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Finite dimensional pointed Hopf algebras over S4

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FINITE DIMENSIONAL POINTED HOPF ALGEBRAS
OVER $\mathbb{S}_4$

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ABSTRACT. Let $\Bbbk$ be an algebraically closed field of characteristic 0. We conclude the classification of finite-dimensional pointed Hopf algebras whose group of group-likes is $\mathbb{S}_4$. We also describe all pointed Hopf algebras over $\mathbb{S}_5$ whose infinitesimal braiding is associated to the rack of transpositions.

1. INTRODUCTION

The classification of finite-dimensional Hopf algebras over an algebraically closed field $\Bbbk$ of characteristic 0 is a widely open problem. Strictly different techniques are employed when dealing with semisimple or non semisimple algebras. In this later case, useful techniques have been developed for the special case of pointed Hopf algebras.

A significant progress has been achieved in [AS3] in the case of pointed Hopf algebras with abelian group of group-likes. When the group of group-likes is not abelian, the problem is far from being completed. Some hope is present in the lack of examples: in this situation, Nichols algebras tend to be infinite dimensional, see for example [AF], [AFGV1], [AZ], [AFZ]. Nevertheless, examples on which the Nichols algebras are finite dimensional do exist. Over $\mathbb{S}_4$ these algebras are determined in [AHS]: there are 3 of them, all arising from racks in $\mathbb{S}_4$, associated to a cocycle. When the cocycle is -1, there are two Nichols algebras, one corresponding to the rack of transpositions and the other to the rack of 4-cycles, presented in [MS] and [AG1], respectively. When the cocycle is non-constant, this Nichols algebra was defined in [MS] and, independently, in [FK] as a quadratic algebra. These three Nichols algebras are an exhaustive list of Nichols algebras in the category of Yetter Drinfeld modules $^{\mathbb{S}_4}_{\mathbb{S}_4}\mathcal{YD}$, and in this paper we prove that all pointed Hopf algebras H with $G(H) \cong \mathbb{S}_4$ are in fact liftings of them. Up to now, only two Nichols algebras over $\mathbb{S}_5$ are known to be finite-dimensional. Here, we describe all their liftings.

In [AG2], a family of pointed Hopf algebras was defined; these were shown, by means of Gröbner basis, to be liftings of the Nichols algebras associated to the class of transpositions and constant cocycle -1. Following ideas from

that paper, we define two more families of Hopf algebras, and show that they are liftings of the other Nichols algebras. We remark that our proofs do not use Gröbner basis, instead, we develop part of the representation theory for these algebras and combine this new knowledge with some techniques on quadratic algebras to prove that these algebras are liftings of $\mathbb{S}_4$, see Proposition 5.1. The main theorem of the present paper is the following:

Main Theorem. *Let H be a finite-dimensional pointed Hopf algebra with $G(H) \cong \mathbb{S}_4$, $H \neq \Bbbk\mathbb{S}_4$. Then H is isomorphic to one and only one of the following algebras:*

(1) $\mathfrak{B}(\mathcal{O}_2^4, -1)\sharp\Bbbk\mathbb{S}_4$;
(2) $\mathfrak{B}(\mathcal{O}_4^4, -1)\sharp\Bbbk\mathbb{S}_4$;
(3) $\mathfrak{B}(\mathcal{O}_2^4, \chi)\sharp\Bbbk\mathbb{S}_4$;
(4) $\mathcal{H}(\mathcal{Q}_4^{-1}[t])$, *for exactly one $t \in \mathbb{P}^1_{\Bbbk}$;*
(5) $\mathcal{H}(\mathcal{D}[t])$, *for exactly one $t \in \mathbb{P}^1_{\Bbbk}$;*
(6) $\mathcal{H}(\mathcal{Q}_4^{\chi}[1])$.

See Subsection 2.4 and Definitions 3.8–3.10 for a description of the algebras mentioned above. We point out that $\mathbb{S}_4$ is the second finite non-abelian group which admits non-trivial finite-dimensional pointed Hopf algebras such that the classification is completed, the first one being $\mathbb{S}_3$ [AHS].

The paper is organized as follows. In Section 2 we recall some basic facts about Hopf algebras and racks, with examples, together with the definitions of three Nichols algebras in $^{\mathbb{S}_4}_{\mathbb{S}_4}\mathcal{YD}$ and two in $^{\mathbb{S}_5}_{\mathbb{S}_5}\mathcal{YD}$. In fact, we present a proof showing that two of these braided Hopf algebras are Nichols algebras, a fact that seems to be absent in the literature. All of these algebras are quadratic. We explicitly describe the set of quadratic relations in a Nichols algebra associated to a rack and a cocycle. In Section 3 we show that a pointed Hopf algebra whose infinitesimal braiding is subject to certain hypothesis is generated by group-likes and skew-primitives. We define families of pointed Hopf algebras associated to quadratic lifting data (Def. 3.5) and explicitly distinguish three of these families, which will be of our interest later on. In Section 4 we use some quadratic-algebra techniques to provide a bound on the dimension of an algebra belonging to one of the families defined previously, and we use this we prove that, under certain conditions, these families give all possible pointed Hopf algebras over a given group. In Section 5 we develop part of the representation theory for the algebras we have remarked in Section 3, in order to prove that for each one the group of group-likes is the symmetric group $\mathbb{S}_4$ or $\mathbb{S}_5$. Thus we show that these algebras are liftings of the Nichols algebras over $\mathbb{S}_n$ from Section 2. Finally, in Section 6 we prove the Main Theorem. We also show that the liftings of $\mathbb{S}_5$ defined exhaust the list of finite-dimensional pointed Hopf algebras over $\mathbb{S}_5$ which are liftings of one of the known finite-dimensional Nichols algebras over this group.

2. Preliminaries

2.1. **Conventions.** Let H be a Hopf algebra over $\Bbbk$, with antipode S. We will use the Sweedler's notation $\Delta(h) = h_1 \otimes h_2$ for the comultiplication [S]. Let $^H_H\mathcal{YD}$ be the category of (left-left) Yetter-Drinfeld modules over H. That is, M is an object of $^H_H\mathcal{YD}$ if and only if there exists an action $\cdot$ such that $(M, \cdot)$ is a (left) H-module and a coaction δ such that (M, δ) is a (left) H-comodule, subject to the following compatibility condition:

$$\delta(h \cdot m) = h_1 m_{-1} S(h_3) \otimes h_2 \cdot m_0, \ \forall m \in M, h \in H,$$

where $\delta(m) = m_{-1} \otimes m_0$. If G is a finite group and $H = \Bbbk G$, we denote $^G_G\mathcal{YD}$ instead of $^H_H\mathcal{YD}$.

2.2. **The Lifting Method for the classification of Hopf algebras.** Let (V, c) be a braided vector space, *i. e.* V is a vector space and $c \in GL(V \otimes V)$ is a *braiding*: $(c \otimes \mathrm{id})(\mathrm{id} \otimes c)(c \otimes \mathrm{id}) = (\mathrm{id} \otimes c)(c \otimes \mathrm{id})(\mathrm{id} \otimes c) \in \mathrm{End}(V \otimes V \otimes V)$. Let $\mathfrak{B}(V)$ be the Nichols algebra of (V, c), see *e. g.* [AS2]. Let $\mathcal{J} = \mathcal{J}_V$ be the kernel of the canonical projection $T(V) \to \mathfrak{B}(V)$; $\mathcal{J} = \oplus_{n \geq 0}\mathcal{J}^n$ is a graded Hopf ideal. If $\mathcal{J}_2$ is the ideal generated by $\mathcal{J}^2$, $\widehat{\mathfrak{B}}_2(V) = T(V)/\mathcal{J}_2$ is the *quadratic* Nichols algebra of (V, c).

The *coradical* H_0 of H is the sum of all simple sub-coalgebras of H. In particular, if $G(H)$ denotes the group of *group-like elements* of H, we have $\Bbbk G(H) \subseteq H_0$. We say that a Hopf algebra is *pointed* if $H_0 = \Bbbk G(H)$. Define, inductively, $H_n = \Delta^{-1}(H \otimes H_{n-1} + H_0 \otimes H)$. The family $\{H_i\}_{i \geq 0}$ is the *coradical filtration* of H, it is a coalgebra filtration and satisfies $H_i \subseteq H_{i+1}$, $H = \bigcup H_i$, see [M], [S]. It is a Hopf algebra filtration if and only if H_0 is a sub-Hopf algebra of H. In this case, for $\mathrm{gr}\,H(n) = H_n/H_{n-1}$ (set $H_{-1} = 0$), $\mathrm{gr}\,H = \oplus_{n \geq 0}\mathrm{gr}\,H(n)$ is the associated graded Hopf algebra. Let $\pi : \mathrm{gr}\,H \to H_0$ be the homogeneous projection. We have the following invariants of H:

- $R = (\mathrm{gr}\,H)^{\mathrm{co}\,\pi}$ is the *diagram* of H; this is a braided Hopf algebra in $^{H_0}_{H_0}\mathcal{YD}$, and it is a graded sub-object of $\mathrm{gr}\,H$.
- $V := R(1) = P(R)$, with the braiding from $^{H_0}_{H_0}\mathcal{YD}$, is called the *infinitesimal braiding* of H.

It follows that the Hopf algebra $\mathrm{gr}\,H$ is the Radford biproduct $\mathrm{gr}\,H \simeq R\#\Bbbk G(H)$. The subalgebra of R generated by V is isomorphic to the Nichols algebra $\mathfrak{B}(V)$.

Let Γ be a finite group and let H_0 be the group algebra of Γ. The main steps of the *Lifting Method* [AS2] for the classification of all finite-dimensional pointed Hopf algebras with group Γ are:

- determine all $V \in {}^{H_0}_{H_0}\mathcal{YD}$ such that the Nichols algebra $\mathfrak{B}(V)$ is finite dimensional,
- for such V, compute all Hopf algebras H such that $\mathrm{gr}\,H \simeq \mathfrak{B}(V)\sharp\Bbbk\Gamma$. We call H a *lifting* of $\mathfrak{B}(V)$ over $\Gamma \cong G(H)$.

