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TECHNIQUES FOR CLASSIFYING HOPF ALGEBRAS AND
APPLICATIONS TO DIMENSION p3

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# TECHNIQUES FOR CLASSIFYING HOPF ALGEBRAS AND APPLICATIONS TO DIMENSION $p^3$

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ABSTRACT. The classification of all Hopf algebras of a given finite dimension over an algebraically closed field of characteristic 0 is a difficult problem. If the dimension is a prime, then the Hopf algebra is a group algebra. If the dimension is the square of a prime then the Hopf algebra is a group algebra or a Taft Hopf algebra. The classification is also complete for dimension $2p$ or $2p^2$, $p$ a prime. Partial results for some other cases are available. For example, for dimension $p^3$ the classification of the semisimple Hopf algebras was done by Masuoka, and the pointed Hopf algebras were classified by Andruskiewitsch and Schneider, Caenepeel and Dăscălescu, and Ştefan and van Oystaeyen independently. Many classification results for the nonsemisimple, nonpointed, non-copointed case have been proved by the second author but the classification in general for dimension $p^3$ is still incomplete, up to now even for dimension 27.

In this paper we outline some results and techniques which have been useful in approaching this problem and add a few new ones. We give some further results on Hopf algebras of dimension $p^3$ and finish the classification for dimension 27.

## 1. INTRODUCTION

The question of classification of all Hopf algebras of a given dimension up to isomorphism dates back over 35 years to Kaplansky's first monograph on bialgebras [K]. Some progress has been made on this problem but, in general, it is a difficult question where there are no standard methods of attack.

Hopf algebras of some dimensions over an algebraically closed field $\Bbbk$ of characteristic zero are completely classified. By the Kac-Zhu Theorem [Z], 1994, the only Hopf algebras of dimension $p$ prime are the group algebras $\Bbbk C_p$. Hopf algebras of dimension $p^2$ are either pointed, and then they are Taft Hopf algebras, or they are semisimple and are group algebras [Ng], [Mk2], [AS1].

For $p = 2$, the Hopf algebras of dimension $p^3 = 8$ were classified by Williams [W]; different proofs appear in [Mk1] and [Ş]. There are 14 isomorphism classes of Hopf algebras of dimension 8. First there are three group algebras for the abelian groups of order 8, then there are the two group algebras for the nonabelian groups of order 8 and their duals which are nonpointed. Finally there is a self-dual semisimple Hopf algebra of dimension 8 which is neither commutative nor cocommutative. This Hopf algebra can be constructed as an extension of $\Bbbk[C_2 \times C_2]$ by $\Bbbk C_2$. Thus every Hopf algebra of dimension 8 is either semisimple or pointed nonsemisimple or the dual is pointed nonsemisimple.

For dimension $p^3$, $p$ odd, the classification question is still open although the semisimple and the pointed Hopf algebras of dimension $p^3$ have been classified. For $H$ semisimple, the classification is due to Masuoka [Mk3]. There are $p + 8$ isomorphism classes of semisimple Hopf algebras of dimension $p^3$, namely the three group algebras of abelian groups, the two group algebras of the nonabelian groups and their duals and $p + 1$ self-dual semisimple Hopf

algebras which are neither commutative nor cocommutative. These Hopf algebras are extensions of $\Bbbk[C_p \times C_p]$ by $\Bbbk C_p$. For $H$ pointed nonsemisimple of dimension $p^3$, the classification is due to Andruskiewitsch and Schneider [AS2], Caenepeel and Dăscălescu [CD], and Ştefan and van Oystaeyen [ŞvO] independently. There are $(p-1)(p+9)/2$ isomorphism classes and two of these have nonpointed duals.

In [G], the second author has proved some classification results for Hopf algebras of dimension $p^3$ and conjectures that Hopf algebras of this dimension are either semisimple, pointed nonsemisimple or the dual is pointed nonsemisimple. It is shown in [G] that every ribbon Hopf algebra of dimension $p^3$ is either a group algebra or a Frobenius-Lusztig kernel. As well, if $H$ is nonsemisimple, nonpointed and the dual is not pointed, then $H$ has no proper normal sub-Hopf algebra and is either of type $(p, 1)$, meaning that the grouplikes of $H$ have order $p$ and those of $H^*$ have order 1, or else is of type $(p, p)$. However, even for dimension 27, only the quasi-triangular Hopf algebras have been classified.

In this note, we first develop some more tools for attacking these classification problems. Some of these are based on work of D. Fukuda [F1], [F2] and use the coradical filtration. Others extend a key theorem of Ştefan [S] on automorphisms of matrix coalgebras. We apply these results in the last section of this paper, but find them interesting and useful in their own right.

In the last section, we present some general results about Hopf algebras of dimension $p^3$ and then in the final subsection we complete the classification for dimension 27 and show that all Hopf algebras of dimension 27 are either semisimple, pointed nonsemisimple or the dual is pointed nonsemisimple. This result adds to the list of small dimensions for which the classification is known. The smallest open dimension is 24. Then the next is 32 since the classification for 27 is done in this note, $29, 31$ are prime, 25 is a prime squared, 26 is twice a prime, 28 was completed in [CNg], and 30 was completed in [F3].

## 2. Preliminaries

2.1. **Conventions.** Throughout, we will work over $\Bbbk$, an algebraically closed field of characteristic zero. Since we are interested in classification, our Hopf algebras will be understood to be finite dimensional unless otherwise stated. Good references for Hopf algebra theory are [Mo], [S]. Throughout $C_n$ will denote the cyclic group of order $n$ and $p$ will denote an odd prime.

It is useful to recall that for $H$ a finite dimensional Hopf algebra, if $B$ is a sub-bialgebra of $H$ then $B$ is a sub-Hopf algebra. To see this, let $\Phi : \mathrm{End}(B) \to \mathrm{End}(B)$ be given by $\Phi(f) = f * \mathrm{id}_B$. Since $f(b_1)b_2 = g(b_1)b_2$ implies $f(b) = g(b)$ by applying $\varepsilon_H$, we have that $\Phi$ is one to one, and since $B$ is finite dimensional then $\Phi$ is onto. Thus there exists $s \in \mathrm{End}(B)$ such that $\Phi(s) = u\varepsilon$, *i.e.* $s$ is a left inverse to the identity. A similar argument yields a right inverse to the identity and these must be equal. Then $s$ and $S_H$ are inverses to the identity in $\mathrm{Hom}(B, H)$ and so must be equal.

If $H$ is any Hopf algebra over $\Bbbk$ then $\Delta$, $\varepsilon$, $S$ denote respectively the comultiplication, the counit and the antipode. Comultiplication and coactions are written using the Sweedler-Heynemann notation with summation sign suppressed. The coradical $H_0$ of $H$ is the coradical of $H$ as a coalgebra, *i.e.* the sum of all simple subcoalgebras of $H$. $H$ is called cosemisimple if $H = H_0$. With our assumptions on $\Bbbk$, by [LR1, LR2] a Hopf algebra $H$ is semisimple if and only if it is cosemisimple if and only if $S^2$ is the identity. $H$ is called pointed if $H$ has only simple subcoalgebras of dimension 1 and $H$ is called copointed if its dual is pointed. We adopt the convention that a pointed Hopf algebra means non-cosemisimple pointed, that is, not a group algebra.

For $D$ a coalgebra, $G(D)$ denotes the grouplike elements of $D$ and $(D_n)_{n \in \mathbb{N}}$ denotes the coradical filtration of $D$. Let $L$ be a coalgebra with a distinguished grouplike 1. If $M$ is a right $L$-comodule via $\delta$, then the space of *right coinvariants* is

$$M^{\mathrm{co}\,\delta} = \{x \in M \mid \delta(x) = x \otimes 1\}.$$

In particular, if $\pi : H \to L$ is a morphism of Hopf algebras, then $H$ is a right $L$-comodule via $(\mathrm{id} \otimes \pi)\Delta$ and in this case $H^{\mathrm{co}\,\pi} := H^{\mathrm{co}\,(\mathrm{id}\otimes\pi)\Delta}$. Left coinvariants, written $^{\mathrm{co}\,\pi}H$ are defined analogously.

$L_h$ (resp. $R_h$) is the left (resp. right) multiplication in $H$ by $h$. The left and right adjoint action $\mathrm{ad}_\ell, \mathrm{ad}_r : H \to \mathrm{End}(H)$, of $H$ on itself are given, in Sweedler notation, by:

$$\mathrm{ad}_\ell(h)(x) = h_1 x S(h_2), \qquad \mathrm{ad}_r(h)(x) = S(h_1)xh_2,$$

for all $h, x \in H$. The set of $(h, g)$-*primitives* (with $h, g \in G(H)$) and *skew-primitives* are:

$$\begin{aligned} \mathcal{P}_{h,g}(H) &:= \{x \in H \mid \Delta(x) = g \otimes x + x \otimes h\}, \\ \mathcal{P}(H) &:= \textstyle\sum_{h,g\in G(H)} \mathcal{P}_{h,g}(H). \end{aligned}$$

We say that $x \in \Bbbk(h - g)$ is a *trivial* skew-primitive; otherwise, it is *nontrivial*.

Let $N$ be a natural number and let $q$ be a primitive $N$-th root of unit. We denote by $T_q$ the Taft Hopf algebra which is generated as an algebra by elements $g$ and $x$ satisfying the relations $x^N = 0 = 1 - g^N$, $gx = qxg$. It is a Hopf algebra of dimension $N^2$ where the comultiplication is determined by $\Delta(g) = g \otimes g$ and $\Delta(x) = x \otimes 1 + g \otimes x$.

2.1.1. *Extensions of Hopf algebras.* We recall the definition of an exact sequence of Hopf algebras.

**Definition 2.1.** [AD]. Let $A \overset{\imath}{\hookrightarrow} H \overset{\pi}{\twoheadrightarrow} B$ be a sequence of Hopf algebra morphisms. We say that the sequence is *exact* if the following conditions hold:

   $(i)$ $\imath$ is injective (and then we identify $A$ with its image);

  $(ii)$ $\pi$ is surjective;

 $(iii)$ $\pi\imath = \varepsilon$;

 $(iv)$ $\ker\pi = A^+ H$ ($A^+$ is the kernel of the counit);

  $(v)$ $A = H^{\mathrm{co}\,\pi}$.

In such a case we will also say that $H$ is an *extension of $A$ by $B$*. An exact sequence is called *central* if $A$ is contained in the center of $H$.

The next theorem is due to S. Natale but it is derived from a result in [Ş]. (See Theorem 2.5 and Proposition 2.6.) Natale's theorem has been a key component in classification results.

**Proposition 2.2.** [N, Prop. 1.3]. *Let $H$ be a finite dimensional nonsemisimple Hopf algebra. Suppose that $H$ is generated by a simple subcoalgebra of dimension $4$ which is stable by the antipode. Then $H$ fits into a central exact sequence $\Bbbk^G \overset{\imath}{\hookrightarrow} H \overset{\pi}{\twoheadrightarrow} A$, where $G$ is a finite group and $A^*$ is a pointed nonsemisimple Hopf algebra.* $\qquad\square$

The following statement condenses some known results and is useful in finding exact sequences; for more details see [GV, Lemma 2.3].

**Lemma 2.3.** *Let $H$ be a finite dimensional Hopf algebra. If $\pi : H \to B$ is an epimorphism of Hopf algebras then $\dim H = \dim H^{\mathrm{co}\,\pi} \dim B$. Moreover, if $A = H^{\mathrm{co}\,\pi}$ is a sub-Hopf algebra then the sequence $A \overset{\imath}{\hookrightarrow} H \overset{\pi}{\twoheadrightarrow} B$ is exact.* $\qquad\square$

2.2. **Matrix-like coalgebras.** Let $\mathcal{M}^*(n, \Bbbk)$ denote the simple coalgebra of dimension $n^2$, dual to the matrix algebra $\mathcal{M}(n, \Bbbk)$. We say that a coalgebra $C$ is a $d \times d$ *matrix-like coalgebra* if $C$ is spanned by elements $(e_{ij})_{1\le i,j\le n}$ called a *matrix-like spanning set* (not necessarily linearly independent) such that $\Delta(e_{ij}) = \sum_{1\le\ell\le n} e_{i\ell} \otimes e_{\ell j}$ and $\varepsilon(e_{ij}) = \delta_{ij}$. If the matrix-like spanning set $(e_{ij})_{1\le i,j\le d}$ is linearly independent, following Ştefan we call $\mathbf{e} = \{e_{ij} : 1 \le i, j \le d\}$ a *multiplicative matrix* and then $C \cong \mathcal{M}^*(d, \Bbbk)$ as coalgebras.

