concurrence ($\rho_{1,2}(E)$) $= \max\{-AE - B, 0\}$, where $A, B > 0$. This extends the

observation of [3] to the whole spectrum. Moreover, this linear regime is reached very rapidly as α grows to infinity; e.g. for $N = 8$ and $\alpha > 6$ one is practically in the linear regime.

For $N = 8$ we have also analyzed if energy is distance selective for two-site entanglement. We find that if at some eigen-energy there is two-site entanglement at nearest or next-nearest neighbor distance then there is no entanglement at all the other possible distances. However, two-site entanglement at maximal and next to maximal distances can be present for the same energy level in certain intervals of α values.

Another way of presenting our results would consist in graphing the concurrence of $\rho_{j,k}(E)$ for fixed N and distance $|j-k|$ as a function of α while keeping the number of the (excited) level E fixed. The level crossings will produce complicated graphs with a number of qualitatively distinct and generally discontinuous behaviors of the concurrence. It is more appropiate to adopt a perturbation theoretic point of view. For fixed N, the operator families $H_N(\alpha)$ and $\widetilde{H}_N(\alpha)$ are holomorphic in $\alpha \in \mathbb{C}$ in the sense of [26]. Thus, for example for $N = 8$, and away from the exceptional points where crossings occur (see figure 1), there are fourty five pairwise orthogonal projectors which depend real analytically on α. The two-site concurrence for any pair of sites obtained by normalizing these orthoprojectors to states will be real analytic functions of α away from the crossings. This alternative way of analyzing two-site entanglement, shows that only very few of the orthoprojectors carry two-site entanglement as is illustrated in the following figure pairs. Figure 6 shows all the non-zero concurrences for nearest neighbors (namely five) for $N = 8$ as functions of α in the top graph, while the lower graph shows which five of the energy curves of figure 1 give rise to these nearest neighbor concurrences. In the concurrence graph of figure 6, only the curve corresponding to maximal nearest-neighbor concurrence (≈ 0.41) is associated to only one level, in this case the ground-state. All other concurrence curves in this graph mix (due to crossings) energy levels of different excitation number. The discontinuities occur at crossings and not all of them are visible in the figure. The following pair of graphs 7 deals with next-nearest neighbors. The energy curves (lower graph) are only drawn where two-site entanglement (at the corresponding distance) is present and they are not drawn through the whole range of values of α (this is repeated for the other distances to be shown below). Observe that the entanglement at next-nearest neighbor distance $|j-k| = 2$ has

a gap: for α betweeen about 2.54 and 3.71 there is no next-nearest neighbor entanglement at all eigen-energies; this is the only distance which shows this feature. The graph pairs of figures 8 and 9, repeat this for the distances $|j-k|=3$, and 4 respectively. Close inspection of figures 8 and 9 show that the maximal concurrence (labeled a in figure 9) at maximal distance $|j-k|=4$ and the monotone increasing concurrence curve (labeled a in figure 8) for the distance $|j-k|=3$ originate in the same energy curve (i.e. its corresponding orthoprojector) namely a in the lower graphs of figures 8 and 9. This feature appears also for nearest neighbor and maximal distance entanglement: the energy vs. α curves labeled c in figures 6 and 9 coincide; however entanglement at maximal distance appears only above $\alpha \approx 3.88$ A careful count shows that of the 45 orthoprojectors only 12 carry two site entanglement, of these 10 do so for one distance only, only one carries two-site entanglement at two distances simultaneously for all values of $\alpha > 0$, and only one carries two-site entanglement at two distance simultaneously albeit above some threshold value of α (as just described).

The following table gives the dimension of the fourty five orthoprojectors appearing for $N=8$ and the number of times each dimension appears.

dimension of orthoprojector	1	2	3	4	5	6	7	8	9	10	11	12	13	14	≥ 15
number of orthoprojectors	6	4	6	0	6	11	1	0	1	7	0	0	0	3	0

All six one-dimensional orthoprojectors carry two-site entanglement and two of these do so for two distinct distances ($|j-k|=3,4$ for all $\alpha>0$ and $|j-k|=1,4$ for $\alpha>3.88$). Moreover, their Meyer-Wallach measure being maximal, they are maximally entangled pure states except at crossings.