- Prove that any finite-dimensional pointed Hopf algebras with group Γ is generated by group-likes and skew-primitives.

2.3. Racks. A *rack* is a pair $(X, \triangleright)$, where X is a non-empty set and $\triangleright : X \times X \to X$ is a function, such that $\phi_i = i \triangleright (\cdot) : X \to X$ is a bijection for all $i \in X$ satisfying:

$$i \triangleright (j \triangleright k) = (i \triangleright j) \triangleright (i \triangleright k), \; \forall i, j, k \in X.$$

The following are examples of racks.

- Let G be a group, $g \in G$, and $\mathcal{O}_g$ the conjugacy class of g. Then, if $x \triangleright y = xyx^{-1}$, $(\mathcal{O}_g, \triangleright)$ is a rack.
- If $G = \mathbb{S}_n$ and τ is a j-cycle, we will denote by $\mathcal{O}_j^n$, or $\mathcal{O}_j$ if n is fixed by the context, the rack induced by its conjugacy class.
- In particular, we will work with the racks $\mathcal{O}_2^n$ of transpositions in $\mathbb{S}_n$ and with the rack $\mathcal{O}_4^4$ of 4-cycles in $\mathbb{S}_4$.
- If $(X, \triangleright)$ is a rack, then the inverse rack $(X^{-1}, \triangleright^{-1})$, see [AFGV2], is given by $X^{-1} = X$ and $i \triangleright^{-1} j = k$, provided $i \triangleright k = j$, $\forall i, j, k \in X^{-1}$. Indeed, note that $i \triangleright^{-1} (j \triangleright^{-1} k) = s \Leftrightarrow i \triangleright s = j \triangleright^{-1} k \Leftrightarrow j \triangleright (i \triangleright s) = k$. While $(i \triangleright^{-1} j) \triangleright^{-1} (i \triangleright^{-1} k) = u \Leftrightarrow (i \triangleright^{-1} j) \triangleright u = i \triangleright^{-1} k \Leftrightarrow i \triangleright ((i \triangleright^{-1} j) \triangleright u) = k \Leftrightarrow (i \triangleright (i \triangleright^{-1} j)) \triangleright (i \triangleright u) = k \Leftrightarrow j \triangleright (i \triangleright u) = k \Leftrightarrow i \triangleright u = i \triangleright s \Leftrightarrow u = s$, since ϕ_j, ϕ_i are bijections.

Note that the racks $\mathcal{O}_4^4$ and $\mathcal{O}_2^4$ on $\mathbb{S}_4$ are not isomorphic, since, for $i \in \mathcal{O}_2^4$, we always have $\phi_i^2 = \mathrm{id}$, and for $i \in \mathcal{O}_4^4$ $\phi_i^4 = \mathrm{id}$, but $\phi_i^2 \neq \mathrm{id}$.

A rack $(X, \triangleright)$ is said to be *indecomposable* if it cannot be decomposed as the disjoint union of two sub-racks. It is said to be *faithful* if $\phi_i = \phi_j$ only for $i = j$. We refer the reader to [AG1] for detailed information on racks.

Let $(X, \triangleright)$ be a rack. A 2-cocycle $q : X \times X \to \Bbbk^*$, $(i, j) \mapsto q_{ij}$ is a function such that

$$q_{i, j \triangleright k} q_{j,k} = q_{i \triangleright j, i \triangleright k} q_{i,k}, \; \forall i, j, k \in X.$$

In this case it is possible to generate a braiding c^q in the vector space $\Bbbk X$ with basis $\{x_i\}_{i \in X}$ by $c^q(x_i \otimes x_j) = q_{ij} x_{i \triangleright j} \otimes x_i$, $\forall i, j \in X$.

Examples 2.1.
- Let $(X, \triangleright)$ be a rack. A constant map $q : X \times X \to \Bbbk^*$, *i. e.* $q_{\sigma \tau} = \xi \in \Bbbk^*$ for every $\sigma, \tau \in X$, is a 2-cocycle.
- [MS, Ex. 5.3] Let $(X, \triangleright) = \mathcal{O}_2^n$. A 2-cocycle is given by a function $\chi : X \times X \to \Bbbk^*$ defined as, if $\tau, \sigma \in X$, $\tau = (ij)$ and $i < j$:

$$\chi(\sigma, \tau) = \begin{cases} 1, & \text{if } \sigma(i) < \sigma(j) \\ -1, & \text{if } \sigma(i) > \sigma(j). \end{cases}$$

The dual braided vector space of $(\Bbbk X, c^q)$ is given by $(\Bbbk X^{-1}, c^{\tilde{q}})$, with

$$(1) \qquad \tilde{q}_{kl} = q_{k \, k \triangleright^{-1} l}, \quad k, l \in X^{-1}.$$

Indeed, let $Y = \{y^i : i \in X\}$ be a dual basis of $\{x_i : i \in X\}$, and consider the pairing $\langle \cdot, \cdot \rangle$ defined as $\langle x_i \otimes x_j, y^k \otimes y^l \rangle = \delta_{i,l} \delta_{j,k}$, for $i, j, k, l \in X$.

The braiding c^* is thus $\langle x_i \otimes x_j, c^*(y^k \otimes y^l) \rangle = \langle c(x_i \otimes x_j), y^k \otimes y^l \rangle = q_{ij}\langle x_{i \triangleright j} \otimes x_i, y^k \otimes y^l \rangle = q_{ij}\delta_{l,i\triangleright j}\delta_{k,i} = q_{k,k\triangleright^{-1}l}\delta_{k\triangleright^{-1}l,j}\delta_{k,i}$ and thus $c^*(y^k \otimes y^l) = q_{k,k\triangleright^{-1}l}y^{k\triangleright^{-1}l} \otimes y^k$.

2.4. **Nichols algebras associated to racks.** Let X be a rack, q a 2-cocycle. We will denote by $\mathfrak{B}(X,q)$ the Nichols algebra associated to the braided vector space $(\Bbbk X, c^q)$. Let $\mathcal{R}$ be the set of equivalence classes in $X \times X$ for the relation generated by $(i,j) \sim (i \triangleright j, i)$. Let $C \in \mathcal{R}$, $(i,j) \in C$. Take $i_1 = j$, $i_2 = i$, and recursively, $i_{h+2} = i_{h+1} \triangleright i_h$. For each C, there exists $n \in \mathbb{N}$ such that $i_{n+k} = i_k$, $k \in \mathbb{N}$. Let $n(C)$ be the minimum such n; thus $C = \{i_1, \ldots, i_{n(c)}\}$ and $n(C) = \#C$. Let $\mathcal{R}'$ be the set of all $C \in \mathcal{R}$ that satisfy

$$(2) \qquad \prod_{h=1}^{n(C)} q_{i_{h+1},i_h} = (-1)^{n(C)}.$$

Lemma 2.2. *A basis of the space $\mathcal{J}^2$ of quadratic relations of $\mathfrak{B}(X,q)$ is given by*

$$(3) \qquad b_C := \sum_{h=1}^{n(C)} \eta_h(C)\, x_{i_{h+1}}x_{i_h}, \quad C \in \mathcal{R}',$$

where $\eta_1(C) = 1$ and $\eta_h(C) = (-1)^{h+1}q_{i_2i_1}q_{i_3i_2}\ldots q_{i_hi_{h-1}}$, $h \geq 2$.

Proof. Let U_C be the subspace of $\Bbbk X \otimes \Bbbk X$ spanned by $x_i \otimes x_j$, $(i,j) \in C$. Then $\mathcal{J}^2 = \ker(c^q + \mathrm{id}) = \oplus_{C \in \mathcal{R}} \ker(c^q + \mathrm{id})|_{U_C}$. We have $\det(c^q + \mathrm{id})|_{U_C} = \prod_{h=1}^{n(C)} q_{i_{h+1},i_h}(-1)^{n(C)+1} + 1$. If this is 0, then b_C spans $\ker(c^q + \mathrm{id})|_{U_C}$. $\square$

Remarks 2.3. The constant cocycle $q \equiv -1$ evidently fulfills (2), and $\eta_h(C) = 1$ for every $C \in \mathcal{R}, h = 1, \ldots, n(C)$.

More generally, the constant cocycle $q \equiv \omega$, for $-\omega$ a primitive l-th root of unity and $l|n(C)$, also fulfills (2), and $\eta_h(C) = (-\omega)^{h-1}$ for every $C \in \mathcal{R}, h = 1, \ldots, n(C)$. See [AG1, Lem. 6.13].

It is easy to see that the cocycle χ in Examples 2.1 fulfills (2) for every $C \in \mathcal{R}$.

When dealing with the racks $\mathcal{O}_2^n, \mathcal{O}_4^4$, an equivalence class $C \in \mathcal{R}$ may have only 1, 2, or 3 elements, and thus a relation on the basis is composed of at most three summands.

Theorem 2.4.   (1) *Let $3 \leq n \leq 5$. The Nichols algebras $\mathfrak{B}(\mathcal{O}_2^n, -1)$ are finite dimensional, of dimensions 12, 576, 8294400, respectively, and it holds $\mathfrak{B}(\mathcal{O}_2^n, -1) = \widehat{\mathfrak{B}}_2(\mathcal{O}_2^n, -1)$.*
     (2) $\dim \mathfrak{B}(\mathcal{O}_4^4, -1) = 576$ *and* $\mathfrak{B}(\mathcal{O}_4^4, -1) = \widehat{\mathfrak{B}}_2(\mathcal{O}_4^4, -1)$.