If $\pi : \mathcal{M}^*(n, \Bbbk) \to D$ is a coalgebra map, then there are various possibilities for the image. From [BD, Thm. 2.1] we have the following theorem that describes them when $n = 2$.

**Theorem 2.4.** *Let $\pi : \mathcal{M}^*(2,\Bbbk) \to D$ be a coalgebra map and denote by $C$ the image of $\mathcal{M}^*(2,\Bbbk)$ in $D$. If $\dim C = 3$, then $C$ has a basis $\{g, h, u\}$ with $g, h$ grouplike and $u$ an $(h,g)$-primitive, and if $\dim C = 1, 2$, then $C$ has a basis consisting of grouplike elements.* $\qquad\square$

The next theorem due to Ştefan has turned out to be crucial for several classification results including the proof of Proposition 2.2. Similar arguments were used before by Larson and Radford in [LR3, Section 6] to study the order of the antipode in a semisimple Hopf algebra.

**Theorem 2.5.** [Ş, Thm. 1.4] *Let $D = \mathcal{M}^*(2,\Bbbk)$.*

(i) *For $f$ an antiautomorphism of $D$ such that $\mathrm{ord}(f^2) = n < \infty$ and $n > 1$, there exists a multiplicative matrix $\boldsymbol{e}$ for $D$ and a root of unity $\omega$ of order $n$ such that*

$$f(e_{12}) = \omega^{-1}e_{12}, \quad f(e_{21}) = \omega e_{21}, \quad f(e_{11}) = e_{22}, \quad f(e_{22}) = e_{11}.$$

(ii) *For $f$ an automorphism of $D$ of finite order $n$, there exist a multiplicative matrix $\boldsymbol{e}$ for $D$ and a root of unity $\omega$ of order $n$ such that $f(e_{ij}) = \omega^{i-j}e_{ij}$.* $\qquad\square$

The proof of the following generalization is due to C. Vay. Although similar results seem to be well-known, we include it for completeness.

**Proposition 2.6.** *Let $D = \mathcal{M}^*(d,\Bbbk)$.*

(i) *For $f$ an automorphism of $D$ of finite order $n$, there exist a multiplicative matrix $\boldsymbol{e}$ for $D$ and $\omega_1, \ldots, \omega_d \in \Bbbk^\times$ such that $f(e_{ij}) = \omega_i \omega_j^{-1} e_{ij}$ and $\omega_i \omega_j^{-1}$ are $n$-roots of unity.*

(ii) *Let $f$ be an anti-automorphism of $D$ with $1 < n = \mathrm{ord}(f^2) < \infty$. Then there exists a multiplicative matrix $\boldsymbol{e}$ for $D$, a symmetric matrix $A_+ \in \mathrm{GL}(\mathbb{C}, a_+)$, an anti-symmetric matrix $A_- \in \mathrm{GL}(\Bbbk, a_-)$, $A_j \in \mathrm{GL}(\Bbbk, a_j)$ and $\lambda_j \in \Bbbk^\times$ for $j = 1, \ldots, s$ such that $d = a_+ + a_- + 2a_1 + \cdots + 2a_s$ and the matrix of $f$ corresponding to $\boldsymbol{e}$ is*

$$A = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & \lambda_1 A_1^t \\ 0 & 0 & 0 & 0 & 0 & 0 & \diagup & 0 \\ 0 & 0 & 0 & 0 & 0 & \lambda_s A_s^t & 0 & 0 \\ 0 & 0 & 0 & A_+ & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & A_- & 0 & 0 & 0 \\ 0 & 0 & A_s & 0 & 0 & 0 & 0 & 0 \\ 0 & \diagup & 0 & 0 & 0 & 0 & 0 & 0 \\ A_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

*Moreover,*

(a) *$\lambda_i^2$, $\lambda_i \lambda_j^{-1}$, $\lambda_i \lambda_j$ are $n$-roots of unity.*

(b) *If $a_+ \neq 0$ then $\lambda_i$ are $n$-roots of unity.*

(c) *If $a_- \neq 0$ then $-\lambda_i$ are $n$-roots of unity.*

*In particular, for $d = 3$ and $n > 2$*

$$A_\lambda = \begin{bmatrix} 0 & 0 & \lambda \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \qquad f(\boldsymbol{e}) = \begin{bmatrix} e_{33} & \lambda e_{23} & \lambda e_{13} \\ \lambda^{-1}e_{32} & e_{22} & e_{12} \\ \lambda^{-1}e_{31} & e_{21} & e_{11} \end{bmatrix} \qquad and \qquad f^2(e_{ij}) = \lambda^{j-i}e_{ij}.$$

*Proof.* Let $f$ be a coalgebra automorphism as in $(i)$. Fix a multiplicative matrix $\tilde{\boldsymbol{e}} = (\tilde{e}_{ij})_{1 \leq i,j \leq n}$ for $D$ and let $\tilde{B}$ be the matrix of $f$ corresponding to $\tilde{\boldsymbol{e}}$. Then $\tilde{B}^n = b\,\mathrm{id}$ for some $b \in \Bbbk^\times$. In particular, there exist $U \in \mathrm{GL}(\Bbbk, d)$ and $\omega_1, \ldots, \omega_d \in \Bbbk^\times$ such that $U\tilde{B}U^{-1} = \mathrm{diag}(\omega_1, \ldots, \omega_d)$. If we take $\boldsymbol{e} = U\tilde{\boldsymbol{e}}U^{-1}$ then $f$ is afforded by $\mathrm{diag}(\omega_1, \ldots, \omega_d)$ and $(i)$ follows.

Now suppose that $f$ is a coalgebra anti-automorphism as in $(ii)$ and $\tilde{A}$ be the matrix of $f$ corresponding to $\tilde{\mathbf{e}}$. Then $f^2$ is afforded by $B = \tilde{A}(\tilde{A}^{-1})^t$. As before, there exist $U \in \mathrm{GL}(\Bbbk, d)$ and $\omega_1, \ldots, \omega_d \in \Bbbk^\times$ such that $UBU^{-1} = \mathrm{diag}(\omega_1, \ldots, \omega_d)$. Since $n > 1$, $\omega_{i_0} \neq 1$ for some $i_0$. If we take $\mathbf{e} = U\tilde{\mathbf{e}}U^{-1}$ then $f$ is afforded by $A = U\tilde{A}U^t$ with respect to $\mathbf{e}$ and $A(A^{-1})^t = UBU^{-1} = \mathrm{diag}(\omega_1, \ldots, \omega_d)$ is the matrix of $f^2$ corresponding to $\mathbf{e}$. Therefore, $a_{ij} = \omega_i a_{ji}$ for all $1 \leq i, j \leq d$ and hence $a_{ij} = \omega_i \omega_j a_{ij}$ for all $1 \leq i, j \leq d$. If $w_j w_{i_0} \neq 1$ for all $1 \leq j \leq d$, we have that $a_{ji_0} = a_{i_0 j} = a_{i_0 i_0} = 0$. Since $A \in \mathrm{GL}(\Bbbk, d)$ it is impossible. Then for all $i$ there exists $j$ such that $\omega_i \omega_j = 1$. If $d = 2$ then $A = \left[\begin{smallmatrix} 0 & a_{12} \\ a_{21} & 0 \end{smallmatrix}\right]$ and Theorem 2.5 follows. We perform a new change of basis using a matrix permutation such that $\{\omega_1, \ldots, \omega_d\} = \{\lambda_1, \ldots, \lambda_1, \ldots, \lambda_s, \ldots, \lambda_s, 1, \ldots, -1, \lambda_s^{-1}, \ldots, \lambda_s^{-1}, \ldots, \lambda_1^{-1}, \ldots, \lambda_1^{-1}\}$. If $I_\pm = \{i \leq d | \omega_i = \pm 1\}$ and $I_{\pm l} = \{i \leq d | \omega_i = \lambda_l^{\pm 1}\}$ then $a_{ij} = 0$ for all $(i, j) \notin \bigcup_{l=1}^s I_{\pm l} \times I_{\mp l} \cup I_+^2 \cup I_-^2$ since $a_{ij} = \omega_i \omega_j a_{ij}$. Then

$$A = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & \lambda_1 A_1^t \\ 0 & 0 & 0 & 0 & 0 & 0 & \diagup & 0 \\ 0 & 0 & 0 & 0 & 0 & \lambda_s A_s^t & 0 & 0 \\ 0 & 0 & 0 & A_+ & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & A_- & 0 & 0 & 0 \\ 0 & 0 & A_s & 0 & 0 & 0 & 0 & 0 \\ 0 & \diagup & 0 & 0 & 0 & 0 & 0 & 0 \\ A_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

where $A_\pm$ is a $|I_\pm| \times |I_\pm|$-matrix and $A_l$ is a $|I_{-l}| \times |I_{+l}|$-matrix. Since $A$ is invertible, we must have that $|I_{-l}| = |I_{+l}|$. Moreover, by the relations above, $A_+$ is symmetric and $A_-$ is a anti-symmetric matrix. Then set $a_\pm := |I_\pm|$ and $a_l := |I_{+l}|$ for all $l = 1, \ldots, s$. The items $(a)$, $(b)$ and $(b)$ follow from $(i)$ for $f^2$.

If $d = 3$ and $n > 2$, the unique possibility up to a change of basis is $\mathrm{diag}\{\omega_1, \omega_2, \omega_3\} = \mathrm{diag}\{\lambda, 1, \lambda^{-1}\}$. If $f$ is an anti-automorphism, we have that $a_+ = 1$, $a_- = 0$, $a_1 = 1$. In such case, we make a new change of basis using $\mathrm{diag}(A_1^{-\frac{1}{2}}, A_+^{-\frac{1}{2}}, A_1^{-\frac{1}{2}})$ and the rest of the claim follows by direct computation. $\qquad\square$

## 3. THE CORADICAL FILTRATION AND SOME CLASSIFICATION RESULTS

In this section we present some results on dimensions of Hopf algebras using a description of the coradical filtration due to Nichols and presented in [AN], and more recent results by D. Fukuda. First we recall the definitions. More detail can be found in [AN, Section 1].

Let $D$ be a coalgebra over $\Bbbk$ and denote by $(D_n)_{n \in \mathbb{N}}$ its coradical filtration. Then there exists a coalgebra projection $\pi : D \to D_0$ from $D$ to the coradical $D_0$ with kernel $I$, see [Mo, 5.4.2]. Define the maps

$$\rho_L := (\pi \otimes \mathrm{id})\Delta : D \to D_0 \otimes D \qquad \text{and} \qquad \rho_R := (\mathrm{id} \otimes \pi)\Delta : D \to D \otimes D_0,$$

and let $P_n$ be the sequence of subspaces defined recursively by

$$P_0 = 0,$$
$$P_1 = \{x \in D : \Delta(x) = \rho_L(x) + \rho_R(x)\} = \Delta^{-1}(D_0 \otimes I + I \otimes D_0),$$
$$P_n = \{x \in D : \Delta(x) - \rho_L(x) - \rho_R(x) \in \sum_{1 \leq i \leq n-1} P_i \otimes P_{n-i}\}, \quad n \geq 2.$$

Then by a result of Nichols, $P_n = D_n \cap I$ for $n \geq 0$, see [AN, Lemma 1.1]. Suppose that $D_0 = \bigoplus_{\tau \in \mathcal{I}} D_\tau$, where the $D_\tau$ are simple coalgebras and $\dim D_\tau = d_\tau^2$. Any $D_0$-bicomodule is a direct sum of simple $D_0$-sub-bicomodules and every simple $D_0$-bicomodule has coefficient coalgebras $D_\tau, D_\gamma$ and has dimension $d_\tau d_\gamma = \sqrt{\dim D_\tau \dim D_\gamma}$ for some $\tau, \gamma \in \mathcal{I}$, where $d_\tau, d_\gamma$ are the dimensions of the associated comodules of $D_\tau$ and $D_\gamma$, respectively.

