ON POINTED HOPF ALGEBRAS ASSOCIATED TO SOME CONJUGACY CLASSES IN $\mathbb{S}_n$

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Abstract. We show that any pointed Hopf algebra with infinitesimal braiding associated to the conjugacy class of $\pi \in \mathbb{S}_n$ is infinite-dimensional, if either the order of π is odd, or π is a product of disjoint cycles of odd order except for exactly two transpositions.

Introduction

The purpose of this article is to contribute to the classification of finite-dimensional complex pointed Hopf algebras H with $G(H) = \mathbb{S}_n$. Although substantial progress has been done in the classification of finite-dimensional complex pointed Hopf algebras with abelian group [AS4], not much is known about the non-abelian case. Our approach fits into the framework of the Lifting Method [AS1, AS3]; we shall freely use the notation and results from *loc. cit.* Given a finite group G, the key step in the classification of finite-dimensional complex pointed Hopf algebras H with $G(H) = G$ is the determination of all Yetter-Drinfeld modules V over the group algebra of G such that the Nichols algebra is finite-dimensional. If G is abelian, this reduces to the study of Nichols algebras of diagonal type; general results were reached in this situation in [AS2, H] for braided vector spaces of Cartan type. If G is not abelian, then very few examples have been computed explicitly in the literature, see [FK, MS, G1, AG2]. Our viewpoint in the present paper is to deduce that some Nichols algebras over non-abelian groups are infinite-dimensional from those results on Nichols algebras of diagonal type. This idea appeared first in [G1]. More precisely, recall that an irreducible Yetter-Drinfeld module over the group algebra of G is determined by a conjugacy class $\mathcal{C}$ of G and an irreducible representation ρ of the centralizer G^s of a fixed $s \in \mathcal{C}$. We seek for conditions on $\mathcal{C}$ and ρ implying that the dimension of the Nichols algebra $\mathfrak{B}(\mathcal{C}, \rho)$ is infinite. We concentrate on the central example $G = \mathbb{S}_n$. The main results in this paper are summarized in the following statement.

Date: November 3, 2005. 2000 Mathematics Subject Classification: 16W30.

The work of the first author partially supported by CONICET, Fundación Antorchas, Agencia Córdoba Ciencia, TWAS (Trieste), ANPCyT and Secyt (UNC).

Theorem 1. *Let $\pi \in \mathbb{S}_n$. If either one of the following conditions holds*

(i) the order of π is odd, or

(ii) all cycles in π have odd order except for exactly two transpositions,

then $\dim \mathfrak{B}(\mathcal{O}_\pi, \rho) = \infty$ *for any* $\rho \in \widehat{\mathbb{S}_n^\pi}$.

See Theorems 2.4 and 2.7. We also apply our main result to determine all irreducible Yetter-Drinfeld over $\mathbb{S}_3$ or $\mathbb{S}_4$ whose Nichols algebra is finite-dimensional. See Theorem 2.8.

1. Preliminaries and conventions

Our references for pointed Hopf algebras are [M], [AS3]. The set of isomorphism classes of irreducible representations of a finite group G is denoted $\widehat{G}$; thus, the group of characters of a finite abelian group Γ is denoted $\widehat{\Gamma}$. We shall often confuse a representant of a class in $\widehat{G}$ with the class itself. If V is a Γ-module then V^χ denotes the isotypic component of type $\chi \in \widehat{\Gamma}$. It is useful to write $g \triangleright h = ghg^{-1}$, $g, h \in G$.

1.1. **Notation on the groups $\mathbb{S}_n$.** Recall that the type of a permutation $\pi \in \mathbb{S}_n$ is a symbol $(1^{m_1}, 2^{m_2}, \dots, n^{m_n})$ meaning that in the decomposition of π as product of disjoint cycles, there are m_j cycles of length j, $1 \le j \le n$. We may omit j^{m_j} when $m_j = 0$. The conjugacy class $\mathcal{O}_\pi$ of π coincides with the set of all permutations in $\mathbb{S}_n$ with the same type as π; we denote it by $\mathcal{O}_{1^{m_1},2^{m_2},\dots,n^{m_n}}$ or by $\mathcal{O}_{2,\dots,2,3\dots,3,\dots}$. For instance, we denote by $\mathcal{O}_j$ the conjugacy class of j-cycles in $\mathbb{S}_n$, $2 \le j \le n$. Also, $\mathcal{O}_{2,2}$ is the conjugacy class of (12)(34) and so on. If $\pi \in \mathbb{S}_n$ and $n < m$ then we also denote by π the natural extension to $\mathbb{S}_m$ that fixes all $i > n$. If some emphasis is needed, we add a superscript n to indicate that we are taking conjugacy classes in $\mathbb{S}_n$, like $\mathcal{O}_j^n$ for the conjugacy class of j-cycles in $\mathbb{S}_n$. It is well-known that the isotropy subgroup $\mathbb{S}_n^\pi$ is isomorphic to a product