Proof. We see (1). Cases $n = 3, 4$ are in [MS, Ex. 6.4]. For $n = 5$, the algebra was introduced as a quadratic algebra in [MS]. Graña established

$\widehat{\mathfrak{B}}_2(V) = \mathfrak{B}(V)$ in [G3]. Its dimension was determined by Roos with computer program `Bergman`. (2) is [AG1, Th. 6.12], using `Bergman`. □

It turns out that the Nichols algebras $\mathfrak{B}(\mathcal{O}_2^4, \chi)$, $\mathfrak{B}(\mathcal{O}_2^5, \chi)$ are also quadratic. To see this, we recall here the notion of differential operators associated to a rack $(X, \triangleright)$ in a Nichols algebra $\mathfrak{B}(X)$ arising from that rack. For $x \in X$, let x^* be the dual element to x in $(\Bbbk X)^*$ and, for $m \neq 1$, extend it as zero in $\mathfrak{B}^m$, the homogeneous component of degree m of $\mathfrak{B}(V)$. Define $\delta_x : \mathfrak{B}^n \to \mathfrak{B}^{n-1}$ by $\delta_x = (\mathrm{id} \otimes x^*)\Delta$, where Δ is the comultiplication of the Nichols algebra. These skew-derivations provide a powerful tool to decide whether an element in the algebra is zero or different from zero, since $\alpha \in \mathfrak{B}^m$ ($m \geq 2$) is zero if and only if $\delta_x(\alpha) = 0 \ \forall \, x \in \Bbbk X$, see [AG1, Section 6]. The following proposition has been assumed to be true in the literature. We thank Matías Graña for providing us the computations needed to finish its proof.

Proposition 2.5. *Let $n = 4, 5$. The Nichols algebra $\mathfrak{B}(\mathcal{O}_2^n, \chi)$ is quadratic and has dimension 576, 8294400, respectively.*

Proof. Let $n = 4$, $B = \widehat{\mathfrak{B}}_2(\mathcal{O}_2^4, \chi)$. According to [FK, Probl. 2.3], the Hilbert polynomial of the algebra B is

$$(4) \qquad\qquad P_B(t) = [2]^2[3]^2[4]^2,$$

where we denote by $[k]$ the polynomial $1 + t + t^2 + \ldots + t^{k-1}$. Thus, $\dim B = P_B(1) = 2^2 3^3 4^2 = 576$. Equation (4) also implies that the top degree (cf. [AG1, Section 6]) of B is 12. If $\mathfrak{B}^{12}(\mathcal{O}_2^4, \chi) \neq 0$ we would get $B = \mathfrak{B}(\mathcal{O}_2^4, \chi)$ by [AG1, Th. 6.4 (2)]. Let

$$a = x_{(12)}, \ b = x_{(13)}, \ c = x_{(14)}, \ d = x_{(23)}, \ e = x_{(24)}, \ f = x_{(34)}.$$

Consider $abacabacdedf \in \mathfrak{B}^{12}(\Bbbk\mathcal{O}_2^4, \chi)$. We use derivations and the help the computer program `Deriva` [G2] developed by Matías Graña to see that

$$\delta_c \delta_b \delta_c \delta_a \delta_b \delta_d \delta_c \delta_b \delta_f \delta_d \delta_f \delta_e (abacabacdedf) \neq 0$$

and thus $\mathfrak{B}^{12}(\mathcal{O}_2^4, \chi) \neq 0$.

For $n = 5$, the result follows similarly. In this case, the Hilbert polynomial is $P(t) = [4]^4[5]^2[6]^4$, see again [FK, Probl. 2.3]. Therefore we have that $\dim \widehat{\mathfrak{B}}_2(\mathcal{O}_2^5, \chi) = 8294400$ and that its top degree is 40. Now, take a, b, c, d, e, f as before and

$$g = x_{(15)}, \ h = x_{(25)}, \ k = x_{(35)}, \ m = x_{(45)}.$$

Again, using `Deriva`, we see that $\mathfrak{B}^{40}(\mathcal{O}_2^5, \chi) \neq 0$ since the operator

$$\begin{aligned}
\nabla =\ & \delta_k \delta_m \delta_f \delta_m \delta_e \delta_h \delta_f \delta_e \delta_c \delta_h \delta_e \delta_h \delta_g \delta_k \delta_c \delta_b \delta_m \delta_d \delta_c \\
& \circ \, \delta_e \delta_m \delta_g \delta_e \delta_k \delta_g \delta_c \delta_h \delta_e \delta_g \delta_k \delta_d \delta_e \delta_g \delta_k \delta_b \delta_a \delta_d \delta_c \delta_a \delta_b
\end{aligned}$$

satisfies $\nabla(abadabadgabadabadgabagdabadgcechcechfkfm) \neq 0$. □

Let us describe explicitly all these Nichols algebras:

$$\mathfrak{B}(\mathcal{O}_2^n, -1) = \Bbbk\langle x_{(ij)}, 1 \le i < j \le n | x_{(ij)}^2, \ x_{(ij)}x_{(kl)} + x_{(kl)}x_{(ij)}, \ (ij) \ne (kl),$$
$$x_{(ij)}x_{(jk)} + x_{(jk)}x_{(ik)} + x_{(ik)}x_{(ij)}, \quad 1 \le i < j < k \le n\rangle;$$
$$\mathfrak{B}(\mathcal{O}_2^n, \chi) = \Bbbk\langle x_{(ij)}, 1 \le i < j \le n | x_{(ij)}^2, \ x_{(ij)}x_{(kl)} - x_{(kl)}x_{(ij)}, \ (ij) \ne (kl),$$
$$x_{(ij)}x_{(jk)} - x_{(jk)}x_{(ik)} - x_{(ik)}x_{(ij)},$$
$$x_{(jk)}x_{(ij)} - x_{(ik)}x_{(jk)} - x_{(ij)}x_{(ik)}, \quad 1 \le i < j < k \le n\rangle;$$
$$\mathfrak{B}(\mathcal{O}_4^4, -1) = \Bbbk\langle x_\sigma, \sigma \in \mathcal{O}_4^4 | x_\sigma^2, \ x_\sigma x_{\sigma^{-1}} + x_{\sigma^{-1}}x_\sigma,$$
$$x_\sigma x_\tau + x_\nu x_\sigma + x_\tau x_\nu, \text{if } \sigma\tau = \nu\sigma \text{ and } \tau \ne \sigma \ne \nu \in \mathcal{O}_4^4\rangle.$$

2.5. **Nichols algebras of Yetter-Drinfeld modules.** Let G be a finite group, $\mathcal{O} = \{t_1, \ldots, t_n\}$ a conjugacy class in G and $\rho : G^s \to \mathrm{GL}(V)$ an irreducible representation of the centralizer G^s of a fixed element $s \in \mathcal{O}$. Assume $s = t_1$, and let $g_i \in G$, $i = 1, \ldots, n$ such that $g_i s g_i^{-1} = t_i$. Irreducible Yetter Drinfeld modules over G are in bijective correspondence with pairs $(\mathcal{O}, \rho)$. For such pair, the module $M(\mathcal{O}, \rho)$ is defined as $\bigoplus_{i \in I} g_i \otimes V$, with action and coaction given by:

$$g \cdot (g_i \otimes v) = g_j \otimes (\rho(\gamma)(v)), \ \delta(g_i \otimes v) = t_i \otimes (g_i \otimes v),$$

for $\gamma \in G^s$ such that $gg_i = g_j\gamma$. When the representation ρ is one dimensional, the underlying braided vector space of $M(\mathcal{O}, \rho)$ is $(\Bbbk X, c^q)$, where X is the rack given by the conjugation on $\mathcal{O}$ and $q : X \times X \to \Bbbk^*$ is a 2-cocycle. The equivalence is given by $q(t_i, t_j) = \rho(\gamma)$, for $\gamma \in G^s$ such that $g_i g_j = g_k\gamma$.

Let $\mathfrak{B}(\mathcal{O}, \rho)$ denote the Nichols algebra of $M(\mathcal{O}, \rho)$. The following theorem lists all irreducible modules $M(\mathcal{O}, \rho)$ in $_{\mathbb{S}_4}^{\mathbb{S}_4}\mathcal{YD}$ such that $\mathfrak{B}(\mathcal{O}, \rho)$ is finite-dimensional. If $G = \mathbb{S}_4, s = (1234)$, let ρ_- be the character of $G^s = \langle s \rangle \cong \mathbb{Z}_4$ given by $\rho_-(s) = -1$. If $s = (12)$, $G^s = \langle(12),(34)\rangle \cong \mathbb{Z}_2 \times \mathbb{Z}_2$; let ϵ and sgn be the trivial and the sign representations of $\mathbb{Z}_2$, respectively. Note that $M(\mathcal{O}_2^n, \mathrm{sgn} \otimes \mathrm{sgn}) \cong (\Bbbk\mathcal{O}_2^n, c^q)$, $M(\mathcal{O}_2^n, \mathrm{sgn} \otimes \epsilon) \cong (\Bbbk\mathcal{O}_2^n, c^\chi)$ and $M(\mathcal{O}_4^4, \rho_-) \cong (\Bbbk\mathcal{O}_4^4, c^q)$, for $n = 4, 5$ and $q \equiv -1$.

Theorem 2.6. [AHS, Th. 4.7] *The only Nichols algebras of Yetter-Drinfeld modules over $\mathbb{S}_4$ with finite dimension are $\mathfrak{B}(\mathcal{O}_2^4, \mathrm{sgn} \otimes \mathrm{sgn})$, $\mathfrak{B}(\mathcal{O}_2^4, \mathrm{sgn} \otimes \epsilon)$ and $\mathfrak{B}(\mathcal{O}_4^4, \rho_-)$, up to isomorphism. All have dimension 576.* $\square$

Remark 2.7. When $n = 5$, if $M \ne M(\mathcal{O}_2^5, \mathrm{sgn} \otimes \mu), M(\mathcal{O}_{2,3}^5, \mathrm{sgn} \otimes \epsilon)$ for $\mu \in \{\epsilon, \mathrm{sgn}\}$, the associated Nichols algebra is infinite dimensional, see [AFZ]. It is not known if the Nichols algebra $\mathfrak{B}(\mathcal{O}_{2,3}^5, \mathrm{sgn} \otimes \epsilon)$ has finite dimension; this is the only open case in $\mathbb{S}_5$.