IV. CONCLUDING REMARKS

We present a study of pair (or two-site) entanglement for the eigenstates of an N spin 1/2 model where the spins (sites) are equidistant in a circular ring and the pair-interaction is inversely proportional to an arbitrary positive power α of the site distance, and proportional to the scalar product of the magnetic moments of the spins. The model interpolates between a nearest neighbor interaction model (of type XXX) and a model where every spin

interacts equally with all the others. The model is solvable for $\alpha = 2$ (Haldane-Shastry model, loc. cit.) where the spectrum is extremely degenerate, for every N, relatively to $\alpha \neq 2$ and same N. We do not include an external magnetic field in the Hamiltonian, and thus exclude a quantum phase transition, i.e. non-analytic behavior of the ground-state energy as a function of the magnetic field strength. Although we have no proposal regarding the experimental realization of the model, we consider it as a simple yet typical example of a Hamiltonian with long-ranged antiferromagnetic spin 1/2 pair interactions decaying smoothly with distance; the features we observe are probably common to such models.
We do not concentrate on ground-state entanglement (the ground-state energy is degenerate for uneven N) but analyze all eigen-energies. The qualitative features reported are independent of N up to $N = 8$, and we have no doubts that they are independent of N for all N.

A key feature is the observed insensitivity to the range of the interaction of two-site entanglement for the ground-state. Not only there is no two-site entanglement beyond nearest neighbors independently of the range of the interaction controlled by α, but also, e,.g. for $N = 8$, taking the limit $\alpha \to 0^+$ and comparing this with the nearest-neighbor concurrence in the nearest neighbor interaction model ($\alpha = \infty$), the variation is only about 2.5% percent over the whole range of values of $\alpha > 0$. Moreover the value is about 94% of the upper bound claimed by [3]. We have no rationalization for the small variation observed in the concurrence, beyond the connection made in section III that this means that the nearest neighbor spin correlation is virtually independent of the range of the interaction in the ground state. As regards the absence of two site entanglement beyond some cut-off distance, we view this as a general feature of anti-ferromagnetic models with pair interactions proportional to $\sigma_j^\omega \sigma_k^\omega$ ($\omega = x, y$ or z) decaying with distance whose two-site spin correlations are bounded in modulus by some positive inverse power of the distance. In the model studied here, as in many others, the cut-off distance is that to the nearest neighbor.

An argument to sustain this goes as follows. Suppose that in some given state ϕ, the two-site spin correlation functions $tr(\phi \sigma_j^\omega \sigma_k^\omega)$ ($\omega = x, y$ and z) are bounded in modulus by some constant times a positive inverse power of the distance between sites j and k. The reduced density matrix for spins j and k can be expressed in terms of these spin correlations and so can its concurrence. If ϕ is the ground-state, the antiferromagnetic nature of the pair

interactions will give negative spin-pair correlations, and pair-entanglement will appear only for sufficently negative values of these spin correlations. The power law will then produce a certain distance beyond which entanglement is impossible because the correlations are not sufficiently negative. This power law behaviour of the two-site spin correlations has been calculated (in the macroscopic limit $N \to \infty$) for the ground-state of some solvable one-dimensional spin 1/2 chains [22–24].

¿From our data for $N = 8$, we obtain (recall Eq. (2)):

$$|tr(\rho_0(\alpha)\sigma_1^\omega\sigma_k^\omega)| \leq \frac{|tr(\rho_0(\infty)\sigma_1^\omega\sigma_2^\omega)|}{r_{1,k}} \ , \quad k = 2, 3, 4, 5 \ ,$$

for every $\alpha > 0$ including $\alpha = \infty$, where $\rho_0(\alpha)$ is the ground-state of $H_8(\alpha)$. The associated cut-off distance d_c determined from Eq. (3), $|tr(\rho_0(\infty)\sigma_1^\omega\sigma_2^\omega)|/d_c = 1/3$, is $d_c = 1.8255$ which is below $r_{1,3}$ (> 1.8477) so there is indeed no entanglement beyond nearest neighbors ($r_{1,2} = 1$) in the ground-state of $H_8(\alpha)$.

Our second observation is that a simple glance at a figure of the type of figures 2-5 allows the onlooker to decide whether $\alpha = 2$ or not. Two-site entanglement of eigenstates is extremely sensitive to a spectral "collapse". If one is able to control the parameter α, one can drastically reduce entanglement by selecting $\alpha = 2$. In order to be useful, this property should be present for some small interval and not just for a point; but the feature is potentially interesting for applications.

Thirdly, one can produce selective two-site entanglement at any required distance by appropiately choosing α and/or the energy-level; a feature which is again of some interest, and is stable under small variations of α.