$$\mathbb{S}_n^\pi \simeq T_1 \dots T_n$$

where $T_i = \Gamma_i \rtimes \mathbb{S}_{m_i}$, $1 \le i \le n$. Here $\Gamma_i \simeq (\mathbb{Z}/i)^{m_i}$ is generated by the i-cycles in π and $\mathbb{S}_{m_i}$ permutes these cycles. Hence any $\rho \in \widehat{\mathbb{S}_n^\pi}$ is of the form $\rho_1 \otimes \dots \otimes \rho_n$ where $\rho_i \in \widehat{T_i}$.

If X is a subset of $\{1, \dots, n\}$ then $\mathbb{S}_X$ denotes the subgroup of bijections in $\mathbb{S}_n$ that fix pointwise $\{1, \dots, n\} - X$. We shall use the following notation for representations of subgroups of $\mathbb{S}_n$. Let ω_j be a fixed primitive j-th root of 1, $2 \le j \le n$. Let $\tau_j = (123\dots j)$.

$$\begin{aligned} \varepsilon &= \text{ trivial character;} \\ \operatorname{sgn} &= \text{ sign character of } \mathbb{S}_X,\ X \subset \mathbb{S}_n; \\ \chi_j &= \text{ character of } \langle \tau_j \rangle \simeq \mathbb{Z}_j \text{ given by } \chi(\tau_j) = \omega_j. \end{aligned}$$

1.2. **Yetter-Drinfeld modules over the group algebra of a finite group.** A Yetter-Drinfeld module over a finite group G is a left G-module and left $\mathbb{C}G$-comodule M such that

$$\delta(g.m) = ghg^{-1} \otimes g.m, \qquad m \in M_h, g, h \in G.$$

Here $M_h = \{m \in M : \delta(m) = h \otimes m\}$; clearly, $M = \oplus_{h \in G} M_h$. Yetter-Drinfeld modules over G are completely reducible. Also, irreducible Yetter-Drinfeld modules over G are parameterized by pairs $(\mathcal{C}, \rho)$ where $\mathcal{C}$ is a conjugacy class and ρ is an irreducible representation of the isotropy subgroup G^s of a fixed point $s \in \mathcal{C}$. As usual, $\deg \rho$ is the dimension of the vector space V affording ρ. We denote the corresponding Yetter-Drinfeld module by $M(\mathcal{C}, \rho)$; see [DPR, W], and also [AG1]. Since $s \in Z(G^s)$, the Schur Lemma says that

$$s \text{ acts by a scalar } q_{ss} \text{ on } V. \tag{1}$$

Here is a precise description of the Yetter-Drinfeld module $M(\mathcal{C}, \rho)$. Let $t_1 = s, \dots, t_M$ be a numeration of $\mathcal{C}$ and let $g_i \in G$ such that $g_i \triangleright s = t_i$ for all $1 \le i \le M$. Then $M(\mathcal{C}, \rho) = \oplus_{1 \le i \le M} g_i \otimes V$. Let $g_i v := g_i \otimes v \in M(\mathcal{C}, \rho)$, $1 \le i \le M$, $v \in V$. If $v \in V$ and $1 \le i \le M$, then the action of $g \in G$ and the coaction are given by

$$\delta(g_i v) = t_i \otimes g_i v, \qquad g \cdot (g_i v) = g_j(\gamma \cdot v),$$

where $g g_i = g_j \gamma$, for some $1 \le j \le M$ and $\gamma \in G^s$. The explicit formula for the braiding is then given by

$$c(g_i v \otimes g_j w) = t_i \cdot (g_j w) \otimes g_i v = g_h(\gamma \cdot v) \otimes g_i v \tag{2}$$

for any $1 \le i, j \le M$, $v, w \in V$, where $t_i g_j = g_h \gamma$ for unique h, $1 \le h \le M$ and $\gamma \in G^s$.