If H is a Hopf algebra with bijective antipode (in particular, if $H = \Bbbk G$), the category $_H^H\mathcal{YD}$ is rigid. Explicitly, let $M \in {}_H^H\mathcal{YD}$ with linear basis $\{e_i\}_{i=1}^n$ and dual basis $\{e^i\}_{i=1}^n$. Its (right) dual M^* is the dual $\Bbbk$-vector space of M

with action and coaction given by:

$$(5)\quad (h \cdot f)(m) = f(S(h)m), \quad f_{(-1)} \otimes f_{(0)} = \sum_{i=1}^{n} S^{-1}((e_i)_{(-1)}) \otimes f((e_i)_{(0)}) e^i,$$

for $f \in M^*$, $m \in M$. The left dual ${}^*M$ is defined analogously. When $H = \Bbbk G$, ${}^*M \cong M^*$. If X is a finite rack and q a 2-cocycle, $\mathfrak{B}^n(X,q)^* \cong \mathfrak{B}^n(X^{-1}, \tilde{q})$, for $\tilde{q}$ as in (1). Besides, if $\dim \mathfrak{B}(X,q) < \infty$, then $\mathfrak{B}(X,q)^* \cong \mathfrak{B}(X^{-1}, \tilde{q})$, see [G4, Lem. 2.6].

3. Pointed Hopf algebras associated to racks.

Let X be a rack, q a 2-cocycle. In this section we are interested in finite-dimensional pointed Hopf algebras H such that its infinitesimal braiding arises from a *principal YD-realization* of (X,q) over a finite group G. We recall from [AG2, Def. 3.2]: this is a collection $(\cdot, g, (\chi_i)_{i \in X})$ where $\cdot$ is an action of G on X; $g : X \to G$ is a function such that $g_{h \cdot i} = h g_i h^{-1}$ and $g_i \cdot j = i \rhd j$; and the family $(\chi_i)_{i \in X}$, with $\chi_i : G \to \Bbbk^*$, is a 1-cocycle, *i. e.* $\chi_i(ht) = \chi_i(t)\chi_{t \cdot i}(h)$, for all $i \in X$, $h, t \in G$, satisfying $\chi_i(g_j) = q_{ji}$. The realization is said to be *faithful* if g is injective. Let K be the subgroup of G generated by the image of X; K acts by rack automorphisms on X. In Theorem 3.1 below we show that, if $\mathfrak{B}(X,q)$ is finite-dimensional and $\mathfrak{B}(X^{-1}, \tilde{q})$ is quadratic, such a pointed Hopf algebra must be generated by group-likes and skew primitive elements. Another relevant result concerning these algebras is Lemma 3.2, where we show that certain relations hold in such an H. Finally, we define families of pointed Hopf algebras associated to a given data, some of which will be shown to be liftings of quadratic Nichols algebras.

3.1. **Generation in degree one.** The following theorem, which agrees with a well-known conjecture [AS1, Conj. 1.4], is a key step in the classification. When $(X,q) = (\mathcal{O}_2^n, -1)$, $n = 3, 4, 5$ the result is [AG2, Th. 2.1].

Theorem 3.1. *Let (X,q) be a rack such that $\mathfrak{B}(X,q)$ is finite-dimensional and $\mathfrak{B}(X^{-1}, \tilde{q})$ is quadratic. Let H be a pointed Hopf algebra such that its infinitesimal braiding arises from a principal YD-realization of (X,q) over a finite group G. Then the diagram of H is a Nichols algebra, and consequently H is generated by its group-like and skew-primitive elements.*

Proof. Let $G = G(H)$, $\operatorname{gr} H = R \sharp \Bbbk G$, $S = R^*$ the graded dual. By [AS1, Lem. 5.5], S is generated by $V = S(1)$ and R is a Nichols algebra if and only if $P(S) = S(1)$, that is, if S is a Nichols algebra. Now, $\mathfrak{B}(V) = \mathfrak{B}(X^{-1}, \tilde{q})$. Therefore, we want to show that the relations in $\mathfrak{B}(V) = T(V)/\mathcal{J}_V$ hold in S. By hypothesis, we deal only with relations in degree 2. It is easily checked that, if $r = b_C \in \mathcal{J}_V^2$ as in (3), then r is primitive and satisfies $c(r \otimes r) = r \otimes r$, where c is the braiding in ${}^G_G\mathcal{YD}$. As $\dim S < \infty$, $r = 0$ for

any such r. Therefore, there is a surjection $\mathfrak{B}(V) \twoheadrightarrow S$, $P(S) = V = S(1)$ and S is a Nichols algebra. $\qquad \square$

3.2. **Lifting of quadratic relations.** Let X be a finite rack, q a 2-cocycle on X. Let H be a pointed Hopf algebra such that its infinitesimal braiding arises from a principal YD-realization $(\cdot, g, (\chi_i)_{i \in X})$ of (X, q) over a group G. Let $\operatorname{gr} H = \bigoplus_{i>0} H_i/H_{i-1} = R \sharp \Bbbk G$. The projection of Hopf bimodules $\pi : H_1 \to H_1/H_0$ is a morphism of Hopf bimodules over $\Bbbk G$. Let σ be a section of Hopf bimodules over $\Bbbk G$ and set $a_i = \sigma(x_i \sharp 1) \in H_1$, then

$$(6) \qquad a_i \text{ is } (g_i, 1) - \text{primitive and } g_i a_j g_i^{-1} = q_{ij} a_{i \triangleright j}, \ \forall \, i, j \in X.$$

Let $\mathcal{R}'$ be as in Subsect. 2.4; recall that the space of quadratic relations $\mathcal{J}^2$ for $\mathfrak{B}(X, q)$ is generated by the relations b_C for $C \in \mathcal{R}'$. For such C, let

$$(7) \qquad \phi_C = \sum_{h=1}^{n(C)} \eta_h(C) \, X_{i_{h+1}} X_{i_h}$$

be a quadratic polynomial in the non-commuting variables $\{X_i : i \in X\}$, with $\eta_h(C)$ as in (3). Thus $b_C = \phi_C(\{x_i\}_{i \in X})$. If $(i, j) \in C$, set

$$a_C = \phi_C(\{a_i\}_{i \in X}) \in H, \quad g_C = g_j g_i \in G.$$

Thus the elements a_C are $(g_C, 1)$-primitives.

Lemma 3.2. *Let a_i be as in (6) and let us suppose that g satisfies:*

$$(8) \qquad g_i \neq g_j g_k, \quad \forall \, i, j, k \in X.$$

Then there exist $\lambda_C \in \Bbbk$, for all $C \in \mathcal{R}'$, normalized by

$$(9) \qquad \lambda_C = 0, \ \text{if} \ g_C = 1,$$

such that:

$$(10) \qquad a_C = \lambda_C(1 - g_C) \ \text{in } H, \ C \in \mathcal{R}',$$

$$(11) \qquad \lambda_C = q_{ki_2} q_{ki_1} \lambda_{k \triangleright C}, \quad \forall \, k \in X,$$

if $C = \{i_1, \ldots, i_n\} \in \mathcal{R}'$ and $k \triangleright C = \{k \triangleright i_1, \ldots, k \triangleright i_n\}$.
    If X is faithful and indecomposable, and q is constant, then the 1-cocycle $(\chi_i)_{i \in X}$ is constant, $\chi_i = \chi$, for all $i \in X$ and a multiplicative character χ of G, and we have $\lambda_C = \chi^2(t) \lambda_{t \cdot C}, \ \forall \, t \in G$.

Proof. Let $C \in \mathcal{R}'$. As a_C is $(g_C, 1)$-primitive, there are λ_C and $\lambda_i \in \Bbbk$, for every $i \in X$ such that $a_C = \lambda_C(1 - g_C) + \sum_{i \in X : \, g_i = g_C} \lambda_i a_i$. Condition (8) forces $\lambda_i = 0$ for every $i \in X$ and thus (10) follows. Relation (11) follows by applying $\operatorname{ad}(g_k)$ to both sides of (10). Indeed, let $C' = k \triangleright C = \{i'_1, \ldots, i'_n\}$

and note first that, as $i'_1 = k \rhd i_1$, $i'_2 = k \rhd i_2$, then $i'_l = k \rhd i_l$. Now,

$$\mathrm{ad}(g_k)(a_C) = \sum_{h=1}^{n} \eta_h(C) q_{k i_{h+1}} q_{k i_h} a_{k \rhd i_{h+1}} a_{k \rhd i_h}$$

$$= \sum_{h=1}^{n} \eta_h(C) q_{k i_{h+1}} q_{k i_h} a_{i'_{h+1}} a_{i'_h}.$$

On the other hand,

$$\mathrm{ad}(g_k)(\lambda_C(1 - g_C)) = \lambda_C(1 - g_{C'}) = \frac{\lambda_C}{\lambda_{C'}} a_{C'} = \frac{\lambda_C}{\lambda_{C'}} \sum_{h=1}^{n} \eta_h(C') a_{i'_{h+1}} a_{i'_h}.$$

but

$$\eta_h(C') = (-1)^{h+1} q_{i'_2 i'_1} \cdots q_{i'_h i'_{h-1}}$$

$$= (-1)^{h+1} q_{k \rhd i_2 k \rhd i_1} \cdots q_{k \rhd i_h k \rhd i_{h-1}}$$

$$\overset{(2)}{=} (-1)^{h+1} \prod_{k=1}^{h-1} q_{k i_{k+1} \rhd i_k} \prod_{k=1}^{h-1} q_{i_{k+1} i_k} \left(\prod_{k=1}^{h-1} q_{k i_k} \right)^{-1}$$

