Restriction of the Fourier transform to bidimensional anisotropically homogeneous hypersurfaces[*†‡]

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Abstract

For $x = (x_1, x_2) \in \mathbb{R}^2$ and $\beta_1, \beta_2 > 1$, let $\varphi : \mathbb{R}^2 \to \mathbb{R}$ be defined by $\varphi(x) = |x_1|^{\beta_1} + |x_2|^{\beta_2}$, let B be the open unit ball in $\mathbb{R}^2$ and let $\Sigma = \{(x, \varphi(x)) : x \in B\}$. For $f \in S(\mathbb{R}^3)$, let $\mathcal{R}f : \Sigma \to \mathbb{C}$ be defined by

$$(\mathcal{R}f)(x, \varphi(x)) = \widehat{f}(x, \varphi(x)) \qquad x \in B,$$

where $\widehat{f}$ denotes the usual Fourier transform of f. Let σ be the Borel measure on Σ defined by $\sigma(A) = \int_B \chi_A(x, \varphi(x))\, dx$ and let E be the type set for the operator $\mathcal{R}$, i.e, the set of the pairs $\left(\frac{1}{p}, \frac{1}{q}\right) \in [0,1] \times [0,1]$ for which there exists $c > 0$ such that $\left\|\widehat{f}\right\|_{L^q(\Sigma)} \leq c \|f\|_{L^p(\mathbb{R}^3)}$ for all $f \in S(\mathbb{R}^3)$. In this paper we give necessary conditions for $\left(\frac{1}{p}, \frac{1}{q}\right) \in E$. We also obtain new points in E.

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1 Introduction

For $x = (x_1, x_2) \in \mathbb{R}^2$ and $1 < \beta_1 \leq \beta_2$, let $\varphi : \mathbb{R}^2 \to \mathbb{R}$ be defined by $\varphi(x) = |x_1|^{\beta_1} + |x_2|^{\beta_2}$, let B be the open unit ball in $\mathbb{R}^2$ and let $\Sigma = \{(x, \varphi(x)) : x \in B\}$. For $f \in S(\mathbb{R}^3)$, let $\mathcal{R}f : \Sigma \to \mathbb{C}$ be defined by

$$(\mathcal{R}f)(x, \varphi(x)) = \widehat{f}(x, \varphi(x)) \qquad x \in B,$$

where $\widehat{f}$ denotes the usual Fourier transform of f defined by

$$\widehat{f}(\xi) = \int f(u)\, e^{-i\langle u, \xi \rangle} du.$$

Let σ be the Borel measure on Σ defined by $\sigma(A) = \int_B \chi_A(x, \varphi(x))\, dx$ and let E be the type set for the operator $\mathcal{R}$, i.e. the set of the pairs $\left(\frac{1}{p}, \frac{1}{q}\right) \in [0,1] \times [0,1]$ for which there exists $c > 0$ such that $\left\| \widehat{f} \right\|_{L^q(\Sigma)} \leq c \|f\|_{L^p(\mathbb{R}^3)}$ for all $f \in S(\mathbb{R}^3)$, where the spaces $L^p(\mathbb{R}^3)$ and $L^q(\Sigma)$ are taken with respect to the Lebesgue measure in $\mathbb{R}^3$ and the measure σ respectively.

In the general n-dimensional case, the $L^p(\mathbb{R}^{n+1}) - L^q(\Sigma)$ boundedness properties of the restriction operator $\mathcal{R}$ have been studied by different authors. A very interesting survey about the recent progress in this research area con be found in [9]. The $L^p(\mathbb{R}^{n+1}) - L^2(\Sigma)$ restriction theorems for the sphere was proved by Stein in 1967, for $1 \leq p < \frac{4n+4}{3n+4}$; for $1 \leq p < \frac{2n+4}{n+4}$ by [11] and then in the same year by Stein for $1 \leq p \leq \frac{4n+4}{3n+4}$. The last argument has been used in several related contexts by Strichartz in [8] and Greenleaf in [6] . This method provides a general tool to obtain, from suitable estimates for $\widehat{\sigma}$, $L^p(\mathbb{R}^{n+1}) - L^2(\Sigma)$ estimates for $\mathcal{R}$. Moreover, a general theorem, due to Stein, holds for smooth enough hypersurfaces with never vanishing Gaussian curvature ([7], pp.386). There it is shown that, in this case, $\left(\frac{1}{p}, \frac{1}{q}\right) \in E$ if $\frac{n+4}{2n+4} \leq \frac{1}{p} \leq 1$ and $\frac{1}{q} \geq -\frac{n+2}{n}\frac{1}{p} + \frac{n+2}{n}$, also that this last relation is the best possible and that no restriction theorem of any kind can hold for $f \in L^p(\mathbb{R}^{n+1})$ when $\frac{1}{p} \leq \frac{n+2}{2n+2}$ ([7], pp.388). The cases $\frac{n+2}{2n+2} < \frac{1}{p} < \frac{n+4}{2n+4}$ are not completely solved. The best results for surfaces with non vanishing curvature like the paraboloid and the sphere are due to Terence Tao [10]. Restriction theorems for the Fourier transform to homogeneous polynomial surfaces in $\mathbb{R}^3$ are obtained in [5].