Let $H$ be a Hopf algebra. Then $H_n, P_n$ are $H_0$-sub-bicomodules of $H$ via $\rho_R$ and $\rho_L$. As in [AN], [F2], for all $n \geq 1$ we denote by $P_n^{\tau,\gamma}$ the isotypic component of the $H_0$-bicomodule of $P_n$ of type the simple bicomodule with coalgebra of coefficients $D_\tau \otimes D_\gamma$. If $D_\tau = \Bbbk g$ for $g$ a grouplike, we may use the superscript $g$ instead of $\tau$. Let $P^{\tau,\gamma} = \sum_{n \geq 0} P_n^{\tau,\gamma}$.

For $\Gamma$ a set of grouplikes of a Hopf algebra $H$, let $P^{\Gamma,\Gamma}$ denote $\sum_{g,h \in \Gamma} P^{g,h}$ and let $H^{\Gamma,\Gamma} := P^{\Gamma,\Gamma} \oplus \Bbbk\Gamma$. If $\mathcal{D}, \mathcal{E}$ are sets of simple subcoalgebras, let $P^{\mathcal{D},\mathcal{E}}$ denote $\sum_{D \in \mathcal{D}, E \in \mathcal{E}} P^{D,E}$.

Let $H_{0,d}$ with $d \geq 1$ denote the direct sum of the simple subcoalgebras of $H$ of dimension $d^2$. It was noted in [AN] that $H_n = H_0 \oplus P_n$ and $|G(H)|$ divides the dimensions of $H_n, P_n$ and $H_{0,d}$ for all $n \geq 0$ and all $d \geq 1$, by the Nichols-Zoeller theorem.

Following D. Fukuda, we say that the subspace $P_n^{\tau,\gamma}$ is *nondegenerate* if $P_n^{\tau,\gamma} \nsubseteq P_{n-1}$. The next results are due to D. Fukuda; note that $(ii)$ is a generalization of [AN, Cor. 1.3] for $n > 1$.

**Lemma 3.1.** *Let $D_\tau, D_\gamma$ be simple subcoalgebras of a Hopf algebra $H$.*

(i) *[F2, Lemma 3.2] If the subspace $P_n^{\tau,\gamma}$ is nondegenerate for some $n > 1$, then there exists a set of simple coalgebras $\{D_1, \cdots, D_{n-1}\}$ such that $P_i^{\tau,D_i}$, $P_{n-i}^{D_i,\gamma}$ are nondegenerate for all $1 \leq i \leq n$.*

(ii) *[F2, Lemma 3.5] For $g \in G(H)$, $\dim P_n^{\tau,\gamma} = \dim P_n^{S\gamma,S\tau} = \dim P_n^{g\tau,g\gamma} = \dim P_n^{\tau g, \gamma g}$, where the superscript $S\alpha$ means that the simple coalgebra is $S(D_\alpha)$ and the superscript $g\alpha$ or $\alpha g$ means that the coalgebra is $gD_\alpha$ or $D_\alpha g$.*

(iii) *[F2, Lemma 3.8] Let $C, D$ be simple subcoalgebras such that $P_m^{C,D}$ is nondegenerate. If $\dim C \neq \dim D$ or $\dim P_m^{C,D} - P_{m-1}^{C,D} \neq \dim C$ then there exists a simple subcoalgebra $E$ such that $P_\ell^{C,E}$ is nondegenerate for some $\ell \geq m + 1$.* $\qquad\square$

The following proposition generalizes [BD, Cor. 4.3] by giving a better lower bound for the dimension of a non-cosemisimple Hopf algebra with no nontrivial skew-primitive elements.

**Proposition 3.2.** *Let $H$ be a non-cosemisimple Hopf algebra with no nontrivial skew-primitives.*

(i) *For any $g \in G(H)$ there exists a simple subcoalgebra $C$ of $H$ of dimension $> 1$ such that $P_1^{C,g} \neq 0$, $P_k^{C,D}$ is nondegenerate for some $k > 1$ and $D$ simple of the same dimension as $C$, and $P_m^{g,h}$ is nondegenerate for some $m > 1$ and $h$ grouplike.*

(ii) *Suppose $H_0 \cong \Bbbk G \oplus \mathcal{M}^*(n, \Bbbk) \oplus \sum_{i=1}^t \mathcal{M}^*(n_i, \Bbbk)$ with $t \geq 0$, $2 \leq n \leq n_1 \leq \ldots \leq n_t$. Then*

$$\dim H \geq \dim(H_0) + (2n+1)|G| + n^2.$$

*Proof.* Part $(i)$ follows from [BD, Cor. 4.2] and Lemma 3.1$(iii)$. Part $(ii)$ follows from [BD, Cor. 4.3], Lemma 3.1 $(ii)$ and part $(i)$. $\qquad\square$

We end this section with a series of results whose proofs use the results of D. Fukuda in Lemma 3.1.

**Proposition 3.3.** *Let $H$ be a Hopf algebra with coradical $H_0 = \Bbbk \cdot 1 \oplus E$ where $E$ is the direct sum of simple coalgebras, each of dimension divisible by $N^2$ for some $N > 1$. Let $\dim H \equiv d$ mod $N$ with $0 \leq d \leq N - 1$. Let $1 \leq e \leq N$ with $e \equiv d - 1$ mod $N$ and assume that $e \neq 1$. Then $\dim H \geq \dim H_0 + 4N + 2N^2 + e$.*

*Proof.* Let $\mathcal{E}$ denote the set of simple subcoalgebras of dimension greater than 1. Throughout, let $X, Y, Z, W$ denote elements of $\mathcal{E}$. Then $N$ divides the dimension of $E, P^{\mathcal{E},1}, P^{1,\mathcal{E}}, P^{\mathcal{E},\mathcal{E}}$ so that by our assumptions and the fact that $P^{1,1} \neq 0$ we have $\dim P^{1,1} \equiv e \mod N$; in particular $\dim P^{1,1} > 1$.

Since by Proposition 3.2$(i)$ there are $X, Y$ such that $P_1^{X,1}, P_1^{1,Y}$ are nonzero, we get that

$$M = \max\{m|\ P_m^{X,1} \text{ is nondegenerate for some } X\}$$
$$= \max\{m|\ P_m^{1,Y} \text{ is nondegenerate for some } Y\} \geq 1.$$

Then by Lemma 3.1$(iii)$, there is an integer $k > M$ such that $P_k^{1,1}$ is nondegenerate. By Lemma 3.1$(i)$, $P_k^{1,1}$ nondegenerate and the fact that $P_1^{1,1} = 0$, implies that for some $X \in \mathcal{E}$, $P_1^{1,X}$ and $P_{k-1}^{X,1}$ are nondegenerate. Thus $k - 1 \leq M < k$ and so $k = M + 1$ and $M + 1$ is the largest integer $m$ such that $P_m^{1,1}$ is nondegenerate.

Suppose that $M = 1$ so $M + 1 = 2$. Then $P_2^{1,1} = P^{1,1}$ and the dimension of $P_2^{1,1}$ must be greater than 1. Thus $\dim P_2^{1,1} - \dim P_1^{1,1} > 1$ and, by Lemma 3.1$(iii)$, $P_t^{1,1}$ is nondegenerate for some $t > 2$, a contradiction. Thus $M \geq 2$ and $M + 1 \geq 3$.

If $P_k^{1,X}$ is nondegenerate then the difference in dimensions between $P_k^{1,X}$ and $P_{k-1}^{1,X}$ is a multiple of $N$. Thus, the fact that we have nondegenerate spaces $P_M^{1,X}, P_M^{Y,1}, P_1^{1,Z}, P_1^{W,1}$ with $X, Y, Z, W$ not necessarily distinct yields that the sum of the dimensions of the $P^{1,\mathcal{E}}$ and $P^{\mathcal{E},1}$ is at least $4N$.

By Lemma 3.1$(iii)$, $P_\ell^{Y,Y'}$ is nondegenerate for some $Y' \in \mathcal{E}$ and $\ell > M$. By Lemma 3.1$(i)$, if $P_M^{1,X}$ is nondegenerate, then $P_{M-1}^{Z,X}$ is nondegenerate for some $Z \in \mathcal{E}$. Thus $\dim P^{\mathcal{E},\mathcal{E}} \geq 2N^2$.

Hence $\dim H = \dim H_0 + \dim P^{1,1} + 2\dim P^{\mathcal{E},1} + \dim P^{\mathcal{E},\mathcal{E}} \geq \dim H_0 + e + 4N + 2N^2$. $\square$

**Example 3.4.** *Suppose that $H$ has dimension $N^3$ with $N > 2$. Then $H$ cannot have coradical $\Bbbk \cdot 1 \oplus \mathcal{M}^*(N, \Bbbk)^t$ where $t = N - 1$ or $t = N - 2$. For then we would have $N^3 \geq 1 + tN^2 + N - 1 + 4N + 2N^2 \geq (N-2)N^2 + 2N^2 + 5N \geq N^3 + 5N$, which is impossible.*

*In particular, if $\dim H = 27$, then $H$ cannot have coradical $\Bbbk \cdot 1 \oplus \mathcal{M}^*(3, \Bbbk)^t$ with $t = 1, 2$ and if $\dim H = 125$, $H$ cannot have coradical $\Bbbk \cdot 1 \oplus \mathcal{M}^*(5, \Bbbk)^t$ with $t = 3, 4$.*

The next proposition applies the methods of this section to say something about Hopf algebras of dimension $p^3$, $p$ an odd prime, leading into the material in the next section.

**Proposition 3.5.** *Let $H$ be a non-copointed Hopf algebra of dimension $p^3$ with $p$ an odd prime. Then $H$ has no simple 4-dimensional subcoalgebra stable under the antipode $S$ so that $H_0 \not\cong \Bbbk \cdot 1 \oplus \mathcal{M}^*(2, \Bbbk)$. Furthermore, if $H_0 \cong \Bbbk \cdot 1 \oplus \mathcal{M}^*(2, \Bbbk)^t$ with $t > 1$, then*

$$\dim H \geq \dim H_0 + 24 \text{ if } p \equiv 1 \mod 4 \quad and \quad \dim H \geq \dim H_0 + 22 \text{ if } p \equiv 3 \mod 4.$$

*Proof.* Let $\mathcal{D}$ be the set of simple subcoalgebras of dimension 4 and suppose that $D \in \mathcal{D}$ is stable under the antipode $S$. Then $D$ must generate all of $H$ since any proper sub-Hopf algebra of $H$ is a group algebra or a Taft Hopf algebra. Thus by Proposition 2.2, $H$ has pointed dual, contradicting our assumptions. Thus for each $D \in \mathcal{D}$, $S(D) \neq D$. This also shows that $H_0 \not\cong \Bbbk \cdot 1 \oplus \mathcal{M}^*(2, \Bbbk)$.

Now suppose that $H_0 \cong \Bbbk \cdot 1 \oplus \mathcal{M}^*(2, \Bbbk)^t$ with $t > 1$. The argument is similar to that in the previous proposition. Since $2\dim P^{\mathcal{D},1}$ and $\dim P^{\mathcal{D},\mathcal{D}}$ are divisible by 4, then $p^3 - 1 \equiv \dim P^{1,1}$ mod 4.

Since $p \equiv p^3 \mod 4$, then $P^{1,1}$ has dimension a nonzero multiple of 4 if $p \equiv 1 \mod 4$ and dimension a nonzero multiple of 2 if $p \equiv 3 \mod 4$. As in the proof of Proposition 3.3, let $M$ be the largest integer such that $P_M^{1,D}$ is nondegenerate for $D \in \mathcal{D}$ and then $\ell = M + 1$ is the largest integer such that $P_\ell^{1,1}$ is nondegenerate. If $M + 1 = \ell = 2$ then $P_2^{1,1} = P^{1,1}$ has dimension 2 or 4 contradicting Lemma 3.1$(iii)$. Thus $M \geq 2$ and $\ell \geq 3$. Suppose that $\ell = 3$ and $M = 2$.

