UNIVERSIDAD NACIONAL DE CÓRDOBA

FACULTAD DE MATEMÁTICA, ASTRONOMÍA Y FÍSICA

SERIE "A"

TRABAJOS DE MATEMÁTICA

Nº 99/2010

REPRESENTATIONS OF TENSOR CATEGORIES COMING FROM QUANTUM LINEAR SPACES

MARTÍN MOMBELLI

Editores: Jorge R. Lauret – Jorge G. Adrover

CIUDAD UNIVERSITARIA – 5000 CÓRDOBA
REPÚBLICA ARGENTINA

REPRESENTATIONS OF TENSOR CATEGORIES COMING FROM QUANTUM LINEAR SPACES

MARTÍN MOMBELLI

Abstract. Exact indecomposable module categories over the tensor category of representations of Hopf algebras that are liftings of quantum linear spaces are classified.

1. Introduction

Given a tensor category $\mathcal{C}$, a very natural object to consider is the family of its *representations*, or *module categories*. A module category over a tensor category $\mathcal{C}$ is an Abelian category $\mathcal{M}$ equipped with an exact functor $\mathcal{C} \times \mathcal{M} \to \mathcal{M}$ subject to natural associativity and unity axioms. In some sense the notion of module category over a tensor category is the categorical version of the notion of module over an algebra. In some works the concept of module category over the tensor category of representations of a quantum group is treated as an idea more closely related to the notion of *quantum subgroup* [Oc], [KO].

The language of module categories has proven to be a useful tool in different contexts, for example in the theory of fusion categories, see [ENO1], [ENO2], in the theory of weak Hopf algebras [O1], in describing some properties of semisimple Hopf algebras [N] and in relation with dynamical twists over Hopf algebras [M1] inspired by ideas of V. Ostrik.

Despite the fact that the notion of module category seems very general, it is implicitly present in diverse areas of mathematics and mathematical physics such as subfactor theory [BEK], affine Hecke algebras [BO], extensions of vertex algebras [KO] and conformal field theory, see for example [BFRS], [FS], [CS1], [CS2].

In [EO1] Etingof and Ostrik propose a class of module categories, called *exact*, and as an interesting problem the classification of such module categories over a given finite tensor category. The first classification results were obtained in [KO], [EO2], where the authors classify semisimple module categories over the semisimple part of the category of representations of $U_q(sl(2))$ for a root of unity q, over the category of corepresentations of $SL_q(2)$ in the case q is not a root of unity and over the fusion category

Date: May 13, 2010.
2000 *Mathematics Subject Classification.* 16W30, 18D10, 19D23.
This work was supported by CONICET, Argentina.

obtained as a semisimple subquotient of the same category in the case q is a root of unity. The main result in those papers is the classification in terms of ADE type Dynkin diagrams, which can be interpreted as a quantum analogue of the McKay's correspondence. The classification for the category of corepresentations of $SL_q(2)$ in the case q is a root of unity was obtained later in [O3] where the results were quite similar as in the semisimple case.

In the case when $\mathcal{C} = \mathrm{Rep}(H)$ is the category of representations of a finite-dimensional Hopf algebra H the first results obtained in the classification of module categories were when the Hopf algebra $H = \Bbbk G$ is the group algebra of a finite group G, see [O1], and in the case when $H = \mathcal{D}(G)$ is the Drinfeld's double of a finite-group G, see [O2]. Moreover, in *loc. cit.* the author classify semisimple module categories over any group-theoretical fusion category. In [EO1] module categories were classify in the case where $H = T_q$ is the Taft Hopf algebra, and also for tensor categories of representations of finite supergroups.

In [AM], [M2] the authors give the first steps towards the understanding of exact module categories over the representation categories of an arbitrary finite-dimensional Hopf algebra. In [M2] the author present a technique to classify module categories over $\mathrm{Rep}(H)$ when H is a finite-dimensional pointed Hopf algebra inspired by the classification results obtained in [EO1]. In particular a classification is obtained when $H = \mathbf{r}_q$ is the Radford Hopf algebra and when $H = u_q(\mathfrak{sl}_2)$ is the Frobenius-Lusztig kernel associated to $\mathfrak{sl}_2$.

The main goal of this paper is the application of the technique presented in [M2] to classify exact indecomposable module categories over representation categories of finite-dimensional pointed Hopf algebras constructed from quantum linear spaces.

Namely, let Γ be a finite Abelian group and V a quantum linear space in ${}^{\Bbbk\Gamma}_{\Bbbk\Gamma}\mathcal{YD}$, $U = \mathfrak{B}(V)\#\Bbbk\Gamma$ the Hopf algebra obtained by bosonization of the Nichols algebra $\mathfrak{B}(V)$ and $\Bbbk\Gamma$. Then if $\mathcal{M}$ is an exact indecomposable module category over $\mathrm{Rep}(U)$ there exists

- a subgroup $F \subseteq \Gamma$,
- a normalized 2-cocycle $\psi \in H^2(F, \Bbbk^\times)$,
- a $\Bbbk\Gamma$-subcomodule $W \subseteq V$ invariant under the action of F,
- scalars $\xi = (\xi_i)$, $\alpha = (\alpha_{ij})$ compatible with V, ψ, and F,

such that $\mathcal{M} \simeq {}_{\mathcal{A}(W,F,\psi,\xi,\alpha)}\mathcal{M}$ is the category of left modules over the left U-comodule algebra $\mathcal{A}(W, F, \psi, \xi, \alpha)$ associated to these data. We also show that module categories ${}_{\mathcal{A}(W,F,\psi,\xi,\alpha)}\mathcal{M}$, ${}_{\mathcal{A}(W',F',\psi',\xi',\alpha')}\mathcal{M}$ are equivalent as module categories over $\mathrm{Rep}(U)$ if and only if $(W, F, \psi, \xi, \alpha) = (W', F', \psi', \xi', \alpha')$.

If H is a lifting of U, that is a Hopf algebra such that the associated graded Hopf algebra $\mathrm{gr}\, H$ is isomorphic to U, then H is a cocycle deformation of U, implying that the categories $\mathrm{Rep}(H^*)$ and $\mathrm{Rep}(U^*)$ are tensor equivalent.

Thus exact indecomposable module categories over $\mathrm{Rep}(H)$ are described by the same data as above.

The organization of the paper is as follows. In Section 2 we recall the definitions of quantum linear spaces and the construction of Andruskiewitsch and Schneider of liftings over quantum linear spaces. In Section 3 we recall the definitions of exact module categories and the description of module categories over finite-dimensional Hopf algebras.

In subsection 3.3 we explain the technique developed in [M2] to describe exact indecomposable module categories over $\mathrm{Rep}(H)$ where H is a finite-dimensional pointed Hopf algebra. The main result is stated as Theorem 3.3.

In section 4 we present a family of module categories constructed explicitly over the representation category of a Hopf algebra constructed from bosonization of a quantum linear space and a group algebra. Then in Theorem 4.6 we show that any module category is equivalent to one of this family. Proposition 4.1 is a key result to the proof of the main result of this section. In subsection 4.1 we prove that any two of those module categories are nonequivalent.

Finally, in section 5 we show an explicit correspondence of comodule algebras over cocycle equivalent Hopf algebras. Since any lifting of a quantum linear space is a cocycle deformation to the Hopf algebra constructed from this quantum linear space, this is Proposition 5.2, the results obtained in Section 4 allows to describe also exact module categories over those liftings.

Acknowledgments. The author thanks César Galindo for pointing out some errors in a previous version of this paper and for some enjoyable and interesting conversations. He also thanks the referee for his constructive comments.

1.1. **Preliminaries and notation.** We shall denote by $\Bbbk$ an algebraically closed field of characteristic zero. All vector spaces, algebras and categories will be considered over $\Bbbk$. For any algebra A, ${}_A\mathcal{M}$ will denote the category of finite-dimensional left A-modules.

If Γ is a finite Abelian group and $\psi \in Z^2(\Gamma, \Bbbk^\times)$ is a 2-cocycle, we shall denote by ψ_g the map defined by

$$\psi_g(h) = \psi(h, g)\psi(g, h)^{-1},$$

for any $g, h \in \Gamma$. Hereafter we shall assume that any 2-cocycle ψ is normalized and satisfies $\psi(g^{-1}, g) = 1$ for all $g \in \Gamma$.

If A is an H-comodule algebra via $\lambda : A \to H \otimes_\Bbbk A$, we shall say that a (right) ideal J is H-costable if $\lambda(J) \subseteq H \otimes_\Bbbk J$. We shall say that A is (right) H-simple, if there is no nontrivial (right) ideal H-costable in A.

If H is a finite-dimensional Hopf algebra then $H_0 \subseteq H_1 \subseteq \cdots \subseteq H_m = H$ will denote the coradical filtration. When $H_0 \subseteq H$ is a Hopf subalgebra then the associated graded algebra $\mathrm{gr}\, H$ is a coradically graded Hopf algebra. If

(A, λ) is a left H-comodule algebra, the coradical filtration on H induces a filtration on A, given by $A_n = \lambda^{-1}(H_n \otimes_{\Bbbk} A)$. This filtration is called the *Loewy series* on A.

Let $U = \oplus_{i=0}^m U(i)$ be a coradically graded Hopf algebra. We shall say that a left U-comodule algebra G, with comodule structure given by $\lambda : G \to U \otimes_{\Bbbk} G$, graded as an algebra $G = \oplus_{i=0}^m G(i)$ is a *graded comodule algebra* if for each $0 \leq n \leq m$

$$\lambda(G(n)) \subseteq \bigoplus_{i=0}^{m} U(i) \otimes_{\Bbbk} G(n-i).$$

A graded comodule algebra $G = \oplus_{i=0}^m G(i)$ is *Loewy-graded* if the Loewy series is given by $G_n = \oplus_{i=0}^n G(i)$ for any $0 \leq n \leq m$.

If A is a left H-comodule algebra the graded algebra $\operatorname{gr} A$ obtained from the Loewy series is a Loewy-graded left $\operatorname{gr} H$-comodule algebra. For more details see [M2].

We shall need the following result. Let $U = \oplus_{i=0}^m U(i)$ be a coradically graded Hopf algebra.

Lemma 1.1. *Let (A, λ) be a left U-comodule algebra with an algebra filtration $A^0 \subseteq A^1 \subseteq \cdots \subseteq A^m = A$ such that A_0 is semisimple and*

$$\lambda(A^n) \subseteq \sum_{i=0}^{n} U(i) \otimes_{\Bbbk} A^{n-i}, \tag{1.1}$$

and such that the graded algebra associated to this filtration $\operatorname{gr}' A$ is Loewy-graded. Then the Loewy filtration on A is equal to this given filtration, that is $A^n = A_n$ for all $n = 0, \ldots, m$.

