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On anosov automorphisms of nilmanifolds

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ON ANOSOV AUTOMORPHISMS OF NILMANIFOLDS

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1. Introduction

Anosov diffeomorphisms play an important and beautiful role in dynamics as the notion represents the most perfect kind of global hyperbolic behavior, giving examples of structurally stable dynamical systems. A diffeomorphism f of a compact differentiable manifold M is called *Anosov* if the tangent bundle $\mathrm{T}M$ admits a continuous invariant splitting $\mathrm{T}M = E^+ \oplus E^-$ such that $\mathrm{d}f$ expands E^+ and contracts E^- exponentially.

Let N be a real simply connected nilpotent Lie group with Lie algebra $\mathfrak{n}$. Let φ be a hyperbolic automorphism of N, that is, all the eigenvalues of its derivative $A = (\mathrm{d}\varphi)_e : \mathfrak{n} \longrightarrow \mathfrak{n}$ have absolute value different from 1. If $\varphi(\Gamma) = \Gamma$ for some lattice Γ of N (i.e. a cocompact discrete subgroup), then φ defines an Anosov diffeomorphism on the nilmanifold $M = N/\Gamma$, which is called an *Anosov automorphism.* The subspaces E^+ and E^- are obtained by left translation of the eigenspaces of eigenvalues of A of absolute value greater than 1 and less than 1, respectively, and so the splitting is differentiable. If more in general, Γ is a cocompact discrete subgroup of $K \ltimes N$, where K is any compact subgroup of $\mathrm{Aut}(N)$, for which $\varphi(\Gamma) = \Gamma$ (recall that φ acts on $\mathrm{Aut}(N)$ by conjugation), then φ also determines an Anosov diffeomorphism of $M = N/\Gamma$. In this case M is called an infranilmanifold and is finitely covered by the nilmanifold $N/(N \cap \Gamma)$.

In [S], S. Smale raised the problem of classifying all compact manifolds (up to homeomorphism) which admit an Anosov diffeomorphism. Curiously enough, the only known examples so far are of algebraic nature, namely Anosov automorphisms of infranilmanifolds described above. It is even conjectured that any Anosov diffeomorphism is topologically conjugate to an Anosov automorphism of an infranilmanifold (see [Mr]). All this certainly highlights the problem of classifying nilmanifolds admitting Anosov automorphisms, which are easily seen to be in correspondence with the following very special class of nilpotent Lie algebras over $\mathbb{Q}$ (see [L1, D, I, De]).

A rational Lie algebra $\mathfrak{n}^{\mathbb{Q}}$ (i.e. with structure constants in $\mathbb{Q}$) of dimension n is said to be *Anosov* if it admits a *hyperbolic* automorphism A (i.e. all their eigenvalues have absolute value different from 1) which is *unimodular*, that is, $[A]_\beta \in \mathrm{GL}_n(\mathbb{Z})$ for some basis β of $\mathfrak{n}^{\mathbb{Q}}$, where $[A]_\beta$ denotes the matrix of A with respect to β. We call a real Lie algebra *Anosov* if it admits a rational form which is Anosov. Unimodularity and hyperbolicity are, together, a rather strong condition to be satisfied by an automorphism of a nilpotent Lie algebra. This is confirmed for instance by the result in [E, 3.5] which asserts that 2-step Anosov Lie algebras live outside of an open dense subset in the moduli space of 2-step nilpotent Lie

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algebras. All this makes of Anosov Lie algebras very distinguished objects, and general existence results are hard to obtain.

We prove in Section 3 a generalization of the construction given in [L1] suggested by F. Grunewald, asserting that $\mathfrak{n}\oplus\dots\oplus\mathfrak{n}$ (s times, $s \geq 2$) is Anosov for any graded nilpotent Lie algebra over $\mathbb{R}$ having a rational form. This in particular shows that at least an explicit classification of Anosov Lie algebras would not be feasible.

It is not true in general that if a direct sum of real Lie algebras is Anosov then each of the direct factors is so, as the example $\mathfrak{h}_3 \oplus \mathfrak{h}_3$ shows, where $\mathfrak{h}_3$ is the 3-dimensional Heisenberg algebra (see [S]). However, we shall see in Section 3 that this actually happens when one of the direct factors is maximal abelian.

The *type* of a nilpotent Lie algebra $\mathfrak{n}$ is the r-tuple $(n_1, ..., n_r)$, where $n_i = \dim C^{i-1}(\mathfrak{n})/C^i(\mathfrak{n})$ and $C^i(\mathfrak{n})$ is the central descending series. In Section 2, by using that any Anosov Lie algebra admits an Anosov automorphism A which is semisimple and some elementary properties of lattices, we obtain some obstructions for the types allowed. Also, we strongly use the fact that the eigenvalues of A are algebraic integers (even units), and prove that the types $(5, 3)$ and $(3, 3, 2)$, in principle allowed as they satisfy the obstructions, are not possible for Anosov Lie algebras (see Section 4).

2. Some obstructions

We give in this section some necessary conditions a real Lie algebra has to satisfy in order to be Anosov (see [M1]).

Proposition 2.1. *Let $\mathfrak{n}$ be a real nilpotent Lie algebra which is Anosov. Then there exist a decomposition $\mathfrak{n} = \mathfrak{n}_1 \oplus \dots \oplus \mathfrak{n}_r$ satisfying $C^i(\mathfrak{n}) = \mathfrak{n}_{i+1} \oplus \dots \oplus \mathfrak{n}_r$, $i = 0, ..., r$, and a hyperbolic $A \in \mathrm{Aut}(\mathfrak{n})$ such that*

(i) *$A\mathfrak{n}_i = \mathfrak{n}_i$ for all $i = 1, ..., r$.*
(ii) *A is semisimple (in particular A is diagonalizable over $\mathbb{C}$).*
(iii) *For each i, there exists a basis β_i of $\mathfrak{n}_i$ such that $[A_i]_{\beta_i} \in \mathrm{SL}_{n_i}(\mathbb{Z})$, where $n_i = \dim \mathfrak{n}_i$ and $A_i = A|_{\mathfrak{n}_i}$.*

Proof. Let β be a $\mathbb{Z}$-basis of $\mathfrak{n}$ for which there is a hyperbolic $A \in \mathrm{Aut}(\mathfrak{n})$ satisfying $[A]_\beta \in \mathrm{GL}_n(\mathbb{Z})$. By using that $\mathrm{Aut}(\mathfrak{n})$ is a linear algebraic group, it is proved in [AS, Section 2] that we can assume that A is semisimple. Thus the existence of the decomposition satisfying (i) follows from the fact that the subspaces $C^i(\mathfrak{n})$ are A-invariant.