1.3. **On Nichols algebras.** Let (V, c) be a braided vector space, that is V is a vector space and $c : V \otimes V \to V \otimes V$ is a linear isomorphism satisfying the braid equation. Then $\mathfrak{B}(V)$ denotes the Nichols algebra of V, see the precise definition in [AS3]. Let G be a finite group, $\mathcal{C}$ a conjugacy class, $s \in \mathcal{C}$ and $\rho \in \widehat{G^s}$. The Nichols algebra of $M(\mathcal{C}, \rho)$ will be denoted simply by $\mathfrak{B}(\mathcal{C}, \rho)$. We collect some general facts on Nichols algebras for further reference.

Remark 1.1. Let (V, c) be a braided vector space. If W is a subspace of V such that $c(W \otimes W) = W \otimes W$ then $\mathfrak{B}(W) \subset \mathfrak{B}(V)$. Thus $\dim \mathfrak{B}(W) = \infty \implies \dim \mathfrak{B}(V) = \infty$. In particular, if there exists $v \in V - 0$ such that $c(v \otimes v) = v \otimes v$ then $\dim \mathfrak{B}(V) = \infty$. This is the case in $\mathfrak{B}(\mathcal{C}, \rho)$ when either the orbit $\mathcal{C}$ is trivial or the representation ρ is trivial, or even if the scalar q_{ss} defined in (1) is 1.

A braided vector space (V, c) is of *diagonal type* if there exists a basis $v_1, \dots, v_\theta$ of V and non-zero scalars q_{ij}, $1 \le i, j \le \theta$, such that

$$c(v_i \otimes v_j) = q_{ij} v_j \otimes v_i, \qquad \text{for all } 1 \le i, j \le \theta.$$

A braided vector space (V, c) is of *Cartan type* if it is of diagonal type, and there exists $a_{ij} \in \mathbb{Z}$, $-\operatorname{ord} q_{ii} < a_{ij} \le 0$ such that

$$q_{ij} q_{ji} = q_{ii}^{a_{ij}}$$

for all $1 \le i \ne j \le \theta$. Set $a_{ii} = 2$ for al $1 \le i \le \theta$. Then $(a_{ij})_{1 \le i,j \le \theta}$ is a generalized Cartan matrix. The following important result was proved in [H, Th. 4], showing that some hypotheses in [AS2, Th. 1] were unnecessary.

Theorem 1.2. *Let (V, c) be a braided vector space of Cartan type. Assume that $q_{ii} \ne 1$ is root of 1 for all $1 \le i \le \theta$. Then $\dim \mathfrak{B}(V) < \infty$ if and only if the Cartan matrix is of finite type.* □

2. On Nichols algebras over $\mathbb{S}_n$

In this section we state some general results about Nichols algebras over $\mathbb{S}_n$. Our main idea is to find out suitable braided subspaces of a Yetter-Drinfeld module over $\mathbb{C}\mathbb{S}_n$ that are diagonal of Cartan type, so that Theorem 1.2 applies. In what follows, G is a finite group, $s \in G$, $\mathcal{C}$ is the conjugacy class of s, $\rho \in \widehat{G^s}$, $\rho : G^s \to GL(V)$. Recall the scalar q_{ss} defined in (1).

Proposition 2.1. [G1, Lemma 3.1] *Assume that* $\dim \mathfrak{B}(\mathcal{C}, \rho) < \infty$. *Then*

- $\deg \rho > 2$ *implies* $q_{ss} = -1$.
- $\deg \rho = 2$ *implies* $q_{ss} = -1$, ω_3 *or* ω_3^2. □

2.1. **Nichols algebras corresponding to permutations of odd order.**

We begin by a general way of finding braided subspaces of rank 2. Assume that

(3) there exists an involution $\sigma \in G$ such that $\sigma s \sigma = s^{-1} \ne s$.

In particular, $s^{-1} \in \mathcal{C}$. Under this hypothesis, we prove the following result that, unlike Proposition 2.1, does not assume any restriction on $\deg \rho$.