$$= \eta_h(C) \prod_{k=3}^{h+1} q_{k i_k} \left(\prod_{k=1}^{h-1} q_{k i_k} \right)^{-1}$$

$$= \eta_h(C) q_{k i_h} q_{k i_{h+1}} q_{k i_1}^{-1} q_{k i_2}^{-1}.$$

Therefore, $\mathrm{ad}(g_k)(\lambda_C(1 - g_C)) = \frac{\lambda_C}{\lambda_{C'}} q_{k i_1}^{-1} q_{k i_2}^{-1} \mathrm{ad}(g_k)(a_C)$. Now, we have that $\mathrm{ad}(g_k)(a_C) = 0 \Leftrightarrow a_C = 0 \Leftrightarrow g_C = 1 \Leftrightarrow g_{C'} = 1$ and in this case $\lambda_C = \lambda_{C'}$ by (9). If $\mathrm{ad}(g_k)(a_C) \neq 0$, we have (11). Analogously, the last relation follows by applying $\mathrm{ad}(H_t)$ to both sides of (10):

$$\mathrm{ad}(H_t)(\sum_{h=1}^{n} a_{i_{h+1}} a_{i_h}) = \sum_{h=1}^{n} \chi^2(t) a_{t \cdot i_{h+1}} a_{t \cdot i_h} = \chi^2(t) a_{t \cdot C}, \quad \text{and}$$

$$\mathrm{ad}(H_t)(\lambda_C(1 - g_C)) = \frac{\lambda_C}{\lambda_{t \cdot C}} \lambda_{t \cdot C}(1 - g_{t \cdot C}) = \frac{\lambda_C}{\lambda_{t \cdot C}} a_{t \cdot C},$$

thus $\lambda_C = \chi^2(t)\lambda_{t \cdot C}$. □

Corollary 3.3. *In the situation above, suppose that (X, q) is $(\mathcal{O}_2^n, -1)$, $(\mathcal{O}_4^4, -1)$ or $(\mathcal{O}_2^n, \chi)$. Let a_i be as in (6). Then $a_C = \lambda_C(1 - g_C)$ in H, for $C \in \mathcal{R}'$ and $\lambda_C \in \Bbbk$ as in Lemma 3.2.*

Proof. For $(X, q) = (\mathcal{O}_2^n, -1)$, see [AG2, Lem. 3.4]. We follow the proof there to check condition (8) in Lemma 3.2 for $(\mathcal{O}_4^4, -1)$. In this case, for every $i, j, k \in X$, g_i acts in the basis X by -1 times a permutation matrix (since the cocycle is $q \equiv -1$) while $g_k g_l$ acts as a permutation matrix in the same basis and thus $g_i \neq g_k g_l$. Consider now $(\mathcal{O}_2^n, \chi)$. Assume that $i, k, l \in X$ are such that $g_i = g_k g_l$. Take $j \in X$, then, $i \rhd j = g_i \cdot j = g_k g_l \cdot j = k \rhd (l \rhd j)$

and therefore $jikl = iklj$, $\forall j \in X$. This would imply $ikl = \mathrm{id}$ in $\mathbb{S}_n$, which is not possible. Thus, the corollary follows by the previous lemma. $\qquad\square$

Remark 3.4. Note that when dealing with the racks of transpositions in $\mathcal{O}_2^n$ or $\mathcal{O}_4^4$, condition (11) determine the existence of at most 3 non-zero scalars λ_C, say $\lambda_1, \lambda_2, \lambda_3$, where the subindex is in correspondence with the number of elements in the class C, since K permutes the classes with the same cardinality.

When $G = \mathbb{S}_4$, the infinitesimal braiding of H, V, is one of the modules $M(\mathcal{O}_2^4, \mathrm{sgn} \otimes \mathrm{sgn})$, $M(\mathcal{O}_2^4, \mathrm{sgn} \otimes \epsilon)$, or $M(\mathcal{O}_4^4, \mathrm{sgn} \otimes \mathrm{sgn})$, by Theorem 2.6. Given V, if (X, q) are the associated rack and cocycle, the relations from Corollary 3.3 hold in H for the realization $(\cdot, \iota, q)$, where $\iota : X \hookrightarrow \mathbb{S}_4$ is the inclusion, $\cdot : \mathbb{S}_4 \times X \to X$ is the action given by conjugation, and q is either the cocycle $q \equiv -1$ or the cocycle $q = \chi$ as in Examples 2.1, as appropriate. In this case, $g_\sigma = H_\sigma$, $\forall \sigma \in X$. Therefore, by the normalization condition (9), we will have $\lambda_1 = 0$ or $\lambda_2 = 0$ in Corollary 3.3 according if we are dealing with $(\mathcal{O}_2^n, q)$ or $(\mathcal{O}_4^4, -1)$, respectively. Furthermore, when the cocycle $q = \chi$, equation (11) forces $\lambda_2 = 0$.

Definition 3.5. A *quadratic lifting datum*, or ql-datum, $\mathcal{Q}$ consists of
- a rack X,
- a 2-cocycle q,
- a finite group G,
- a principal YD-realization $(\cdot, g, (\chi_i)_{i \in X})$ of (X, q) over G such that g satisfies (8),
- a collection $(\lambda_C)_{C \in \mathcal{R}'}$ satisfying (9) and (11).

Definition 3.6. Given a ql-datum $\mathcal{Q}$, we define the algebra $\mathcal{H}(\mathcal{Q})$ by generators $\{a_i, H_t : i \in X, t \in G\}$ and relations:

$$(12) \qquad H_e = 1, \quad H_t H_s = H_{ts}, \ t, s \in G;$$

$$(13) \qquad H_t a_i = \chi_i(t) a_{t \cdot i} H_t, \ t \in G, \ i \in X;$$

$$(14) \qquad \phi_C(\{a_i\}_{i \in X}) = \lambda_C(1 - H_{g_j g_i}), \ C \in \mathcal{R}', \ (i, j) \in C.$$

Here $i_1 = j$, $i_2 = i$, $i_{h+2} = i_{h+1} \triangleright i_h$ and ϕ_C is as in (7). We shall denote by a_C the left-hand side of (14).

Remark 3.7. $\mathcal{H}(\mathcal{Q})$ is a pointed Hopf algebra, setting $\Delta(H_t) = H_t \otimes H_t$, $\Delta(a_\sigma) = g_\sigma \otimes a_\sigma + a_\sigma \otimes 1$, $t \in G$, $\sigma \in X$. Indeed, it is readily checked that the comultiplication is well-defined. In this way, $\mathcal{H}(\mathcal{Q})$ is generated by group-likes and skew-primitives; therefore it is pointed by [M, Lem. 5.5.1].

3.3. **Pointed Hopf algebras over $\mathbb{S}_n$.** Take A a finite-dimensional pointed Hopf algebra over $\mathbb{S}_4$. As said in Remark 3.4, in this case the ql-datums
- $\mathcal{Q}_n^{-1}[t] = (\mathbb{S}_n, \mathcal{O}_2^n, -1, \cdot, \iota, \{0, \Lambda, \Gamma\})$,
- $\mathcal{Q}_n^\chi[\lambda] = (\mathbb{S}_n, \mathcal{O}_2^n, \chi, \cdot, \iota, \{0, 0, \lambda\})$ and
- $\mathcal{D}[t] = (\mathbb{S}_4, \mathcal{O}_4^4, -1, \cdot, \iota, \{\Lambda, 0, \Gamma\})$;

for $\Lambda, \Gamma, \lambda \in \mathbb{k}$, $t = (\Lambda, \Gamma)$, are of particular interest. We will write down the algebras $\mathcal{H}(\mathcal{Q})$ for these datums in detail. In this case, relations (10) for each $C \in \mathcal{R}'$ with the same cardinality are $\mathbb{S}_4$-conjugated. Thus it is enough to consider a single relation for each C with a given number of elements. See Lemma 6.1 for isomorphisms between algebras in the same family.

Definition 3.8. [AG2, Def. 3.7] $\mathcal{H}(\mathcal{Q}_n^{-1}[t])$ is the algebra presented by generators $\{a_i, H_r : i \in \mathcal{O}_2^n, r \in \mathbb{S}_n\}$ and relations:

$$H_e = 1, \quad H_r H_s = H_{rs}, \ r, s \in \mathbb{S}_n;$$
$$H_j a_i = -a_{jij} H_j, \ i, j \in \mathcal{O}_2^n;$$
$$a_{(12)}^2 = 0;$$
$$a_{(12)} a_{(34)} + a_{(34)} a_{(12)} = \Gamma(1 - H_{(12)} H_{(34)});$$
$$a_{(12)} a_{(23)} + a_{(23)} a_{(13)} + a_{(13)} a_{(12)} = \Lambda(1 - H_{(12)} H_{(23)}).$$

Definition 3.9. $\mathcal{H}(\mathcal{Q}_n^\chi[\lambda])$ is the algebra presented by generators $\{a_i, H_r : i \in \mathcal{O}_2^n, r \in \mathbb{S}_n\}$ and relations:

$$H_e = 1, \quad H_r H_s = H_{rs}, \ r, s \in \mathbb{S}_n;$$
$$H_j a_i = \chi_i(j) a_{jij} H_j, \ i, j \in \mathcal{O}_2^n;$$
$$a_{(12)}^2 = 0;$$
$$a_{(12)} a_{(34)} - a_{(34)} a_{(12)} = 0;$$
$$a_{(12)} a_{(23)} - a_{(23)} a_{(13)} - a_{(13)} a_{(12)} = \lambda(1 - H_{(12)} H_{(23)}).$$