By Lemma 3.1, there exist $D, E \in \mathcal{D}$ so that $P_1^{1,D}, P_2^{D,1}, P_1^{D,E}, P_1^{E,1}$ are nondegenerate. Thus twice the dimension of $P^{\mathcal{D},1}$ is at least 8.

Consider the dimension of $P^{\mathcal{D},\mathcal{D}}$. If $D = E$ in the paragraph above, then for some $X, Y \in \mathcal{D}$, and some $k > 1$, the spaces $P_1^{D,D}, P_1^{S(D),S(D)}, P_3^{D,X}, P_k^{S(D),Y}$ are nondegenerate so the dimension of $P^{\mathcal{D},\mathcal{D}}$ is at least 16. If $E = S(D)$, then there exists $X \in \mathcal{D}$ and $k > 1$, such that $P_1^{D,S(D)}, P_k^{S(D),X}, P_3^{D,X}$ is a set of nondegenerate spaces. Then the dimension of $P^{\mathcal{D},\mathcal{D}}$ is at least 12. Finally, if $D \neq E \neq S(D)$, then the spaces $P_1^{D,E}, P_1^{S(E),S(D)}, P_3^{D,X}, P_k^{E,Y}$ for some $k > 1$ and $X, Y \in \mathcal{D}$ are nondegenerate. Again in this case the dimension of $P^{\mathcal{D},\mathcal{D}}$ is at least 16.

Thus the dimension of $H$ is at least $\dim H_0 + \dim P^{\mathcal{D},\mathcal{D}} + 2 \dim P^{1,\mathcal{D}} + \dim P^{1,1} \geq \dim H_0 + 20 + \dim P^{1,1}$ and the statement is proved. $\qquad\square$

**Example 3.6.** *Let $H$ be a non-copointed Hopf algebra of dimension 27, then $H_0 \not\cong \Bbbk \cdot 1 \oplus \mathcal{M}^*(2,\Bbbk)^t$ for any $t \geq 1$.*

The last example in this section uses similar techniques to prove the existence of nontrivial skew-primitive elements in a Hopf algebra with nontrivial grouplikes.

**Example 3.7.** *Let $H$ be a nonsemisimple nonpointed and non-copointed Hopf algebra of dimension $5^3$. Then by [G] (see Proposition 4.3), $H$ has no simple subcoalgebra of dimension 4 fixed by the antipode. Moreover, if $H_0 \cong \Bbbk C_5 \oplus \mathcal{M}^*(2,\Bbbk)^5$, then $H$ must have a nontrivial skew-primitive element.*

*To see this, assume $H$ has no nontrivial skew-primitive element. By Proposition 3.2$(i)$, $P_1^{1,D} \neq 0$ for some simple subcoalgebra $D$ of dimension 4. Since $S^{4p}$ is the identity by Radford's formula for $S^4$ and $S(D) \neq D$, we must have that the simple subcoalgebras of dimension 4 are $D_0 := D, D_1 = S(D), \ldots, D_4 = S^4(D)$. Thus $P_1^{1,D}, P_1^{D_1,1}, P_1^{1,D_2}, \ldots, P_1^{1,D_4}, P_1^{D,1}, \ldots, P_1^{1,D_4}$ are all nonzero and of equal dimension. By Lemma 3.1$(ii)$ and the fact that $G(H) = C_5$, $\dim P^{C_5,\mathcal{D}} + \dim P^{\mathcal{D},C_5} \geq 2 \cdot 25 \cdot \dim P^{1,D} \geq 100$ where $\mathcal{D}$ denotes the set of simple 4-dimensional coalgebras. By Lemma 3.1$(iii)$, $P^{D_i,\mathcal{D}}$ is nonzero for every $i$ so that the dimension of $P^{\mathcal{D},\mathcal{D}}$ is at least 20. Also $\dim P^{C_5,C_5}$ must be divisible by 4 and also by 5 so is at least 20. Thus $\dim H_0 + 2 \dim P^{C_5,\mathcal{D}} + \dim P^{\mathcal{D},\mathcal{D}} + \dim P^{C_5,C_5} \geq 165$, a contradiction.*

## 4. Hopf algebras of dimension $p^3$

Throughout this section, $H$ will be a Hopf algebra of dimension $p^3$. Since semisimple and nonsemisimple pointed Hopf algebras have already been classified, we will assume that $H$ is nonsemisimple, nonpointed and non-copointed. We denote by $T_q$ the Taft Hopf algebra of dimension $p^2$ for $q$ some primitive $p$-th root of unity. For $y$ a $(1, g)$-primitive in $T_q$ we will write $T_q = \Bbbk\langle g, y \rangle$; here $gy = qyg$. We first summarize some results from [G].

4.1. **Known results for dimension $p^3$.** The general classification of Hopf algebras of dimension $p^3$ was studied by the second author in [G] where it was conjectured that a Hopf algebra of dimension $p^3$ is semisimple or pointed or copointed. Since these Hopf algebras have all been

classified, a complete list of isomorphism types can be given if the conjecture holds. In particular, it is shown in [G] that ribbon Hopf algebras of dimension $p^3$ are group algebras or Frobenius-Lusztig kernels.

Also by [G], we know that if a Hopf algebra $H$ of dimension $p^3$ is nonsemisimple, nonpointed and non-copointed, then $H$ is simple as a Hopf algebra, meaning that it has no normal proper sub-Hopf algebras. Furthermore the only possible types are $(p,p)$ and $(p,1)$ so that $H$ has grouplikes of order $p$, and the dual has grouplikes of order $p$ or 1. In particular, $S^{4p} = \mathrm{id}$ by Radford's formula.

**Theorem 4.1.** [G, Thm. 2.1] *Let $L$ be a finite dimensional Hopf algebra which fits into an extension $A \xhookrightarrow{i} L \xtwoheadrightarrow{\pi} \Bbbk C_p$, where $A^*$ is pointed and $|G(A^*)| \leq p$. Then $L$ is copointed.* $\square$

**Proposition 4.2.** [G, Cor. 3.14] *Let $L$ be a Hopf algebra of dimension $p^3$ and type $(p,p)$ such that $L$ contains a nontrivial skew-primitive element. Then $L$ is a bosonization of $\Bbbk C_p$.* $\square$

**Proposition 4.3.** [G, Prop. 3.15] *Let $L$ be a nonsemisimple Hopf algebra of dimension $p^3$ and assume that $L$ contains a simple subcoalgebra of dimension 4 fixed by the antipode. Then $L$ is copointed and furthermore, $L$ cannot be of type $(p,p)$.* $\square$

In [G] the complete classification of the quasi-triangular Hopf algebras of dimension 27 was given. We are able to complete the classification for dimension 27 in the general case in Subsection 4.3.

4.2. **Some general results for dimension $p^3$.** In this subsection we prove some general results for Hopf algebras of dimension $p^3$ and grouplikes of order $p$, especially those containing a sub-Hopf algebra isomorphic to $T_{q'}$ for some $q'$ and/or having a Hopf algebra projection to some $T_q$, with $q$ not necessarily equal to $q'$, that is, the Hopf algebra contains a nontrivial skew-primitive element and/or its dual does. Since by Proposition 3.2$(ii)$ every non-cosemisimple Hopf algebra of dimension 27 with grouplikes of order 3 has a nontrivial skew-primitive element, these results will lead to the classification for dimension 27.

The next proposition is interesting itself but also has a number of useful consequences.

**Proposition 4.4.** *Let $L$ be a Hopf algebra of dimension $p^3$. Assume $L$ contains a sub-Hopf algebra $T \cong T_q$ and suppose that for some $n \geq 1$ and grouplike $g$, $L$ contains an element $y \in P_n^{g,1}$ with $y \notin P_{n-1}^{g,1}$ and $y \notin T$ such that $\Delta(y) = g \otimes y + y \otimes 1 + y'$ where $y' \in T \otimes T$. Then $L$ is pointed.*

*Proof.* Since $T \subsetneq L$, $p \leq |G(L)| < p^3$. If $|G(L)| = p^2$, then $L$ is pointed by [G, Prop. 3.3]. Assume then that $G(L) = G(T) \cong C_p$. Let $A_0 := T$ and $A_1$ be the vector space spanned by $T \cup \{ayb : a, b \in T\}$. Then $A_0 A_1 \subseteq A_1$ and $A_1 A_0 \subseteq A_1$. Let $K$ be the algebra generated by $A_1$, in other words by $T$ and $y$. Since the comultiplication is an algebra map, $K$ is a sub-bialgebra of $L$, and thus a sub-Hopf algebra. Since $p^2 < \dim(K) \leq p^3$, we have that $K = L$. Furthermore, $\Delta(A_1) \subseteq A_0 \otimes A_1 + A_1 \otimes A_0$ so that by [Mo, 5.5.1], it follows that $A_0 \supseteq L_0$ and so $L$ is pointed. $\square$

Recall that throughout this section $H$ denotes a nonsemisimple nonpointed non-copointed Hopf algebra of dimension $p^3$.

**Corollary 4.5.** *Assume $H$ contains a pointed sub-Hopf algebra $T \cong T_q$. Then all skew-primitive elements of $H$ lie in $T$. In particular, the dimension of the space of nontrivial skew-primitive elements is $p$.* $\square$

**Proposition 4.6.** *Suppose that $H$ has grouplikes of order $p$ and $P_1^{g,E} = 0 = P_1^{E,g}$ for all $g \in G(H)$ and all simple subcoalgebras $E$ of dimension greater than 1. Then $P^{g,E} = 0 = P^{E,g}$ for all $g \in G(H)$ and all simple subcoalgebras $E$ of dimension greater than 1.*

*If we assume further that $H$ contains a pointed sub-Hopf algebra $T \cong T_q$, then $H^{C_p,C_p} = T$ and if every simple subcoalgebra of dimension greater than 1 of $H$ has dimension divisible by $d^2 < p^2$, then $d^2$ must divide $p - 1$.*

*Proof.* Suppose that $P_k^{g,E} = 0$ for all $k < n$ and $P_n^{g,E}$ is nonzero. Then by Lemma 3.1$(i)$, there are simple subcoalgebras $D_i$, $1 \le i \le n - 1$ such that $P_i^{g,D_i}$ and $P_{n-i}^{D_i,E}$ are nondegenerate. But the first condition implies that the $D_i$ all have dimension 1 and the second condition then gives a contradiction to the induction assumption.

Since clearly $T \subseteq H^{C_p,C_p}$, it remains to show that $P^{g,h} \subset T$ for all $g, h \in G(H)$. By Corollary 4.5, $P_1^{g,h} \subset T$. Suppose that $P_k^{g,h} \subset T$ for all $k < n$ and let $0 \ne y \in P_n^{g,h}$ with $y \notin P_{n-1}^{g,h}$. Then $\Delta(y) = g \otimes y + y \otimes h + y'$ where $y' \in \sum_{1 \le i \le n-1} P_i \otimes P_{n-i}$. Since, by $(i)$, for $g$ any grouplike, $P_i^{g,D} = 0 = P_{n-i}^{D,g}$ for $1 \le i < n$ and for $D$ simple of dimension greater than 1, then $y' \in P_i^{C_p,C_p} \otimes P_{n-i}^{C_p,C_p} \subset T \otimes T$ by the induction assumption. By Proposition 4.4, $y \in T$, which implies that $P^{g,h} \subset T$ and hence $H^{C_p,C_p} = T$.

Suppose now that $H_0 = \Bbbk C_p \oplus E$ where $E$ is the sum of simple subcoalgebras of dimension divisible by $d^2 > 1$. Let $\mathcal{E}$ denote this set of simple subcoalgebras. Then as a coalgebra $H \cong H^{C_p,C_p} \oplus E \oplus P^{\mathcal{E},\mathcal{E}} = T \oplus E \oplus P^{\mathcal{E},\mathcal{E}}$. Thus $\dim H - \dim T = p^3 - p^2 = p^2(p-1)$ must be divisible by $d^2$ and since $d < p$, $d^2$ must divide $p - 1$. □

*Remark* 4.7. Suppose that $G(H) \cong C_p$ and $H$ contains a pointed sub-Hopf algebra $T \cong T_q$ such that $T = H^{C_p,C_p}$. If for some $d > 1$, any simple subcoalgebra of dimension greater than 1 has dimension divisible by $d^2$, then $d$ must divide $p - 1$ and if $d$ is even, then $2d$ divides $p - 1$. The argument is the same as that above, except that here possibly $P^{C_p,\mathcal{E}}$ is nonzero.