If $\beta = \{X_1, ..., X_n\}$ then the discrete (additive) subgroup

$$\mathfrak{n}^{\mathbb{Z}} = \left\{ \sum_{i=1}^{n} a_i X_i : a_i \in \mathbb{Z} \right\}$$

of $\mathfrak{n}$ is closed under the Lie bracket of $\mathfrak{n}$ and A-invariant, and $C^i(\mathfrak{n}^{\mathbb{Z}})$ is a discrete subgroup of $C^i(\mathfrak{n})$ of maximal rank. Since $AC^i(\mathfrak{n}^{\mathbb{Z}}) = C^i(\mathfrak{n}^{\mathbb{Z}})$ for any i we have that A induces an invertible map

$$C^{i-1}(\mathfrak{n}^{\mathbb{Z}})/C^i(\mathfrak{n}^{\mathbb{Z}}) \mapsto C^{i-1}(\mathfrak{n}^{\mathbb{Z}})/C^i(\mathfrak{n}^{\mathbb{Z}}),$$

and it follows from $C^i(\mathfrak{n}^{\mathbb{Z}}) \otimes \mathbb{R} = C^i(\mathfrak{n})$ that $C^{i-1}(\mathfrak{n}^{\mathbb{Z}})/C^i(\mathfrak{n}^{\mathbb{Z}}) \simeq \mathbb{Z}^{n_i}$ is a discrete subgroup of $C^{i-1}(\mathfrak{n})/C^i(\mathfrak{n}) \simeq \mathfrak{n}_i$ which is leaved invariant by A, proving the existence of the basis β_i of $\mathfrak{n}_i$ in (iii). Recall that by considering A^2 rather than A if necessary, we can assume that $\det A_i = 1$ for all i. □

Let $\mathfrak{n}$ be a nilpotent Lie algebra over K, where K is $\mathbb{R}$, $\mathbb{Q}$ or $\mathbb{C}$.

Definition 2.2. Consider the central descendent series of $\mathfrak{n}$ defined by $C^0(\mathfrak{n}) = \mathfrak{n}$, $C^i(\mathfrak{n}) = [\mathfrak{n}, C^{i-1}(\mathfrak{n})]$. When $C^r(\mathfrak{n}) = 0$ and $C^{r-1}(\mathfrak{n}) \neq 0$, $\mathfrak{n}$ is said to be r-step nilpotent, and we denote by $(n_1, ..., n_r)$ the *type* of $\mathfrak{n}$, where

$$n_i = \dim C^{i-1}(\mathfrak{n})/C^i(\mathfrak{n}).$$

We also take a decomposition $\mathfrak{n} = \mathfrak{n}_1 \oplus ... \oplus \mathfrak{n}_r$, a direct sum of vector spaces, such that $C^i(\mathfrak{n}) = \mathfrak{n}_{i+1} \oplus ... \oplus \mathfrak{n}_r$ for all i.

Proposition 2.3. *Let $\mathfrak{n}$ be a real r-step nilpotent Lie algebra of type $(n_1, ..., n_r)$. If $\mathfrak{n}$ is Anosov then at least one of the following is true:*

(i) $n_1 \geq 4$ *and* $n_i \geq 2$ *for all* $i = 2, ..., r$.
(ii) $n_1 = n_2 = 3$ *and* $n_i \geq 2$ *for all* $i = 3, ..., r$.

In particular, $\dim \mathfrak{n} \geq 2r + 2$.

Proof. We know from Proposition 2.1 that $A_i \in \mathrm{SL}_{n_i}(\mathbb{Z})$ is hyperbolic, which implies that $n_i \geq 2$ for any i. Assuming (i) does not hold means then that $n_1 = 3$. If $n_2 = 2$ and $\{\lambda_1, \lambda_2, \lambda_3\}$ are the eigenvalues of A_1 then the eigenvalues of A_2 are of the form $\lambda_i\lambda_j$, say $\{\lambda_1\lambda_2, \lambda_1\lambda_3\}$, and hence $\lambda_1 = \lambda_1^2\lambda_2\lambda_3 = 1$, which contradicts the fact that A_1 is hyperbolic. This implies that $n_2 = 3$. □

In [L1, Question (ii)] there are examples of real Anosov Lie algebras of type $(4, 2, ..., 2)$ for any $r \geq 2$. We shall prove in Section 4 that in part (ii) of the above proposition one actually needs $n_3 \geq 3$. Also, we do not know of any example of type of the form $(3, 3, ...)$.

3. Abelian factors and a general construction

An *abelian factor* of a Lie algebra $\mathfrak{n}$ is an abelian ideal $\mathfrak{a}$ for which there exists an ideal $\tilde{\mathfrak{n}}$ of $\mathfrak{n}$ such that $\mathfrak{n} = \tilde{\mathfrak{n}} \oplus \mathfrak{a}$ (i.e. $[\tilde{\mathfrak{n}}, \mathfrak{a}] = 0$). Let $m(\mathfrak{n})$ denote the maximum dimension over all abelian factors of $\mathfrak{n}$. If $\mathfrak{z}$ is the center of $\mathfrak{n}$ then the maximal abelian factors are precisely the linear direct complements of $\mathfrak{z} \cap [\mathfrak{n}, \mathfrak{n}]$ in $\mathfrak{z}$, that is, those subspaces $\mathfrak{a} \subset \mathfrak{z}$ such that $\mathfrak{z} = \mathfrak{z} \cap [\mathfrak{n}, \mathfrak{n}] \oplus \mathfrak{a}$. Therefore

$$m(\mathfrak{n}) = \dim \mathfrak{z} - \dim \mathfrak{z} \cap [\mathfrak{n}, \mathfrak{n}].$$