Finally, we point out that for $N = 8$ and every $2 \neq \alpha > 0$, including the nearest neighbor model (see figure 5), there is an excited eigen-state which is pure (except for special values of α where a crossing occurs; see figure 9) which presents entanglement at maximal distance and at next to maximal distance simultaneously. This state corresponds to the energy vs. α curve labeled a in figures 8 and 9. The concurrence for maximal distance in this state is the maximal concurrence for all possible distances. It would be interesting to give an

experimental procedure to prepare this state.

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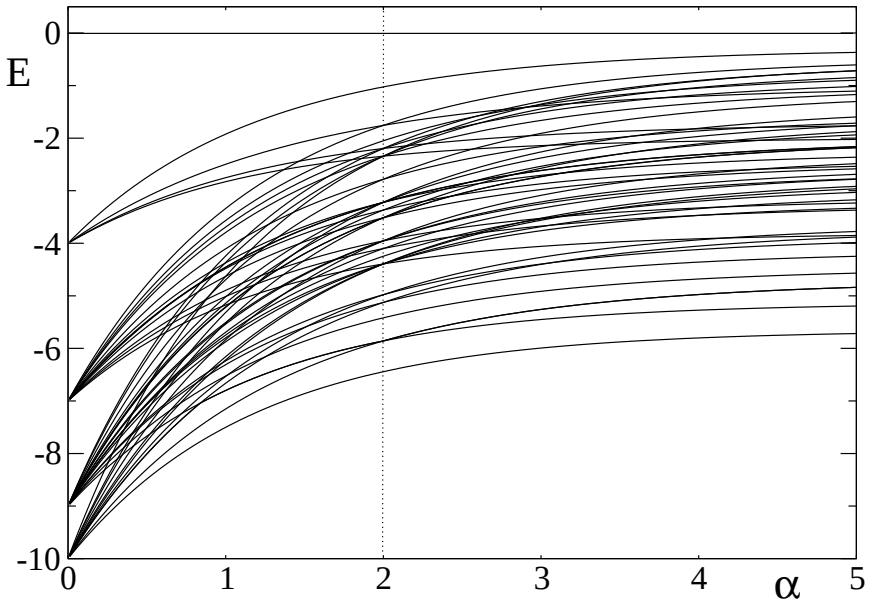

FIG. 1: Spectrum of $\widetilde{H}_8(\alpha)$. 0 is the largest eigen-energy and has degeneracy 9 ($= N + 1$) for all $\alpha \geq 0$. Beyond $\alpha \approx 7.29$ there are no further crossings. The monotone increase of the (differentiable) energy vs. α curves is a consequence of the Hellman-Feynman formula. Here and in all figures the energies are dimensionless.

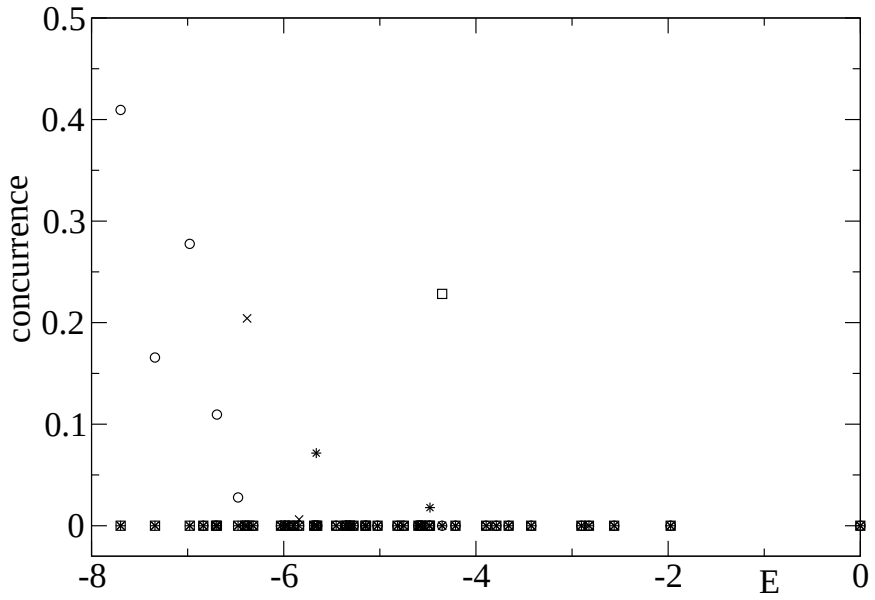

FIG. 2: Concurrence of $\rho_{j,k}(E)$ for $N = 8$ and $\alpha = 1$. $\circ$ corresponds to $|j-k| = 1$; $\times$ to $|j-k| = 2$; $*$ to $|j-k| = 3$; and $\square$ to $|j-k| = 4$.