Proof. Straightforward. □

We shall need the following important theorem due to Skryabin. The statement does not appear explicitly in [Sk] but is contained in the proof of [Sk, Theorem 3.5]. Let H be a finite dimensional Hopf algebra.

Theorem 1.2. *If A is a finite dimensional H-simple left H-comodule algebra and $M \in {}^H\mathcal{M}_A$, then there exists $t \in \mathbb{N}$ such that M^t is a free A-module.* □

The following Lemma will be useful to distinguish equivalence classes of module categories. Let $\sigma : H \otimes H \to \Bbbk$ be a Hopf 2-cocycle and K be a left H-comodule algebra.

Lemma 1.3. *There is an equivalence of categories ${}^H\mathcal{M}_K \simeq {}^{H^\sigma}\mathcal{M}_{K_\sigma}$. In particular if $K \subseteq H$ is a left coideal subalgebra, $Q = H/HK^+$ and σ is cocentral then the categories ${}^H\mathcal{M}_{K_\sigma}$, ${}^Q\mathcal{M}$ are equivalent.*

Proof. See [M2, Lemma 2.1]. □

2. Liftings of quantum linear spaces

In this section we recall some results from [AS1]. More precisely, we shall recall the definition of a certain family of finite-dimensional Hopf algebras such that the associated graded Hopf algebras are the bosonization of a quantum linear space and a group algebra of an Abelian group.

2.1. **Quantum linear spaces.** We shall use the notation from [AS1], [AS2]. Let $\theta \in \mathbb{N}$ and Γ be a finite Abelian group. A *datum for a quantum linear space* consists of elements $g_1, \dots, g_\theta \in \Gamma$, $\chi_1, \dots, \chi_\theta \in \widehat{\Gamma}$ such that

$$q_i = \chi_i(g_i) \neq 1, \text{ for all } i, \tag{2.1}$$

$$\chi_i(g_j)\chi_j(g_i) = 1, \text{ for all } i \neq j. \tag{2.2}$$

Let us denote $q_{ij} = \chi_j(g_i)$ and for any i let $N_i > 1$ denote the order of q_i. Denote $V = V(g_1, \dots, g_\theta, \chi_1, \dots, \chi_\theta)$ the Yetter-Drinfeld module over $\Bbbk\Gamma$ generated as a vector space by $x_1, \dots, x_\theta$ with structure given by

$$\delta(x_i) = g_i \otimes x_i, \; h \cdot x_i = \chi_i(h)\, x_i, \tag{2.3}$$

for all $i = 1, \dots, \theta$, $h \in \Gamma$. The associated Nichols algebra $\mathfrak{B}(V)$ is the graded braided Hopf algebra generated by elements $x_1, \dots, x_\theta$ subject to relations

$$x_i^{N_i} = 0, \quad x_i x_j = q_{ij}\, x_j x_i \text{ if } i \neq j. \tag{2.4}$$

This algebra is called the *quantum linear space* associated to V, or to $(g_1, \dots, g_\theta, \chi_1, \dots, \chi_\theta)$ and it is denoted by $\mathfrak{R} = \mathfrak{R}(g_1, \dots, g_\theta, \chi_1, \dots, \chi_\theta)$. The gradation on $\mathfrak{R} = \oplus_n \mathfrak{R}(n)$ is given as follows. If $n \in \mathbb{N}$ then

$$\mathfrak{R}(n) =< \{x_1^{r_1} \dots x_\theta^{r_\theta} : r_1 + \cdots + r_\theta = n\} >_{\Bbbk}$$

Remark 2.1. The space V decomposes as $V = \oplus_{i=1}^{\theta} V_{g_i}$, where $V_{g_i} = \{v \in v : \delta(v) = g_i \otimes v\}$ is the isotypic component of type g_i. Since it can happen that for some $k \neq l$, $g_k = g_l$, then $\dim V_{g_k} \geq 1$. If $\dim V_{g_k} > 2$ for some k then there are at least two $g_i's$ equal to g_k. Using equation (2.2) this implies that $q_k^2 = 1$, hence $N_k = 2$.

Remark 2.2. If $g, h \in \Gamma$ and $v \in V_g$, $w \in V_h$ then there exists a scalar $q_{h,g} \in \Bbbk$ that only depends on g and h such that $wv = q_{h,g}\, vw$. Indeed it is enough to prove that if $x_i, x_k \in V_g$ and $x_j, x_l \in V_h$ then $q_{ij} = q_{kl}$. Since $g = g_i = g_k$ then $q_{ij} = q_{kj}$, and since $h = g_j = g_l$ then $q_{jk} = q_{lk}$. Using that $q_{ij}q_{ji} = 1$ we deduce that $q_{ij} = q_{kl}$.

Let us denote $U = \mathfrak{R}\#\Bbbk\Gamma$ the Hopf algebra obtained by bosonization. The Hopf algebra U is coradically graded with gradation given by $U = \oplus_n U(n)$, $U(n) = \mathfrak{R}(n)\#\Bbbk\Gamma$, see for example [AS1, Lemma 3.4]. Next we will describe a family of Loewy-graded U-comodule algebras.

Definition 2.3. If $W \subseteq V$ is a $\Bbbk\Gamma$-subcomodule, we shall denote by $\mathcal{K}(W)$ the subalgebra of $\mathfrak{R}$ generated by elements $\{w : w \in W\}$. Clearly $\mathcal{K}(W)$ is a left coideal subalgebra of U.

Let $F \subseteq \Gamma$ be a subgroup, $\psi \in Z^2(F, \Bbbk^\times)$ a 2-cocycle and $W \subseteq V$ a $\Bbbk\Gamma$-subcomodule invariant under the action of F. Set $\mathcal{K}(W,\psi,F) = \mathcal{K}(W)\otimes_{\Bbbk}\Bbbk_\psi F$ with product and left U-comodule structure $\lambda : \mathcal{K}(W,\psi,F) \to U\otimes_{\Bbbk}\mathcal{K}(W,\psi,F)$ given as follows. If $g \in G$, $w \in W_g$, $v, v' \in W$ and $f, f' \in F$, then

$$(v\otimes f)(v'\otimes f') = vf \cdot v'\otimes\psi(f,f')\, ff',$$

$$\lambda(w\otimes f) = (w\# f)\otimes 1\otimes f + (1\# gf)\otimes w\otimes f.$$

There is a natural inclusion of vector spaces $\mathcal{K}(W,\psi,F) \hookrightarrow U$. Using this inclusion the coaction λ coincides with the coproduct of U. Clearly $\mathcal{K}(W,1,F)$ is a coideal subalgebra of U. For any $n \in \mathbb{N}$ set $\mathcal{K}(W,\psi,F)(n) = \mathcal{K}(W,\psi,F) \cap U(n)$.

Lemma 2.4. *With the above given gradation the algebra $\mathcal{K}(W,\psi,F)$ is a Loewy-graded U-comodule algebra.*

Proof. Let be $x \in \mathcal{K}(W,\psi,F)$, then $x = \sum_i x_i$, where each $x_i \in U(i)$. Since the coproduct of U coincides with the coaction λ and U is a graded Hopf algebra, then

$$\lambda(x_i) \in \bigoplus_{j=0}^{i} U(j)\otimes_{\Bbbk} U(i-j) \cap U\otimes_{\Bbbk}\mathcal{K}(W,\psi,F).$$

Applying ϵ to the first tensorand we obtain that

$$x_i = (\epsilon\otimes\mathrm{id}\,)\lambda(x_i) \in U(i) \cap \mathcal{K}(W,\psi,F),$$

thus for any i, $x_i \in \mathcal{K}(W,\psi,F)(i)$, hence we conclude that $\mathcal{K}(W,\psi,F) = \oplus_i \mathcal{K}(W,\psi,F)(i)$. It is straightforward to prove that this gradation is an algebra gradation and since U is coradically graded then $\mathcal{K}(W,\psi,F)$ is a Loewy-graded U-comodule algebra. □

2.2. **Liftings of quantum linear spaces.** Given a datum for a quantum linear space $\mathfrak{R} = \mathfrak{R}(g_1,\dots,g_\theta,\chi_1,\dots,\chi_\theta)$ for the group Γ, a *compatible datum* for $\mathfrak{R}$ and Γ is a pair $\mathcal{D} = (\mu,\lambda)$ where $\mu = (\mu_i)$, $\mu_i \in \{0,1\}$ for $i = 1\dots\theta$ and $\lambda = (\lambda_{ij})$ where $\lambda_{ij} \in \Bbbk$ for $1 \le i < j \le \theta$, satisfying

(1) μ_i is arbitrary if $g_i^{N_i} \neq 1$ and $\chi_i^{N_i} = 1$, and $\mu_i = 0$ otherwise.
(2) λ_{ij} is arbitrary if $g_i g_j \neq 1$ and $\chi_i\chi_j = 1$, and 0 otherwise.

The algebra $\mathcal{A}(\Gamma,\mathfrak{R},\mathcal{D})$ is generated by Γ and elements a_i, $i = 1\dots\theta$ subject to relations

$$ga_i = \chi_i(g)\, a_i g, \quad a_i^{N_i} = \mu_i(1 - g_i^{N_i}),\ i = 1\dots\theta, \tag{2.5}$$

$$a_i a_j = \chi_j(g_i)\, a_j a_i + \lambda_{ij}(1 - g_i g_j), 1 \le i < j \le \theta. \tag{2.6}$$

The following result is [AS1, Lemma 5.1, Thm. 5.5].

Theorem 2.5. *The algebra $\mathcal{A}(\Gamma, \mathfrak{R}, \mathcal{D})$ has a Hopf algebra structure with coproduct determined by*

$$\Delta(g) = g \otimes g, \quad \Delta(a_i) = a_i \otimes 1 + g_i \otimes a_i,$$

for any $g \in \Gamma$, $i = 1, \dots, \theta$. It is a pointed Hopf algebra with coradical $\Bbbk\Gamma$ and the associated graded Hopf algebra with respect to the coradical filtration $\operatorname{gr} \mathcal{A}(\Gamma, \mathfrak{R}, \mathcal{D})$ is isomorphic to $\mathfrak{R} \# \Bbbk\Gamma$. □

3. Representations of tensor categories

We shall recall the basic definitions of exact module categories over a tensor category and we shall describe the strategy to classify exact module categories over the tensor category of representations of a finite-dimensional pointed Hopf algebra.