Theorem 3.1. *Let $\mathfrak{n}$ be a rational Lie algebra with $m(\mathfrak{n}) = r$ and let $\mathfrak{n} = \tilde{\mathfrak{n}} \oplus \mathbb{Q}^r$ be any decomposition in ideals, that is, $\mathbb{Q}^r$ is a maximal abelian factor of $\mathfrak{n}$. Then $\mathfrak{n}$ is Anosov if and only if $\tilde{\mathfrak{n}}$ is Anosov and $r \geq 2$.*

Proof. If $\tilde{\mathfrak{n}}$ is Anosov and $r \geq 2$ then we consider the automorphism A of $\mathfrak{n}$ defined on $\tilde{\mathfrak{n}}$ as an Anosov automorphism of $\tilde{\mathfrak{n}}$ and on $\mathbb{Q}^r$ as any hyperbolic matrix in $\mathrm{GL}_r(\mathbb{Z})$. Thus A is an Anosov automorphism of $\mathfrak{n}$.

Conversely, let A be an Anosov automorphism of $\mathfrak{n}$. As in the proof of Proposition 2.1 we may assume that A is semisimple and consider the discrete (additive) subgroup

$$\mathfrak{n}_{\mathbb{Z}} = \left\{ \sum_{i=1}^{n} a_i X_i, \ a_i \in \mathbb{Z} \right\}$$

which is A-invariant. Since the center $\mathfrak{z}$ of $\mathfrak{n}$ and $\mathfrak{z}_1 = \mathfrak{z} \cap [\mathfrak{n}, \mathfrak{n}]$ are both leaved invariant by A, there exist A-invariant subspaces V and $\mathfrak{a} \subset \mathfrak{z}$ such that

$$\mathfrak{n} = V \oplus \mathfrak{z} = V \oplus \mathfrak{z}_1 \oplus \mathfrak{a}.$$

Thus $\mathfrak{a}$ is a maximal abelian factor, $\dim \mathfrak{a} = r$ and A has the form

$$A = \begin{bmatrix} A_1 & & \\ & A_2 & \\ & & A_3 \end{bmatrix}, \quad A_1 = A|_V, \quad A_2 = A|_{\mathfrak{z}_1}, \quad A_3 = A|_{\mathfrak{a}}.$$

The subgroup $\mathfrak{z}(\mathfrak{n}_{\mathbb{Z}}) = \{X \in \mathfrak{n}_{\mathbb{Z}} : [X, Y] = 0 \ \ \forall\, Y \in \mathfrak{n}_{\mathbb{Z}}\}$ is also A-invariant and it is a lattice of $\mathfrak{z}$ (i.e. a discrete subgroup of maximal rank) since for any $Z \in \mathfrak{z}$ there exist $k \in \mathbb{Z}$ such that $kZ \in \mathfrak{z}(\mathfrak{n}_{\mathbb{Z}})$ and $Z = \frac{1}{k}(kZ)$, that is, $\mathfrak{z}(\mathfrak{n}_{\mathbb{Z}}) \otimes \mathbb{Q} = \mathfrak{z}$. Since $\mathfrak{n}_{\mathbb{Z}}/\mathfrak{z}(\mathfrak{n}_{\mathbb{Z}})$ is A-invariant and $(\mathfrak{n}_{\mathbb{Z}}/\mathfrak{z}(\mathfrak{n}_{\mathbb{Z}})) \otimes \mathbb{Q} \simeq V$ we get that A_1 is unimodular. Analogously, A_2 and A_3 are unimodular since $\mathfrak{z}_1(\mathbb{Z}) = \mathfrak{z}(\mathfrak{n}_{\mathbb{Z}}) \cap [\mathfrak{n}_{\mathbb{Z}}, \mathfrak{n}_{\mathbb{Z}}]$ and $\mathfrak{z}(\mathfrak{n}_{\mathbb{Z}})/\mathfrak{z}_1(\mathbb{Z})$ are also discrete subgroups of maximal rank of $\mathfrak{z}_1$ and $\mathfrak{z}/\mathfrak{z}_1 \simeq \mathfrak{a}$, respectively.

The hyperbolicity of A guaranties the one of A_1, A_2 and A_3 and so $\tilde{\mathfrak{n}} \simeq V \oplus \mathfrak{z}_1$ is Anosov and $\dim \mathfrak{a} \geq 2$, as we wanted to show. □

We have recently became aware that there is another proof for Theorem 3.1 in [G, Proposition 7].

We now give a simple procedure to construct explicit examples of Anosov Lie algebras. This result is a generalization of [L1, Theorem 3.1] suggested by F. Grunewald.

A Lie algebra $\mathfrak{n}$ is said to be *graded* (over $\mathbb{N}$) if there exist subspaces $\mathfrak{n}_i$ of $\mathfrak{n}$ such that

$$\mathfrak{n} = \mathfrak{n}_1 \oplus \mathfrak{n}_2 \oplus ... \oplus \mathfrak{n}_k \qquad \text{and} \qquad [\mathfrak{n}_i, \mathfrak{n}_j] \subset \mathfrak{n}_{i+j}.$$

Equivalently, $\mathfrak{n}$ is graded when there are nonzero subspaces $\mathfrak{n}_{d_1}, ..., \mathfrak{n}_{d_r}$, $d_1 < ... < d_r$, such that $\mathfrak{n} = \mathfrak{n}_{d_1} \oplus ... \oplus \mathfrak{n}_{d_r}$ and if $0 \neq [\mathfrak{n}_{d_i}, \mathfrak{n}_{d_j}]$ then $d_i + d_j = d_k$ for some k and $[\mathfrak{n}_{d_i}, \mathfrak{n}_{d_j}] \subset \mathfrak{n}_{d_k}$. Recall that any graded Lie algebra is necessarily nilpotent.