Lemma 2.2. *If* $\dim \mathfrak{B}(\mathcal{C}, \rho) < \infty$ *then* $q_{ss} = -1$. *In particular s has even order.*

Proof. Let $N = \operatorname{ord} q_{ss}$; clearly $N > 1$. Let $t_1 = s$, $t_2 = s^{-1}$, $g_1 = e$, $g_2 = \sigma$. By (1), (2) and (3), we have for any $v, w \in V$

$$\begin{aligned}
c(g_1 v \otimes g_1 w) &= g_1(s \cdot w) \otimes g_1 v = q_{ss}\, g_1 w \otimes g_1 v,\\
c(g_1 v \otimes g_2 w) &= g_2(s^{-1} \cdot w) \otimes g_1 v = q_{ss}^{-1}\, g_2 w \otimes g_1 v,\\
c(g_2 v \otimes g_1 w) &= g_1(s^{-1} \cdot w) \otimes g_2 v = q_{ss}^{-1}\, g_1 w \otimes g_2 v,\\
c(g_2 v \otimes g_2 w) &= g_2(s \cdot w) \otimes g_2 v = q_{ss}\, g_2 w \otimes g_2 v.
\end{aligned}$$

Let $v \in V$, $v \neq 0$. Then the subspace of $M(\mathcal{C}, \rho)$ spanned by $v_1 := g_1 v$, $v_2 := g_2 v$ is a braided subspace of Cartan type: $c(v_i \otimes v_j) = q_{ij} v_j \otimes v_i$, for $1 \leq i, j \leq 2$, where $\begin{pmatrix} q_{11} & q_{12} \\ q_{21} & q_{22} \end{pmatrix} = \begin{pmatrix} q_{ss} & q_{ss}^{-1} \\ q_{ss}^{-1} & q_{ss} \end{pmatrix}$, with Cartan matrix $\begin{pmatrix} 2 & a_{12} \\ a_{21} & 2 \end{pmatrix}$, $a_{12} = a_{21} \equiv -2 \mod N$. Now $a_{12} = a_{21} = 0$ or -1, by hypothesis and Theorem 1.2. Thus, necessarily $a_{12} = a_{21} = 0$ and $N = 2$. $\square$

Lemma 2.3. *Let $\pi \in \mathbb{S}_n$, $\operatorname{ord} \pi > 2$, $\rho \in \widehat{\mathbb{S}_n^\pi}$. If $\dim \mathfrak{B}(\mathcal{O}_\pi, \rho) < \infty$ then $q_{\pi,\pi} = -1$.*

Proof. It is well-known that (3) holds for any $\pi \in \mathbb{S}_n$. Namely, assume that $\pi = t_j$ for some j and take

$$g_2 = \begin{cases} (1\, j-1)(2\, j-2) \cdots (k-1\, k+1), & \text{if } j = 2k \text{ is even,} \\ (1\, j-1)(2\, j-2) \cdots (k\, k+1), & \text{if } j = 2k+1 \text{ is odd.} \end{cases}$$

It is easy to see that $g_2 t_j g_2 = t_j^{-1}$. The general case follows using that any π is a product of disjoint cycles. We conclude from Lemma 2.2. $\square$

Theorem 2.4. *If $\pi \in \mathbb{S}_n$ has odd order then $\dim \mathfrak{B}(\mathcal{O}_\pi, \rho) = \infty$ for any $\rho \in \widehat{\mathbb{S}_n^\pi}$.* $\square$

If n es even, the Nichols algebras $\mathfrak{B}(\mathcal{O}_n^n, \rho)$ cannot be treated by similar arguments as above. For, the isotropy subgroup $\mathbb{S}_n^{\tau_n}$ is cyclic of order n and we can assume that $\rho(\tau_n) = -1$. Assume that $\tau \in \mathcal{O}_n^n \cap \mathbb{S}_n^{\tau_n}$ then $\tau = \tau_n^j$ with $(n, j) = 1$ hence $\rho(\tau) = -1$. But there are Nichols algebras like these that are finite-dimensional. For instance, the Nichols algebra $\mathfrak{B}(\mathcal{O}_4, \chi_4^2)$ was computed in [AG2, Th. 6.12] and has dimension 576.