3.1. **Exact module categories.** Given $\mathcal{C} = (\mathcal{C}, \otimes, a, \mathbf{1})$ a tensor category a *module category* over $\mathcal{C}$ is an abelian category $\mathcal{M}$ equipped with an exact bifunctor $\otimes : \mathcal{C} \times \mathcal{M} \to \mathcal{M}$ and natural associativity and unit isomorphisms $m_{X,Y,M} : (X \otimes Y) \otimes M \to X \otimes (Y \otimes M)$, $\ell_M : \mathbf{1} \otimes M \to M$ satisfying natural associativity and unit axioms, see [EO1], [O1]. We shall assume, as in [EO1], that all module categories have only finitely many isomorphism classes of simple objects.

A module category is *indecomposable* if it is not equivalent to a direct sum of two non trivial module categories. A module category $\mathcal{M}$ over a finite tensor category $\mathcal{C}$ is *exact* ([EO1]) if for any projective $P \in \mathcal{C}$ and any $M \in \mathcal{M}$, the object $P \otimes M$ is again projective in $\mathcal{M}$.

3.2. **Exact module categories over Hopf algebras.** We shall give a brief account of the results obtained in [AM] on exact module categories over the category $\operatorname{Rep}(H)$, where H is a finite-dimensional Hopf algebra.

If $\lambda : A \to H \otimes_{\Bbbk} A$ is a left H-comodule algebra the category ${}_A\mathcal{M}$ is a module category over $\operatorname{Rep}(H)$. When A is right H-simple then ${}_A\mathcal{M}$ is an indecomposable exact module category [AM, Prop 1.20]. Moreover any module category is of this form.

Theorem 3.1. [AM, Theorem 3.3] *If $\mathcal{M}$ is an exact idecomposable module category over $\operatorname{Rep}(H)$ then $\mathcal{M} \simeq {}_A\mathcal{M}$ for some right H-simple left comodule algebra A with $A^{\operatorname{co} H} = \Bbbk$.* □

The proof of this result uses in a significant way the results of [EO1], [O1]. The main ingredient here is the H-simplicity of the comodule algebra that helps in the classification results.

Two left H-comodule algebras A and B are *equivariantly Morita equivalent*, and we shall denote it by $A \sim_M B$, if the module categories ${}_A\mathcal{M}$, ${}_B\mathcal{M}$ are equivalent as module categories over $\operatorname{Rep}(H)$.

Proposition 3.2. [AM, Prop. 1.24] *The algebras A and B are Morita equivariant equivalent if and only if there exists $P \in {}^H\mathcal{M}_B$ such that $A \simeq \mathrm{End}_B(P)$ as H-comodule algebras.* □

The left H-comodule structure on $\mathrm{End}_B(P)$ is given by $\lambda : \mathrm{End}_B(P) \to H \otimes_{\Bbbk} \mathrm{End}_B(P)$, $\lambda(T) = T_{(-1)} \otimes T_{(0)}$ where

$$\langle \alpha, T_{(-1)} \rangle \, T_0(p) = \langle \alpha, T(p_{(0)})_{(-1)} \mathcal{S}^{-1}(p_{(-1)}) \rangle \, T(p_{(0)})_{(0)}, \tag{3.1}$$

for any $\alpha \in H^*$, $T \in \mathrm{End}_B(P)$, $p \in P$. It is easy to prove that $\mathrm{End}_B(P)^{\mathrm{co}\, H} = \mathrm{End}^H_B(P)$.

3.3. **Exact module categories over pointed Hopf algebras.** We shall explain the technique developed in [M2] to compute explicitly exact indecomposable module categories over some families of pointed Hopf algebras.

Let H be a finite-dimensional Hopf algebra. Denote by $H_0 \subseteq H_1 \subseteq \cdots \subseteq H_m = H$ the coradical filtration. Let us assume that $H_0 = \Bbbk\Gamma$, where Γ is a finite Abelian group, and that the associated graded Hopf algebra $U = \mathrm{gr}\, H$. We shall further assume that $U = \mathfrak{B}(V)\#\Bbbk\Gamma$, where V is a Yetter-Drinfeld module over $\Bbbk\Gamma$ with coaction given by $\delta : V \to \Bbbk\Gamma \otimes_{\Bbbk} V$, and $\mathfrak{B}(V)$ is the Nichols algebra associated to V.

The technique presented in [M2] to find all right H-simple left H-comodule algebras is the following. Let $\lambda : A \to H \otimes_{\Bbbk} A$ be a right H-simple left H-comodule algebra with trivial coinvariants. Consider the Loewy filtration $A_0 \subseteq \cdots \subseteq A_m = A$ and the associated right U-simple left U-comodule graded algebra $\mathrm{gr}\, A$.

There is an isomorphism $\mathrm{gr}\, A \simeq \mathfrak{B}_A \# A_0$ of U-comodule algebras, where $\mathfrak{B}_A \subseteq \mathrm{gr}\, A$ is a certain U-subcomodule algebra, and A_0 happens to be a right H_0-simple left H_0-comodule algebra.

Since $H_0 = \Bbbk\Gamma$ then $A_0 = \Bbbk_\psi F$ where $F \subseteq \Gamma$ is a subgroup and $\psi \in Z^2(F, \Bbbk^\times)$ is a 2-cocycle. The algebra $\mathfrak{B}_A$ can be seen as a subalgebra in $\subseteq \mathfrak{B}(V)$ and under this identification $\mathfrak{B}_A$ is an homogeneous left U-coideal subalgebra.

In conclusion, to determine all possible right H-simple left H-comodule algebras we have to, first, find all homogeneous left U-coideal subalgebras K inside the Nichols algebra $\mathfrak{B}(V)$, and then all *liftings* of $K\#\Bbbk_\psi F$, that is all left H-comodule algebras A such that $\mathrm{gr}\, A \simeq K\#\Bbbk_\psi F$.

The problem of finding coideal subalgebras can be a very difficult one. Some work has been done in this direction for the small quantum groups $u_q(\mathfrak{sl}_n)$ [KL] and for $U_q^+(\mathfrak{so}_{2n+1})$ [K] also very beautiful results are obtained for right coideal subalgebras inside Nichols algebras [HS] and for the Borel part of a quantized enveloping algebra [HK].

The following result summarizes what we have explained before.

Theorem 3.3. *Under the above assumptions there exists a graded subalgebra* $\mathfrak{B}_A = \oplus_{i=0}^m \mathfrak{B}_A(i) \subseteq \mathfrak{B}(V)$*, a subgroup* $F \subseteq \Gamma$*, and a 2-cocycle* $\psi \in Z^2(F, \Bbbk^\times)$ *such that*

1. $\mathfrak{B}_A(0) = \Bbbk$*,* $\mathfrak{B}_A(i) \subseteq \mathfrak{B}^i(V)$ *for all* $i = 0, \dots, m$*,*
2. $\mathfrak{B}_A(1) \subseteq V$ *is a* $\Bbbk\Gamma$*-subcomodule stable under the action of* F*,*
3. *for any* $n = 1, \dots, m$*,* $\Delta(\mathfrak{B}_A(n)) \subseteq \oplus_{i=0}^n U(i) \otimes_\Bbbk \mathfrak{B}_A(n-i)$*,*
4. $\operatorname{gr} A \simeq \mathfrak{B}_A \# \Bbbk_\psi F$ *as left* U*-comodule algebras.*

Proof. The proof of [M2, Proposition 7.3] extends *mutatis mutandis* to the case when the group F is arbitrary. □

The algebra structure and the left U-comodule structure of $\mathfrak{B}_A \# \Bbbk_\psi F$ is given as follows. If $x, y \in \mathcal{K}$, $f, g \in F$ then

$$(x\#g)(y\#f) = x(g \cdot y)\#\psi(g,f)\, gf,$$
$$\lambda(x\#g) = (x_{(1)}g) \otimes (x_{(2)}\#g),$$

where the action of F on $\mathfrak{B}_A$ is the restriction of the action of Γ on $\mathfrak{B}(V)$ as an object in ${}^\Gamma_\Gamma\mathcal{YD}$. Observe that if $\mathfrak{B}_A = \mathcal{K}(W)$ for some $\Bbbk\Gamma$-subcomodule W of V invariant under the action of F, then $\mathfrak{B}_A \# \Bbbk_\psi F = \mathcal{K}(W, \psi, F)$.

Lemma 3.4 below will be useful to find liftings of comodule algebras over Hopf algebras coming from quantum linear spaces.

Let us further assume that there is a basis $\{x_1, \dots, x_\theta\}$ of V such that there are elements $g_i \in \Gamma$ and characters $\chi_i \in \widehat{\Gamma}$, $i = 1, \dots, \theta$ such that $g \cdot x_i = \chi_i(g)\, x_i$, $\delta(x_i) = g_i \otimes x_i$ for all $i = 1, \dots, \theta$.

Let (G, λ_0) be a Loewy-graded U-comodule algebra with grading $G = \oplus_{i=0}^m G(i)$ such that $G(1) \simeq V \# \Bbbk_\psi F$ under the isomorphism in Theorem 3.3 (4) and there is a subgroup $F \subseteq \Gamma$ such that $G(0) \simeq \Bbbk_\psi F$ as U-comodules, that is there is a basis $\{e_f : f \in F\}$ of $G(0)$ such that

$$e_f e_h = \psi(f,h)\, e_{fh}, \quad \lambda_0(e_f) = f \otimes e_f,$$

for all $f, h \in F$, and there are elements $y_i \in G(1)$ such that $\lambda_0(y_i) = x_i \otimes 1 + g_i \otimes y_i$ for any $i = 1, \dots, \theta$. Also let $\lambda : A \to H \otimes_\Bbbk A$ be a left H-comodule algebra such that $\operatorname{gr} A = G$.