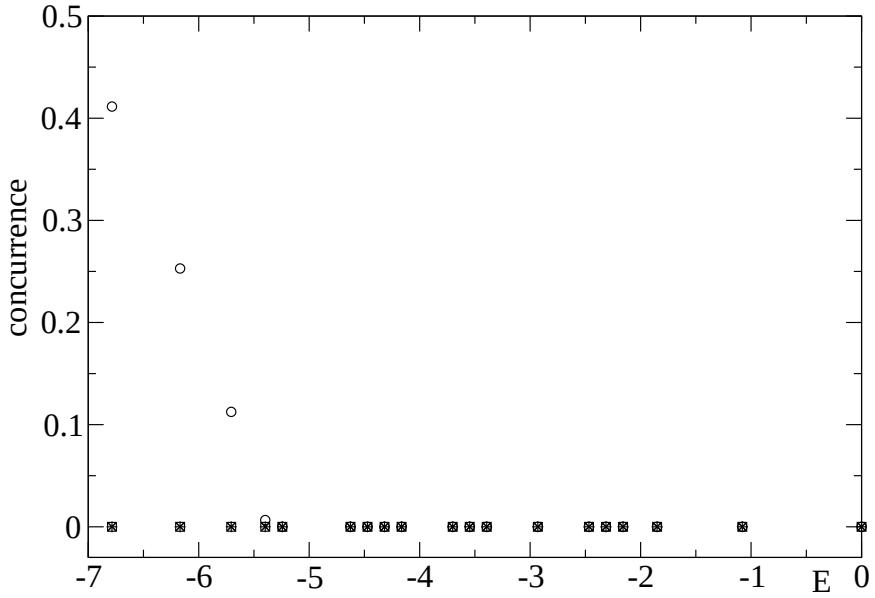

FIG. 3: Concurrence of $\rho_{j,k}(E)$ for $N = 8$ and $\alpha = 2$. $|j-k| = 1(\circ), 2(\times), 3(*), 4(\square)$.

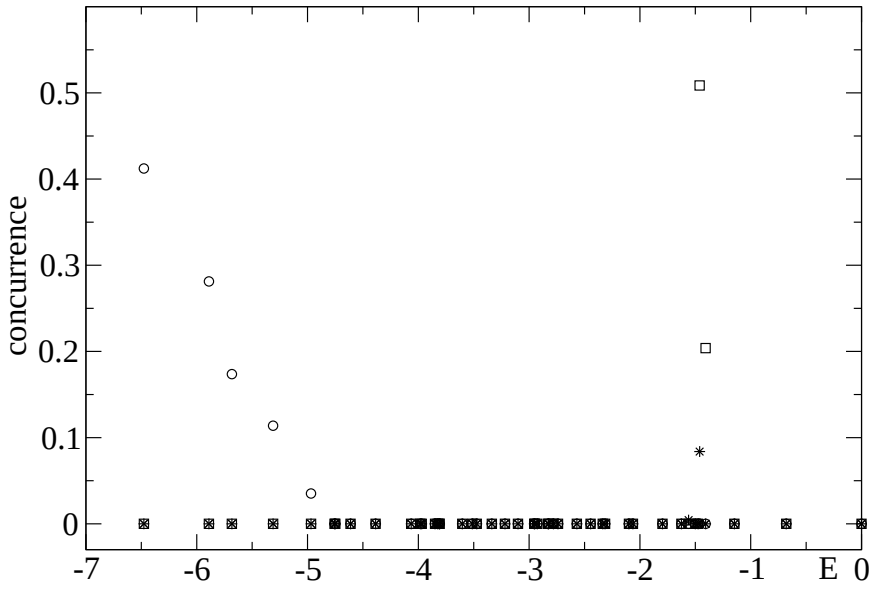

FIG. 4: Concurrence of $\rho_{j,k}(E)$ for $N = 8$ and $\alpha = 3$. $|j - k| = 1(\circ), 2(\times), 3(*), 4(\square)$.