Theorem 3.2. *Let $\mathfrak{n}^{\mathbb{Q}}$ be a graded rational Lie algebra, and consider the direct sum $\tilde{\mathfrak{n}}^{\mathbb{Q}} = \mathfrak{n}^{\mathbb{Q}} \oplus ... \oplus \mathfrak{n}^{\mathbb{Q}}$ (s times, $s \geq 2$). Then the real Lie algebra $\tilde{\mathfrak{n}} = \tilde{\mathfrak{n}}^{\mathbb{Q}} \otimes \mathbb{R}$ is Anosov. In other words, if $\mathfrak{n}$ is a graded real Lie algebra admitting a rational form, then $\tilde{\mathfrak{n}} = \mathfrak{n} \oplus ... \oplus \mathfrak{n}$ (s-times, $s \geq 2$) is Anosov.*

Remark 3.3. We note that the existing Anosov rational form of $\tilde{\mathfrak{n}}$ is not necessarily $\mathfrak{n}^{\mathbb{Q}} \oplus ... \oplus \mathfrak{n}^{\mathbb{Q}}$, as the case $\mathfrak{h}_3 \oplus \mathfrak{h}_3$ shows.

Proof. Let $\{X_1, ..., X_n\}$ be a $\mathbb{Z}$-basis of $\mathfrak{n}^{\mathbb{Q}}$ compatible with the gradation $\mathfrak{n}^{\mathbb{Q}} = \mathfrak{n}^{\mathbb{Q}}_{d_1} \oplus ... \oplus \mathfrak{n}^{\mathbb{Q}}_{d_r}$, that is, a basis with integer structure constants and such that each $X_i \in \mathfrak{n}^{\mathbb{Q}}_{d_j}$ for some j. We will denote this basis by $\{X_{l1}, ..., X_{ln}\}$ when we need to make clear that it is a basis of the l-th copy of $\mathfrak{n}^{\mathbb{Q}}$ in $\tilde{\mathfrak{n}}^{\mathbb{Q}}$, so the Lie bracket of $\tilde{\mathfrak{n}}^{\mathbb{Q}}$ is given by $[X_{li}, X_{l'j}] = 0$ for all $l \neq l'$, and for any $l = 1, ..., s$

$$(1) \qquad [X_{li}, X_{lj}] = \sum_{k=1}^{n} m_{ij}^k X_{lk}, \qquad m_{ij}^k \in \mathbb{Z}.$$

Every nonzero $\lambda \in \mathbb{R}$ defines an automorphism A_λ of $\mathfrak{n}^{\mathbb{Q}} \otimes \mathbb{R}$ by

$$A_\lambda|_{\mathfrak{n}^{\mathbb{Q}}_{d_i} \otimes \mathbb{R}} = \lambda^{d_i} I.$$

Let B be a matrix in $\mathrm{GL}_s(\mathbb{Z})$ with eigenvalues $\lambda_1, ..., \lambda_s$ and assume that all of them are real numbers different from ± 1 (we are using here that $s \geq 2$). This determines an automorphism A of $\tilde{\mathfrak{n}}$ in the following way: A leaves the decomposition $\tilde{\mathfrak{n}}^{\mathbb{Q}} = (\mathfrak{n}^{\mathbb{Q}} \otimes \mathbb{R}) \oplus ... \oplus (\mathfrak{n}^{\mathbb{Q}} \otimes \mathbb{R})$ invariant and on the l-th copy of $\mathfrak{n}^{\mathbb{Q}} \otimes \mathbb{R}$ coincides with A_{λ_l}.

Consider the new basis of $\tilde{\mathfrak{n}}$ defined by

$$\begin{aligned}\beta = \quad & \{X_{11} + X_{21} + ... + X_{s1}, \lambda_1 X_{11} + \lambda_2 X_{21} + ... + \lambda_s X_{s1}, ..., \\ & \lambda_1^{s-1} X_{11} + \lambda_2^{s-1} X_{21} + ... + \lambda_s^{s-1} X_{s1}, ..., ..., X_{1n} + X_{2n} + ... + X_{sn}, \\ & \lambda_1 X_{1n} + \lambda_2 X_{2n} + ... + \lambda_s X_{sn}, ..., \lambda_1^{s-1} X_{1n} + \lambda_2^{s-1} X_{2n} + ... + \lambda_s^{s-1} X_{sn}\}.\end{aligned}$$

In order to prove that β is also a $\mathbb{Z}$-basis we take two generic elements of it, say $X = \lambda_1^t X_{1i} + \lambda_2^t X_{2i} + ... + \lambda_s^t X_{si}$ and $Y = \lambda_1^u X_{1j} + \lambda_2^u X_{2j} + ... + \lambda_s^u X_{sj}$ for some $0 \leq t, u \leq s-1$ and $1 \leq i, j \leq n$. Since the λ_l's are all roots of the characteristic polynomial $f(x) = a_0 + a_1 x + ... + a_{s-1} x^{s-1} + x^s$ of B (with $a_i \in \mathbb{Z}$ and $a_0 = \pm 1$), there exist $b_0, ..., b_{s-1} \in \mathbb{Z}$ (independent from l) such that $\lambda_l^{t+u} = b_0 + b_1 \lambda_l + ... + b_{s-1} \lambda_l^{s-1}$ for any $l = 1, ..., s$. Now, by using (1) we obtain that

$$\begin{aligned}[X, Y] \quad & = \lambda_1^{t+u} [X_{1i}, X_{1j}] + ... + \lambda_s^{t+u} [X_{si}, X_{sj}] \\ & = \sum_{k=1}^{n} m_{ij}^k \lambda_1^{t+u} X_{1k} + ... + \sum_{k=1}^{n} m_{ij}^k \lambda_s^{t+u} X_{sk} \\ & = \sum_{k=1}^{n} m_{ij}^k b_0 (X_{1k} + ... + X_{sk}) + \sum_{k=1}^{n} m_{ij}^k b_1 (\lambda_1 X_{1k} + ... + \lambda_s X_{sk}) \\ & +... + \sum_{k=1}^{n} m_{ij}^k b_{s-1} (\lambda_1^{s-1} X_{1k} + ... + \lambda_s^{s-1} X_{sk}),\end{aligned}$$