Turning back to our problem, the type set E is studied in [1]. We prove results about E that improve those obtained by Dellanegra. In section 2 we obtain a better necessary condition, wich is a consequence of the characterization of the type set concerning the convolution operatos with σ, that is described in [4]. In section 3, using results obtained in [2] and [3], we obtain new points that belong to E.

2 Necessary conditions

It is well known that for a manifold Σ as above, we have that if $\left(\frac{1}{p},\frac{1}{q}\right)\in E$ then $\frac{1}{q}\geq-\frac{2}{p}+2$.

A standard homogeneity argument gives the following result

Proposition 1 *If* $\left(\frac{1}{p},\frac{1}{q}\right)\in E$ *then*

$$\frac{1}{q}\geq-\frac{\beta_1+\beta_2+\beta_1\beta_2}{\beta_1+\beta_2}\frac{1}{p}+\frac{\beta_1+\beta_2+\beta_1\beta_2}{\beta_1+\beta_2}. \tag{1}$$

Proof. Let

$$t.\left(x_1,x_2\right)=\left(t^{\beta_2}x_1,t^{\beta_1}x_2\right),$$
$$t\circ\left(x_1,x_2,x_3\right)=\left(t^{\beta_2}x_1,t^{\beta_1}x_2,t^{\beta_1\beta_2}x_3\right).$$

For a fixed $l\in\mathbb{Z}$ we define

$$A_0=\left\{x=\left(x_1,x_2\right)\in\mathbb{R}^2:2^{l-1}\leq\|x\|\leq 2^l\right\} \tag{2}$$

where $\|(x_1,x_2)\|=|x_1|^{\frac{1}{\beta_2}}+|x_2|^{\frac{1}{\beta_1}}$, and for $j\in\mathbb{N}$

$$A_j=2^{-j}.A_0. \tag{3}$$

We choose $l\in\mathbb{Z}$ such that $\left\{x\in\mathbb{R}^2:\|x\|\leq 2^l\right\}\subset B$. We denote

$$\Sigma_j=\left\{\left(x,\varphi\left(x\right)\right)\in\mathbb{R}^3:x\in A_j\right\}.$$

and, for $f\in S(\mathbb{R}^3)$ we define $\mathcal{R}^{A_j}f=\widehat{f}_{|\Sigma_j}$ and $f_{2^j}\left(x\right)=f\left(2^j\circ x\right)$. Thus

$$\left\|\mathcal{R}^{A_j}f\right\|_{L^q(\Sigma_j)}=2^{-j\frac{\beta_1+\beta_2}{q}}\left(\int_{A_0}\left|\widehat{f}\left(2^{-j}.x,\varphi\left(2^{-j}.x\right)\right)\right|^q dx\right)^{\frac{1}{q}}$$
$$=2^{-j\left(\frac{\beta_1+\beta_2}{q}-(\beta_1+\beta_2+\beta_1\beta_2)\right)}\left\|\mathcal{R}^{A_0}f_{2^j}\right\|_{L^q(\Sigma_0)}$$

From this, it follows that

$$\left\| \mathcal{R}^{A_j} \right\|_{p,q} = 2^{-j\left(\frac{\beta_1+\beta_2}{q} - (\beta_1+\beta_2+\beta_1\beta_2) + \frac{\beta_1+\beta_2+\beta_1\beta_2}{p} \right)} \left\| \mathcal{R}^{A_0} \right\|_{p,q} \tag{4}$$