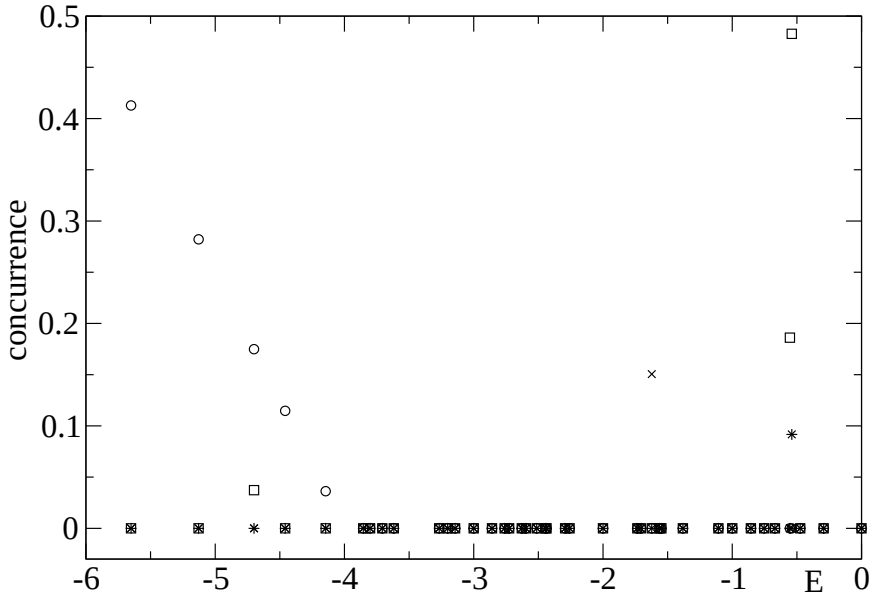

FIG. 5: Concurrence of $\rho_{j,k}(E)$ for $N = 8$ in the nearest-neighbor model. $|j-k| = 1(\circ), 2(\times), 3(*), 4(\square)$.

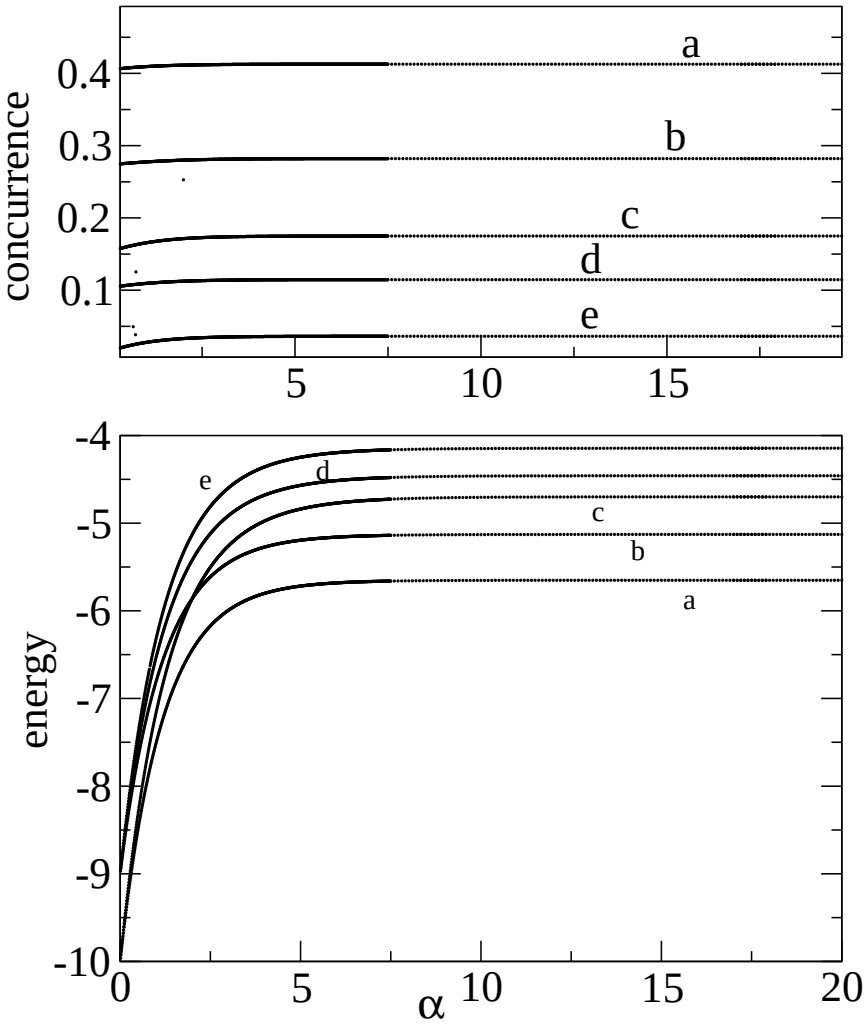

FIG. 6: Top graph: Nearest neighbor concurrence vs α for $N = 8$. The top curve labeled a corresponds to the ground-state. Not all discontinuities are visible in the figure.