2.2. **A reduction argument.** We now discuss a general reduction argument. Let $n, p \in \mathbb{N}$ and let $m = n + p$. Let $\# : \mathbb{S}_n \times \mathbb{S}_p \to \mathbb{S}_m$ be the group homomorphism given by

$$\pi \# \tau(i) = \begin{cases} \pi(i), & 1 \leq i \leq n; \\ \tau(i-n) + n, & n+1 \leq i \leq m, \end{cases}$$

$\pi \in \mathbb{S}_n$, $\tau \in \mathbb{S}_p$. If also $g \in \mathbb{S}_n$, $h \in \mathbb{S}_p$ then $(g \# h) \triangleright (\pi \# \tau) = (g \triangleright \pi) \# (h \triangleright \tau)$. Thus $\mathcal{O}_{\pi\#\tau} \supset \mathcal{O}_\pi \# \mathcal{O}_\tau$. Let us say that π and τ are *orthogonal*, denoted $\pi \perp \tau$, if there is no j such that both π and τ contain a j-cycle. In other words, if π has type $(1^{a_1}, 2^{a_2}, \dots, n^{a_n})$ and τ has type $(1^{b_1}, 2^{b_2}, \dots, p^{b_p})$ then either $a_j = 0$ or $b_j = 0$ for any j. If $\pi \perp \tau$ then $\mathbb{S}_m^{\pi\#\tau} = \mathbb{S}_n^\pi \# \mathbb{S}_p^\tau$, say by a counting argument. Hence any $\mu \in \widehat{\mathbb{S}_m^{\pi\#\tau}}$ is of the form $\mu = \rho \otimes \lambda$, for unique

$\rho \in \widehat{\mathbb{S}_n^\pi}$, $\lambda \in \widehat{\mathbb{S}_p^\tau}$. Say that V, W are the vector spaces affording ρ, λ. Let $q_{\tau\tau}$, resp. $q_{\pi\pi}$, be the scalar as in (1) for the representation ρ, resp. λ.

Lemma 2.5. *Let $\pi \in \mathbb{S}_n$, $\tau \in \mathbb{S}_p$.*

(1) Assume that $\pi \perp \tau$ and $\operatorname{ord}(\pi\#\tau) > 2$. Let $\mu \in \widehat{\mathbb{S}_m^{\pi\#\tau}}$ of the form $\mu = \rho \otimes \lambda$, for $\rho \in \widehat{\mathbb{S}_n^\pi}$, $\lambda \in \widehat{\mathbb{S}_p^\tau}$. If $\dim \mathfrak{B}(\mathcal{O}_{\pi\#\tau}, \mu) < \infty$ then $q_{\pi\pi}q_{\tau\tau} = -1$.

(2) Assume that the orders of π and τ are relatively prime, that $\operatorname{ord} q_{\tau\tau}$ is odd and that $\operatorname{ord}(\pi\#\tau) > 2$. If $\dim \mathfrak{B}(\mathcal{O}_{\pi\#\tau}, \mu) < \infty$ then $q_{\tau\tau} = 1$.

Clearly, if the orders of π and τ are relatively prime then $\pi \perp \tau$. On the other hand, if $\operatorname{ord}(\pi\#\tau) = 2$ and the orders of π and τ are relatively prime, then we can assume $\tau = e$, and $q_{\tau\tau} = 1$ anyway.

Proof. Since $q_{\pi\#\tau,\pi\#\tau} = q_{\pi,\pi}q_{\tau\tau}$, (1) follows from Lemma 2.3. Then (2) follows from (1). $\square$

Our aim is to obtain information on $\mathfrak{B}(\mathcal{O}_{\pi\#\tau}, \mu)$ from $\mathfrak{B}(\mathcal{O}_\pi, \rho)$ and $\mathfrak{B}(\mathcal{O}_\tau, \lambda)$. For this, we fix $g_1 = e, \dots, g_P \in \mathbb{S}_n$, $h_1 = e, \dots, h_T \in \mathbb{S}_p$, such that

$$g_1 \triangleright \pi = \pi, \dots, g_P \triangleright \pi \text{ is a numeration of } \mathcal{O}_\pi,$$
$$h_1 \triangleright \tau = \tau, \dots, h_T \triangleright \tau \text{ is a numeration of } \mathcal{O}_\tau.$$