Now, since $\left\| \mathcal{R}^{A_j} \right\|_{p,q} \leq \| \mathcal{R} \|_{p,q}$ the proposition follows. ■

Let T be the given by $Tf = \sigma * f$, $f \in S(\mathbb{R}^3)$ and let E_σ be the associated type set, i.e. the set of the pairs $\left(\frac{1}{p}, \frac{1}{q} \right) \in [0,1] \times [0,1]$ for which there exists $c > 0$ such that $\|Tf\|_{L^q(\mathbb{R}^3)} \leq c \|f\|_{L^p(\mathbb{R}^3)}$ for all $f \in S(\mathbb{R}^3)$. In [3] we give neccesary conditions for $\left(\frac{1}{p}, \frac{1}{q} \right) \in E_\sigma$. The next proposition relates the type sets E and E_σ.

Proposition 2 *If* $\left(\frac{1}{p}, \frac{1}{q} \right) \in E$ *for some* $1 \leq p \leq 2$ *and* $1 \leq q \leq \infty$, *then* $\left(\frac{3p-2}{2p}, \frac{1}{2} \right) \in E_\sigma$.

Proof. If for some $1 \leq p \leq 2$ and $1 \leq q \leq \infty$ we have $\|\mathcal{R}f\|_{L^q(\Sigma)} \leq c_p \|f\|_{L^p(\mathbb{R}^3)}$ then $\widehat{\sigma} \in L^{p'}(\mathbb{R}^3)$. So

$$\|\sigma * f\|_2 = \left\| \widehat{\sigma * f} \right\|_2 = \left\| \widehat{\sigma} \widehat{f} \right\|_2 = \left(\int_{\mathbb{R}^3} |\widehat{\sigma}(x)|^2 \left| \widehat{f(x)} \right|^2 dx \right)^{\frac{1}{2}}$$

$$\leq \left(\int_{\mathbb{R}^3} |\widehat{\sigma}(x)|^{p'} dx \right)^{\frac{1}{p'}} \left(\int_{\mathbb{R}^3} \left| \widehat{f(x)} \right|^{2\left(\frac{p'}{2}\right)'} dx \right)^{\frac{1}{2\left(\frac{p'}{2}\right)'}}$$

$$\leq c \left(\int_{\mathbb{R}^3} \left| \widehat{f(x)} \right|^{2\left(\frac{p'}{2}\right)'} dx \right)^{\frac{1}{2\left(\frac{p'}{2}\right)'}}$$

$$= c \left\| \widehat{f} \right\|_{\frac{2p}{2-p}} \leq c \|f\|_{\frac{2p}{3p-2}},$$

and the proposition follows. ■

The following lemma is contained in Lemma 1 in [2].

Lemma 3 *If* $\left(\frac{1}{p}, \frac{1}{q} \right) \in E_\sigma$ *then* $\frac{1}{q} \geq \frac{2\beta_2+1}{\beta_2+1} \frac{1}{p} - 1$.

From Proposition 2 and this lemma we obtain the following necessary condition.

Corollary 4 *If* $\left(\frac{1}{p},\frac{1}{q}\right) \in E$ *for some* $1 \leq p \leq 2$ *and* $1 \leq q \leq \infty$, *then* $\frac{1}{p} \geq \frac{3\beta_2}{4\beta_2+2}$.

Proof. From Proposition 2 we know that if $\left(\frac{1}{p},\frac{1}{q}\right) \in E$ for some $1 \leq p \leq 2$ and $1 \leq q \leq \infty$, then $\left(\frac{3p-2}{2p},\frac{1}{2}\right) \in E_\sigma$ but then by Lemma 3, $\frac{1}{2} \geq \frac{2\beta_2+1}{\beta_2+1}\frac{3p-2}{2p} - 1$ and then $\frac{1}{p} \geq \frac{3\beta_2}{4\beta_2+2}$. ■