Lemma 3.4. *Under the above assumptions for any* $i = 1, \dots, \theta$ *there are elements* $v_i \in A_1$ *such that the class of* v_i *in* $A_1/A_0 = G(1)$ *is* $\overline{v}_i = y_i$ *and*

$$\lambda(v_i) = x_i \otimes 1 + g_i \otimes v_i, \quad e_f v_i = \chi_i(f)\, v_i e_f, \tag{3.2}$$

for any $i = 1, \dots, \theta$ *and any* $f \in F$*.*

Proof. The existence of elements v_i such that $\lambda(v_i) = x_i \otimes 1 + g_i \otimes v_i$ is [M2, Lemma 5.5]. For any $i = 1, \dots, \theta$ and any $f \in F$ set

$$\mathcal{P}_{i,f} = \{y \in A_1 : \lambda(y) = \mu\, f x_i \otimes e_f + g_i \otimes y, \;\; \mu \in \Bbbk\}.$$

The sets $\mathcal{P}_{i,f}$ are non-zero vector spaces since $e_f v_i \in \mathcal{P}_{i,f}$, thus $\dim \mathcal{P}_{i,f} \geq 1$. It is evident that if $(i,f) \neq (i',f')$ then $\mathcal{P}_{i,f} \cap \mathcal{P}_{i',f'} = \{0\}$. Since $\dim A_1 = \dim G(0) + \dim G(1) =\mid F \mid (1+\theta)$ this forces to $\dim \mathcal{P}_{i,f} = 1$. Hence, since $e_f v_i e_{f^{-1}}, v_i \in \mathcal{P}_{i,1}$ there exists $\nu \in \Bbbk$ such that $e_f v_i e_{f^{-1}} = \nu\ v_i$, but this scalar must be equal to $\chi_i(f)$. □

4. Module categories over quantum linear spaces

In this section we shall apply the technique explained above to describe exact module categories over Hopf algebras coming from quantum linear spaces. Let $\theta \in \mathbb{N}$ and Γ be a finite Abelian group, $(g_1, \dots, g_\theta, \chi_1, \dots, \chi_\theta)$ be a datum of a quantum linear space, $V = V(g_1, \dots, g_\theta, \chi_1, \dots, \chi_\theta)$ the Yetter-Drinfeld module over $\Bbbk\Gamma$ and $\mathfrak{R} = \mathfrak{B}(V)$ its Nichols algebra. Let $U = \mathfrak{R}\#\Bbbk\Gamma$ denote the bosonization. Elements in U will be denoted by $v\#g$ instead of $v \otimes g$ to emphasize the presence of the semidirect product.

To describe all exact indecomposable module categories over $\mathrm{Rep}(U)$ we will describe all possible right U-simple left U-comodule algebras. For this description we shall need first the following crucial result which essentially says that such comodule algebras are generated in degree 1.

Proposition 4.1. *Let $K = \oplus_{i=0}^{m} K(i) \in \mathfrak{R}$ be a graded subalgebra such that*

1. *for all $i = 0, \dots, m$, $K(i) \subseteq \mathfrak{R}(i)$,*
2. *$K(1) = W \subseteq V$ is a $\Bbbk\Gamma$-subcomodule,*
3. *$\Delta(K(n)) \subseteq \oplus_{i=0}^{n} U(i) \otimes_{\Bbbk} K(n-i)$.*

Then K is generated as an algebra by $K(1)$, in another words $K \simeq \mathcal{K}(W)$.

Proof. Let $n \in \mathbb{N}$, $0 < n \leq m$. Since W is a $\Bbbk\Gamma$-subcomodule then $W = \oplus_{i=0}^{\theta} W_{g_i}$. Let $z \in K(n)$ be a nonzero element and let $1 \leq d \leq \theta$ be the number of $x_i's$ appearing in z with non-zero coefficient. We shall prove by induction on $n+d$ that $z \in \mathcal{K}(W)$. If $n + d = 2$ there is nothing to prove because in that case $d = 1$ and $n = 1$, so assume that every time that $y \in K(n)$ is an element with d different variables and $n + d < l$ then $y \in \mathcal{K}(W)$. We shall use the following claim as the main tool for the induction.

Claim 4.1. *If $2 \leq n$ and $z \in K(n)$, $z = \sum_{j=1}^{n-1} w^j y_j$, where $w \in W_h$, for some $h \in \Gamma$ and for any $j = 1, \dots, n-1$ the elements $y_j \in \mathfrak{R}(n-j)$ are such that the $x_i's$ appearing in the decomposition of y_j does not appear in w. Then $y_j \in K(n-j)$ for any $j = 1, \dots, n-1$.*

Proof of Claim. Let $z \in K(n)$ such that $z = \sum_{j=1}^{n-1} w^j y_j$, as above. Let $p : U \to U(1)$ be the linear map defined by: $p(wg) = wg$ for any $g \in \Gamma$ and $p(x) = 0$ if $x \notin < wg : g \in \Gamma >_{\Bbbk}$.

Using (3) we obtain that $(p\otimes\mathrm{id}\,)\Delta(z) \in U(1)\otimes K(n-1)$ and using that $\Delta(w) = w\otimes 1 + h\otimes w$, a simple computation shows that

$$(p\otimes\mathrm{id}\,)\Delta(z) = \sum_{j=1}^{n-1} w\beta_j\otimes w^{j-1}y_j,$$

for some $\beta_j \in \Bbbk\Gamma$. The second equality follows because $p(y_j) = 0$ for any $j = 1,\dots,n-1$. Therefore the element $\sum_{j=1}^{n-1} w^{j-1}y_j \in K(n-1)$. Repeating this process we deduce that $y_{n-1} \in K(n-1)$ and arguing inductively we conclude that each $y_j \in K(n-j)$ for any $j = 1,\dots,n-1$. □

Let $z \in K(n)$ be a nonzero element and let $1 \le d \le \theta$ be the number of $x_i's$ appearing in z with non-zero coefficient. Assume that $n + d = l$. Since $\mathfrak{R}(n)$ is generated by the monomials $\{x_1^{l_1}\dots x_\theta^{l_\theta} : l_1 + \cdots + l_\theta = n\}$, and $K(n) \subseteq \mathfrak{R}(n)$ we can write $z = \sum_{l_1+\cdots+l_\theta=n} \alpha_{l_1,\dots,l_\theta}\, x_1^{l_1}\dots x_\theta^{l_\theta}$ where $\alpha_{l_1,\dots,l_\theta} \in \Bbbk$ and $0 \le l_i \le N_i$. There is no harm to assume that the monomial x_1 appears, that is there exists $l_1,\dots,l_\theta$ with $0 < l_1$ such that $\alpha_{l_1,\dots,l_\theta} \neq 0$, since otherwise we can repeat the argument with x_2 or x_3 and so on.

Under this mild assumption the space W_{g_1} is not zero. Moreover there is an element $0 \neq w \in W_{g_1}$ where $w = \sum_{i=1}^{\theta} a_i\, x_i$ and $a_1 \neq 0$. Indeed, if $\pi : U \to V_{g_1}$ denotes the canonical projection, the quantum binomial formula implies that for any $j = 1,\dots,\theta$

$$\Delta(x_j^{l_j}) = \sum_{k_j=0}^{l_j} \binom{l_j}{k_j}_{q_j} x_j^{l_j-k_j} g_j^{k_j} \otimes x_j^{k_j}, \tag{4.1}$$

where $\binom{l_j}{i_j}_{q_j}$ denotes the quantum Gaussian coefficients. Using (3) we know that $(\mathrm{id}\otimes\pi)\Delta(z) \in H(n-1)\otimes_{\Bbbk} W_{g_1}$ and equals to

$$\sum_{\substack{j=1,\dots,\theta\\ l_1+\cdots+l_\theta=m}} \alpha_{l_1,\dots,l_\theta} \binom{l_j}{1}_{q_j} x_1^{l_1}\dots x_j^{l_j-1} g_j \dots x_\theta^{l_\theta}\otimes x_j.$$

Since there exists $l_1,\dots,l_\theta$ such that $l_1+\cdots+l_\theta = m$ and $0 < l_1$, $\alpha_{l_1,\dots,l_\theta} \neq 0$ then $(\mathrm{id}\otimes\pi)\Delta(z) = \sum h_j\otimes w_j$ where at least one $w_j \neq 0$ written in the basis $\{x_1,\dots,x_\theta\}$ has positive coefficient in x_1.

Up to reordering the variables we can assume that $g_1 = g_2 = \cdots = g_{r_1}$ and if $r_1 < j$ then $g_j \neq g_1$. In this case $\dim V_{g_1} = r_1$. We shall treat separately the following three cases: Case (A) $r_1 = 1$, Case (B) $r_1 = 2$, Case (C) $r_1 > 2$.

Since $W \subseteq V$ is a $\Bbbk\Gamma$-subcomodule, in case (A) $W_{g_1} = \{0\}$ or $W_{g_1} = V_{g_1}$. We have proven that W_{g_1} is not zero, hence $W_{g_1} = V_{g_1}$. Let us write $z = \sum_{i=0}^{N_1} x_1^i y_i$, where $y_i \in \mathfrak{R}(n-i)$, and x_1 does not appear in any factor of y_i, that is, for any $i = 0,\dots,N_1$

$$y_i = \sum \gamma^i_{l_2,\dots,l_\theta}\, x_2^{l_2}\dots x_\theta^{l_\theta},$$

for some $\gamma^i_{l_2,\dots,l_\theta}\in\Bbbk$. The projection $V\to V$ that maps V_{g_1} to zero, extends to an algebra map $q:\mathfrak{R}\to\mathfrak{R}$. Using (3) we get that $(q\otimes q)\Delta(z)=\Delta(y_0)$, thus $y_0\in K(n)$, and therefore $z-y_0=\sum_{i=1}^{N_1}x_1^iy_i\in K(n)$. Using Claim 4.1 we deduce that for any $i=1,\dots,N_1$ the element $y_i\in K(n-i)$ and by inductive hypothesis each $y_i\in\mathcal{K}(W)$ for all $i=0,\dots,N_1$.

Now we proceed to the case (B). In this case V_{g_1} has basis $\{x_1,x_2\}$, and $W_{g_1}=0$, $\dim W_{g_1}=1$ or $W_{g_1}=V_{g_1}$. The first case is impossible. If $W_{g_1}=V_{g_1}$ then $x_1\in W_{g_1}$ and we proceed as in case (A). Let us assume that $\dim W_{g_1}=1$, that is W_{g_1} is generated by an element $a\,x_1+b\,x_2$, for some $a,b\in\Bbbk$, where we can assume that $b\neq 0$ because otherwise $x_1\in W_{g_1}$.