**Corollary 4.8.** *Suppose that $G(H) \cong C_p$. Then $H_0 \not\cong \Bbbk C_p \oplus \mathcal{M}^*(p-1,\Bbbk)^{sp} \oplus E$ as coalgebras where $E = 0$ or $E = \mathcal{M}^*(p,\Bbbk)^t$ with $s, t \ge 1$ .*

*Proof.* Suppose first that $E = 0$ and $H_0 \cong \Bbbk C_p \oplus \mathcal{M}^*(p-1,\Bbbk)^{sp}$. By Proposition 3.2$(ii)$, if $H$ has no nontrivial skew-primitives then $\dim H \ge p + sp(p-1)^2 + 2p(p-1) + (p-1)^2 + p = p^3 + (p-1)^2 + p > p^3$, a contradiction. Thus, $H$ must contain a pointed sub-Hopf algebra of dimension $p^2$.

Next suppose that $P_1^{g,D} \ne 0$ for some $g \in G(H)$, $D \in \mathcal{D}$, where $\mathcal{D}$ is the set of simple coalgebras of dimension $(p-1)^2$. Then by Lemma 3.1$(ii)$, $\dim P^{C_p,\mathcal{D}} = \dim P^{\mathcal{D},C_p} \ge p(p-1)$. By Lemma 3.1$(iii)$, $\dim P^{\mathcal{D},\mathcal{D}}$ is a nonzero multiple of $(p-1)^2$. In fact, since $p$ divides $\dim H_0$, $\dim P^{C_p,C_p}$ and $\dim P^{C_p,\mathcal{D}}$, then $\dim P^{\mathcal{D},\mathcal{D}}$ is a nonzero multiple of $p(p-1)^2$. Thus

$$\dim H \ge p^2 + sp(p-1)^2 + 2p(p-1) + p(p-1)^2 \ge 2p^3 - p^2 > p^3,$$

a contradiction. Hence $P^{C_p,\mathcal{D}} = 0$ and by Proposition 4.6, $(p-1)^2$ must divide $p - 1$, an impossibility. Thus $H_0 \not\cong \Bbbk C_p \oplus \mathcal{M}^*(p-1,\Bbbk)^{sp}$.

Now suppose that $E \cong \mathcal{M}^*(p,\Bbbk)^t$. The usual argument shows that $H$ has a sub-Hopf algebra isomorphic to $T_q$. But then $p^3 \ge p^2 + sp(p-1)^2 + tp^2 \ge p^3 + (t-1)p^2 + p$, a contradiction. □

*Remark* 4.9. Suppose that $H$ has coradical $\Bbbk C_p \oplus \mathcal{M}^*(p,\Bbbk)^n$. Then by Proposition 3.2$(ii)$, if $H$ has no nontrivial skew-primitive, the dimension of $H$ is at least $(n+3)p^2 + 2p$. Thus if

$(n+3)p^2 + 2p > p^3$, then $H$ contains a pointed sub-Hopf algebra isomorphic to $T_q$. In particular, if $H_0 \cong \Bbbk C_p \oplus \mathcal{M}^*(p, \Bbbk)^n$ with $n \geq p - 3$, then $H$ has a skew-primitive element.

**Corollary 4.10.** *Assume $p \geq 7$, then for $s \geq 1$, $H_0 \not\cong \Bbbk C_p \oplus \mathcal{M}^*(p-2, \Bbbk)^{sp}$ as coalgebras.*

*Proof.* We assume $s = 1$; the proof for $s > 1$ is the same. First we show that $H$ must have a skew-primitive element. Suppose not, then by Proposition 3.2(*i*), $P^{C_p, \mathcal{E}} \neq 0$ and $P^{\mathcal{E}, \mathcal{E}} \neq 0$ where $\mathcal{E}$ is the set of simple subcoalgebras of dimension $p - 2$. By the usual dimension arguments $\dim P^{\mathcal{E}, \mathcal{E}}$ is divisible by $p$ so that $\dim H_0 + 2 \dim P^{C_p, \mathcal{E}} + \dim P^{\mathcal{E}, \mathcal{E}} + \dim P^{C_p, C_p} = p + p(p-2)^2 + 2(p)(p-2) + p(p-2)^2 = 2p^3 + -6p^2 + 5p = p^3 + [p^6(p-6) + 5p] > p^3$, which is impossible. Thus $H$ has a sub-Hopf algebra $T \cong T_q$ for some $q$.

Now a similar argument shows that $P^{C_p, \mathcal{E}} = 0$. If not, then the dimension of $H$ is $p^2 + 2p(p-2)^2 + 2p(p-2) = p^3 + [p^2(p-5) + 4p] > p^3$, a contradiction. Thus we may apply Proposition 4.6 to obtain the result. $\qquad\square$

**Corollary 4.11.** *Let $p \geq 19$ be such that $p - 1$ is not divisible by $4$. Then $H_0 \not\cong \Bbbk C_p \oplus \mathcal{M}^*(p-5, \Bbbk)^p \oplus \mathcal{M}^*(p-3, \Bbbk)^p$.*

*Proof.* By Proposition 3.2(*ii*), if $H$ has no skew-primitives, then

$$p^3 \geq p + p(p-5)^2 + p(p-3)^2 + (2p - 10 + 1)p + (p-5)^2 = 2p^3 - 13p^2 + 16p + 25$$

which is impossible if $p \geq 19$. Thus $T_q \subsetneq H$ for some $q$. If $P_1^{g, E}$ for some $E$ simple of dimension at least $(p-5)^2$, then the same counting argument as in Corollary 4.8 gives a contradiction. The statement then follows from Proposition 4.6. $\qquad\square$

The next propositions give some information about the coradical of Hopf algebras of type $(p, p)$ such that the dual contains a Taft Hopf algebra.

**Proposition 4.12.** *Let $H$ be of type $(p, p)$ with $G(H) = \langle c \rangle$ and suppose there is a Hopf algebra projection $\pi : H \twoheadrightarrow T_q$. Then $\pi(c) \neq 1$. If, also, $H$ contains a sub-Hopf algebra $T_{q'} = \Bbbk\langle c, x \rangle$, then $\pi(x) = 0$ and $H^{co\pi}$ has basis $\{1, x, \ldots, x^{p-1}\}$.*

*Proof.* If $\pi(c) = 1$, then $\Bbbk\langle c \rangle \cong \Bbbk C_p \subseteq H^{co\,\pi}$. Since $\dim H^{co\,\pi} = p$, by Lemma 2.3 the sequence of Hopf algebras $\Bbbk C_p \hookrightarrow H \twoheadrightarrow T_q$ is exact, which implies by Theorem 4.1 that $H$ is pointed.

If $H$ has a skew-primitive element $x$ and $\pi(x) \neq 0$, then necessarily $T_q \cong T_{q'}$ and hence $H$ is a bosonization of $T_q$. This implies that it is pointed by [AS2, Thm. 8.8]. $\qquad\square$

**Proposition 4.13.** *Let $H$ be of type $(p, p)$ with $G(H) = \langle c \rangle$ and suppose there is a Hopf algebra projection $\pi : H \twoheadrightarrow T_q$. Assume further that $H_0 \cong \Bbbk\langle c \rangle \oplus \mathcal{M}^*(2, \Bbbk)^{tp} \oplus E$ where $t \geq 1$ and $E$ is either $0$ or the sum of simple coalgebras of dimension strictly greater than $4$. Then*

*(i) The simple subcoalgebras of dimension $4$ are nontrivially permuted by $L_c$ ($R_c$), left (right) multiplication by $c$ and also by the antipode.*

*(ii) The number of simple subcoalgebras of dimension $4$ must be strictly greater than $p$.*

*(iii) If $t = 2$ then the simple subcoalgebras of dimension $4$ are stable under $S_H^4$.*

*(iv) If some simple subcoalgebra is fixed by $S_H^4$, then there is a simple subcoalgebra $D$ of dimension $4$ with multiplicative matrix $\mathbf{e}$ such that $e_{22}, e_{21}$ are in $^{co\pi}H$ and $c^{-1}e_{11}, c^{-1}e_{21}$ are in $H^{co\pi}$.*

*Proof.* (*i*) For any simple subcoalgebra $D$ of $H$, the algebra generated by $D$ is a bialgebra, thus a sub-Hopf algebra, and hence all of $H$. Therefore $\pi(D)$, the image of $D$ in $T_q$ must generate $T_q$ as an algebra. Suppose $T_q = \Bbbk\langle g, y\rangle$ with $\Delta(y) = g \otimes y + y \otimes 1$.

If $\dim D = 4$, by Theorem 2.4, $\pi(D)$ must have dimension 3 and have basis $\{g^i y, g^i, g^{i+1}\}$ for some $i \geq 0$. Denote this subcoalgebra of $T_q$ by $\langle g^i y\rangle$. By Proposition 4.12, we may assume that $\pi(c) = g$ and thus $\pi(c^j D) = \pi(Dc^j) = \langle g^{i+j} y\rangle$. Thus $L_c$ and $R_c$ nontrivially permute the simple subcoalgebras of dimension 4. Also $S_{T_q}(y) = -g^{-1}y$, so that $S_{T_q}(\langle g^i y\rangle) = \langle g^{-1}yg^{-i}\rangle = \langle g^{-i-1}y\rangle$ and $S$ also properly permutes the simple subcoalgebras of dimension 4.

(*ii*) Suppose that $t = 1$ so that for each $i$, there is exactly one 4-dimensional simple subcoalgebra which maps by $\pi$ to $\langle g^i y\rangle$. Since $\langle g^i y\rangle$ is fixed by $S_{T_q}^2$ then the simple subcoalgebras of dimension 4 are fixed by $S_H^2$ but permuted nontrivially by $S_H$. Since there is an odd number of simple subcoalgebras of dimension 4, then one of these must be fixed by $S_H$, a contradiction by Proposition 4.3.

(*iii*) If $t = 2$, then for each $i$ there are exactly two 4-dimensional simple subcoalgebras which map by $\pi$ to any $\langle g^i y\rangle$. If these are not stable under $S_H^2$, then they are exchanged by $S_H^2$ and thus stable by $S_H^4$.

(*iv*) Suppose $D$ is a simple 4-dimensional subcoalgebra stable under $S_H^4$. By (*i*), we may assume that $\pi(D) = \langle y\rangle$. By Radford's formula for $S^4$, since $H$ and $H^*$ have grouplikes of order $p$, $S_H^{4p} = \mathrm{id}_H$. Also $S_{T_q}$ has order $2p$. Thus $\mathrm{ord}\, S_H^4 = p$ and also $\mathrm{ord}\, S_{T_q}^4 = p$. Since $gy = qyg$, $S^4(y) = q^{-2}y$. Thus the subcoalgebra $\langle y\rangle$ of $T_q$ is the direct sum of the eigenspace for 1 with basis $\{1, g\}$ and the eigenspace for $q^{-2}$ with basis $y$. Now apply Theorem 2.5 to obtain a multiplicative matrix $\mathbf{e}$ for $D$ such that $e_{11}, e_{22}$ are a basis for the eigenspace for 1 for $S_H^4$ and $e_{12}$ is an eigenvector for $q^{-2}$. Then $e_{21}$ is an eigenvector for $q^2$ and thus $\pi(e_{21}) = 0$. Then $\pi(e_{ii})$ is grouplike in $\langle y\rangle$ and $\pi(e_{12})$ is $(\pi(e_{22}), \pi(e_{11}))$- primitive. Thus $\pi(e_{22}) = 1, \pi(e_{11}) = g, \pi(e_{12}) = \alpha y$ where $\alpha$ is a scalar, and by rescaling if necessary we can assume that $\pi(e_{12}) = y$. It is now immediate that $e_{22}, e_{21}$ are in ${}^{co\pi}H$ and $c^{-1}e_{11}, c^{-1}e_{21}$ are in $H^{co\pi}$. $\qquad\square$

In the next proposition, we assume that both $H$ and $H^*$ have nontrivial skew-primitive elements and we can then show that $H$ cannot contain a 4-dimensional simple subcoalgebra.