Remark 5 *In the article [3] we also obtain two additional necessary conditions, contained in Lemmas 2.1 and 2.2. Using these conditions, Proposition 2 and proceeding as in the Corollary 4, we get* $\frac{1}{p} \geq \frac{2}{3}$ *and* $\frac{1}{p} \geq \frac{\beta_1\beta_2}{\beta_1\beta_2+\beta_1+\beta_2}$ *respectively. The first inequality is a known result as we mentioned in the Introduction and the last inequality can also be obtained from Proposition 1, taking* $q = 1$. *The corollary adds, in some cases, necessary conditions not contained in the before mentioned results. For example, if we take* $\beta_1 = 2$ *and* $\beta_2 = 5$ *the corollary says* $\frac{1}{p} \geq \frac{15}{22}$ *and proposition 1, with* $q = 1$ *implies* $\frac{1}{p} \geq \frac{10}{17}$. *We note that* $\frac{10}{17} < \frac{2}{3} < \frac{15}{22} < \frac{3}{4}$.

3 Sufficient Conditions

If V is a measurable set in $\mathbb{R}^3$ we denote $\Sigma^V = \{(x,\varphi(x)) : x \in V\}$ and σ^V the associated surface measure. Also, for $f \in S(\mathbb{R}^3)$, we define $\mathcal{R}^V f : \Sigma^V \to \mathbb{C}$ by

$$\left(\mathcal{R}^V f\right)(x,\varphi(x)) = \widehat{f}(x,\varphi(x)) \qquad x \in V,$$

we note that $\mathcal{R}^B = \mathcal{R}$ and $\Sigma^B = \Sigma$.

Remark 6 *We take* $l \in \mathbb{Z}$ *such that* $B \subset \left\{x \in \mathbb{R}^2 : \|x\| \leq 2^l\right\}$. *We define* A_0 *and* A_j, $j \in \mathbb{N}$ *by* (2) *and* (3) *respectively. If*

$$\frac{1}{q} > -\frac{\beta_1+\beta_2+\beta_1\beta_2}{\beta_1+\beta_2}\frac{1}{p} + \frac{\beta_1+\beta_2+\beta_1\beta_2}{\beta_1+\beta_2}$$

and

$$\left\|\mathcal{R}^{A_0} f\right\|_{L^q\left(\Sigma^{A_0}\right)} \leq c_p \|f\|_{L^p(R^3)},$$

for all $f \in S(\mathbb{R}^3)$, *then, from* (4), *by summing over* $j \in \mathbb{N} \cup \{0\}$, *we obtain that* $\left(\frac{1}{p},\frac{1}{q}\right) \in E$.

Remark 7 *In [5], following a Strichartz's theorem (see [8]), we prove that if*

$$\left|\left(\sigma^V\right)^\wedge(\xi)\right| \leq A\left(1+|\xi_3|\right)^{-\tau}$$

for some $\tau>0$ and for all $\xi=(\xi_1,\xi_2,\xi_3)\in\mathbb{R}^3$, then

$$\left\|\mathcal{R}^V\right\|_{L^p(\mathbb{R}^3),L^2(\Sigma^V)} \leq c_\tau A^{\frac{1}{2(1+\tau)}}$$

for $p=\frac{2+2\tau}{2+\tau}$ and c_τ a positive constant depending only on τ.

Theorem 8 *Let E be the type set defined in the Introduction and suppose $2\leq\beta_1\leq\beta_2$.*

a) If $\frac{(\beta_2+2)(\beta_1+\beta_2+\beta_2\beta_1)}{(\beta_1+\beta_2)(6\beta_2+4)}<\frac{1}{q}\leq 1$ then $\left(\frac{5\beta_2+2}{6\beta_2+4},\frac{1}{q}\right)\in E$.

b) If $\beta_2\geq 6$ and $\frac{\beta_2}{\beta_2+2}<\frac{1}{p}\leq 1$ then $\left(\frac{1}{p},1\right)\in E$.

c) If $\beta_2\geq 6$ and $\frac{1}{p}>\frac{\beta_2}{\beta_2+2}$ then the open segment with vertices $\left(\frac{1}{p},1\right)$ and $\left(\frac{3\beta_2^2+\beta_1\beta_2^2-5\beta_1\beta_2+2\beta_2-2\beta_1}{4\beta_2^2+\beta_1\beta_2^2-4\beta_1\beta_2-4\beta_1},\frac{(\beta_2-2)(\beta_1+\beta_2+\beta_2\beta_1)}{4\beta_2^2+\beta_1\beta_2^2-4\beta_1\beta_2-4\beta_1}\right)$ is contained in E.