Let $p:K\to V_{g_1}$ be the canonical projection. Follows from (4.1) that $(\mathrm{id}\otimes p)\Delta(z)$ equals to

$$\sum_{l_1+\dots+l_\theta=n}\alpha_{l_1,\dots,l_\theta}\binom{l_1}{1}_{q_1}x_1^{l_1-1}g_1x_2^{l_2}\dots x_\theta^{l_\theta}\otimes x_1+ \\ +\sum_{l_1+\dots+l_\theta=n}\alpha_{l_1,\dots,l_\theta}\binom{l_2}{1}_{q_2}x_1^{l_1}x_2^{l_2-1}g_2x_3^{l_3}\dots x_\theta^{l_\theta}\otimes x_2.$$

Using (3) we obtain that $(\mathrm{id}\otimes p)\Delta(z)\in H(n-1)\otimes W_{g_1}$, hence there exists an element $v\in H(n-1)$ such that $(\mathrm{id}\otimes p)\Delta(z)=av\otimes x_1+bv\otimes x_2$, thus

$$av=\sum_{l_1+\dots+l_\theta=n}\alpha_{l_1,\dots,l_\theta}\binom{l_1}{1}_{q_1}x_1^{l_1-1}g_1x_2^{l_2}\dots x_\theta^{l_\theta},$$

$$bv=\sum_{l_1+\dots+l_\theta=n}\alpha_{l_1,\dots,l_\theta}\binom{l_2}{1}_{q_2}x_1^{l_1}x_2^{l_2-1}g_2x_3^{l_3}\dots x_\theta^{l_\theta}.$$

Comparing coefficients from the above equations and using that $g_1=g_2$, $x_2x_1=q_1\,x_1x_2$, $g_1x_1=q_1\,x_1g_1$, $g_1x_2=q_1^{-1}\,x_2g_1$ we obtain that

$$\alpha_{l_1+1,l_2,\dots,l_\theta}=\frac{a}{b}\,\frac{q_1^{l_2+1}-1}{q_1^{l_1+1}-1}\,\alpha_{l_1,l_2+1,l_3,\dots,l_\theta}. \tag{4.2}$$

For any $m\in\mathbb{N}$ set $\gamma^m_{l_3,\dots,l_\theta}=\alpha_{0,m,l_3,\dots,l_\theta}$, then if $l_1+l_2=m$ we deduce from equation (4.2) that

$$\alpha_{l_1,l_2,\dots,l_\theta}=\binom{m}{l_1}_{q_1}a^{l_1}b^{m-l_1}\frac{\gamma^m_{l_3,\dots,l_\theta}}{b^m}. \tag{4.3}$$

Then we can write the element z as

$$\sum_{m\geq 0}\sum_{\substack{l_3+\dots+l_\theta=n-m\\ l_1+l_2=m}}\alpha_{l_1,\dots,l_\theta}\,x_1^{l_1}\dots x_\theta^{l_\theta}= \\ =\sum_{m\geq 1}\sum_{\substack{l_1+l_2=m\\ l_3+\dots+l_\theta=n-m}}\alpha_{l_1,\dots,l_\theta}\,x_1^{l_1}\dots x_\theta^{l_\theta}+\sum_{l_3+\dots+l_\theta=n}\alpha_{0,0,l_3,\dots,l_\theta}\,x_3^{l_3}\dots x_\theta^{l_\theta}.$$

Using the same argument as before and the inductive hypothesis we deduce that the element $y_0 = \sum_{l_3+\dots+l_\theta=n} \alpha_{0,0,l_3,\dots,l_\theta}\, x_3^{l_3} \dots x_\theta^{l_\theta} \in \mathcal{K}(W)$ and $z - y_0 \in K(n)$. From (4.3) we conclude that

$$z - y_0 = \sum_{m\geq 1} \sum_{l_3+\dots+l_\theta=n-m} \frac{\gamma^m_{l_3,\dots,l_\theta}}{b^m} (ax_1 + bx_2)^m x_3^{l_3} \dots x_\theta^{l_\theta},$$

hence by Claim (4.1) $z - y_0 \in \mathcal{K}(W)$ thus $z \in \mathcal{K}(W)$. Case (C) can be treated in a similar way as case (B). □

Remark 4.2. The above result uses in an essential way the structure of $\mathfrak{R}$ and it is no longer true for arbitrary Nichols algebras. It is worth to mention that this is one of the main difficulties to classify module categories over, for example, $\mathrm{Rep}(u_q(\mathfrak{sl}_3))$, since there are homogeneous coideal subalgebras that are not generated in degree 1.

Let us define now a family of right U-simple left U-comodule algebras. Let $F \subseteq \Gamma$ be a subgroup, $\psi \in Z^2(F, \Bbbk^\times)$ a 2-cocycle and $\xi = (\xi_i)_{i=1\dots\theta}$, $\alpha = (\alpha_{ij})_{1\leq i<j\leq\theta}$ be two families of elements in $\Bbbk$ satisfying

$$\xi_i = 0 \text{ if } g_i^{N_i} \notin F \text{ or } \chi_i^{N_i}(f) \neq \psi_{g_i^{N_i}}(f), \tag{4.4}$$

$$\alpha_{ij} = 0 \text{ if } g_i g_j \notin F \text{ or } \chi_i\chi_j(f) \neq \psi_{g_i g_j}(f), \tag{4.5}$$

for all $f \in F$. In this case we shall say that the pair (ξ, α) is *compatible comodule algebra datum* with respect to the quantum linear space $\mathfrak{R}$, the 2-cocycle ψ and the group F.

Definition 4.3. The algebra $\mathcal{A}(V, F, \psi, \xi, \alpha)$ is the algebra generated by elements in $\{v_i : i = 1 \dots \theta\}$, $\{e_f : f \in F\}$ subject to relations

$$e_f e_g = \psi(f, g)\, e_{fg}, \quad e_f v_i = \chi_i(f)\, v_i e_f, \tag{4.6}$$

$$v_i v_j - q_{ij}\, v_j v_i = \begin{cases} \alpha_{ij}\, e_{g_i g_j} & \text{if } g_i g_j \in F \\ 0 & \text{otherwise,} \end{cases} \tag{4.7}$$

$$v_i^{N_i} = \begin{cases} \xi_i\, e_{g_i^{N_i}} & \text{if } g_i^{N_i} \in F \\ 0 & \text{otherwise,} \end{cases} \tag{4.8}$$

for any $1 \leq i < j \leq \theta$. Observe that we are abusing of the notation since we are changing the name of the variables x_i by v_i to emphasize that the elements no longer belong to U. If $W \subseteq V$ is a $\Bbbk\Gamma$-subcomodule invariant under the action of F, we define $\mathcal{A}(W, F, \psi, \xi, \alpha)$ as the subalgebra of $\mathcal{A}(V, F, \psi, \xi, \alpha)$ generated by W and $\{e_f : f \in F\}$.

The algebra $\mathcal{A}(V, F, \psi, \xi, \alpha)$ is a left U-comodule algebra with structure map $\lambda : \mathcal{A}(V, \psi, \xi, \alpha) \to U \otimes_\Bbbk \mathcal{A}(V, F, \psi, \xi, \alpha)$ given by:

$$\lambda(v_i) = x_i \otimes 1 + g_i \otimes v_i, \quad \lambda(e_f) = f \otimes e_f,$$

It is clear that the map λ is well defined and is an algebra morphism and that the subalgebra $\mathcal{A}(W,F,\psi,\xi,\alpha)$ is a U-subcomodule.

Remark 4.4.
1. The algebra $\mathcal{A}(W,F,\psi,\xi,\alpha)$ does not depend on the class of the 2-cocycle ψ.
2. If $W=0$ then $\mathcal{A}(W,F,\psi,\xi,\alpha)=\Bbbk_\psi F$.

Proposition 4.5. *Under the above assumptions the following assertions hold.*

(1) *For any 2-cocycle ψ of Γ and any compatible comodule algebra datum (ξ,α) the algebra $\mathcal{A}(V,\Gamma,\psi,\xi,\alpha)$ is a Hopf-Galois extension over the field $\Bbbk$.*

(2) *The Loewy filtration on $\mathcal{A}=\mathcal{A}(V,F,\psi,\xi,\alpha)$ is given as follows*

$$\mathcal{A}_n = < \{e_f v_1^{r_1}\dots v_\theta^{r_\theta} : r_1+\dots+r_\theta=m : m\le n, f\in F\} >_{\Bbbk} . \tag{4.9}$$

(3) *The graded algebra $\operatorname{gr}\mathcal{A}(W,F,\psi,\xi,\alpha)$ is isomorphic to $\mathcal{K}(W,\psi,F)$.*

Proof. The proof of (1) is standard. One must show that the canonical map $\beta:\mathcal{A}(V,G,\psi,\xi,\alpha)\otimes_{\Bbbk}\mathcal{A}(V,G,\psi,\xi,\alpha)\to U\otimes_{\Bbbk}\mathcal{A}(V,G,\psi,\xi,\alpha)$, $\beta(a\otimes b)=a_{(-1)}\otimes a_{(0)}b$ is bijective. For this it is enough to show that the elements $g\otimes 1$ and $x_i\otimes 1$ are in the image of β for all $g\in G$, $i=1,\dots,\theta$, and this follows because $\beta(e_g\otimes e_{g^{-1}})=g\otimes 1$, $\beta(v_i\otimes 1-e_{h_i}\otimes e_{h_i^{-1}}v_i)=x_i\otimes 1$.

The filtration on $\mathcal{A}=\mathcal{A}(V,F,\psi,\xi,\alpha)$ defined by (4.9) satisfies the hypothesis in Lemma 1.1, hence it coincides with the Loewy filtration. This proves (2).

The algebra $\operatorname{gr}\mathcal{A}(W,F,\psi,\xi,\alpha)$ is a Loewy-graded U-comodule algebra satisfying

$$\operatorname{gr}\mathcal{A}(W,F,\psi,\xi,\alpha)(0)=\Bbbk_\psi F,\quad \operatorname{gr}\mathcal{A}(W,F,\psi,\xi,\alpha)(1)=W\otimes_{\Bbbk}\Bbbk F.$$

Thus (3) follows from Theorem 3.3 and Proposition 4.1. □

Now we can state the main result of this section.

Theorem 4.6. *Let $\theta\in\mathbb{N}$, Γ be a finite Abelian group, $g_1,\dots,g_\theta\in\Gamma$, $\chi_1,\dots,\chi_\theta\in\widehat{\Gamma}$ be a datum for a quantum linear space, with associated Yetter-Drinfeld module over $\Bbbk\Gamma$ $V=V(g_1,\dots,g_\theta,\chi_1,\dots,\chi_\theta)$ and $U=\mathfrak{B}(V)\#\Bbbk\Gamma$.*

If $\mathcal{M}$ is an exact indecomposable module category over $\operatorname{Rep}(U)$ then there exists a subgroup $F\subseteq\Gamma$, a 2-cocycle $\psi\in Z^2(F,\Bbbk^\times)$, a compatible datum (ξ,α) and $W\subseteq V$ a subcomodule invariant under the action of F such that $\mathcal{M}\simeq {}_{\mathcal{A}(W,F,\psi,\xi,\alpha)}\mathcal{M}$ as module categories.