**Proposition 4.14.** *Suppose $H$ contains a sub-Hopf algebra $T \cong T_{q'}$ and suppose also that there is a Hopf algebra projection $\pi : H \to T_q = \Bbbk\langle g, y\rangle$. Then $H$ cannot contain a simple subcoalgebra of dimension* 4.

*Proof.* Suppose $D \cong \mathcal{M}^*(2, \Bbbk)$ is a subcoalgebra of $H$. Then, as in the proof of Proposition 4.13, $\pi(D)$ generates $T_q$ so that $\pi(D) = \langle g^i y\rangle$ has basis $\{g^i, g^i y, g^{i+1}\}$ for some $0 \leq i \leq p-1$.

Then $\ker \pi_{|_D} = \Bbbk z$ for some $z \in D$ and is a coideal in $D$. Since $\pi(z) = 0$, then $\varepsilon(z) = 0$ too. Thus $\Delta(z) = a \otimes z + z \otimes b$ where $a, b \in H$ with $\varepsilon(a) = \varepsilon(b) = 1$. Then $(\Delta \otimes \mathrm{id}_D) \circ \Delta(z) = (\mathrm{id}_D \otimes\Delta) \circ \Delta(z)$ implies that

$$\Delta(a) \otimes z + \Delta(z) \otimes b = a \otimes \Delta(z) + z \otimes \Delta(b).$$

Applying $\mathrm{id}_D \otimes\pi \otimes \pi$ and $\pi \otimes \pi \otimes \mathrm{id}_D$ to the above, we obtain that

$$z \otimes \pi(b) \otimes \pi(b) = z \otimes (\pi \otimes \pi)\Delta(b); \quad (\pi \otimes \pi)\Delta(a) \otimes z = \pi(a) \otimes \pi(a) \otimes z.$$

Thus $\pi(a), \pi(b)$ are grouplike in $\pi(D)$. If $\pi(b) = g^j$, since by Proposition 4.12 there is some $c \in G(H)$ with $\pi(c) = g^{-j}$, then

$$(\mathrm{id}_D \otimes\pi) \circ \Delta(cz) = cz \otimes \pi(cb) = cz \otimes 1,$$

and $cz \in H^{co\pi}$, which is a contradiction by Proposition 4.12. $\qquad\square$

In the next example we give another proof of the impossibility of coradical $\Bbbk C_3 \oplus \mathcal{M}^*(2, \Bbbk)^3$ for $H$ of dimension 27 since the argument uses the dimension of projective covers of simple $H$-comodules and illustrates different techniques for these problems.

**Example 4.15.** *Assume* $\dim H = 27$ *and* $H$ *is of type* $(3, 3)$. *Then* $H_0 \not\cong \Bbbk C_3 \oplus \mathcal{M}^*(2, \Bbbk)^3$ *as coalgebras.*

*To see this, suppose* $H_0 \cong \Bbbk C_3 \oplus \mathcal{M}^*(2, \Bbbk)^3$ *as coalgebras. Then the isomorphism classes of the simple* $H^*$-*modules are the three one-dimensional modules* $\Bbbk_\varepsilon, \Bbbk_g, \Bbbk_{g^2}$ *and three two-dimensional modules* $V_1, V_2$ *and* $V_3$. *If we denote by* $P(V)$ *the projective cover of a simple module* $V$, *then we have that* $27 = 3 \dim P(\Bbbk_\varepsilon) + \sum_{i=1}^3 2 \dim P(V_i)$. *Since by* [EG, *Proof of Prop. 2.1*], $\dim P(\Bbbk_{g^j}) \geq \dim P(\Bbbk_\varepsilon)$ *for* $j = 1, 2$, *and* $P(\Bbbk_\varepsilon) \otimes \Bbbk_{g^j}$ *is projective, it follows that* $P(\Bbbk_\varepsilon) \otimes \Bbbk_{g^j} \cong P(\Bbbk_{g^j})$ *for* $j = 1, 2$. *Analogously,* $P(V_1) \otimes \Bbbk_{g^{j-1}} \cong P(V_j)$ *for* $j = 1, 2$. *Indeed, as the simple modules* $V_j$ *are not stable by right tensoring with* $\Bbbk_g$, *since otherwise 3 would divide* $\dim V_i = 2$ *by* [EG, *Prop. 2.5*], *we may assume that* $V_j \cong V_1 \otimes \Bbbk_{g^{j-1}}$ *for* $j = 1, 2$. *Then there is a surjection from the projective module* $P(V_1) \otimes \Bbbk_{g^{j-1}}$ *to* $V_1 \otimes \Bbbk_{g^{j-1}} \cong V_j$ *which implies that it also has a projection to* $P(V_j)$ *and in particular,* $\dim P(V_1) \geq \dim P(V_j)$. *Since the action given by right tensoring with* $\Bbbk_g$ *is transitive, we have that* $\dim P(V_j) \geq \dim P(V_1)$ *for* $j = 1, 2$ *which implies that* $P(V_1) \otimes \Bbbk_{g^{j-1}} \cong P(V_j)$ *for* $j = 1, 2$. *Consequently we have that* $27 = 3 \dim P(\Bbbk_\varepsilon) + 6 \dim P(V_1)$.

*We claim now that* $\operatorname{ord} S = 6$. *Indeed, by Radford's formula for the antipode we know that* $\operatorname{ord} S | 12$. *Assume that* $\operatorname{ord} S \neq 6$ *and let* $D$ *be a simple 4-dimensional subcoalgebra of* $H$. *Then* $D$ *generates* $H$ *as an algebra since the subalgebra generated by* $D$ *is a sub-Hopf algebra and there is no nonsemisimple and nonpointed Hopf algebra of dimension* $p$ *or* $p^2$ *by* [Z] *and* [Ng]. *Since the coradical of* $H$ *contains 3 simple subcoalgebras of dimension 4, we have that* $S^6(D) = D$ *and by Theorem 2.5, there exists a comatrix basis* $(e_{ij})_{1 \leq i,j \leq 2}$ *such that* $S^6(e_{ij}) = (-1)^{i-j} e_{ij}$. *On the other hand, since* $H^*$ *contains a sub-Hopf algebra isomorphic to a Taft Hopf algebra* $T_q$, *there exists a Hopf algebra surjection* $\pi : H \to T_q$. *But then,* $(-1)^{i-j} \pi(e_{ij}) = \pi(S^6(e_{ij})) = S^6(\pi(e_{ij})) = \pi(e_{ij})$, *which implies that* $\pi(e_{ij}) = 0$ *for* $i \neq j$ *and consequently* $\pi(D) \subseteq \Bbbk G(T_q)$. *This is impossible since* $D$ *generates* $H$ *as an algebra and therefore* $\pi(D)$ *generates* $T_q$ *as an algebra too.*

*As* $\operatorname{ord} S = 6$, *we have that 3 divides* $\dim P(\Bbbk_\varepsilon)$ *by* [CNg, *Cor. 1.5*]. *Since* $27 = 3 \dim P(\Bbbk_\varepsilon) + 6 \dim P(V_1)$ *and* $\dim P(V_1) \geq 2$, *the only possibility is that* $\dim P(\Bbbk_\varepsilon) = 3$ *and* $\dim P(V_1) = 3$. *But this cannot occur, for if* $P(V) \neq V$ *for a simple module* $V$, *then* $\dim P(V) \geq 2 \dim V$. *Hence, there is no Hopf algebra of dimension 27 whose coradical contains 3 simple subcoalgebras of dimension 4.*

Next we consider the number of simple subcoalgebras of dimension 9 in a Hopf algebra $H$ of dimension $p^3$ where both $H$ and $H^*$ have a nontrivial skew-primitive element.

**Proposition 4.16.** *Let* $p > 3$. *Assume* $H$ *has a pointed sub-Hopf algebra* $T \cong T_{q'}$ *and suppose there is a Hopf algebra projection* $\pi : H \twoheadrightarrow T_q = \Bbbk\langle g, y \rangle$. *Then the number of simple subcoalgebras of* $H$ *of dimension 9 is a multiple of* $p^2$.

*Proof.* If $H$ does not contain a simple subcoalgebra of dimension 9 there is nothing to prove. Assume then that $H$ contains a simple subcoalgebra $D \cong \mathcal{M}^*(3, \Bbbk)$. As usual, by Proposition 4.12, there is a grouplike element $c \in H$ such that $\pi(c) = g$. We show that the set $\mathcal{D}$ of

subcoalgebras $c^i D c^j$, $0 \le i, j \le p-1$, must have $p^2$ distinct elements. Since if $E$ is another simple subcoalgebra of dimension 9 such that $E \notin \mathcal{D}$, then for $\mathcal{E}$ the set of coalgebras $c^i E c^j$, $\mathcal{D} \cap \mathcal{E} = \emptyset$, this will prove the statement.

Suppose that two coalgebras in the set $\{c^i D c^j | 0 \le i, j \le p-1\}$ are equal. Thus $D = c^i D c^j$ for some $i, j$, not both 0. Let $\phi : D \to D$ denote the coalgebra automorphism of order $p$ defined by $\phi(d) = c^i d c^j$ and then $\pi \circ \phi(d) = g^i \pi(d) g^j$. First we show that $i + j \equiv 0 \mod p$.

Suppose not. For the coalgebra automorphism $\phi'$ of $T_q$ given by $w \mapsto g^i w g^j$, let $T_q^{[\ell]}$ be the eigenspace for $q^\ell$, $0 \le \ell < p$. Then a basis for $T_q^{[\ell]}$ is $\{(\sum_{t=0}^{p-1} q^{-t(kj+\ell)} g^{t(i+j)}) y^k | 0 \le k < p\}$. Since $\pi(D)$ generates $T_q$, then $\pi(D) \nsubseteq \Bbbk \langle g \rangle$ and thus some element $(\sum_{t=0}^{p-1} q^{-t(kj+\ell)} g^{t(i+j)}) y^k \in \pi(D)$ with $k > 0$ lies in $\pi(D)$. But then $g^m y^k$ and $g^m$ lie in $\pi(D)$ for all $0 \le m \le p-1$ so that the dimension of $\pi(D)$ is at least $2p$. But since $p > 3$, $2p > 9$ so this is impossible and thus $i+j \equiv 0 \mod p$. Then $\phi = \mathrm{ad}_\ell c^i$ and $D$ is also stable under $\mathrm{ad}_\ell c$. Thus we assume that $i = 1$.

Now let $A := (a_{ij})$ be the $3 \times 3$ matrix of eigenvalues for the eigenvectors $e_{ij}$ in the multiplicative matrix $\mathbf{e}$ for $\mathrm{ad}_\ell c$ on $D$ so that in the notation of Proposition 2.6, $a_{ij} = \omega_i \omega_j^{-1}$. In $T_q$, a basis of eigenvectors for $\mathrm{ad}_\ell g$ for the eigenvalue $q^m$ is $\{g^n y^m | 0 \le n \le p-1\}$. Again, we denote this eigenspace by $T_q^{[m]}$. If every entry in the first row of $A$ is 1, then $\omega_1 = \omega_2 = \omega_3$. Since $T_q^{[0]} = \Bbbk \langle g \rangle$ and $\pi(D)$ generates $T_q$ we cannot have that every entry of $A$ is 1 and so this is impossible. Suppose that exactly two of the entries in the first row of $A$ equal 1, say $a_{11} = a_{12} = 1$. Then $\omega_1 = \omega_2$ and so the first two rows of $A$ are equal and the first two columns of $A$ are equal. Since $\pi(D)$ generates $T_q$ as an algebra, some $a_{ij}$ must be $q$. Suppose that $a_{13} = a_{23} = q$ and then $a_{31} = a_{32} = q^{-1}$. (The argument will be the same if we swap $q$ and $q^{-1}$.)