showing that β is also a $\mathbb{Z}$-basis of $\tilde{\mathfrak{n}}$. Thus the linear combinations over $\mathbb{Q}$ of β determine a rational form of $\tilde{\mathfrak{n}}$, denoted by $\mathfrak{n}_\beta^{\mathbb{Q}}$, which will be now showed to be Anosov. Indeed, it is easy to see that, written in terms of β, the hyperbolic automorphism A of $\tilde{\mathfrak{n}}$ has the form

$$[A]_\beta = \begin{bmatrix} B' & & \\ & \ddots & \\ & & B' \end{bmatrix} \in \mathrm{GL}_{ns}(\mathbb{Z}),$$

where

$$B' = \begin{bmatrix} 0 & 0 & & & -a_0 \\ 1 & 0 & & & -a_1 \\ 0 & 1 & & & \\ & & \ddots & & \\ 0 & 0 & & 1 & -a_{s-1} \end{bmatrix} \in \mathrm{GL}_s(\mathbb{Z})$$

is the rational form of the matrix B, concluding the proof of the theorem. $\square$

Remark 3.4. Different choices of matrices B in the above proof can eventually give non-isomorphic Anosov rational forms of $\tilde{\mathfrak{n}}$, as in the case $\tilde{\mathfrak{n}} = \mathfrak{h}_3 \oplus \mathfrak{h}_3$ and $\tilde{\mathfrak{n}} = \mathfrak{l}_4 \oplus \mathfrak{l}_4$ (see [L2]).

Recall that two-step nilpotent Lie algebras are graded, so Theorem 3.2 shows that an explicit classification of Anosov Lie algebras up to isomorphism is a wild problem, not only in the rational case but even in the real case (see [L1] for further information).

Remark 3.5. The explicit examples of real Anosov Lie algebras in the literature so far which are not covered by Theorem 3.2 are the following: the free k-step nilpotent Lie algebras on n generators with $k < n$ (see [D], and also [DeM, De] for a different approach); certain k-step nilpotent Lie algebras of dimension $d+\binom{d}{2}+...+\binom{d}{k}$ with $d \geq k^2$ (see [F]); the 2-step nilpotent Lie algebra of type $(d, \binom{d}{2}-1)$ with center of codimension d for $d \geq 5$ (see [DeD]); certain 2-step algebras associated with graphs (see [DM]), where $\mathfrak{h}$ is attained; and the Lie algebra in [L1, Example 3.3]). For the known examples of infranilmanifolds which are not nilmanifolds and admit Anosov automorphisms we refer to [M2] and the references therein.

The *signature* of an Anosov diffeomorphism is the pair of natural numbers $\{p, q\} = \{\dim E^+, \dim E^-\}$. It is known that signature $\{1, n-1\}$ is only possible for torus and their finitely covered spaces: compact flat manifolds (see [Fr]).

If $\dim \mathfrak{n}^{\mathbb{Q}} = n$ then the signature of the Anosov automorphism of $\tilde{\mathfrak{n}}^{\mathbb{Q}} \otimes \mathbb{R}$ ($\tilde{\mathfrak{n}}^{\mathbb{Q}} = \mathfrak{n}^{\mathbb{Q}} \oplus ... \oplus \mathfrak{n}^{\mathbb{Q}}$, s times) in the proof of Theorem 3.2 is $\{np', nq'\}$, $p' + q' = s$, where p', q' are the number of eigenvalues of $B \in \mathrm{GL}_s(\mathbb{Z})$ having module greater and smaller than 1, respectively. In the nonabelian case n is necessarily ≥ 3 and so the signature $\{2, q\}$ is not allowed for this construction. We do not actually know of any nonabelian example of signature $\{2, q\}$. We may choose $\{p', q'\} = \{1, s-1\}$ and $\mathfrak{n}^{\mathbb{Q}} \otimes \mathbb{R} = \mathfrak{h}_3$ in order to obtain signature $\{3, 3(s-1)\}$ for any $s \geq 2$.

4. Two nonexistence results

In this section, we give two examples on how one can use algebraic number theory to prove that certain types are not allowed for Anosov Lie algebras. Recall that eigenvalues of an Anosov automorphism are algebraic integers. An overview on several basic properties of algebraic numbers is given in the Appendix.

Let $\mathfrak{n}$ be a real nilpotent Lie algebra which is Anosov, and let A and $\mathfrak{n} = \mathfrak{n}_1 \oplus \mathfrak{n}_2 \oplus \cdots \oplus \mathfrak{n}_r$ be as in Proposition 2.1. If $A_i = A|_{\mathfrak{n}_i}$ then the corresponding eigenvalues $\lambda_1, \dots, \lambda_{n_i}$, are algebraic units such that $1 < \mathrm{dgr}\, \lambda_i \leq n_i$ and $\lambda_1 ... \lambda_{n_i} = 1$. This follows from the fact that $[A_i]_{\beta_i} \in \mathrm{SL}_{n_i}(\mathbb{Z})$ and so its characteristic polynomial $p_{A_i}(x) \in \mathbb{Z}[x]$ is a monic polynomial with constant coefficient $a_0 = (-1)^n \det A_i = \pm 1$, satisfying $p_{A_i}(\lambda_j) = 0$ for all $j = 1, \dots, n_i$.

Case $(5, 3)$. We shall prove that there are no Lie algebras of this type with no abelian factor admitting a hyperbolic automorphism.

Suppose that A is as in Proposition 2.1. Hence as we have already pointed out, the eigenvalues of A_1, $\lambda_1, \dots, \lambda_5$ are algebraic integers with $2 \leq \mathrm{dgr}\, \lambda_j \leq 5$ for all $1 \leq j \leq 5$. If two of them coincide then, after reordering, we can assume that $\lambda_1 = \lambda_2$. This implies that $2 \leq 2\, \mathrm{dgr}\, \lambda_1 \leq 5$ and therefore $\mathrm{dgr}\, \lambda_1 = \mathrm{dgr}\, \lambda_2 = 2$. From this it is easy to see that there exist $i \in \{3, 4, 5\}$ such that $\mathrm{dgr}\, \lambda_i = 1$, contradicting the hyperbolicity of A_1. Therefore, we obtain that $\lambda_i \neq \lambda_j$, for all $i \neq j$. In this situation it is easy to see that

$$\text{if } \sharp\,(\{X_i, X_j\} \cap \{X_k, X_l\}) = 1 \qquad \text{then} \qquad [X_i, X_j] \notin \mathbb{C}[X_k, X_l]. \tag{2}$$

Moreover, since $2 \leq \mathrm{dgr}\, \mu_k \leq 3$ we have that $\mu_k \neq \mu_l$ for all $1 \leq k \neq l \leq 3$ and then for all i, j there exist k such that $[X_i, X_j] \in \mathbb{C}Z_k$.