Definition 3.10. $\mathcal{H}(\mathcal{D}[t])$ is the algebra generated by elements $\{a_i, H_r : i \in \mathcal{O}_4^4, r \in \mathbb{S}_n\}$ and relations:

$$H_e = 1, \quad H_r H_s = H_{rs}, \ r, s \in \mathbb{S}_n;$$
$$H_j a_i = -a_{jij} H_j, \ i \in \mathcal{O}_4^4, j \in \mathcal{O}_2^4;$$
$$a_{(1234)}^2 = \Gamma(1 - H_{(13)} H_{(24)});$$
$$a_{(1234)} a_{(1432)} + a_{(1432)} a_{(1234)} = 0;$$
$$a_{(1234)} a_{(1243)} + a_{(1243)} a_{(1423)} + a_{(1423)} a_{(1234)} = \Lambda(1 - H_{(12)} H_{(13)}).$$

4. Quadratic algebras

Finite dimensional Nichols algebras in $\mathop{}_{\mathbb{S}_4}^{\mathbb{S}_4}\mathcal{YD}$ are all defined by quadratic relations, see Theorem 2.6. We need to know the dimension of the algebras in Definitions 3.8, 3.9 and 3.10, in order to show that they are liftings of these Nichols algebras. To do this, we first develop a technique on quadratic algebras to obtain a bound on the dimensions. We follow [BG] for our exposition. We fix a vector space W, and we let $T = T(W)$ be the tensor algebra. This algebra presents a natural grading $T = \oplus_{n \geq 0} T^n$, with $T^0 = \mathbb{k}$ and $T^n = W^{\otimes n}$ and an induced increasing filtration F^i, with $F^i = \oplus_{j \leq i} T^j$. Let $R \subset W \otimes W$ be a subspace and denote by $J(R)$ the two-sided ideal in T generated by R. A (homogeneous) quadratic algebra $Q(W, R)$ is the

quotient $T(W)/J(R)$. Analogously, for a subspace $P \subset F^2 = \Bbbk \oplus W \oplus W \otimes W$, we denote by $J(P)$ the two-sided ideal in P generated by R. A (nonhomogeneous) quadratic algebra $Q(W, P)$ is the quotient $T(W)/J(P)$.

Let $A = Q(W, P)$ be a nonhomogeneous quadratic algebra. It inherits an increasing filtration A_n from $T(W)$. Explicitly, $A_n = F^n/J(P) \cap F^n$; let $\operatorname{Gr} A = \oplus_{n \geq 0} A_n/A_{n-1}$ be the associated graded algebra, where $A_{-1} = 0$. Consider the natural projection $\pi : F^2 \to W \otimes W$ with kernel F^1 and set $R = \pi(P) \subset W \otimes W$. Let $B = Q(W, R)$ be the homogeneous quadratic algebra defined by R. Then we have a natural epimorphism $\rho : B \to \operatorname{Gr} A$. Explicitly, let $\rho' : T(W) \to \operatorname{Gr} A$ be induced by the map $W \hookrightarrow A_1 \twoheadrightarrow A_1/A_0$. Let $x \in R \subset T^2$. Then, there exist $x_0 \in \Bbbk$, $x_1 \in W$ such that $x - x_1 - x_0 \in P$ and therefore $x = x_1 + x_0 \in F^2/F^2 \cap J(P) = A_2$, thus $\rho'(x) = 0 \in A_2/A_1$, since $x_1 + x_0 \in A_1$. Hence ρ' induces $\rho : B = T/J(R) \twoheadrightarrow \operatorname{Gr} A$.

Lemma 4.1. *Let $x = x_0 - x_1 \in P \cap F^1$ be such that $x_0 \in \Bbbk$ and $x_1 \in W$. Then $\rho(x_1) = 0$, so that ρ factors through a morphism of graded algebras*

$$\rho_x : \widetilde{B} = B/Bx_1B \to \operatorname{Gr} A,$$

and therefore, $\dim \operatorname{Gr}^n A \leq \dim \widetilde{B}^n$, $\forall\, n \geq 0$.

Proof. Indeed, $x_1 + F^1 \cap J(P) = x_0 \in \Bbbk$ and thus $\rho(x_1) = 0$. The last claim follows since from ρ_x is an epimorphism of graded algebras. $\square$

Proposition 4.2. *Let $\mathcal{Q}$ be a ql-datum and $\mathcal{H}(\mathcal{Q})$ be as in Def. 3.6. Then*

$$\dim \operatorname{Gr}^n \mathcal{H}(\mathcal{Q}) \leq \dim \widehat{\mathfrak{B}}_2^n(X, q)\, |G|.$$

In particular, $\dim \mathcal{H}(\mathcal{Q}) \leq \dim \widehat{\mathfrak{B}}_2(X, q)|G|$.

For $H = \mathcal{H}(\mathcal{Q}_n^{-1}[t])$ or $\mathcal{H}(\mathcal{Q}_n^\chi[\lambda])$ and $n = 4, 5$, the above proposition implies:

- $\dim \mathcal{H}(\mathcal{Q}_4^{-1}[t]), \dim \mathcal{H}(\mathcal{Q}_4^\chi[\lambda]), \dim \mathcal{H}(\mathcal{D}[t]) \leq 24^3 < \infty$,
- $\dim \mathcal{H}(\mathcal{Q}_5^{-1}[t]), \dim \mathcal{H}(\mathcal{Q}_5^\chi[\lambda]) \leq 2^{15}3^5 5^3 < \infty$.

Proof. $\mathcal{H}(\mathcal{Q})$ is the nonhomogeneous quadratic algebra $Q(W, P)$ defined by W and P, for $W = \Bbbk\{a_i, H_t : i \in X, t \in G\}$ and $P \subset \Bbbk \oplus W \oplus W \otimes W$ the subspace generated by

$$\{H_e - 1, H_t \otimes H_s - H_{ts}, H_t \otimes a_i - \chi_i(t)a_{t \cdot i} \otimes H_t,$$
$$a_C - \lambda_C 1 + \lambda_C H_{g_j g_i}, \ C \in \mathcal{R}', t, s \in G, i \in X\}.$$

Let $R = \pi(P)$. Explicitly, $R \subset W \otimes W$ is the subspace generated by

$$\{H_t \otimes H_s, \ H_t \otimes a_i - \chi_i(t)a_{t \cdot i} \otimes H_t, \ a_C, \ C \in \mathcal{R}', t, s \in G, i \in X\}.$$

Let $B = Q(W, R)$ be the homogeneous quadratic algebra defined by W and R. Let Y_G be the algebra linearly spanned by the set $\{1, y_t : t \in G\}$, with unit 1 and multiplication table:

$$y_t y_s = 0, \quad s, t \in G.$$

If $\widehat{\mathfrak{B}}_2 = \widehat{\mathfrak{B}}_2(X, q)$, then $B \cong \widehat{\mathfrak{B}}_2 \sharp Y_G$ where $\sharp$ stands for the commutation relation $(1\sharp y_t)(a_i\sharp 1) = \chi_i(t)a_{t\cdot i}\sharp y_t$, $(1\sharp 1)(a_i\sharp 1) = a_i\sharp 1$, $t \in G$, $i \in X$. Thus we have an epimorphism $\rho : \widehat{\mathfrak{B}}_2 \sharp Y_G \to \operatorname{Gr}\mathcal{H}(\mathcal{Q})$.

Now, note that $P \cap F^1 = \Bbbk\{H_e - 1\}$ and thus, by Lemma 4.1, we have $\rho(H_e) = 0$ and an epimorphism $\rho_e : \widetilde{B} \to \operatorname{Gr}\mathcal{H}(\mathcal{Q})$, with $\widetilde{B} = B/By_eB$. The commutation relation and the fact that the elements $\{y_t\}_{t\in G}$ are pairwise orthogonal, give $By_eB = \widehat{\mathfrak{B}}_2y_e \subset B$. This implies $\dim B^n - \dim(\widehat{\mathfrak{B}}_2^n y_e) \geq \dim\operatorname{Gr}^n\mathcal{H}(\mathcal{Q})$, and, as $\dim B^n = \dim\widehat{\mathfrak{B}}_2^n(|G| + 1)$, we have $\dim\widehat{\mathfrak{B}}_2^n|G| \geq \dim\operatorname{Gr}^n\mathcal{H}(\mathcal{Q})$. □

We now apply Proposition 4.2 to show that all liftings of some quadratic Nichols algebras are of the form $\mathcal{H}(\mathcal{Q})$.

Theorem 4.3. *Let X be a rack, q a 2-cocycle. Let H be a finite-dimensional pointed Hopf algebra, such that its infinitesimal braiding is a principal YD-realization $(\cdot, g, (\chi_i)_{i\in X})$ of (X, q) over $G := G(H)$ that g satisfies (8). Assume that $\mathfrak{B}(X, q)$, $\mathfrak{B}(X^{-1}, \tilde{q})$ are quadratic and finite-dimensional. Then there exists a collection $(\lambda_C)_{C\in\mathcal{R}'}$ satisfying (9) and (11) such that $H \cong \mathcal{H}(\mathcal{Q})$, for the ql-datum $\mathcal{Q} = (X, q, G, (\cdot, g, (\chi_i)_{i\in X}), (\lambda_C)_{C\in\mathcal{R}'})$.*

Proof. By Th. 3.1, $\operatorname{gr}(H) \cong \mathfrak{B}(X, q)\sharp\Bbbk G$. Therefore, we have $\dim H = \dim\mathfrak{B}(X, q)|G|$. On the other hand, by Lemma 3.2, there exists a collection $\{\lambda_C\}_{C\in\mathcal{R}'}$ and an epimorphism $\mathcal{H}(\mathcal{Q}) \twoheadrightarrow H$ for the ql-datum $\mathcal{Q} = (X, q, G, (\cdot, g, (\chi_i)_{i\in X}), \{\lambda_C\}_{C\in\mathcal{R}'})$. Thus $\dim H \leq \dim\mathcal{H}(\mathcal{Q})$. Now, by Prop. 4.2, $\dim\mathcal{H}(\mathcal{Q}) \leq \dim\mathfrak{B}(X, q)|G|$. Therefore, $H \cong \mathcal{H}(\mathcal{Q})$. □

5. Representation theory

We have seen that liftings of some quadratic Nichols algebras are of the form $\mathcal{H}(\mathcal{Q})$ and now we investigate the converse, namely when $\mathcal{H}(\mathcal{Q})$ is actually a lifting.