Then we can extend $(g_i\#h_j) \triangleright (\pi\#\tau)$, $1 \le i \le P$, $1 \le j \le T$ to a numeration of $\mathcal{O}_{\pi\#\tau}$. Let $v, u \in V$, $w, z \in W$, $1 \le i, k \le P$, $1 \le j, l \le T$. Then the braiding in $M(\mathcal{O}_{\pi\#\tau}, \rho \otimes \lambda)$ has the form

$$\begin{aligned} c\left((g_i\#h_j)(v \otimes w) \otimes (g_k\#h_l)(u \otimes z)\right) \\ = \left((g_k\#h_l)(\gamma \cdot u \otimes \beta \cdot z) \otimes (g_i\#h_j)(v \otimes w)\right) \end{aligned} \tag{4}$$

where $g_i \triangleright g_k = g_r\gamma$, $h_j \triangleright h_l = h_p\beta$ for unique $1 \le r \le P$, $1 \le p \le T$, $\gamma \in \mathbb{S}_n^\pi$ and $\beta \in \mathbb{S}_p^\tau$. Assume in (4) that $j = l = 1$; then $p = 1$, $\beta = \tau$ and (4) takes the form

$$\begin{aligned} c\left((g_i\#h_1)(v \otimes w) \otimes (g_k\#h_1)(u \otimes z)\right) \\ = q_{\tau\tau}\left((g_k\#h_1)(\gamma \cdot u \otimes z) \otimes (g_i\#h_1)(v \otimes w)\right). \end{aligned} \tag{5}$$

Our aim is to spell out some consequences of formula (5).

Proposition 2.6. *Let $\pi \in \mathbb{S}_n$, $\tau \in \mathbb{S}_p$. Assume that the orders of π and τ are relatively prime and that $\operatorname{ord} q_{\tau\tau}$ is odd. Let $\mu \in \widehat{\mathbb{S}_m^{\pi\#\tau}}$ of the form $\mu = \rho \otimes \lambda$, for $\rho \in \widehat{\mathbb{S}_n^\pi}$, $\lambda \in \widehat{\mathbb{S}_p^\tau}$.*

(1) If $\dim \mathfrak{B}(\mathcal{O}_{\pi\#\tau}, \mu) < \infty$, then $\dim \mathfrak{B}(\mathcal{O}_\pi, \rho) < \infty$.

(2) If $\dim \mathfrak{B}(\mathcal{O}_\pi, \rho) = \infty$ for any $\rho \in \widehat{\mathbb{S}_n^\pi}$, then $\dim \mathfrak{B}(\mathcal{O}_{\pi\#\tau}, \mu) = \infty$ for any $\mu \in \widehat{\mathbb{S}_m^{\pi\#\tau}}$.

In particular, let $\pi \in \mathbb{S}_n$ with no fixed points. If $\dim \mathfrak{B}(\mathcal{O}_\pi^n, \rho) = \infty$ for any $\rho \in \widehat{\mathbb{S}_n^\pi}$, then $\dim \mathfrak{B}(\mathcal{O}_\pi^m, \rho') = \infty$ for any $\rho' \in \widehat{\mathbb{S}_m^\pi}$, $m > n$.

Proof. Assume that $\dim \mathfrak{B}(\mathcal{O}_{\pi\#\tau}, \mu) < \infty$. By Lemma 2.5, $q_{\tau\tau} = 1$. Let $0 \neq w = z \in W$. The linear map $\psi : M(\mathcal{O}_\pi, \rho) \to M(\mathcal{O}_{\pi\#\tau}, \mu)$ given by $\psi(g_i v) = (g_i \# h_1)(v \otimes w)$ is a morphism of braided vector spaces because of (5). Now apply Remark 1.1. □

2.3. Nichols algebras of orbits with exactly two transpositions.

Theorem 2.7. *Let $\pi \in \mathbb{S}_n$. If π has type $(1^a, 2^2, h_1^{m_1}, \dots, h_r^{m_r})$ where $h_1, \dots, h_r$ are odd, then $\dim \mathfrak{B}(\mathcal{O}_\pi^n, \rho) = \infty$ for any $\rho \in \widehat{\mathbb{S}_n^\pi}$.*

For instance, if $n \geq 4$ then $\dim \mathfrak{B}(\mathcal{O}_{2,2}^n, \rho) = \infty$ for any ρ.