On the other hand, it is clear that we can split the set of Lie algebras of this type according to the following condition:

(3) There are two disjoint pairs of $\{X_i\}$ such that the corresponding Lie brackets are linearly independent.

Note that if $\mathfrak{n}$ does not satisfy this condition, we will have that

$$\{X_i, X_j\} \cap \{X_l, X_k\} = \emptyset \quad \Rightarrow \quad [X_i, X_j] \in \mathbb{C}[X_l, X_k]. \tag{4}$$

If (4) holds, we can assume without any lost of generality that

$$[X_1, X_2] = Z_1 \quad [X_1, X_3] = Z_2, \tag{5}$$

and for Z_3 we have two possibilities

$$\text{a) } [X_1, X_4] = Z_3, \qquad \text{b) } [X_2, X_3] = Z_3.$$

We will now show that any of this assumptions leads to a contradiction.

Concerning a), we have that $[X_5, X_k] \neq 0$ for some $1 \leq k \leq 4$, but since $\lambda_i \neq \lambda_j$, when $i \neq j$ it is clear that $k \neq 1$. We can assume then that $k = 2$, since every other choice (i.e. $k = 3, 4$) is entirely analogous. Now, since $\{5, 2\} \cap \{1, 3\} = \emptyset$, by (4) we have that $[X_5, X_2] \in \mathbb{C}Z_2$, and analogously, $\{5, 2\} \cap \{1, 4\} = \emptyset$ and then $[X_5, X_2] \in \mathbb{C}Z_3$, giving the contradiction $[X_5, X_2] = 0$.

In case b) $\lambda_1\lambda_2\lambda_3 = 1$, and therefore $\lambda_4\lambda_5 = 1$. Thus $[X_5, X_4] = 0$, and we may assume that $0 \neq [X_4, X_1] \in \mathbb{C}Z_3$ and $0 \neq [X_5, X_2] \in \mathbb{C}Z_2$. Therefore, $\lambda_5\lambda_2 = \lambda_1\lambda_3$ and $\lambda_4\lambda_1 = \lambda_2\lambda_3$, and since $\lambda_4\lambda_5 = 1$, we get to the contradiction $\lambda_3 = 1$.

We can assume then that $\mathfrak{n}$ satisfies condition (3) and thus without any lost of generality we can suppose that

$$[X_1, X_2] = Z_1 \qquad [X_3, X_4] = Z_2. \tag{6}$$

Note that we can not have $[X_5, X_j] = Z_3$ because this would imply $\lambda_j = 1$ by using that $\lambda_1 \dots \lambda_5 = 1$. Let us say then that $[X_5, X_j] = aZ_1$, $a \neq 0$. From (2) we have that $j \neq 1, 2$, and since both cases $j = 3$ and $j = 4$ are completely analogous, we will just analyze the case $j = 3$. This is

$$[X_1, X_2] = Z_1, \;\; [X_3, X_4] = Z_2, \;\; [X_5, X_3] = aZ_1.$$

Also, since $Z_3 \in [\mathfrak{n}, \mathfrak{n}]$ there is $1 \leq k, k' \leq 4$ such that $[X_k, X_{k'}] = Z_3$, and by the above observations, it is easy to see that

$$\{k, k'\} = \begin{cases} \{1, 3\} \text{ or (equivalently) } \{2, 3\} \\ \\ \{1, 4\} \text{ or (equivalently) } \{2, 4\} \end{cases}$$

To finish the proof, we will see that both cases leads to a contradiction. The idea is to show that one of the λ_i is equal to one of the μ_j, and since the conjugated numbers are uniquely determined, this implies that every μ_j appears as a λ_k. From here it is easy to check in both cases that this is not possible.

Indeed, if $[X_1, X_3] = Z_3$, since $1 = \lambda_5\lambda_3\lambda_3\lambda_4\lambda_1\lambda_3$, we have that $\lambda_3^2 = \lambda_2$. Therefore, $\lambda_5\lambda_3 = \lambda_1\lambda_2 = \lambda_1\lambda_3^2$ and so $\lambda_5 = \lambda_1\lambda_3 = \mu_3$. Hence, there exists i such that $\mu_1 = \lambda_1\lambda_3^2 = \lambda_i$. It is clear that $i \neq 1, 2, 3, 5$ and if $\lambda_1\lambda_3^2 = \lambda_4$, since $1 = \lambda_1\lambda_2\lambda_3\lambda_4\lambda_1\lambda_3 = \lambda_1^2\lambda_3^4\lambda_4$, then $1 = \lambda_1^3\lambda_3^6 = \mu_1^3$ contradicting the fact that A_2 is hyperbolic.

Now, if $[X_1, X_4] = Z_3$, then

(i) $1=\lambda_5\lambda_3\lambda_3\lambda_4\lambda_1\lambda_4$, and from there $\lambda_2=\lambda_3\lambda_4=\mu_2$, and
(ii) $1=\lambda_1\lambda_2\lambda_3\lambda_4\lambda_1\lambda_4$, hence $\lambda_5=\lambda_1\lambda_4=\mu_3$.

