On RA and GM Estimates for Spatial Autoregressive Models

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Abstract

This paper is concerned with the computational implementation of the robust RA estimates for spatial autoregressive (AR-2D) models with three parameters. The computational behavior of the RA estimators for AR-2D models which depends on the number of parameters in the model, has been studied for those models containing at most two parameters. A simulation study is carried out to compare the relative performance of GM and RA estimators under additive contamination, and to compare the RA and GM estimators with the M and LS estimators. This implementation can be explored in models with more than three parameters.

Keywords: AR-2D model, RA and GM estimators, Additive Outlier.

1 Introduction and Motivation

Most of the real images of interest, e.g. the images of cultivated fields and concentration of population are naturally rich in texture, level of gray, etc.. The same thing happens to the images of geographical regions that allows the making of maps and, in general, almost all the images of the earth. In this sense the AR-2D model is an adequate model to show the diversity of the real sceneries. Another desirable feature of a model for images is parsimony, which is, the capability to represent different real sceneries without requiring a large number of parameters. To illustrate the capability of this model, in Figure 1 we show four images generated from AR-2D models with different sets of parameters.

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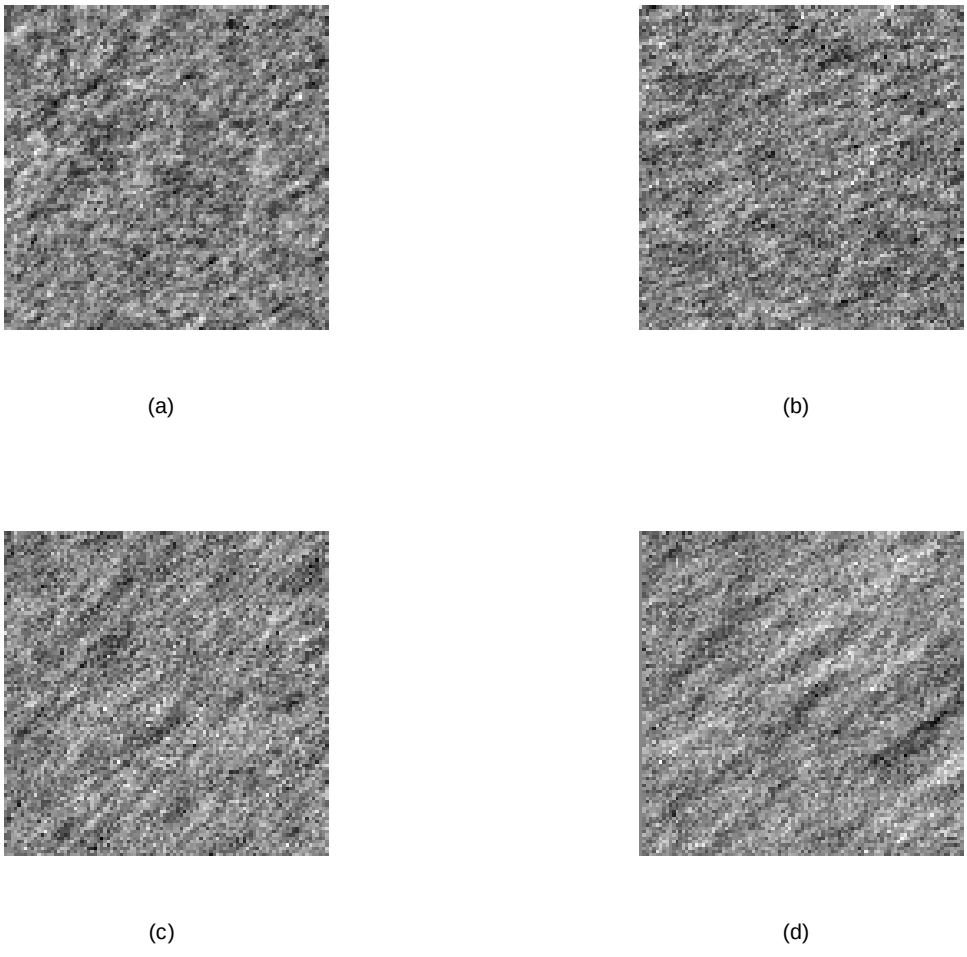

Figure 1: (a) Image from an AR-2D model with 3 parameters, (b) Image from an AR-2D model with 5 parameters (c) Image from an AR-2D model with 8 parameters, (d) Image from an AR-2D model with 10 parameters.