Proof. By Theorem 3.1 there exists a right U-simple left U-comodule algebra $\lambda:\mathcal{A}\to H\otimes_{\Bbbk}\mathcal{A}$ with trivial coinvariants such that $\mathcal{M}\simeq {}_{\mathcal{A}}\mathcal{M}$ as module categories over $\operatorname{Rep}(U)$. Since $U_0=\Bbbk\Gamma$, and $\mathcal{A}_0$ is a right U_0-simple left U_0-comodule algebra [M2, Proposition 4.4] then $\mathcal{A}_0=\Bbbk_\psi F$ for some subgroup $F\subseteq G$ and a 2-cocycle $\psi\in Z^2(F,\Bbbk^\times)$. Thus we may assume that $\mathcal{A}\neq\mathcal{A}_0$. By Theorem 3.3 there exists an homogeneous coideal subalgebra $\mathfrak{B}_A\subseteq\mathfrak{R}$ such that $\operatorname{gr}\mathcal{A}\simeq\mathfrak{B}_A\#\Bbbk_\psi F$. Proposition 4.1 implies that $\mathfrak{B}_A=\mathcal{K}(W)$

for some $\Bbbk\Gamma$-subcomodule $W \subseteq V$ invariant under the action of F, thus $\operatorname{gr}\mathcal{A} \simeq \mathcal{K}(W, \psi, F)$. Since $\mathcal{A} \neq \mathcal{A}_0$ the space W is not zero.

Let us assume first that $\operatorname{gr}\mathcal{A} \simeq \mathcal{K}(V, \psi, F)$. By Lemma 3.4, there are elements $\{v_i : i = 1 \dots \theta\}$ in $\mathcal{A}$ such that for all $f \in F$

$$\lambda(v_i) = x_i \otimes 1 + g_i \otimes v_i, \quad e_f v_i = \chi_i(f)\, v_i e_f.$$

Since $\operatorname{gr}\mathcal{A}$ is generated as an algebra by V and $\Bbbk_\psi F$ then $\mathcal{A}$ is generated as an algebra by the elements $\{v_i : i = 1 \dots \theta\}$ and $\Bbbk_\psi F$. Since $x_i \otimes 1$ and $g_i \otimes v_i$ q_i-commute then the quantum binomial formula implies that

$$\lambda(v_i^{N_i}) = g_i^{N_i} \otimes v_i^{N_i},$$

thus $v_i^{N_i} \in \mathcal{A}_0$ and there exists $\xi_i \in \Bbbk$ such that $v_i^{N_i} = \xi_i\, e_{g_i^{N_i}}$ if $g_i^{N_i} \in F$, otherwise $v_i^{N_i} = 0$. If $i \neq j$ then $\lambda(v_i v_j - q_{ij}\ v_j v_i) = g_i g_j \otimes (v_i v_j - q_{ij}\ v_j v_i)$. Hence $v_i v_j - q_{ij}\ v_j v_i \in \mathcal{A}_0$, and therefore $v_i v_j - q_{ij}\ v_j v_i = \sum_{f \in F} \zeta_f\, e_f$. Thus we conclude that if $g_i g_j \in F$ then there exists $\alpha_{ij} \in \Bbbk$ such $v_i v_j - q_{ij}\ v_j v_i = \alpha_{ij}\ e_{g_i g_j}$, and if $g_i g_j \notin F$ then $v_i v_j - q_{ij}\ v_j v_i = 0$. It is clear that (ξ, α) is compatible with the quantum linear space and ψ, therefore there is a projection $\mathcal{A}(V, G, \psi, \xi, \alpha) \twoheadrightarrow \mathcal{A}$ of U-comodule algebras, but both algebras have the same dimension, since by Proposition 4.5 (3) $\operatorname{gr}\mathcal{A} \simeq \operatorname{gr}\mathcal{A}(V, G, \psi, \xi, \alpha)$, thus they are isomorphic.

If $\operatorname{gr}\mathcal{A} \simeq \mathcal{K}(W, \psi, F)$ for some $\Bbbk\Gamma$-subcomodule $W \subseteq V$ invariant under the action of F we proceed as follows. We shall define an U-comodule algebra $\mathcal{D}$ such that $\operatorname{gr}\mathcal{D} = \mathcal{K}(V, \psi, F)$ such that $\mathcal{A}$ is a U-subcomodule algebra of $\mathcal{D}$, and this will finish the proof of the theorem since $\mathcal{D} \simeq \mathcal{A}(V, F, \psi, \xi, \alpha)$ and by definition $\mathcal{A}(W, F, \psi, \xi, \alpha)$ is the subcomodule algebra of $\mathcal{A}(V, F, \psi, \xi, \alpha)$ generated by W and $\Bbbk F$.

Using again Lemma 3.4 there is an injective map $W \hookrightarrow \mathcal{A}_1$ such that for any $h \in \Gamma$, $w \in W_h$

$$\lambda(w) = w\#1 \otimes 1 + 1\#h \otimes w.$$

Observe that here we are abusing of the notation since the element w also denotes the element in $\mathcal{A}_1$ under the above inclusion. Using this identification $\mathcal{A}$ is generated as an algebra by W and $\Bbbk F$.

Let $W' \subseteq V$ be a $\Bbbk\Gamma$-subcomodule and an F-submodule such that $V = W' \oplus W$. Set $\mathcal{D} = \mathcal{K}(W') \otimes_\Bbbk \mathcal{A}$, with algebra structure determined by

$$(1 \otimes a)(1 \otimes b) = 1 \otimes ab, \ \ (x \otimes 1)(y \otimes 1) = xy \otimes 1, \ \ (x \otimes 1)(1 \otimes a) = x \otimes a,$$

$$(1 \otimes e_f)(v \otimes 1) = f \cdot v \otimes e_f, \ \ (1 \otimes w)(v \otimes 1) = q_{h,g}(v \otimes w),$$

for any $a, b \in \mathcal{A}$, $x, y \in \mathcal{K}(W')$, $f \in F$, $h, g \in \Gamma$, $w \in W'_h$, $v \in W_g$. Here the scalar $q_{h,g} \in \Bbbk$ is determined by the equation in U: $wv = q_{h,g}\, vw$, see Remark 2.2. Let us define $\widetilde{\lambda} : \mathcal{D} \to U \otimes_\Bbbk \mathcal{D}$ the coaction by:

$$\widetilde{\lambda}(x \otimes a) = x_{(-1)} a_{(-1)} \otimes x_{(0)} \otimes a_{(0)},$$

for all $a \in \mathcal{A}$, $x \in \mathcal{K}(W')$. By a direct computation one can see that $\widetilde{\lambda}$ is an algebra map. It is not difficult to see that $\mathcal{D}_0 = \Bbbk_\psi F$ and that $\operatorname{gr}\mathcal{D}(1) = V \otimes_\Bbbk \Bbbk F$, thus $\operatorname{gr}\mathcal{D} \simeq \mathcal{K}(V, \psi, F)$. □

Example 4.7. This example is a particular case of a classification result obtained in [EO1, §4.2] for the representation category of finite supergroups.

Let $\theta \in \mathbb{N}$, Γ be an Abelian group and $u \in \Gamma$ be an element of order 2. Set $g_1 = \cdots = g_\theta = u$ and for any $i = 1, \ldots, \theta$ let $\chi_i \in \widehat{\Gamma}$ be characters such that $\chi_i(u) = -1$. If $V = V(g_1, \ldots, g_\theta, \chi_1, \ldots, \chi_\theta)$ then the associated quantum linear space is the exterior algebra $\wedge V$. In this case $U = \wedge V \# \Bbbk\Gamma$.

Let $F \subseteq \Gamma$ be a subgroup and $\psi \in Z^2(F, \Bbbk^\times)$ a 2-cocycle. A compatible comodule algebra datum in this case is a pair (ξ, α) satisfying

$$\xi_i = 0 \text{ if } \chi_i^2(f) \neq 1, \quad \alpha_{ij} = 0 \text{ if } \chi_i\chi_j(f) \neq 1, \text{ for all } f \in F. \tag{4.10}$$

If $W \subseteq V$ is a subspace stable under the action of F the algebra $\mathcal{A}(W, F, \psi, \xi, \alpha)$ is isomorphic to the semidirect product $Cl(W, \beta)\#\Bbbk_\psi F$ where $Cl(W, \beta)$ is the Clifford algebra associated to the symmetric bilinear form $\beta : V \times V \to \Bbbk$ invariant under F defined by

$$\beta(v_i, v_j) = \begin{cases} \frac{\alpha_{ij}}{2} & \text{if } i \neq j \\ \xi_i & \text{if } i = j. \end{cases} \tag{4.11}$$

Reciprocally, if $W \subseteq V$ is a F-submodule, any symmetric bilinear form $\beta : W \times W \to \Bbbk$ invariant under F defines a comodule algebra datum (ξ, α). Indeed, take $U \subseteq V$ a F-submodule such that $V = W \oplus U$ and define $\widehat{\beta} : V \times V \to \Bbbk$ such that $\widehat{\beta}(w_1, w_2) = \beta(w_1, w_2)$ if $w_1, w_2 \in W$ and $\widehat{\beta}(v, u) = 0$ for any $v \in V$, $u \in U$. Follows that the pair (ξ, α) defined by equation (4.11) using $\widehat{\beta}$ gives a compatible comodule algebra datum.

Remark 4.8. It would be interesting to give an explicit description of data $(W, F, \psi, \xi, \alpha)$ such that the algebra $\mathcal{A}(W, F, \psi, \xi, \alpha)$ is simple. This would give a description of twists over U, *i.e.* fiber functors for $\operatorname{Rep}(U)$.

4.1. Equivariant equivalence classes of algebras $\mathcal{A}(W, F, \psi, \xi, \alpha)$.

In this section we shall distinguish equivalence classes of module categories of Theorem 4.6, that is equivariant Morita equivalence classes of the algebras $\mathcal{A}(W, F, \psi, \xi, \alpha)$.

Let U be the Hopf algebra as in the previous section. Let $W, W' \subseteq V$ be subcomodules, $F, F' \subseteq \Gamma$ be two subgroups, $\psi \in Z^2(F, \Bbbk^\times)$, $\psi' \in Z^2(F', \Bbbk^\times)$ 2-cocycles and (ξ, α), (ξ', α') compatible comodule algebra datum with respect to the quantum linear space $\mathfrak{R}$, the 2-cocycles ψ, ψ' and the groups F, F' respectively.