Therefore, as we have observed before, there is k such that $\mu_1=\lambda_k$. This implies that $\lambda_1\lambda_2=\lambda_5\lambda_3=\lambda_k$ for some $1\leq k\leq 5$. Again, it is clear that $k\neq 1,2,3,5$, and if $\lambda_1\lambda_2=\lambda_5\lambda_3=\lambda_4$, then by (ii) $\lambda_1\lambda_4\lambda_3=\lambda_5\lambda_3=\lambda_4$ and hence $\lambda_1\lambda_3=1$. From this, using that $1=\det A_2=\lambda_4\lambda_2\lambda_5$, we obtain that $\lambda_1\lambda_2=\lambda_5\lambda_3=\frac{1}{\lambda_2\lambda_4}.\frac{1}{\lambda_1}$, or equivalently $\lambda_4^2=(\lambda_1\lambda_2)^2=\frac{1}{\lambda_4}$ and then $\lambda_4=1$ contradicting the fact that A_1 is hyperbolic, and concluding the proof of case $(5,3)$.

Case $(3,3,2)$. We will show in this case that there is no Anosov Lie algebra. We will begin by noting that since $\mathfrak{n}_2$ has dimension three, we may assume that

$$[X_1,X_2]=Y_3,\quad [X_1,X_3]=Y_2,\quad [X_2,X_3]=Y_1,$$

where $\{X_1,X_2,X_3\}$ and $\{Y_1,Y_2,Y_3\}$ are basis of $(\mathfrak{n}_1)_{\mathbb{C}}$ and $(\mathfrak{n}_2)_{\mathbb{C}}$ of eigenvectors of A_1 and A_2, respectively.

It follows that

$$[X_1,Y_1]=0,\quad [X_2,Y_2]=0,\quad [X_3,Y_3]=0, \tag{7}$$

since any of them would be an eigenvector of A of eigenvalue $\lambda_1\lambda_2\lambda_3=1$ and then A_3 would not be hyperbolic.

On the other hand, since $Z_1,Z_2\in\mathfrak{n}_3$ we have that for some i,j,k,l

$$[X_i,Y_j]=Z_1,\qquad [X_k,Y_l]=Z_2,$$

and thus $i\neq k$. Indeed, if $i=k$ then $j\neq l$ and by (7) $j,l\neq i$. This would imply that $\lambda_i.\lambda_i\lambda_j.\lambda_i.\lambda_i\lambda_l=1$ and so $\lambda_i^3=1$, a contradiction.

Hence we can assume that

$$[X_1,Y_j]=Z_1\qquad [X_2,Y_l]=Z_2.$$

For the pairs (j,l) we have four possibilities as follows: $(2,1),(2,3),(3,1)$ and $(3,3)$. In order to discard some of them, we recall that since $\dim\mathfrak{n}_1=3$, $\lambda_i\neq\lambda_j$ for all $1\leq i,j\leq 3$ and from this, it follows that $(j,l)\neq(3,1)$ or $(2,3)$. Indeed, if $(j,l)=(3,1)$ (or $(2,3)$) we have that $\lambda_1\lambda_1\lambda_2\lambda_2\lambda_2\lambda_3=1$ (or $\lambda_1\lambda_1\lambda_3\lambda_2\lambda_1\lambda_2=1$). Hence $\lambda_1\lambda_2^2=1$ (or $\lambda_1^2\lambda_2=1$) and we get to the contradiction $\lambda_2=\lambda_3$ (or $\lambda_1=\lambda_3$).

It is also easy to see that $(j,l)\neq(3,3)$ since this implies $\lambda_1\lambda_1\lambda_2\lambda_2\lambda_1\lambda_2$ and so $\lambda_1\lambda_2=1$, contradicting the fact that A_2 is hyperbolic. Finally, assume that $(j,l)=(2,1)$, that is, in $\mathfrak{n}_{\mathbb{C}}$ we have at least the following non trivial brackets:

$$\begin{aligned}&[X_1,X_2]=Y_3,\quad [X_1,X_3]=Y_2,\quad [X_2,X_3]=Y_1,\\ &[X_1,Y_2]=Z_1,\quad [X_2,Y_1]=Z_2.\end{aligned} \tag{8}$$

Let $\lambda_1=\lambda$ and $\lambda_2=\nu$, then the matrix of A is given by

$$[A]=\begin{bmatrix} B &&&\\ & B^{-1} &&\\ && \frac{\lambda}{\nu} &\\ &&& \frac{\nu}{\lambda}\end{bmatrix},\qquad\text{where}\qquad B=\begin{bmatrix}\lambda &&\\ &\nu&\\ &&\frac{1}{\lambda\nu}\end{bmatrix},$$

and B is conjugated to an element of $\mathrm{SL}_3(\mathbb{Z})$. Thus $\frac{\lambda}{\nu}$ is an algebraic unit with $|\frac{\lambda}{\nu}|\neq 1$ and $\operatorname{dgr}\frac{\lambda}{\nu}=2$. It is easy to see that under such conditions $\frac{\lambda}{\nu}$ is necessarily a real number. Since the possibilities for ν are either $\nu=\overline{\lambda}$ or $\frac{1}{|\lambda|^2}$, we obtain that $\lambda,\nu\in\mathbb{R}$, which is a contradiction by the following lemma applied to λ^2,ν^2. This concludes the proof of this case.

Lemma 4.1. Let λ_1, λ_2 be two positive totally real algebraic integers of degree 3. If λ_1 and λ_2 are conjugated and units then $\frac{\lambda_1}{\lambda_2}$ can never have degree two.