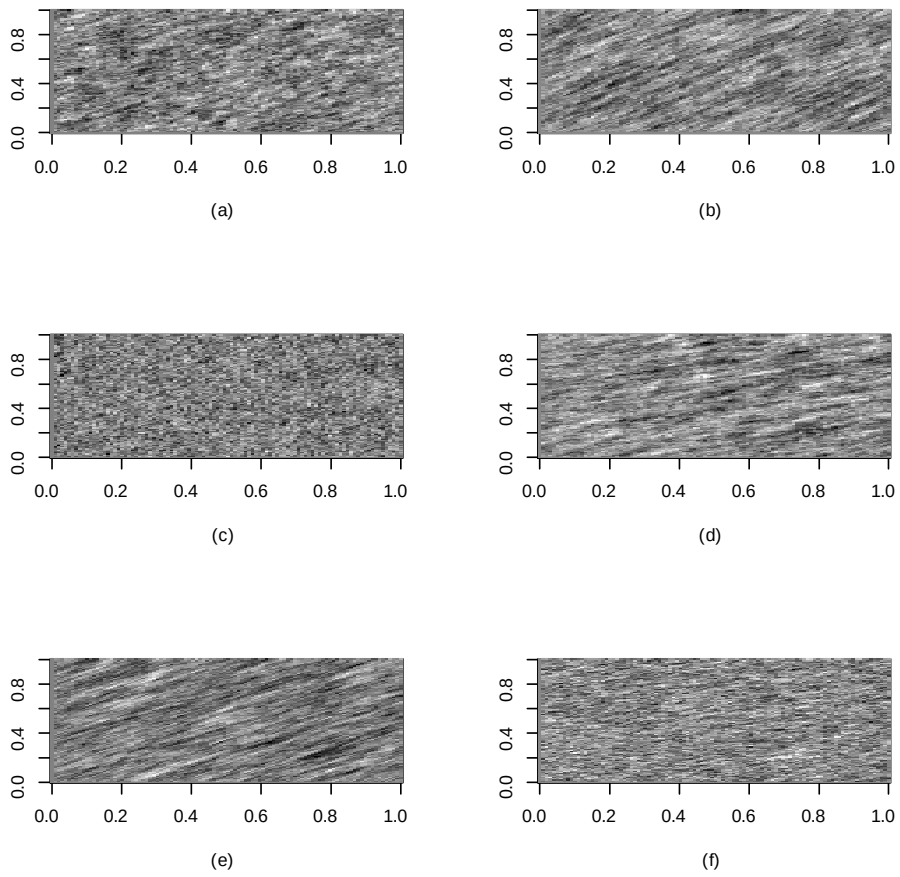

Figure 2: Images generated from an AR-2D model with 3 parameters.

The spatial autoregressive models have been used extensively to represent images (Bennett and Khotanzad, 1999). In particular, the first order AR-2D model has been used to represent real scenaries (see Kashyap and Eom, 1988). Basu and Reinsel (1993), derived the correlation structure and the maximum likelihood estimators of the parameters for a first order AR-2D model. In this study, we provide an example to illustrate the expressivity of the first order AR-2D model to represent several different textures (see Figure 2).