Theorem 4.9. *The associated right simple left U-comodule algebras to these data $\mathcal{A}(W, F, \psi, \xi, \alpha)$, $\mathcal{A}(W', F', \psi', \xi', \alpha')$ are equivariantly Morita equivalent if and only if $(W, F, \psi, \xi, \alpha) = (W', F', \psi', \xi', \alpha')$.*

We shall need first the following result.

Lemma 4.10. *The algebras* $\mathcal{A}(W,F,\psi,\xi,\alpha)$, $\mathcal{A}(W',F',\psi',\xi',\alpha')$ *are isomorphic as left U-comodule algebras if and only if* $W=W'$, $F=F'$, $\psi=\psi'$, $\xi=\xi'$ *and* $\alpha=\alpha'$.

Proof. Let $\Phi:\mathcal{A}(W,F,\psi,\xi,\alpha)\to\mathcal{A}(W',F',\psi',\xi',\alpha')$ be an isomorphism of U-comodule algebras. The map Φ induces an isomorphism between $\Bbbk_\psi F$ and $\Bbbk_{\psi'}F'$ that must be the identity, thus F is equal to F' and $\psi=\psi'$ in $H^2(F,\Bbbk^\times)$.

Let $\widetilde{W}\in {}^{\Bbbk\Gamma}_{\Bbbk F}\mathcal{M}$ be a complement of W in V, that is $V=W\oplus\widetilde{W}$. Let us define a map $\widetilde{\Phi}:\mathcal{A}(V,F,\psi,\xi,\alpha)\to\mathcal{A}(V,F,\psi,\xi',\alpha')$ such that $\widetilde{\Phi}(a)=\Phi(a)$ whenever $a\in\mathcal{A}(W,F,\psi,\xi,\alpha)$.

It is enough to define $\widetilde{\Phi}$ on V and $\{e_f: f\in F\}$ since $\mathcal{A}(V,F,\psi,\xi,\alpha)$ is generated as an algebra by these elements. Set

$$\widetilde{\Phi}(w)=\Phi(w),\quad \widetilde{\Phi}(u)=u,\quad \widetilde{\Phi}(e_f)=\Phi(e_f),$$

for any $w\in W$, $u\in\widetilde{W}$, $f\in F$. It is straightforward to prove that $\widetilde{\Phi}$ is an U-comodule algebra map, and necessarily $\widetilde{\Phi}$ is the identity map, whence Φ is the identity and the Lemma follows. □

Proof of Theorem 4.9. Let us assume that $\mathcal{A}=\mathcal{A}(W,F,\psi,\xi,\alpha)$ and $\mathcal{A}'=\mathcal{A}(W',F',\psi',\xi',\alpha')$ are equivariantly Morita equivalent. Thus there exists an equivariant Morita context (P,Q,f,g), see [AM]. That is $P\in {}^{U}_{\mathcal{A}'}\mathcal{M}_{\mathcal{A}}$, $Q\in {}^{U}_{\mathcal{A}}\mathcal{M}_{\mathcal{A}'}$ and $f:P\otimes_{\mathcal{A}}Q\to\mathcal{A}'$, $g:Q\otimes_{\mathcal{A}'}P\to\mathcal{A}$ are bimodule isomorphisms and $\mathcal{A}'\simeq\operatorname{End}_{\mathcal{A}}(P)$ as comodule algebras, where the comodule structure on $\operatorname{End}_{\mathcal{A}}(P)$ is given in (3.1).

Let us denote by $\delta:P\to U\otimes_\Bbbk P$ the coaction. Consider the filtration on P given by $P_i=\delta^{-1}(U_i\otimes_\Bbbk P)$ for any $i=0\dots m$. This filtration is compatible with the Loewy filtration on $\mathcal{A}$, that is $P_i\cdot\mathcal{A}_j\subseteq P_{i+j}$ for any i,j and for any $n=0\dots m$, $\delta(P_n)\subseteq\sum_{i=0}^n U_i\otimes_\Bbbk P_{n-i}$.

The space $P_0\cdot\mathcal{A}$ is a subobject of P in the category ${}^U\mathcal{M}_{\mathcal{A}}$, thus we can consider the quotient $\overline{P}=P/P_0\cdot\mathcal{A}$. Let us denote by $\overline{\delta}$ the coaction of $\overline{P}$. Clearly $\overline{P}_0=0$, therefore $\overline{P}=0$. Indeed, if $\overline{P}\neq 0$ there exists an element $q\in\overline{P}_n$ such that $q\notin\overline{P}_{n-1}$, but $\overline{\delta}(q)\subseteq\sum_{i=0}^n U_i\otimes_\Bbbk\overline{P}_{n-i}$. Since $\overline{P}_0=0$ then $\overline{\delta}(q)\in U_{n-1}\otimes_\Bbbk\overline{P}$ which contradicts the assumption. Hence $P=P_0\cdot\mathcal{A}$.

Since $P_0\in {}^{\Bbbk\Gamma}\mathcal{M}_{\Bbbk_\psi F}$ then by Lemma 1.3 there exists an object $N\in {}^C\mathcal{M}$, $C=\Bbbk\Gamma/\Bbbk\Gamma(\Bbbk F)^+$ such that $P_0\simeq N\otimes_\Bbbk\Bbbk_\psi F$ as objects in ${}^{\Bbbk\Gamma}\mathcal{M}_{\Bbbk_\psi F}$. The right $\Bbbk_\psi F$-module structure on $N\otimes_\Bbbk\Bbbk_\psi F$ is the regular action on the second tensorand and the left $\Bbbk\Gamma$-comodule structure is given by $\delta:N\otimes_\Bbbk\Bbbk_\psi F\to\Bbbk\Gamma\otimes_\Bbbk N\otimes_\Bbbk\Bbbk_\psi F$, $\delta(v\otimes e_f)=v_{(-1)}f\otimes v_{(0)}\otimes e_f$, $v\in N$, $f\in F$. Here we are identifying the quotient C with $\Bbbk\Gamma/F$. Observe that $P=(N\otimes 1)\cdot\mathcal{A}$. It is not difficult to prove that the action $(N\otimes 1)\otimes\mathcal{A}\to P$ is injective, thus $\dim P=\dim N\dim\mathcal{A}$. In a similar way one may prove that $\dim Q=s\dim\mathcal{A}'$ for some $s\in\mathbb{N}$.

If $\dim N = 1$ then there exists an element $g \in \Gamma$ and a non-zero element v such that $\delta(v) = g\otimes v$ and $P \simeq v \cdot \mathcal{A}$, where the left U-comodule structure is given by $\delta(v \cdot a) = ga_{(-1)}\otimes v \cdot a_{(0)}$, for all $a \in \mathcal{A}$. In this case the map $\varphi : g\mathcal{A}g^{-1} \to \mathrm{End}_{\mathcal{A}}(P)$ given by

$$\varphi(gag^{-1})(v \cdot b) = v \cdot ab,$$

for all $a, b \in \mathcal{A}$ is an isomorphism of U-comodule algebras. Hence $\mathcal{A}' \simeq \mathcal{A}$. Thus, the proof of the Theorem follows from Lemma 4.10 once we prove that $\dim N = 1$.

Using Theorem 1.2 there exists $t, s \in \mathbb{N}$ such that P^t is a free right $\mathcal{A}$-module, *i.e.* there is a vector space M such that $P^t \simeq M\otimes_{\Bbbk}\mathcal{A}$, hence

$$t \dim N = \dim M. \tag{4.12}$$

Since $P\otimes_{\mathcal{A}}Q \simeq \mathcal{A}'$ then $P^t\otimes_{\mathcal{A}}Q \simeq M\otimes_{\Bbbk}Q \simeq \mathcal{A}'^t$, then $\dim M \dim Q = s\,\mathcal{A}' \dim M = t \dim \mathcal{A}'$ and using (4.12) we obtain that $s \dim N = 1$ whence $\dim N = 1$ and the Theorem follows. $\square$

5. A correspondence for twist equivalent Hopf algebras

We shall present an explicit correspondence between module categories over twist equivalent Hopf algebras. For this we shall use the notion of biGalois extension. A (L, H)-biGalois extension B, for two Hopf algebras L, H is a right H-Galois structure and a left L-Galois structure on B such that the coactions make B an (L, H)-bicomodule. For more details on this subject we refer to [Sch].

Let L, H be finite-dimensional Hopf algebras and B a (L, H)-biGalois extension. We denote by $\widetilde{B}$ the (H, L)-biGalois extension with underlying algebra B^{op}, and comodule structure given as in [Sch, Theorem 4.3]. This new biGalois extension satisfies that $B\Box_H\widetilde{B} \simeq L$ as (L, L)-biGalois extensions and $\widetilde{B}\Box_H B \simeq H$ as (H, H)-biGalois extensions. Here $\Box_H$ denotes the cotensor product over H.

Let us recall that a Hopf 2-cocycle for H is a map $\sigma : H\otimes_{\Bbbk}H \to \Bbbk$, invertible with respect to convolution, such that for all $x, y, z \in H$

$$\sigma(x_{(1)}, y_{(1)})\sigma(x_{(2)}y_{(2)}, z) = \sigma(y_{(1)}, z_{(1)})\sigma(x, y_{(2)}z_{(2)}), \tag{5.1}$$

$$\sigma(x, 1) = \varepsilon(x) = \sigma(1, x). \tag{5.2}$$

Using this cocycle there is a new Hopf algebra structure constructed over the same coalgebra H with the product described by

$$x{\cdot_{[\sigma]}}y = \sigma(x_{(1)}, y_{(1)})\sigma^{-1}(x_{(3)}, y_{(3)})\; x_{(2)}y_{(2)}, \qquad x, y \in H.$$

This new Hopf algebra is denoted by H^σ. If K is a left H-comodule algebra, then we can define a new product in K by

$$a{.}_\sigma b = \sigma(a_{(-1)}, b_{(-1)})\, a_{(0)}.b_{(0)}, \tag{5.3}$$

$a, b \in K$. We shall denote by K_σ this new left comodule algebra. We shall say that the cocycle σ is *compatible* with K if for any $a, b \in K$, $\sigma(a_{(2)}, b_{(2)})\, a_{(1)}b_{(1)} \in K$. In that case we shall denote by ${}_\sigma K$ the left comodule algebra with underlying space K and algebra structure given by

$$a_\sigma.b = \sigma(a_{(-1)}, b_{(-1)})\, a_{(0)}.b_{(0)} \quad a, b \in K. \tag{5.4}$$

The algebra H_σ is a left H-comodule algebra with coaction given by the coproduct of H and it is a (H^σ, H)-biGalois extension and ${}_\sigma H$ is a (H, H^σ)-biGalois extension.