Lower graph: The corresponding five energies vs α curves which carry nearest neighbor entanglement for $N = 8$. The curve labels a-e correspond to the concurrence curves of the same label in the top graph. The dimension of the associated orthoprojectors are: 1 (a), 3 (b), 1 (c), 6 (d) and 6 (e).

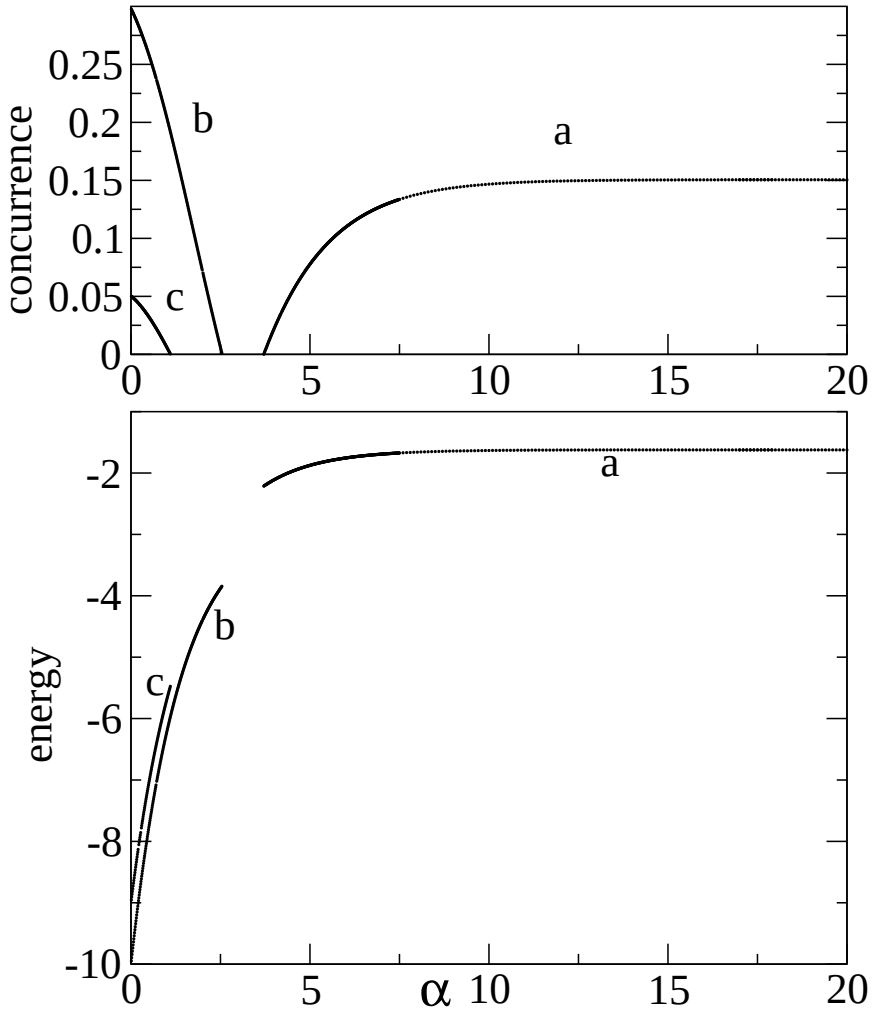

FIG. 7: Top graph: Next-nearest neighbor concurrence vs α for $N = 8$.

Lower graph: The three energies vs α curves which carry next-nearest neighbor entanglement for $N = 8$. The labels on the curves correspond to those on the concurrence curves of the top graph. The dimensions of the associated orthoprojectors are: 1 (a), 1 (b), and 6 (c).