d) If $\beta_2<6$ and $\frac{\beta_2+2}{8}<\frac{1}{q}\leq 1$ then $\left(\frac{3}{4},\frac{1}{q}\right)\in E$.

e) If $\beta_2<6$ and $\frac{\beta_2+2}{8}<\frac{1}{q}\leq 1$ then the open segment with vertices $\left(\frac{3}{4},\frac{1}{q}\right)$ and $\left(\frac{5\beta_2+\beta_1}{6\beta_2+2\beta_1},\frac{\beta_2+\beta_2\beta_1+\beta_1}{6\beta_2+2\beta_1}\right)\in E$.

f) $\left(\frac{\beta_1+\beta_2+2\beta_1\beta_2}{2\beta_1+2\beta_2+2\beta_1\beta_2},\frac{1}{2}\right)\in E$.

Proof. We take $l\in\mathbb{Z}$ such that $B\subset\left\{x\in\mathbb{R}^2:\|x\|\leq 2^l\right\}$. Without loss of generality, we suppose that x belongs to the first quadrant, $\mathbb{R}_{\geq 0}\times\mathbb{R}_{\geq 0}$. For $j,k\in\mathbb{N}$ we define $Q_{j,k}=\left[2^{-j+l-1},2^{-j+l}\right]\times\left[2^{-k+l-1},2^{-k+l}\right]$ and $Q_j=\left[2^{-j+l-1},2^{-j+l}\right]\times\left[0,2^l\right]$. From (3.2) in [2] we have, for $\xi=(\xi_1,\xi_2,\xi_3)\in\mathbb{R}^3$,

$$\left|\widehat{\sigma^{Q_{j,k}}}(\xi)\right| \leq c\frac{2^{j\left(\frac{\beta_1-2}{2}\right)+k\left(\frac{\beta_2-2}{2}\right)}}{1+|\xi_3|}$$

and

$$\left|\widehat{\sigma^{Q_j}}(\xi)\right| \leq c\frac{2^{j\left(\frac{\beta_1-2}{2}\right)}}{(1+|\xi_3|)^{\frac{1}{2}+\frac{1}{\beta_2}}}$$

so, from Remark 7 it follows that

$$\left\|\mathcal{R}^{Q_{j,k}}\right\|_{L^{\frac{4}{3}}(\mathbb{R}^3),L^2\left(\Sigma^{Q_{j,k}}\right)} \leq c2^{j\left(\frac{\beta_1-2}{8}\right)+k\left(\frac{\beta_2-2}{8}\right)} \tag{5}$$

and

$$\left\|\mathcal{R}^{Q_j}\right\|_{L^{p_1}(\mathbb{R}^3),L^2\left(\Sigma^{Q_j}\right)} \leq c2^{j\frac{(\beta_1-2)\beta_2}{6\beta_2+4}} \tag{6}$$

for $p_1 = \frac{6\beta_2+4}{5\beta_2+2}$. Also, if we define $Q'_k = \left[0, 2^l\right] \times \left[2^{-k+l-1}, 2^{-k+l}\right]$, in a similar way we obtain

$$\left|\widehat{\sigma^{Q'_k}}(\xi)\right| \leq c\frac{2^{k\left(\frac{\beta_2-2}{2}\right)}}{(1+|\xi_3|)^{\frac{1}{2}+\frac{1}{\beta_1}}} \leq c\frac{2^{k\left(\frac{\beta_2-2}{2}\right)}}{(1+|\xi_3|)^{\frac{1}{2}+\frac{1}{\beta_2}}},$$

since A_0 is contained in a finite union of Q_j's and Q'_k's, again from Remark 7, it follows that

$$\left\|\mathcal{R}^{A_0}\right\|_{L^{p_1}(\mathbb{R}^3),L^2\left(\Sigma^{Q_j}\right)} \leq c.$$

From the Hölder's inequality, we also get

$$\left\|\mathcal{R}^{A_0}\right\|_{L^{p_1}(\mathbb{R}^3),L^q\left(\Sigma^{Q_j}\right)} \leq c, \tag{7}$$

for $1 \leq q \leq 2$. Since $\left(\frac{1}{p_1}, \frac{(\beta_2+2)(\beta_1+\beta_2+\beta_2\beta_1)}{(\beta_1+\beta_2)(6\beta_2+4)}\right)$ satisfies (1), from (7) and Remark 6, a) follows.