Next note that if $a_{ij} = q^n$ with $n \ge 3$ then $\pi(e_{ij}) = 0$. For if $\pi(e_{ij}) \ne 0$ then $\pi(e_{ij}) = \psi(g) y^n$ where $\psi(g) \in \Bbbk \langle g \rangle$ so that $g^m y^n \in \pi(D)$ for some $0 \le m < p$. Since $\pi$ is a coalgebra map this implies that $g^m y^i \in \pi(D)$ for $0 \le i \le n$ so that $\dim \pi(D) \ge n+1 \ge 4$. Also $g^m y \in \pi(D)$ so that $g^{m+1} y^{n-1} \in \pi(D)$ and thus $g^{m+1} y^i \in \pi(D)$ for $0 \le i \le n-1$. Similarly $g^{m+2} y^{n-2}, g^{m+2} y^{n-3} \in \pi(D)$ so $\dim \pi(D) \ge 2n+3 \ge 9$, a contradiction.

Now assuming that $a_{11} = a_{12} = 1$ and $a_{13} = a_{23} = q$ implies that $a_{31} = a_{32} = q^{-1}$ so that $\pi(e_{31}) = \pi(e_{32}) = 0$. Then $\pi(e_{33})$ is grouplike and $\pi \otimes \mathrm{id} \circ \Delta(e_{33}) = \pi(e_{33}) \otimes e_{33}$. Thus $c^m e_{33} \in {}^{co\pi} H$ for some $m$, contradicting Proposition 4.12.

Therefore exactly one entry in the first row of $A$, namely $a_{11}$, equals 1. Thus $\pi(e_{11})$ is grouplike in $\pi(D)$ and $\pi(e_{12}) \otimes \pi(e_{21}) + \pi(e_{13}) \otimes \pi(e_{31}) = 0$. If $\pi(e_{12}), \pi(e_{13})$ are both 0 (both nonzero) then $e_{11} \in {}^{co\pi} H$ ($H^{co\pi}$ respectively), and again Proposition 4.12 provides a contradiction. Suppose $\pi(e_{12}) = 0$ and $\pi(e_{13}) \ne 0$ so that $\pi(e_{31}) = 0$. But $\pi(e_{12}) = 0$ implies that $\pi(e_{13}) \otimes \pi(e_{32}) = 0$ so that $\pi(e_{32}) = 0$. Then $\pi(e_{22})$ is grouplike and $e_{22} \in H^{co\pi}$, contradiction.

Thus the set $\mathcal{D}$ contains $p^2$ distinct coalgebras of dimension 9 and the proof is finished. $\square$

**Corollary 4.17.** *If $p = 5$ or $7$ in Proposition 4.16 then $H$ has no simple subcoalgebras of dimension 9.*

*Proof.* Since $T \subset H$, if $H$ has a simple subcoalgebra of dimension 9, then the dimension of $H$ is at least $p^2 + 9p^2 = 10p^2$ and this is impossible for $p < 10$. $\square$

**Corollary 4.18.** *Suppose $H$ and $H^*$ are of type $(p, p)$ and each contains a skew-primitive element. Let $\mathcal{D}$ be the set of simple subcoalgebras of dimension 9 in $H$.*

    *(i) Then $P^{C_p, \mathcal{D}} = 0$.*

(ii) If $p - 1$ is not divisible by 9 then $H_0 \ncong \Bbbk C_p \oplus \mathcal{M}^*(3, \Bbbk)^{tp^2}$.

*Proof.* (i) By Proposition 4.16 and Lemma 3.1(ii), the dimension of $P^{C_p, \mathcal{D}}$ is at least $p^3$.

(ii) Suppose $H_0 \cong \Bbbk C_p \oplus \mathcal{M}^*(3, \Bbbk)^{tp^2}$. Then by Proposition 4.6(iii), 9 must divide $p - 1$. □

Finally consider the simple subcoalgebras of dimension $p^2$ in a Hopf algebra of dimension $p^3$.

**Lemma 4.19.** *Assume $H$ is of type $(p, p)$ with $G(H) = \langle c \rangle$ and suppose there is a Hopf algebra projection $\pi : H \to T_q$. If $D$ is a simple subcoalgebra of $H$ of dimension $p^2$ then $L_c$, left multiplication by $c$, nontrivially permutes either the right or left simple subcomodules of $D$.*

*Proof.* Let $T_q = \Bbbk\langle g, y \rangle$. By Proposition 4.12 we may assume that $\pi(c) = g$. Since $L_c$, left multiplication by $c$, is a coalgebra bijection of order $p$ in $\text{End}(H)$ and since there are fewer than $p$ simple subcoalgebras of dimension $p^2$ in $H$, then $L_c$ maps $D$ to $D$. Similarly, $\text{ad}_\ell c$, the left adjoint action of $c$ on $H$ is a coalgebra automorphism of $D$. By Proposition 2.6, there is a multiplicative matrix $\mathbf{e}$ in $D$ for $\text{ad}_\ell c$. Then $\pi \circ \text{ad}_\ell c = \text{ad}_\ell g \circ \pi$. Since $gy = qyg$, with $q$ a primitive $p$th root of unity, the eigenspace for the eigenvalue $q^i$ for $\text{ad}_\ell g$ in $T_q$ is $T_q^{[i]} := \{g^j y^i | 0 \le j \le p - 1\}$. Using similar notation, let $D^{[i]}$ denote the eigenspace for $q^i$ for $\text{ad}_\ell c$ in $D$. The Hopf algebra projection $\pi$ maps $D^{[i]}$ to $T_q^{[i]}$.

Let $M_j$ denote the simple left subcomodule of $D$ with basis $\{e_{1j}, \dots, e_{pj}\}$ and let $R_j$ denote the simple right subcomodule of $D$ with basis $\{e_{j1}, \dots, e_{jp}\}$. Suppose that both the $R_i$ and the $M_i$ are stable under $L_c$. Then for all $i, j$, $L_c(e_{ij}) = \alpha e_{ij}$ for some nonzero scalar $\alpha$; in other words, $\mathbf{e}$ is a multiplicative matrix of eigenvectors both for $\text{ad}_\ell c$ and $L_c$. If $i = j$, by applying $\varepsilon$, we see that $\alpha = 1$ and since $e_{jj} \in D^{[0]}$ and $\pi(e_{jj}) = g\pi(e_{jj})$ then for all $j$, $\pi(e_{jj}) = t = (1/p) \sum_i g^i$, the integral in $\Bbbk\langle g \rangle$. Similarly for $i \ne j$, if $e_{ij} \in D^{[0]}$, then $\pi(e_{ij}) = \beta f_\lambda$ where $f_\lambda := [1 + \lambda g + \lambda^2 g^2 + \dots + \lambda^{p-1} g^{p-1}]$ for $\lambda$ a primitive $p$th root of 1. Note that $gf_\lambda = \lambda^{-1} f_\lambda$.

Suppose that $R_1$ contains exactly $n \ge 1$ eigenvectors for the eigenvalue 1. Then since $(\pi \otimes \pi) \circ \Delta(e_{11}) = \Delta \circ \pi(e_{11}) = \Delta(t) \in T_q^{[0]} \otimes T_q^{[0]}$, and since for $e_{ij} \in D^{[0]}$, $i \ne j$, each $\pi(e_{ij})$ is a scalar multiple of $f_\lambda$ for some primitive $p$th root of unity $\lambda$, we have that

$$(4.1) \qquad \Delta(t) = t \otimes t + \beta_2 f_{\lambda_2} \otimes f_{\lambda_2'} + \dots + \beta_n f_{\lambda_n} \otimes f_{\lambda_n'}.$$

By comparing coefficients of $1 \otimes 1$ on both sides of (4.1) we obtain:

$$1 = 1 + \beta_2 + \beta_3 + \dots + \beta_n,$$

and comparing coefficients of $g^i \otimes 1$ for $i > 0$ on both sides of (4.1) we get:

$$0 = 1 + \beta_2 \lambda_2^i + \beta_3 \lambda_3^i + \dots + \beta_n \lambda_n^i.$$

Then, adding the coefficients of $g^i \otimes 1$ for $0 \le i \le p - 1$ on both sides of (4.1), we have:

$$1 = p + \beta_2 [1 + \lambda_2^1 + \lambda_2^2 + \dots + \lambda_2^{(p-1)}] + \dots + \beta_n [1 + \lambda_n^1 + \lambda_n^2 + \dots + \lambda_n^{(p-1)}] = p,$$

a contradiction. Thus $L_c$ properly permutes either the $R_i$ or the $M_i$ and the proof is complete. □

The next proposition shows that Hopf algebras of dimension $p^3$ of type $(p, p)$ where both $H$ and $H^*$ have nontrivial skew-primitives cannot have simple subcoalgebras of dimension $p^2$.

**Proposition 4.20.** *Assume $H$ has a pointed sub-Hopf algebra $T \cong T_{q'}$ and there is a Hopf algebra projection $\pi : H \twoheadrightarrow T_q = \Bbbk\langle g, y \rangle$. Then $H$ has no simple subcoalgebra of dimension $p^2$.*

*Proof.* Let $c \in G(H)$ be such that $\pi(c) = g$. Let $D$ be a simple subcoalgebra of $H$ of dimension $p^2$, and we seek a contradiction. As in the proof of Lemma 4.19, let $T_q^{[i]} := \{g^j y^i | 0 \le j \le p - 1\}$ and $D^{[i]}$ denote the eigenspaces for the eigenvalue $q^i$ for the coalgebra morphisms $\operatorname{ad}_\ell g$ and $\operatorname{ad}_\ell c$ in $T_q$ and $D$ respectively. Let $\mathbf{e}$ be a multiplicative matrix for the coalgebra isomorphism $\operatorname{ad}_\ell c$ of $D$ and let $R_i \subset D$ be the simple right $D$-comodule with basis $\{e_{ij} | 1 \le j \le p\}$.

Let $A = (a_{ij}) = (\omega_i \omega_j^{-1})$ be the $p \times p$ matrix whose entries are the eigenvalues for the multiplicative matrix $\mathbf{e}$ from Proposition 2.6. Since by Proposition 2.6 the $a_{ij}$ are $p$th roots of unity, each $a_{ij} = q^k$ for some $k$. First we wish to show that each row of $A$ contains the $p$ distinct entries $\{1, q, \dots, q^{p-1}\}$ in some order so that the dimension of $D^{[i]}$ is $p$ for every $i$. By Lemma 4.19, $L_c$ permutes either the simple left $D$-subcomodules $M_i$ or the simple right $D$-subcomodules $R_i$ nontrivially. We assume that $L_c$ permutes the $R_i$ nontrivially; the argument is the same if the $M_i$ are permuted nontrivially. Thus each row of $A$ contains exactly the same entries, perhaps in a different order. We note that each row of $A$ contains an entry 1 since $a_{ii} = 1$ and also must contain an entry $q$ since $\pi(D)$ generates $T_q$ as an algebra.

Let $N$ be the maximum number of equal entries in a row and suppose that $a_{1i_1} = a_{1i_2} = \dots = a_{1i_N}$. Then $\omega_{i_1} = \omega_{i_2} = \dots = \omega_{i_N}$ so that rows $\omega_{i_1}, \dots, \omega_{i_N}$ are equal as vectors in $\Bbbk^p$ and the columns $\omega_{i_1}, \dots, \omega_{i_N}$ are also equal as vectors in $\Bbbk^p$. Thus $1 = a_{i_1,i_1} = a_{i_1,i_2} = \dots, a_{i_1,i_N}$ and so since row $i_1$ has $N$ entries equal to 1, each row of $A$ also contains $N$ entries equal to 1. By the maximality of $N$, the rows must contain exactly $N$ entries equal to 1.

Let $a_{i_1 k}$ be an entry in row $i_1$ different from 1. Then $a_{i_1 k}^{-1} = a_{k i_1}$ lies in column $i_1$ and since columns $i_1, \dots, i_N$ are equal, the $k$th row contains $N$ entries equal to $a_{k i_1}$. Thus row $i_1$ also has $N$ entries equal to $a_{k i_1}$. The same argument with entry $a_{k i_1}$ in row $i_1$ shows that row $i_1$ has $N$ entries equal to $a_{k i_1}^{-1} = a_{i_1 k}$. Thus every distinct entry in row $i_1$ occurs exactly $N$ times and so $N$ divides $p$. Thus $N = p$ or $N = 1$. If $N = p$ then all entries in $A$ equal 1 contradicting the fact that $\pi(D)$ generates $T_q$. Thus $N = 1$.