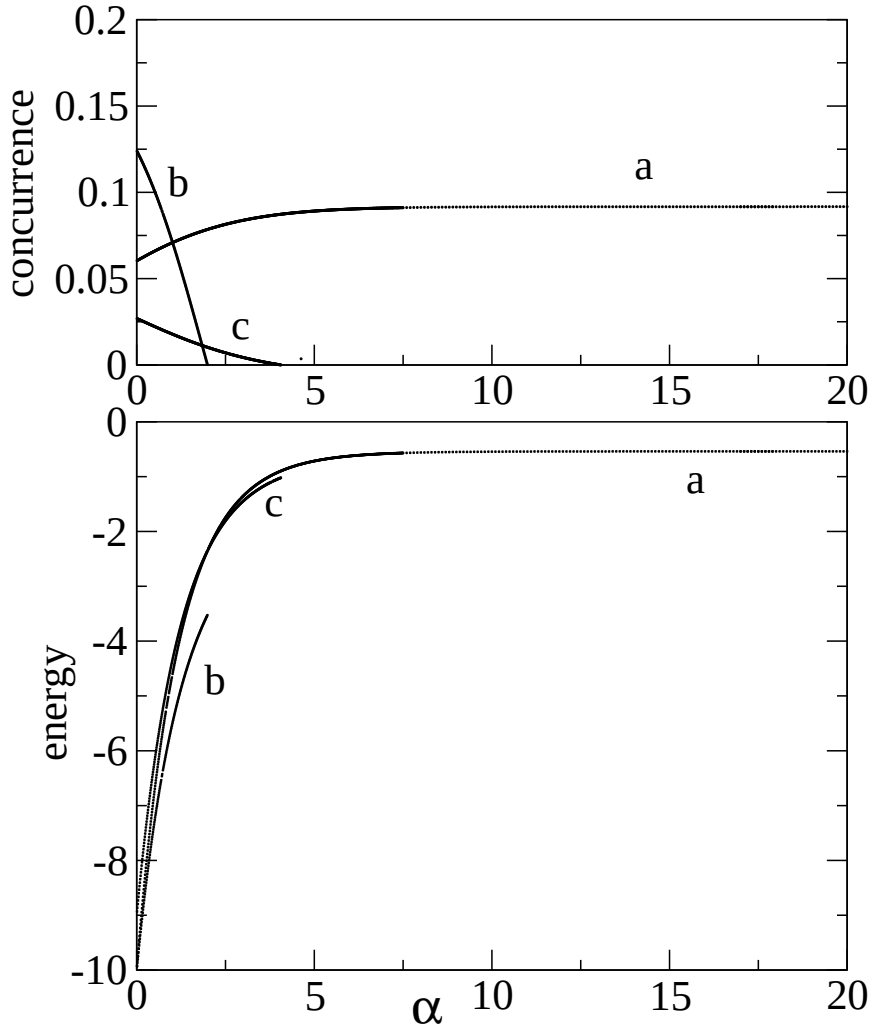

FIG. 8: Top graph: Concurrence vs α for $N = 8$ and $|j - k| = 3$.

Lower graph: The three energies vs α curves which carry entanglement for $N = 8$ at distance $|j - k| = 3$. The curve labels correspond to those of the top graph. Curves a and c cross at $\alpha \approx 2.35$. The dimensions of the associated orthoprojectors are: 1 (a), 1 (b), and 3 (c).