Proof. Let λ_1 and λ_2 be as in the lemma, then the minimal polynomial of λ_i is given by $m_{\lambda_i}(x) = (x-\lambda_1)(x-\lambda_2)(x-\lambda_3)$, where $\lambda_1\lambda_2\lambda_3 = \pm 1$. Since m_{λ_i} has its coefficients in $\mathbb{Z}$, we have that

$$\lambda_1 + \lambda_2 + \lambda_3 \in \mathbb{Z}, \qquad \frac{1}{\lambda_1} + \frac{1}{\lambda_2} + \frac{1}{\lambda_3} \in \mathbb{Z},$$

and hence

$$\lambda_1^2 + \lambda_2^2 + \lambda_3^2 = d \in \mathbb{Z}. \tag{9}$$

On the other hand, if we assume that λ_1/λ_2 has degree two then $\frac{\lambda 1}{\lambda_2} + \frac{\lambda_2}{\lambda_1} = a \in \mathbb{Z}$, and thus

$$\frac{\lambda_1}{\lambda_2} = \frac{a}{2} + \sqrt{\frac{a^2}{4} - 1} \quad \text{and} \quad \frac{\lambda_2}{\lambda_1} = \frac{a}{4} - \sqrt{\frac{a^2}{2} - 1}.$$

Recall that $a \geq 2$. We also note that $\frac{\lambda_1}{\lambda_2} = \pm\lambda_1^2\lambda_3$ and $\frac{\lambda_2}{\lambda_1} = \pm\lambda_2^2\lambda_3$, and hence $\lambda_1^2 = \pm\frac{1}{\lambda_3}\left(\frac{a}{2} + \sqrt{\frac{a^2}{4} - 1}\right)$ and $\lambda_2^2 = \pm\frac{1}{\lambda_3}\left(\frac{a}{2} - \sqrt{\frac{a^2}{4} - 1}\right)$. By replacing this in (9) we obtain $\pm\frac{1}{\lambda_3}a + \lambda_3^2 = d$, or equivalently,

$$\lambda_3^3 - \lambda_3 d \pm a = 0.$$

This means that $p(x) = x^3 - dx \pm a$ is a monic polynomial of degree 3 with coefficient in $\mathbb{Z}$ which is annihilated by λ_3. Hence it is equal to the minimal polynomial of λ_3 and then $a = \pm 1$, which is a contradiction since as we have observed above, $a \geq 2$. □

We would like to point out that in this lemma, we are strongly using the fact that λ_1 and λ_2 are totally real algebraic numbers and units. Indeed, if we consider $p(x) = x^3 - 2$, the roots of p are $\left\{\lambda_1 = 2^{1/3},\ \lambda_2 = \omega 2^{1/3},\ \lambda_3 = \omega^2 2^{1/3}\right\}$, where $\omega^2 + \omega + 1 = 0$. Since $x^3 - 2$ is indecomposable over $\mathbb{Q}$, we have that $\operatorname{dgr}\lambda_i = 3$ for all $i = 1, 2, 3$, and however $\lambda_2 . \frac{1}{\lambda_1} = \omega$ has degree two.

5. Appendix: Algebraic numbers

We give in this section a short summary of some results about algebraic numbers over $\mathbb{Q}$ that are used in Section 4. We are mainly following [La, Chapter V]. Note that we will omit information on numberfields since we are not going to need it.

An element $\lambda \in \mathbb{C}$ is called *algebraic over* $\mathbb{Q}$ if there exist a polynomial $p(x) \in \mathbb{Q}[x]$ such that $p(\lambda) = 0$. It is easy to see that the set D of all such polynomials form an ideal in $\mathbb{Q}[x]$ and since this is a principal ideal domain, D is generated by a single polynomial. This polynomial can be chosen to be monic, and in that case it is uniquely determined by λ and will be called the *minimal polynomial of* λ, denoted by $m_\lambda(x)$. Therefore, if we have an algebraic number λ then we can define *the degree of* λ as the degree of $m_\lambda(x)$. It will be denoted by $\operatorname{dgr}\lambda$. The minimal polynomial $m_\lambda(x)$ is irreducible over $\mathbb{Q}$ and λ is not a double root of $m_\lambda(x)$.

If $\lambda \neq \mu$ are two algebraic numbers, we say that they are *conjugated* if $m_\lambda(\mu) = 0$. Note that the numbers which are conjugated to λ are uniquely determined by λ and have the same degree.

An algebraic number λ is said to be an *algebraic integer* if there exists a monic polynomial $p(x) \in \mathbb{Z}[x]$ such that $p(\lambda) = 0$. It can be seen that in this case,

$m_\lambda(x) \in \mathbb{Z}[x]$, and moreover, these conditions are actually equivalent. An algebraic number is called *totally real* if $m_\lambda(x)$ has only real roots, that is, $m_\lambda(x) = \prod_{i=1}^{r}(x-\lambda_i)$ with $\lambda_i \in \mathbb{R}$, $\lambda_1 = \lambda$. If λ is a totally real algebraic number with $\operatorname{dgr}\lambda = r$, set $A_\lambda = \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_r \end{bmatrix}$. The characteristic polynomial of A_λ is $m_\lambda(x)$ and then the rational form of A_λ is given by

$$\begin{bmatrix} 0 & 0 & \dots & 0 & -a_0 \\ 1 & 0 & \dots & 0 & -a_1 \\ 0 & 1 & & 0 & -a_2 \\ \vdots & & \ddots & & \vdots \\ 0 & & & 1 & -a_{r-1} \end{bmatrix},$$

where $m_\lambda(x) = x^r + a_{r-1}x^{r-1} + \dots + a_1x + a_0$. If λ is an algebraic integer then $a_i \in \mathbb{Z}$ for all $i = 0, \dots, r-1$ and then this shows that A_λ is conjugated to an element in $\mathrm{GL}_r(\mathbb{Z})$.

Conversely, if $A = \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_r \end{bmatrix}$ is conjugated to an element of $\mathrm{GL}_r(\mathbb{Z})$, then if $p_A(x)$ is the characteristic polynomial of A, $p_A(x) \in \mathbb{Z}[x]$, and therefore λ_i is an algebraic integer for all $i = 1, \dots, r$. Concerning the degree of the λ_i's as algebraic numbers in such a case, we can only say that $1 \leq \operatorname{dgr}\lambda_i \leq r$. Moreover, if $\lambda_i = \lambda_j$ for some $i \neq j$, and since λ is not a double root of $m_\lambda(x)$, we will have that $m^2_{\lambda_i}(x)|p_A(x)$ and hence $1 \leq 2\operatorname{dgr}\lambda_i \leq r$.

If λ is an algebraic integer, we say that λ is a *unit* if $1/\lambda$ is an algebraic integer as well. If it is so, then the constant coefficient a_0 of $m_\lambda(x)$ is $(-1)^n$, where $n = \operatorname{dgr}\lambda$. Conversely, if $a_0 = \pm 1$ then λ is a unit.

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