If $\lambda : \mathcal{A} \to H \otimes_{\Bbbk} \mathcal{A}$ is a left H-comodule algebra then $B \Box_H \mathcal{A}$ is a left L-comodule algebra. The left coaction is the induced by the left coaction on B with the following algebra structure, if $\sum x \otimes a, \sum y \otimes b \in B \Box_H \mathcal{A}$ then

$$(\sum x \otimes a)(\sum y \otimes b) := \sum xy \otimes ab.$$

A direct computation shows that $B \Box_H \mathcal{A}$ is a left L-comodule algebra.

Proposition 5.1. *The following assertions hold.*

1. *If $\mathcal{A}$ is right H-simple, then $B \Box_H \mathcal{A}$ is right L-simple.*
2. *If $\mathcal{A} \sim_M \mathcal{A}'$ then $B \Box_H \mathcal{A} \sim_M B \Box_H \mathcal{A}'$.*
3. *If $\sigma : H \otimes_{\Bbbk} H \to \Bbbk$ is an invertible 2-cocycle and $L = H^\sigma$, $B = H_\sigma$ then $B \Box_H \mathcal{A} \simeq \mathcal{A}_\sigma$.*
4. *If $K \subseteq H$ is a left coideal subalgebra, $\tau : H \otimes_{\Bbbk} H \to \Bbbk$ is an invertible 2-cocycle compatible with K, $\sigma : H \otimes_{\Bbbk} H \to \Bbbk$ is an invertible 2-cocycle and $B = H_\sigma$ then $B \Box_H ({}_\tau K) \simeq ({}_\tau K)_\sigma$.*

As a consequence we obtain that the application $\mathcal{A} \to B \Box_H \mathcal{A}$ gives a explicit bijective correspondence between indecomposable exact module categories over $\operatorname{Rep}(H)$ *and over* $\operatorname{Rep}(L)$.

Proof. 1. If $I \subseteq \mathcal{A}$ is a right ideal H-costable then $B \Box_H I$ is a right ideal L-costable of $B \Box_H \mathcal{A}$.

2. Let $P \in {}^H\mathcal{M}_{\mathcal{A}}$ such that $\mathcal{A}' \simeq \operatorname{End}_{\mathcal{A}}(P)$ as comodule algebras. The object $B \Box_H P$ belongs to the category ${}^L\mathcal{M}_{B \Box_H \mathcal{A}}$. The result follows since there is a natural isomorphism $B \Box_H \operatorname{End}_{\mathcal{A}}(P) \simeq \operatorname{End}_{B \Box_H \mathcal{A}}(B \Box_H P)$.

3. and 4. follow by a straightforward computation. □

5.1. **BiGalois extensions for quantum linear spaces.** Let $\theta \in \mathbb{N}$ and Γ be a finite Abelian group, $(g_1, \dots, g_\theta, \chi_1, \dots, \chi_\theta)$ be a datum of a quantum linear space, $V = V(g_1, \dots, g_\theta, \chi_1, \dots, \chi_\theta)$ and $\mathfrak{R}$ the quantum linear space associated to this data. Let $U = \mathfrak{R} \# \Bbbk\Gamma$.

Let $\mathcal{D} = (\mu, \lambda)$ be a compatible datum for $\mathfrak{R}$ and Γ, and $H = \mathcal{A}(\Gamma, \mathfrak{R}, \mathcal{D})$ be the Hopf algebra as described in section 2.2. We shall present a (H, U)-biGalois object.

The pair $(-\mu, -\lambda)$ is a compatible comodule algebra datum with respect to $\mathfrak{R}$ and the trivial 2-cocycle. In this case the left U-comodule algebra

$\mathcal{A}(V,\Gamma,1,-\mu,-\lambda)$ is also a right H-comodule algebra with structure ρ : $\mathcal{A}(V,\Gamma,1,-\mu,-\lambda) \to \mathcal{A}(V,\Gamma,1,-\mu,-\lambda)\otimes_{\Bbbk} H$ determined by

$$\rho(e_g) = e_g \otimes g, \quad \rho(v_i) = v_i \otimes 1 + e_{g_i} \otimes a_i, \quad g \in \Gamma, \quad i = 1, \dots, \theta.$$

The following result seems to be part of the folklore.

Proposition 5.2. *The algebra $\mathcal{A}(V,\Gamma,1,-\mu,-\lambda)$ with the above coactions is a (H,U)-biGalois object.*

Proof. Straightforward. □

References

[AM] N. Andruskiewitsch and M. Mombelli, *On module categories over finite-dimensional Hopf algebras*, J. Algebra **314** (2007), 383–418.

[AS1] N. Andruskiewitsch and H.-J. Schneider, *Lifting of quantum linear spaces and pointed Hopf algebras of order p^3*, J. Algebra **209** (1998), 658–691.

[AS2] N. Andruskiewitsch and H.-J. Schneider, *Pointed Hopf Algebras*, in New directions in Hopf algebras, 1–68, Math. Sci. Res. Inst. Publ. 43, Cambridge Univ. Press, Cambridge, 2002.

[Be] Bénabou, J., *Introduction to bicategories*, Reports of the Midwest Category Seminar (1967), Lecture Notes in Math. **47**, pp. 1–77, Springer, Berlin.

[BEK] J. Böckenhauer, D. E. Evans and Y. Kawahigashi, *Chiral Structure of Modular Invariants for Subfactors,* Commun. Math. Phys. **210** (2000), 733–784.

[BFRS] T. Barmeier, J. Fuchs, I. Runkel and C. Schweigert, *Module categories for permutation modular invariants*, `arXiv:0812.0986`.

[BO] R. Bezrukavnikov and V. Ostrik, *On tensor categories attached to cells in affine Weyl groups II*, `math.RT/0102220`.

[CS1] R. Coquereaux and G. Schieber, *Orders and dimensions for $\mathfrak{sl}_2$ and $\mathfrak{sl}_3$ module categories and boundary conformal field theories on a torus*, J. of Mathematical Physics, **48**, 043511 (2007).

[CS2] R. Coquereaux and G. Schieber, *From conformal embeddings to quantum symmetries: an exceptional $SU(4)$ example*, Journal of Physics- Conference Series Volume **103** (2008), 012006.

[EN] P. Etingof and D. Nikshych, *Dynamical twists in group algebras*, Int. Math. Res. Not. **13** (2001), 679–701.

[EO1] P. Etingof and V. Ostrik, *Finite tensor categories*, Mosc. Math. J. **4** (2004), no. 3, 627–654.

[EO2] P. Etingof and V. Ostrik, *Module categories over representations of $SL_q(2)$ and graphs*, Math. Res. Lett. (1) **11** (2004) 103–114.

[ENO1] P. Etingof, D. Nikshych and V. Ostrik, *On fusion categories*, Ann. Math. **162**, 581–642 (2005).

[ENO2] P. Etingof, D. Nikshych and V. Ostrik, *Weakly group-theoretical and solvable fusion categories*, preprint `arXiv:0809.3031`.

[FS] J. Fuchs and C. Schweigert, *Category theory for conformal boundary conditions*, in Vertex Operator Algebras in Mathematics and Physics, (2000). Preprint `math.CT/0106050`.

[Ga] C. Galindo, *Clifford theory for tensor categories*, preprint `arXiv:0902.1088`.

[HK] I. Heckenberger and S. Kolb, *Right coideal subalgebras of the Borel part of a quantized enveloping algebra*, preprint `arXiv:0910.3505`.

[HS] I. Heckenberger and H.-J. Schneider, *Right coideal subalgebras of Nichols algebras and the Duflo order on the Weyl groupoid*, preprint `arXiv:0909.0293`.

[K] V.K. Kharchenko, *Right coideal subalgebras in $U_q^+(\mathfrak{so}_{2n+1})$*, preprint arxiv:0908.4235.

[KL] V.K. Kharchenko and A.V. Lara Sagahon, *Right coideal subalgebras in $U_q(\mathfrak{sl}_{n+1})$*, J. Algebra **319** (2008) 2571–2625.

[KO] A. Kirillov Jr. and V. Ostrik, *On a q-analogue of the McKay correspondence and the ADE classification of sl_2 conformal field theories*, Adv. Math. **171** (2002), no. 2, 183–227.

[Ma] A. Masuoka, *Abelian and non-abelian second cohomologies of quantized enveloping algebras*, J. Algebra **320** (2008), 1–47.

[M1] M. Mombelli, *Dynamical twists in Hopf algebras*, Int. Math. Res. Not. (2007) vol.2007.

[M2] M. Mombelli, *Module categories over pointed Hopf algebras*, Math. Z. to appear, preprint arXiv:0811.4090.

[N] D. Nikshych, *Non group-theoretical semisimple Hopf algebras from group actions on fusion categories*, Selecta Math. **14** (2008), 145–161.

[Oc] A. Ocneanu, *The classification of subgroups of quantum $SU(N)$*, in Quantum symmetries in theoretical physics and mathematics (Bariloche, 2000), Contemp. Math. 294 (2002), 133–159.

[O1] V. Ostrik, *Module categories, Weak Hopf Algebras and Modular invariants*, Transform. Groups, 2 **8**, 177–206 (2003).

[O2] V. Ostrik, *Module categories over the Drinfeld double of a Finite Group*, Int. Math. Res. Not. **2003**, no. 27, 1507–1520.

[O3] V. Ostrik, *Module Categories Over Representations of $SL_q(2)$ in the Non-Semisimple Case*, Geom. funct. anal. Vol. **17** (2008), 2005–2017.

[Sch] P. Schauenburg, *Hopf Bigalois extensions*, Comm. in Algebra **24** (1996) 3797–3825.

[Sk] S. Skryabin, *Projectivity and freeness over comodule algebras*, Trans. Am. Math. Soc. **359**, No. 6, 2597-2623 (2007).

Facultad de Matemática, Astronomía y Física, Universidad Nacional de Córdoba, CIEM, Medina Allende s/n, (5000) Ciudad Universitaria, Córdoba, Argentina

E-mail address: mombelli@mate.uncor.edu, martin10090@gmail.com

URL: http://www.mate.uncor.edu/~mombelli