Now, from (5) and the Hölder's inequality, we obtain, for $1 \leq q \leq 2$,

$$\left\|\mathcal{R}^{Q_{j,k}}\right\|_{L^{\frac{4}{3}}(\mathbb{R}^3),L^q\left(\Sigma^{Q_{j,k}}\right)} \leq c2^{j\left(\frac{\beta_1-2}{8}+\frac{q-2}{2q}\right)+k\left(\frac{\beta_2-2}{8}+\frac{q-2}{2q}\right)}. \tag{8}$$

It is also easy to check that

$$\left\|\mathcal{R}^{Q_{j,k}}\right\|_{L^1(\mathbb{R}^3),L^1\left(\Sigma^{Q_{j,k}}\right)} \leq c2^{-j-k}. \tag{9}$$

We use (8) with $q = 1$ and (9), to obtain, from the Riesz Thorin theorem, that for $0 \leq t \leq 1$ and $\frac{1}{p} = t\frac{3}{4} + (1-t)$,

$$\left\|\mathcal{R}^{Q_{j,k}}\right\|_{L^p(\mathbb{R}^3),L^1\left(\Sigma^{Q_{j,k}}\right)} \leq c2^{j\left(t\frac{\beta_1+2}{8}-1\right)}2^{k\left(t\frac{\beta_2+2}{8}-1\right)}. \tag{10}$$

Now if $\frac{1}{p} > \frac{\beta_2}{\beta_2+2}$, $t\frac{\beta_2+2}{8} - 1 < 0$ and then we can sum in (10) over j and k and b) follows, since $B \subset \cup_{j,k} Q_{j,k}$.

To obtain c), we sum over k in (10) for $\frac{\beta_2}{2+\beta_2} < \frac{1}{p} \leq 1$ and we apply the Riesz Thorin theorem, interpolating between $\left(\frac{1}{p}, 1\right)$, $\frac{\beta_2}{2+\beta_2} < \frac{1}{p} \leq 1$, and $\left(\frac{1}{p_1}, \frac{1}{2}\right)$ with the estimation just obtained and (6). Then we sum over j.

If $\beta_2 < 6$ in the estimation (8) we can sum over j and k for $\frac{\beta_2+2}{8} < \frac{1}{q} \leq 1$, so d) follows.

Since $B \subset \cup_j Q_j$, to obtain e), for $\frac{\beta_2+2}{8} < \frac{1}{q} \leq 1$ we sum over k in (8), and we proceed as in the proof of c), interpolating between $\left(\frac{3}{4}, \frac{1}{q}\right)$ and $\left(\frac{1}{p_1}, \frac{1}{2}\right)$ with the estimation just obtained and (6). Then we sum over j.

The statement f) follows straighforward from Remark 7 and the inequality

$$|\widehat{\sigma}(\xi)| \leq \frac{c}{(1+|\xi_3|)^{\frac{1}{\beta_1}+\frac{1}{\beta_2}}}.$$

(See Lemma 2.5 in [3]). ■

Theorem 9 *Let E be the type set defined in the Introduction.*

a) If $1 < \beta_1 \leq \beta_2 \leq 2$, then $\left(\frac{3}{4}, \frac{1}{2}\right) \in E$.

b) If $1 < \beta_1 \leq 2 \leq \beta_2$, then $\left(\frac{5\beta_2+2}{6\beta_2+4}, \frac{1}{2}\right) \in E$.

c) If $1 < \beta_1 \leq 2 \leq \beta_2 < 6$, then $\left(\frac{3}{4}, \frac{1}{q}\right) \in E$ for $\frac{\beta_2+2}{8} < \frac{1}{q} \leq 1$.

d) If $\beta_1 \leq 2$ and $\beta_2 \geq 6$, then $\left(\frac{1}{p}, 1\right) \in E$ for $\frac{\beta_2}{\beta_2+2} < \frac{1}{p} \leq 1$.