Let $\mathcal{Q} = (X, q, G, (\cdot, g, (\chi_i)_{i\in X}), \{\lambda_C\}_{C\in\mathcal{R}'})$ be a ql-datum; assume that $\mathfrak{B}(X, q)$, $\mathfrak{B}(X^{-1}, \tilde{q})$ are quadratic and finite-dimensional. Let V be the corresponding YD-module. By definition the group of group-likes $G(\mathcal{H}(\mathcal{Q}))$ is a quotient of the group G in the datum. Therefore, if $\pi : G \twoheadrightarrow G(\mathcal{H}(\mathcal{Q}))$ is the induced epimorphism, any $\mathcal{H}(\mathcal{Q})$-module W becomes a $\Bbbk G$-module by letting $t \cdot w = \pi(t) \cdot w$, $\forall w \in W, t \in G$. Thus any $\mathcal{H}(\mathcal{Q})$-module is a direct sum of irreducible $\Bbbk G$-modules. For $i \in X$, set $J_i = \{k \in X : g_i = g_k\}$.

Proposition 5.1. *If there is a representation $\rho : \mathcal{H}(\mathcal{Q}) \to \operatorname{End} M$ such that*

(i) *$\rho_{|G(\mathcal{H}(\mathcal{Q}))} \circ \pi : G \to \operatorname{End}(M)$, is faithful;*

(ii) *$\rho(a_i) \notin \Bbbk\rho(G(\mathcal{H}(\mathcal{Q})))$, for all $i \in X$; and*

(iii) *the sets $\{\rho(a_j)\}_{j\in J_i}$ are linearly independent for each $i \in X$;*

then $\operatorname{gr}\mathcal{H}(\mathcal{Q}) = \mathfrak{B}(X, q)\sharp\Bbbk G$.

Proof. By (i), $G(\mathcal{H}(\mathcal{Q})) \cong G$. Now, $\operatorname{gr}\mathcal{H}(\mathcal{Q}) \cong \mathfrak{B}(W)\sharp\Bbbk G$, for $\mathfrak{B}(W) \in {}^G_G\mathcal{YD}$ a finite-dimensional Nichols algebra, by Th. 3.1; and $\dim\mathfrak{B}(W) \leq$

$\dim \mathfrak{B}(X, q)$ by Prop. 4.2. The map $\mathfrak{B}(X, q) \ni x_i \mapsto \bar{a}_i \in \mathcal{H}(\mathcal{Q})_1/\mathcal{H}(\mathcal{Q})_0$ defines a morphism $\phi : V \to W \in {}_G^G\mathcal{YD}$. Conditions (ii) and (iii) make ϕ injective. Thus, we have a monomorphism $\mathfrak{B}(X, q) \hookrightarrow \mathfrak{B}(W)$, see [AS2, Cor 3.3]; so $V \cong W$. $\qquad\square$

Remark 5.2. If $i, j \in X$, then $\rho(a_i) \notin \Bbbk \rho(G)$ implies that $\rho(a_{j \triangleright i}) \notin \Bbbk \rho(G)$ by (13). Thus, if X is indecomposable, then (ii) is equivalent to

(ii') $\exists i \in X$ such that $\rho(a_i) \notin \Bbbk \rho(G(\mathcal{H}(\mathcal{Q})))$.

On the other hand, if the realization is faithful (for instance, if X is faithful), then (iii) is automatic.

5.1. **The standard representation of** $\mathbb{S}_n$. Recall that $\mathbb{S}_n$ acts on $\Bbbk^n$ permuting the vectors from the standard basis $\{e_1, \ldots, e_n\}$. The vector $e = e_1 + \ldots + e_n$ is fixed by this action. The orthogonal complement to the space generated by this vector, with the restricted action from $\mathbb{S}_n$, is the *standard representation* of $\mathbb{S}_n$, that is the Specht module $S^{(n-1,1)}$ associated to the partition $n = n - 1 + 1$. The canonical basis for this space is given by $\{v_1, \ldots, v_{n-1}\}$ for $v_i = e_i - e_n$. The action on the corresponding basis is described by:

$$(i, i+1) \cdot v_j = v_{(i,i+1)(j)}, \text{ if } i < n - 1,$$

$$(n-1, n) \cdot v_j = \begin{cases} v_j - v_n & \text{if } j < n, \\ -v_n & \text{if } j = n, \end{cases}$$

This representation is faithful. If $n \geq 4$, there is another $(n-1)$-dimensional representation of $\mathbb{S}_n$, namely the Specht module $S^{(2,1^{n-2})} \cong S^{(n-1,1)} \otimes S^{(1^n)}$ associated to the partition $n = 2 + 1 + \cdots + 1$, where $S^{(1^n)}$, the module associated to the partition $n = 1 + \cdots + 1$, is the sign representation. For this module, we fix the basis $\{w_i\}$, with $w_i = v_i \otimes z$, $i = 1, \ldots, n-1$, if $S^{(1^n)} = \Bbbk\{z\}$.

Let now $\mathcal{Q}$ be one of the ql-data $\mathcal{Q}_n^{-1}[t]$, $\mathcal{Q}_n^\chi[\lambda]$, $n = 4, 5$, or $\mathcal{D}[t]$. We will construct:

- a $\mathcal{H}(\mathcal{Q}_n^{-1}[t])$-module $W(n)$ supported on $S^{(n-1,1)} \oplus S^{(2,1^{n-2})}$;
- a $\mathcal{H}(\mathcal{Q}_n^\chi[\lambda])$-module $U(n)$ supported on $S^{(n-1,1)}$;
- a $\mathcal{H}(\mathcal{D}[t])$-module V supported on $S^{(3,1)} \oplus S^{(2,1^2)}$.

Note that for $\mathcal{Q} = \mathcal{Q}_n^\chi[\lambda]$ a single irreducible $\Bbbk\mathbb{S}_n$ module is enough. This is related to the fact that these algebras only depend on one parameter λ. We have used the computer program $\mathtt{Mathematica}^{\copyright}$ to find the representations. Here we limit ourselves to give a basis $\mathcal{B}$ of these modules and write down the base-exchange matrix between $\mathcal{B}$ and the canonical basis of the $\Bbbk\mathbb{S}_4$-module. We also write down the matrix defining the action of $a_{(12)}$ or $a_{(1234)}$ with respect to the basis $\mathcal{B}$. The action of the other elements a_i, $i \in X$ may be deduced from this one through commutation relations (13). Thus, we will have the following proposition, which provides a Gröbner basis-free proof of the analogous result in [AG2, Th. 3.8], for $\mathcal{H}(\mathcal{Q}_n^{-1}[t])$.

Proposition 5.3. *Let $n = 4, 5$, $t \in \Bbbk^2, \lambda \in \Bbbk$. Let $A = \mathcal{H}(\mathcal{Q}_n^{-1}[t])$ or $\mathcal{H}(\mathcal{Q}_n^\chi[\lambda])$.*

- *If $n = 4$, $G(A) \cong \mathbb{S}_4$, $A \not\cong \Bbbk\mathbb{S}_4$ and $\dim A = 24^3$.*
- *If $n = 5$, $G(A) \cong \mathbb{S}_5$, $A \not\cong \Bbbk\mathbb{S}_5$ and $\dim A = 2^{15}3^5 5^3$.*
- *$G(\mathcal{H}(\mathcal{D}[t])) \cong \mathbb{S}_4$, $\mathcal{H}(\mathcal{D}[t]) \not\cong \Bbbk\mathbb{S}_4$ and $\dim \mathcal{H}(\mathcal{D}[t]) = 24^3$.*

Therefore, the graded Hopf algebra associated to the coradical filtration of $\mathcal{H}(\mathcal{Q}_n^{-1}[t])$ (respectively $\mathcal{H}(\mathcal{Q}_n^\chi[\lambda]), \mathcal{H}(\mathcal{D}[t])$) is isomorphic to $\mathfrak{B}(\mathcal{O}_2^n, -1) \sharp \Bbbk\mathbb{S}_n$ (respectively $\mathfrak{B}(\mathcal{O}_2^n, \chi) \sharp \Bbbk\mathbb{S}_n$, $\mathfrak{B}(\mathcal{O}_4^4, -1) \sharp \Bbbk\mathbb{S}_4$).

Proof. In view of Proposition 5.1, it follows from Th. 2.6 and Rem. 2.7, together with Th. 2.4 and Prop. 2.5, using the representations defined on Propositions 5.4, 5.5 and 5.6, as appropriate. $\qquad\square$

5.2. Construction of $\mathcal{H}(\mathcal{Q})$-modules. Let $t = (\Gamma : \Lambda) \neq (0, 0)$.

Proposition 5.4. *There is a $\mathcal{H}(\mathcal{Q}_n^{-1}[t])$-module $W(n)$ such that*

- *$W(n)$ has a basis $\{\xi_i, \zeta_i\}_{i=1}^{n-1}$ such that*

$$
\begin{aligned}
a_{12}\xi_1 &= 2\zeta_1, & a_{12}\zeta_1 &= 0, \\
a_{12}\xi_2 &= 0, & a_{12}\zeta_2 &= \alpha_n(t)\xi_2, \\
a_{12}\xi_j &= 0, & a_{12}\zeta_j &= \Gamma\xi_j, j \geq 3.
\end{aligned}
$$

- *$W(n) \cong_{\mathbb{S}_n} S^{(n-1,1)} \oplus S^{(2,1^{n-2})}$.*
- *$W(n)$ is irreducible.*

Here, $\alpha_4(t) = \dfrac{4\Lambda - 2\Gamma}{4}$ and $\alpha_5(t) = \dfrac{6\Lambda - 4\Gamma}{5}$.