Proof. By Proposition 2.6, we can assume that $n = 4$ and π is of type $(2, 2)$. Let us consider the irreducible Yetter-Drinfeld modules corresponding to $\mathcal{O}_{2,2} = \{a = (13)(24), b = (12)(34), d = (14)(23)\}$. The isotropy subgroup of a is $\mathbb{S}_4^a = \langle (1234), (13) \rangle \simeq \mathcal{D}_4$. Let $A = (1234)$, $B = (13)$; $\mathcal{O}_{2,2} \subseteq \mathbb{S}_4^a$ and $a = A^2$, $b = BA$, $d = BA^3$. Hence the irreducible representations of $\mathbb{S}_4^a$ are (1) the characters given by $A \mapsto \varepsilon_1$, $B \mapsto \varepsilon_2$ where $\varepsilon_j \in \{\pm 1\}$, $1 \leq j \leq 2$, and (2) the 2-dimensional representation $\rho : \mathbb{D}_4 \to GL(2, \mathbb{C})$ given by

$$\rho(A) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \qquad \rho(B) = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Let μ be a one-dimensional representation of $\mathbb{S}_4^a$. Then $M(\mathcal{O}_{2,2}, \mu)$ has a basis v_a, v_b, v_d with $\delta(v_a) = a \otimes v_a$, etc., and $c(v_a \otimes v_a) = \mu(a) v_a \otimes v_a = v_a \otimes v_a$. Hence $\dim \mathfrak{B}(\mathcal{O}_{2,2}, \mu) = \infty$.

Let $\sigma_0 = 1$, $\sigma_1 = (12)$, $\sigma_2 = (23)$. Then

$$\sigma_1 \triangleright a = d, \quad \sigma_2 \triangleright a = b, \quad \sigma_1 \triangleright b = b, \quad \sigma_2 \triangleright d = d.$$

Let us consider the Yetter-Drinfeld module $M(\mathcal{O}_{2,2}, \rho)$. We have

$$\rho(a) = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \rho(b) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \rho(d) = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}.$$

Let $\sigma_j v := \sigma_j \otimes v$, $v \in V$, $0 \leq j \leq 2$. The coaction is given by $\delta(\sigma_j v) = \sigma_j \triangleright a \otimes \sigma_j v$; we need the action of the elements a, b, d, which is

$$\begin{aligned}
&a \cdot \sigma_0 v = \sigma_0 \rho(a)(v), && a \cdot \sigma_1 v = \sigma_1 \rho(d)(v), && a \cdot \sigma_2 v = \sigma_2 \rho(b)(v), \\
&b \cdot \sigma_0 v = \sigma_0 \rho(b)(v), && b \cdot \sigma_1 v = \sigma_1 \rho(b)(v), && b \cdot \sigma_2 v = \sigma_2 \rho(a)(v), \\
&d \cdot \sigma_0 v = \sigma_0 \rho(d)(v), && d \cdot \sigma_1 v = \sigma_1 \rho(a)(v), && d \cdot \sigma_2 v = \sigma_2 \rho(d)(v).
\end{aligned}$$

Orbit	**Isotropy group**	**Representation**	$\dim \mathfrak{B}(V)$	**Reference**
e	$\mathbb{S}_3$	any	∞	Remark 1.1
$\mathcal{O}_3$	$\mathbb{Z}_3$	any	∞	Theorem 2.4
$\mathcal{O}_2$	$\mathbb{Z}_2$	ε	∞	Remark 1.1
$\mathcal{O}_2$	$\mathbb{Z}_2$	sgn	12	[MS]

TABLE 1. Nichols algebras of irreducible Yetter-Drinfeld modules over $\mathbb{S}_3$

Hence the braiding is given, for all $0 \le j \le 2$ and $v, w \in V$, by

$$c(\sigma_j v \otimes \sigma_j w) = (\sigma_j \triangleright a) \cdot \sigma_j w \otimes \sigma_j v = \sigma_j \rho(a)(w) \otimes \sigma_j v = -\sigma_j w \otimes \sigma_j v$$

and

$$\begin{aligned} c(\sigma_0 v \otimes \sigma_1 w) &= \sigma_1 \rho(d)(w) \otimes \sigma_0 v, & c(\sigma_0 v \otimes \sigma_2 w) &= \sigma_2 \rho(b)(w) \otimes \sigma_0 v, \\ c(\sigma_1 v \otimes \sigma_0 w) &= \sigma_0 \rho(d)(w) \otimes \sigma_1 v, & c(\sigma_1 v \otimes \sigma_2 w) &= \sigma_2 \rho(d)(w) \otimes \sigma_1 v, \\ c(\sigma_2 v \otimes \sigma_0 w) &= \sigma_0 \rho(b)(w) \otimes \sigma_2 v, & c(\sigma_2 v \otimes \sigma_1 w) &= \sigma_1 \rho(b)(w) \otimes \sigma_2 v. \end{aligned}$$