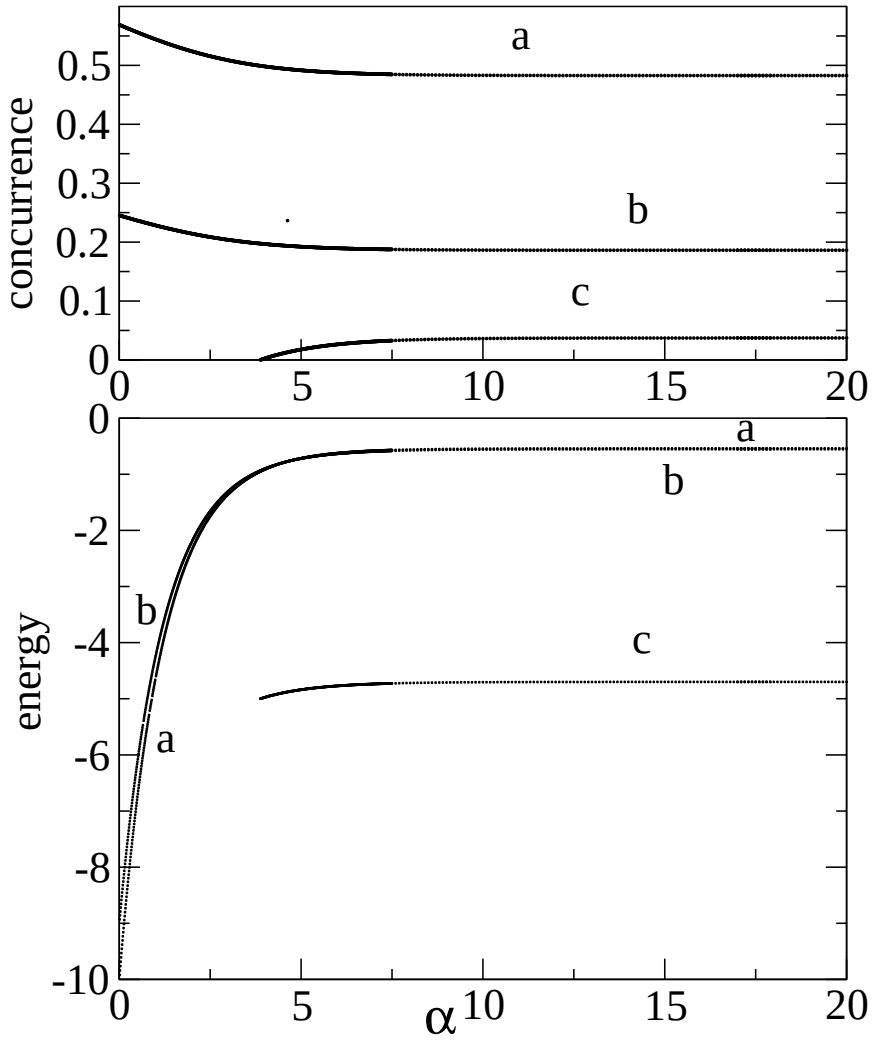

FIG. 9: Top graph: Concurrence vs α for $N = 8$ and maximal distance $|j - k| = 4$.

Lower graph: The three energies vs α curves which carry entanglement for $N = 8$ at maximal distance. The curve labels correspond to those of the top graph. Curves a and b cross at $\alpha \approx 4.63$ where the concurrence curves a and b have a discontinuity. The dimensions of the associated orthoprojectors are: 1 (a), 6 (b), and 1 (c).

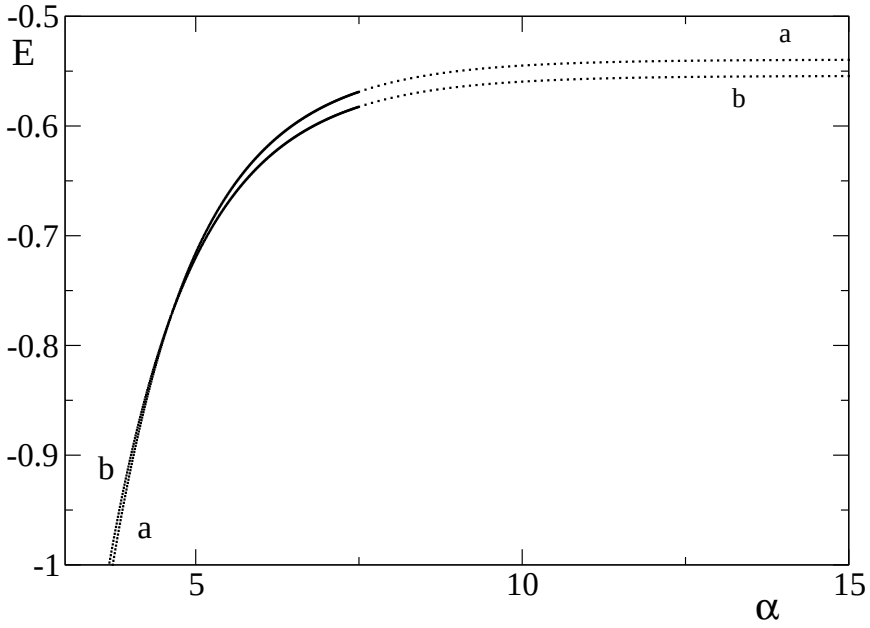

FIG. 10: Detail of the crossing of the energy curves a and b in the lower graph of the previous figure 9. The curves do not coincide to the right of the crossing, a feature which is not resolved at the scale of figure 9.