Proof. a) follows straighforward from Remark 6 and the inequality

$$|\widehat{\sigma}(\xi)| \leq \frac{c}{(1+|\xi_3|)}.$$

(See Lemma 3 in [2]). Similarily, if $1 < \beta_1 \leq 2 \leq \beta_2$, from the same lemma we get the estimate

$$|\widehat{\sigma}(\xi)| \leq \frac{c}{(1+|\xi_3|)^{\frac{1}{2}+\frac{1}{\beta_2}}},$$

so, again from Remark 6, we obtain b). Statements c) and d) follows as d) and b) in Theorem 8, respectively. ■

Comments. By the Riesz Thorin interpolation theorem, the Hölder's inequality and Theorem 8 we obtain that in the case $\beta_1 \geq 2$, $\beta_2 \geq 6$, and $\frac{\beta_1\beta_2}{\beta_1\beta_2+\beta_1+\beta_2} < \frac{5\beta_2+2}{6\beta_2+4}$ (the point $\left(\frac{\beta_1\beta_2}{\beta_1\beta_2+\beta_1+\beta_2}, 1\right)$ satisfies the equality in (1)) the polygonal region given as the open convex hull of the points $(1,1)$, $(1,0)$, $\left(\frac{\beta_1+\beta_2+2\beta_1\beta_2}{2\beta_1+2\beta_2+2\beta_1\beta_2}, \frac{1}{2}\right)$, $\left(\frac{5\beta_2+2}{6\beta_2+4}, \frac{(\beta_2+2)(\beta_1+\beta_2+\beta_2\beta_1)}{(\beta_1+\beta_2)(6\beta_2+4)}\right)$, $\left(\frac{5\beta_2+2}{6\beta_2+4}, 1\right)$, $\left(\frac{\beta_2}{\beta_2+2}, 1\right)$ and $\left(\frac{3\beta_2^2+\beta_1\beta_2^2-5\beta_1\beta_2+2\beta_2-2\beta_1}{4\beta_2^2+\beta_1\beta_2^2-4\beta_1\beta_2-4\beta_1}, \frac{(\beta_2-2)(\beta_1+\beta_2+\beta_2\beta_1)}{4\beta_2^2+\beta_1\beta_2^2-4\beta_1\beta_2-4\beta_1}\right)$ is contained in E. We observe that in some cases, the point $\left(\frac{5\beta_2+2}{6\beta_2+4}, 1\right)$ is located at the left of the point $\left(\frac{\beta_2}{\beta_2+2}, 1\right)$, and that in other cases we have the oposite situation. Indeed, $\frac{5\beta_2+2}{6\beta_2+4} < \frac{\beta_2}{\beta_2+2}$ if and only if $\beta_2 > 4 + 2\sqrt{5}$.

In the case $\beta_1 \geq 2$, $\beta_2 \geq 6$, and $\frac{5\beta_2+2}{6\beta_2+4} \leq \frac{\beta_1\beta_2}{\beta_1\beta_2+\beta_1+\beta_2}$ we completely characterize E° as the open triangle with vertices $(1,1)$, $(1,0)$ and $\left(\frac{\beta_1\beta_2}{\beta_1\beta_2+\beta_1+\beta_2}, 1\right)$. These cases correspond to the relation $\beta_1\beta_2^2 - 3\beta_1\beta_2 - 5\beta_2^2 - 2\beta_1 - 2\beta_2 > 0$, for example, $\beta_1 = 6$ and $\beta_2 > 10 + 4\sqrt{7}$.

When $2 \leq \beta_1 \leq \beta_2 < 6$, the corresponding polygonal region is the open convex hull of the points $(1,1)$, $(1,0)$, $\left(\frac{\beta_1+\beta_2+2\beta_1\beta_2}{2\beta_1+2\beta_2+2\beta_1\beta_2}, \frac{1}{2}\right)$, $\left(\frac{3}{4}, \frac{\beta_2+2}{8}\right)$, $\left(\frac{3}{4}, 1\right)$ and $\left(\frac{5\beta_2+\beta_1}{6\beta_2+2\beta_1}, \frac{\beta_2+\beta_2\beta_1+\beta_1}{6\beta_2+2\beta_1}\right)$.

We observe that some pieces of the border of these polygonal regions are also contained in E.

In a similar way, for the cases $\beta_1 < 2$ from Theorem 9 we obtain some polygonal regions contained in E.

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