Proof. Let $\phi_n : W(n) \to S^{(n-1,1)} \oplus S^{(2,1^{n-2})}$ be the linear isomorphism defined, on the basis $\{\xi_i, \zeta_i\}_{i=1}^{n-1}$ and $\{v_i, w_i\}_{i=1}^{n-1}$ by

$$
[\phi_4] = \begin{pmatrix}
1 & 1 & 0 & 0 & 0 & 0 \\
-1 & 1 & 0 & 0 & 0 & 0 \\
0 & -1 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 1 & 0 \\
0 & 0 & 0 & -1 & 1 & 0 \\
0 & 0 & 0 & 0 & -1 & 1
\end{pmatrix}, \text{ and}
$$

$$
[\phi_5] = \begin{pmatrix}
1 & 3 & 0 & 0 & 0 & 0 & 0 & 0 \\
-1 & 3 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -2 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & -2 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 3 & 0 & 0 \\
0 & 0 & 0 & 0 & -1 & 3 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -2 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & -2 & 0 & 1
\end{pmatrix}.
$$

We define a structure of $\Bbbk\mathbb{S}_n$-module on $W(n)$ to make ϕ_n into an isomorphism of $\Bbbk\mathbb{S}_n$-modules. We have thus defined the action of the elements H_t on this module. From the definition of the action for $a_{(12)}$ it is easy to

describe the action of $a_{(13)}, a_{(23)}, a_{(34)}$ and to check that it defines in fact an $\mathcal{H}(\mathcal{Q}_n^{-1}[t])$-module. Since $a_{(12)}$ permutes the $\mathbb{S}_n$-modules $S^{(n-1,1)}$ and $S^{(2,1^{n-2})}$, $W(n)$ is irreducible. $\qquad\square$

Proposition 5.5. *There is an $\mathcal{H}(\mathcal{Q}_n^\chi[\lambda])$-module $U(n)$ such that*

- $U(n) \cong_{\mathbb{S}_n} S^{(n-1,1)}$.
- $U(n)$ *has a basis $\{\xi_i\}_{i=1}^{n-1}$ such that*
$$a_{12}\xi_1 = 2\sqrt{-\lambda}\xi_2, \quad a_{12}\xi_j = 0, \ j \geq 3.$$
- $U(n)$ *is irreducible.*

Proof. Let $\phi_n : U(n) \to S^{(n-1,1)}$ be the linear isomorphism defined, on the basis $\{\xi_i\}_{i=1}^{n-1}$ and $\{v_i\}_{i=1}^{n-1}$ by

$$[\phi_4] = \begin{pmatrix} 1 & 1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad [\phi_5] = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

We define a structure of $\mathbb{k}\mathbb{S}_n$-module on $U(n)$ to make ϕ_n into an isomorphism of $\mathbb{k}\mathbb{S}_n$-modules. We have thus defined the action of the elements H_t on this module. From the definition of the action for $a_{(12)}$ it is easy to describe the action of $a_{(13)}, a_{(23)}, a_{(34)}$ and to check that it defines in fact an $\mathcal{H}(\mathcal{Q}_n^\chi[\lambda])$-module. $U(n)$ is clearly irreducible. $\qquad\square$

Proposition 5.6. *There is a $\mathcal{H}(\mathcal{D}[t])$-module V such that*

- $V \cong_{\mathbb{S}_n} S^{(3,1)} \oplus S^{(2,1^2)}$.
- V *has a basis $\{\xi_i, \zeta_i\}_{i=1}^3$ such that*
$$\begin{aligned} a_{1234}\xi_1 &= 2\zeta_1, & a_{1234}\zeta_1 &= \Gamma\xi_1, \\ a_{1234}\xi_2 &= 2\zeta_2, & a_{1234}\zeta_2 &= \Gamma\xi_2, \\ a_{1234}\xi_3 &= 0, & a_{1234}\zeta_3 &= (\Lambda - \Gamma)\xi_3. \end{aligned}$$
- V *is irreducible.*

Proof. Let $\phi : V \to S^{(3,1)} \oplus S^{(2,1^2)}$ be the linear isomorphism defined, on the basis $\{\xi_i, \zeta_i\}_{i=1}^3$ and $\{v_i, w_i\}_{i=1}^3$ by

$$[\phi] = \begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 & -1 & 1 \end{pmatrix}.$$

We define a structure of $\mathbb{k}\mathbb{S}_4$-module on V to make ϕ into an isomorphism of $\mathbb{k}\mathbb{S}_4$-modules. We have thus defined the action of the elements H_t on this module. From the definition of the action for $a_{(1234)}$ it is easy to describe the action of $a_{(1432)}, a_{(1243)}, a_{(1423)}$ and to check that it defines in fact an $\mathcal{H}(\mathcal{D}[t])$-module. Since $a_{(1234)}$ permutes $S^{(3,1)}$ and $S^{(2,1^2)}$, V is irreducible. $\qquad\square$

6. Pointed Hopf algebras over $\mathbb{S}_4$

In this section we prove the Main Theorem. We first need the following lemma, which establishes the isomorphism classes between the algebras belonging to one of the families defined on Definitions 3.8–3.10.

Lemma 6.1. *Let $t, t' \in \Bbbk^2$. Then $\mathcal{H}(\mathcal{Q}_n^{-1}[t]) \cong \mathcal{H}(\mathcal{Q}_n^{-1}[t'])$ if and only if $t \neq 0$ and $t = t' \in \mathbb{P}_\Bbbk^1$ or if $t = t' = (0,0)$, and the same holds for $\mathcal{H}(\mathcal{D}[t])$. Finally, $\mathcal{H}(\mathcal{Q}_n^X[\lambda]) \cong \mathcal{H}(\mathcal{Q}_n^X[1])$, $\forall \lambda \in \Bbbk^*$ and $\mathcal{H}(\mathcal{Q}_n^X[1]) \not\cong \mathcal{H}(\mathcal{Q}_n^X[0])$.*

Proof. Let $\mathcal{Q} = \mathcal{Q}_n^{-1}[t]$, $\mathcal{Q}_n^X[\lambda]$ or $\mathcal{D}[t]$; $\mathcal{Q}' = \mathcal{Q}_n^{-1}[t']$, $\mathcal{Q}_n^X[\lambda']$ or $\mathcal{D}[t']$. Let $H = \mathcal{H}(\mathcal{Q})$, $H = \mathcal{H}(\mathcal{Q}')$. An isomorphism $\phi : H \to H'$ induces $\phi_{|\mathbb{S}_n} \in \operatorname{Aut}(\mathbb{S}_n)$ for $n = 4, 5$ as appropriate, and thus we have a permutation $\varphi : X \to X$ such that $\phi(H_i) = H_{\varphi(i)}$ and $\phi(a_i)$ is $(H_{\varphi(i)}, 1)$-primitive, for $i \in X$. Furthermore, ϕ restricts to $\phi_1 : H_1 \to H_1'$ between the second factors from the coradical filtration, and thus $\phi(a_i) = \eta a'_{\varphi(i)} + \mu(1 - H_{\varphi(i)})$, $\eta, \mu \in \Bbbk$. When $X = \mathcal{O}_2^n$, relation $a_i^2 = 0$ makes $\mu = 0$, and for $X = \mathcal{O}_4^4$ relation $a_i^2 = \Gamma(1 - H_i^2)$ makes $\mu = 0$ and $\Gamma = \eta^2 \Gamma'$. Therefore, $\phi(a_i) = \eta a_{\varphi(i)}$. Now, taking into account the rest of the quadratic relations, we have that, for $\mathcal{Q}_n^{-1}[t]$ as well as for $\mathcal{D}[t]$, $t = \eta^2 t'$; while for $\mathcal{Q}_n^{-1}[\lambda]$ we get $\lambda = \eta^2 \lambda'$. Thus the result follows. $\square$

Proof of the Main Theorem. In Remark 3.7 we have already seen in that the algebras listed are pointed Hopf algebras. We have computed their dimension in Proposition 5.3. Reciprocally, let H be a finite-dimensional pointed Hopf algebra with $G(H) \cong \mathbb{S}_4$. By Th. 2.6, the infinitesimal braiding of H is isomorphic either to $M(\mathcal{O}_2^4, \operatorname{sgn} \otimes \operatorname{sgn})$, to $M(\mathcal{O}_2^4, \operatorname{sgn} \otimes \epsilon)$ or to $M(\mathcal{O}_4^4, \rho_-)$. Since all these modules are self-dual, H is isomorphic to one, and only one, of the Hopf algebras in the list by Th. 4.3, Rem. 3.4 and Lemma 6.1. $\square$

Remark 6.2. The classification of finite-dimensional pointed Hopf algebras with $G(H) \cong \mathbb{S}_3$, done in [AHS] using [AG2], can be alternatively proved in this way.

We end this paper by describing in the following corollary a sub-class of finite-dimensional pointed Hopf algebras over $\mathbb{S}_5$. It follows in the same fashion as the Main Theorem.

Corollary 6.3. *Let A be a finite-dimensional pointed Hopf algebra with $G(A) \cong \mathbb{S}_5$ and let $M \in {}^{\mathbb{S}_5}_{\mathbb{S}_5}\mathcal{YD}$ be its infinitesimal braiding.*

- *If $M \cong M(\mathcal{O}_2^5, \operatorname{sgn} \otimes \operatorname{sgn})$, then $A \cong \mathcal{H}(\mathcal{Q}_5^{-1}[t])$, for exactly one $t \in \mathbb{P}_\Bbbk^1 \cup \{(0,0)\}$.*
- *If $M \cong M(\mathcal{O}_2^5, \operatorname{sgn} \otimes \epsilon)$, then $A \cong \mathcal{H}(\mathcal{Q}_5^X[\lambda])$, for one $\lambda \in \{0, 1\}$.* $\square$

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