Let $v_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$, $v_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$. Then $\rho(b)(v_1) = v_1$, $\rho(b)(v_2) = -v_2$, $\rho(d)(v_1) = -v_1$, $\rho(d)(v_2) = v_2$. Hence the braiding is diagonal of Cartan type in the basis

$$w_1 = \sigma_0 v_1,\ w_2 = \sigma_0 v_2,\ w_3 = \sigma_1 v_1,\ w_4 = \sigma_1 v_2,\ w_5 = \sigma_2 v_1,\ w_6 = \sigma_2 v_2.$$

The corresponding Dynkin diagram is not connected; its connected components are $\{1, 4, 6\}$ and $\{2, 3, 5\}$, each of them supporting the affine Dynkin diagram $A_2^{(1)}$. Then $\dim \mathfrak{B}(\mathcal{O}_{2,2}, \rho) = \infty$ by Theorem 1.2. $\square$

2.4. **Nichols algebras over $\mathbb{S}_3$ and $\mathbb{S}_4$.** We apply the main result of this paper to classify finite-dimensional Nichols algebras over $\mathbb{S}_3$ or $\mathbb{S}_4$ with irreducible module of primitive elements.

Theorem 2.8. *Let $M(\mathcal{C}, \lambda)$ be an irreducible Yetter-Drinfeld module over $\mathbb{S}_n$ such that $\mathfrak{B}(\mathcal{C}, \lambda)$ is finite-dimensional.*

(i). If $\mathbb{S}_n = \mathbb{S}_3$ then $M(\mathcal{C}, \lambda) \simeq M(\mathcal{O}_2, \mathrm{sgn})$.

(ii). If $\mathbb{S}_n = \mathbb{S}_4$ then $M(\mathcal{C}, \lambda)$ is isomorphic either to $\mathfrak{B}(\mathcal{O}_4, \chi_4^2)$ or to $\mathfrak{B}(\mathcal{O}_2, \mathrm{sgn} \oplus \varepsilon)$ or to $\mathfrak{B}(\mathcal{O}_2, \mathrm{sgn} \oplus \mathrm{sgn})$.

Orbit	**Isotropy group**	**Representation**	$\dim \mathfrak{B}(V)$	**Reference**
e	$\mathbb{S}_4$	any	∞	Remark 1.1
$\mathcal{O}_{2,2}$	$\mathcal{D}_4$	any	∞	Theorem 2.7
$\mathcal{O}_4$	$\mathbb{Z}_4$	ε	∞	Remark 1.1
$\mathcal{O}_4$	$\mathbb{Z}_4$	χ_4 or χ_4^3	∞	Lemma 2.3
$\mathcal{O}_4$	$\mathbb{Z}_4$	χ_4^2	576	[AG2, 6.12]
$\mathcal{O}_3$	$\mathbb{Z}_3$	any	∞	Theorem 2.4
$\mathcal{O}_2$	$\mathbb{Z}_2 \oplus \mathbb{Z}_2$	ε or $\varepsilon \oplus \operatorname{sgn}$	∞	Remark 1.1
$\mathcal{O}_2$	$\mathbb{Z}_2 \oplus \mathbb{Z}_2$	$\operatorname{sgn} \oplus \varepsilon$	576	[FK]
$\mathcal{O}_2$	$\mathbb{Z}_2 \oplus \mathbb{Z}_2$	$\operatorname{sgn} \oplus \operatorname{sgn}$	576	[MS]

TABLE 2. Nichols algebras of irreducible Yetter-Drinfeld modules over $\mathbb{S}_4$

Proof. See tables 1 and 2. □

Acknowledgements. The first author thanks István Heckenberger and Matías Graña for interesting e-mail exchanges.

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