Thus each $R_i$ contains precisely one eigenvector for each eigenvalue $q^i$. By relabelling if necessary we may assume that $e_{1i}$ is an eigenvector for $q^{i-1}$ and then $e_{i1}$ is an eigenvector for $q^{1-i}$. Then $e_{jk}$ is an eigenvector for $q^{1-j}q^{k-1} = q^{k-j}$ and $D^{[i]} = \{c^j e_{1,i+1} | 0 \le j \le p - 1\}$.

Let $0 \le i \le p - 1$. Suppose $0 \ne \pi(e_{1,i+1}) = \phi(g)y^i$ where $\phi(g) = a_0 + a_1 g + \dots a_{p-1} g^{p-1}$. Suppose that $a_k \ne 0$. Then $g^k y^i \otimes g^k \in \pi(D) \otimes \pi(D)$ and so $g^k y^i \in \pi(D)$. Since $\pi(D)$ is invariant under left multiplication by $g$, this means that all $g^j y^i \in \pi(D)$ and so if the dimension of $\pi(D^{[i]})$ is nonzero, it is $p$. Note that the dimensions of $\pi(D^{[0]})$ and $\pi(D^{[1]})$ must be $p$ and that if the dimension of $\pi(D^{[i]})$ is $p$, then the dimension of $\pi(D^{[j]})$ is also $p$ for $j < i$.

Let $M$ be the maximum such that $\pi(D^{[M]}) \ne 0$ so that $1 \le M \le p - 2$ and then $0 = \pi(e_{1,n})$ for $n > M + 1$. Then

$$0 = \Delta(\pi(e_{1,M+1})) = \sum_{i=1}^{M} \pi(e_{1,i}) \otimes \pi(e_{i,M+1}) \in \sum \pi(D^{[i-1]}) \otimes \pi(D^{[M+1-i]}).$$

But if any of the terms $\pi(e_{1,i})$ or $\pi(e_{i,M+1})$ is 0 then the dimension of $\pi(D^{[i-1]})$ or $\pi(D^{[M+1-i]})$ is less than $p$, so we have a contradiction. $\square$

**Corollary 4.21.** *If $p = 3$ and $H$ is of type $(3,3)$, then $H$ has no subcoalgebra $\mathcal{M}^*(3, \Bbbk)$.*

*Proof.* By Proposition 3.2$(ii)$, both $H$ and $H^*$ contain a 9-dimensional Taft Hopf algebra. Now apply Proposition 4.20. $\square$

**Corollary 4.22.** *Let $H$ have grouplikes of order $p$ and suppose $H_0 \cong \Bbbk G(H) \oplus \mathcal{M}^*(p, \Bbbk)^t$. Then if $t \geq p - 3$, $H^*$ has no skew-primitives.*

*Proof.* By Remark 4.9, $H$ has a skew-primitive element and so has a sub-Hopf algebra isomorphic to a Taft Hopf algebra. If $H^*$ also had a skew-primitive element, there would be a contradiction to Proposition 4.20. $\square$

**Example 4.23.** *Suppose $H$ has dimension $5^3$, and is of type $(5, 5)$. Then $H$ and $H^*$ cannot both have a skew-primitive element.*

For suppose that $H$ contains a sub-Hopf algebra $T_{q'}$ and there is also a Hopf algebra projection $\pi$ from $H$ to $T_q = \Bbbk \langle g, y \rangle$. Then by Proposition 4.14, $H$ has no simple subcoalgebra of dimension $2^2$, by Corollary 4.17, $H$ has no simple subcoalgebra of dimension $3^2$, and by Proposition 4.20, $H$ has no simple subcoalgebra of dimension $5^2$. Thus if $H$ has any simple subcoalgebra of dimension greater than $1$ it must have dimension $4^2$, i.e., $H_0 \cong \Bbbk C_5 \oplus \mathcal{M}^*(4, \Bbbk)^{tp}$ and this is impossible by Corollary 4.8.

4.3. **Hopf algebras of dimension** 27. In this short subsection we complete the classification for dimension 27.

**Theorem 4.24.** *A Hopf algebra of dimension 27 is semisimple, pointed or copointed.*

*Proof.* Assume $H$ is nonsemisimple, nonpointed and non-copointed. Then by [G] we need only consider types $(3, 3)$ and $(3, 1)$. First consider type $(3, 3)$. By Proposition 3.2, if $H$ is of type $(3, 3)$ then $H$ and $H^*$ both contain a Taft Hopf algebra of dimension 9. By Corollary 4.21, $H$ has no subcoalgebra isomorphic to $\mathcal{M}^*(3, \Bbbk)$ and by Proposition 4.14, $H$ has no subcoalgebra isomorphic to $\mathcal{M}^*(2, \Bbbk)$, which implies that $H_0 = \Bbbk G(H)$, a contradiction.

To complete the proof, we show that no Hopf algebra of dimension 27 can have only trivial grouplikes. If $|G(H)| = 1$, all possible coradicals are listed in the table below. We show that each leads to a contradiction.

| **Case** | $H_0$ | dim $H_0$ |
|---|---|---|
| $(i)$ | $\Bbbk \cdot 1 \oplus \mathcal{M}^*(2, \Bbbk)^n$, $1 \leq n \leq 6$ | $1 + 4n$ |
| $(ii)$ | $\Bbbk \cdot 1 \oplus \mathcal{M}^*(3, \Bbbk)^n$, $1 \leq n \leq 2$ | $1 + 9n$ |
| $(iii)$ | $\Bbbk \cdot 1 \oplus \mathcal{M}^*(4, \Bbbk)$ | $1 + 16$ |
| $(iv)$ | $\Bbbk \cdot 1 \oplus \mathcal{M}^*(5, \Bbbk)$ | $1 + 25$ |
| $(v)$ | $\Bbbk \cdot 1 \oplus \mathcal{M}^*(2, \Bbbk)^n \oplus \mathcal{M}^*(3, \Bbbk)^m$, $0 < n, m$, $0 < 4n + 9m < 26$ | $1 + 4n + 9m$ |
| $(vi)$ | $\Bbbk \cdot 1 \oplus \mathcal{M}^*(2, \Bbbk)^n \oplus \mathcal{M}^*(3, \Bbbk)^m \oplus \mathcal{M}^*(4, \Bbbk)$, $0 \leq n, m$, $0 < 4n + 9m + 16 < 26$ | $1 + 4n + 9m + 16$ |

TABLE 1. Coradicals dim $H = 27$, $|G(H)| = 1$

Case $(i)$ is not possible by Example 3.6. Case $(ii)$ is impossible by Example 3.4. Cases $(iii)$, $(iv)$ are impossible by Proposition 3.2$(ii)$.

Only cases $(v)$, $(vi)$ remain. By Proposition 3.5, $H$ cannot have a simple 4-dimensional subcoalgebra stable under the antipode so the case $n = 1$ is impossible in either $(v)$ or $(vi)$. The remainder of case $(vi)$ is impossible by Proposition 3.2.

The last remaining case is coradical $H_0 \cong \Bbbk \cdot 1 \oplus \mathcal{M}^*(2, \Bbbk)^2 \oplus \mathcal{M}^*(3, \Bbbk)$ since one sees immediately that if $n > 2$ or $m > 1$ then the dimension of $H_0$ is impossibly big and Proposition 3.2 will give a contradiction. So suppose that $H_0 \cong \Bbbk \cdot 1 \oplus D \oplus E \oplus \mathcal{M}^*(3, \Bbbk)$ where $D \cong \mathcal{M}^*(2, \Bbbk)$, $E = S(D)$ and $S^2(D) = D$; in particular $H_0$ has dimension 18. If $P^{-, \mathcal{M}^*(3, \Bbbk)} \neq 0$ then the usual

dimension arguments give a contradiction. So suppose that these spaces are 0. By Proposition 3.2$(i)$, $P^{1,1}$ is nonzero and this implies by Lemma 3.1$(i)$ that $P^{1,D}, P^{D,1}$ are nonzero. But then $P^{1,E}, P^{E,1}$ are nonzero also and so are $P^{D,X}, P^{E,Y}$ where $X, Y \in \{D, E\}$. Then the dimension of $H$ is at least $18 + 1 + 8 + 8 = 35$, a contradiction.                                    □

We end this note by giving the complete list of Hopf algebras of dimension 27.

*Remark* 4.25. By Theorem 4.24, if $H$ is a Hopf algebra of dimension 27 then $H$ is either semisimple or pointed or the dual is pointed. By the classification of semisimple and pointed Hopf algebras of dimension $p^3$, $H$ is isomorphic to exactly one Hopf algebra in the following list:

Semisimple Hopf algebras of dimension 27 were classified by Masuoka [Mk3]; there are 11 isomorphism types, namely

$(a)$ Three group algebras of abelian groups.

$(b)$ Two group algebras of nonabelian groups, and their duals.

$(c)$ 4 self-dual Hopf algebras which are neither commutative nor cocommutative. They are extensions of $\Bbbk[C_3 \times C_3]$ by $\Bbbk C_3$.

Pointed Hopf algebras of dimension 27 were classified independently by different authors, see [AS2], [CD] and [ŞvO]. Here $q$ is a primitive 3-rd root of unity and $T_q$ the Taft Hopf algebra of dimension 9. Note that in $(d)$ the grouplikes are isomorphic to $C_3 \times C_3$, in $(e), (f), (g)$ to $C_9$, in $(h), (i)$ to $C_3$ and $(j), (k)$ are copointed but not pointed.

$(d)$ The tensor-product Hopf algebra $T_q \otimes \Bbbk C_3$.

$(e)$ $\widetilde{T_q} := \Bbbk\langle g,\ x|\ gxg^{-1} = q^{1/3}x,\ g^9 = 1,\ x^3 = 0\rangle$ ($q^{1/3}$ a 3-th root of $q$), with comultiplication $\Delta(x) = x \otimes g^3 + 1 \otimes x$, $\Delta(g) = g \otimes g$.

$(f)$ $\widehat{T_q} := \Bbbk\langle g,\ x|\ gxg^{-1} = qx,\ g^9 = 1,\ x^3 = 0\rangle$, with comultiplication $\Delta(x) = x \otimes g + 1 \otimes x$, $\Delta(g) = g \otimes g$.

$(g)$ $\mathbf{r}(q) := \Bbbk\langle g,\ x|\ gxg^{-1} = qx,\ g^9 = 1,\ x^3 = 1 - g^3\rangle$, with comultiplication $\Delta(x) = x \otimes g + 1 \otimes x$, $\Delta(g) = g \otimes g$.

$(h)$ The Frobenius-Lusztig kernel $\mathbf{u}_q(\mathfrak{sl}_2) := \Bbbk\langle g,\ x,\ y|\ gxg^{-1} = q^2x,\ gyg^{-1} = q^{-2}y,\ g^3 = 1,\ x^3 = 0,\ y^3 = 0,\ xy - yx = g - g^{-1}\rangle$, with comultiplication $\Delta(x) = x \otimes g + 1 \otimes x$, $\Delta(y) = y \otimes 1 + g^{-1} \otimes y$, $\Delta(g) = g \otimes g$.

$(i)$ The book Hopf algebra $\mathbf{h}(q, m) := \Bbbk\langle g,\ x,\ y|\ gxg^{-1} = qx,\ gyg^{-1} = q^m y,\ g^3 = 1,\ x^3 = 0,\ y^3 = 0,\ xy - yx = 0\rangle$, $m \in C_3 \smallsetminus \{0\}$, with comultiplication $\Delta(x) = x \otimes g + 1 \otimes x$, $\Delta(y) = y \otimes 1 + g^m \otimes y$, $\Delta(g) = g \otimes g$.

$(j)$ The dual of the Frobenius-Lusztig kernel, $\mathbf{u}_q(\mathfrak{sl}_2)^*$.

$(k)$ The dual of the case $(g)$, $\mathbf{r}(q)^*$.

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