The importance of contaminated models has been addressed by several authors in the area of image processing and image analysis (Kashyap and Eom, 1988 and Zhu and Beex 1994). Since a single outlier can produce bias and a large variance, most of the proposals in this direction are oriented to provide robust estimations for parametric models that represent the image intensity of a given picture by a small number of parameters. In this paper we deal with robust parameter estimation of contaminated weakly stationary random processes. These models have been extensively studied in the context of image processing for modelling blurred

images. Allende, Galbiati and Vallejos (2001) describe an algorithm base on GM estimation to restore additively contaminated images. Later a modified version of this algorithm was studied by Vallejos and Mardesic (2004).

In the literature the M, GM, and RA estimators have been addressed by several authors. The RA estimators for spatial AR-2D models (Ojeda, Vallejos and Lucini, 2002) are extensions of the RA estimators introduced by Bustos and Yohai (1986) in the context of time series. The RA estimator is less sensitive than the M and GM estimators when the process has been contaminated with additive outliers. Another advantage of the RA estimator is that its consistency and asymptotic normality have been studied in the context of spatial AR models. However, the asymptotic properties of the M and GM estimators in this context were not studied.

In this paper the implementation of the robust RA estimator on AR-2D models with three parameters is studied. We found two difficulties in the computational implementation of the RA estimator. First, the RA estimator is the solution of a highly nonlinear system of equations, hence a large number of parameters in the model will require more computational time. Second, the definition of the RA estimator requires a causal representation of the process. In general this representation can involve complex coefficients. In fact, a spatial AR-2D model is defined (Martin, 1996) as,

$$\Phi(B_1, B_2)X(i,j) = e(i,j), \tag{1}$$

where

$$\Phi(B_1, B_2) = \sum_k \sum_l \phi(k,l) B_1^k B_2^l,$$

with $B_1X(i,j) = X(i-1,j)$, $B_2X(i,j) = X(i,j-1)$ and $e(i,j)$ is a collection of uncorrelated random variables with mean zero and the same variance. If the sums for $\Phi(B_1, B_2)$ begin at $k = l = 0$, the process is a top (left) quadrant process with $\phi(0,0) = 1$. Assume that the complex valued polynomial $\Phi(z_1, z_2)$ is not zero for any z_1, and z_2, which simultaneously satisfy $|z_1| < 1$ and $|z_2| < 1$. Such a process is called causal autoregressive process. Notice that in this case,

$$X(i,j) = \Phi(B_1, B_2)^{-1}\epsilon(i,j), \tag{2}$$

where $\Phi(z_1, z_2)^{-1}$ can be written using a Laurent expansion as

$$\Phi(z_1, z_2)^{-1} = \sum_{k,l} \psi_{kl} z_1^k z_2^l.$$

To define a finite version of (1) in a more general context, let $X = \{X(s) : s \in S \subset Z^2\}$ be a real random process over the probability space (Ω, F, μ). Without loss of generality suppose that $E(X(s)) = 0$, for all $s \in S$, and let $\boldsymbol{e} = \{e(s) : s \in S\}$ be a gaussian white noise process over (Ω, F, μ). X will be called a real AR-2D process with three parameters if it is stationary, and

$$X(m,n) - \sum_{(k,l)\in T} \phi(k,l)X(m-k, n-l) = e(m,n),$$

where $\phi(k,l)$ are the parameters of the model such that the polynomial $P(z_1,z_2) = 1-\sum_{(k,l)\in T}\phi(k,l)z_1^k z_2^l$ is not null in the set $D^* = \{(z_1,z_2)\in C^2 : |z_1| = |z_2| = 1\}$, and T is given by {(1,0),(1,1),(0,1) }. Because of the uniqueness of representation (2), the coefficients ψ_{kl} can be obtained using a multinomial expansion for $\Phi(z_1,z_2)^{-1}$. Basu and Reinsel (1993) found conditions on the parameters to have an infinite moving average representation of X. These conditions are as follows: i) $|\phi(k,l)| < 1$, for $k,l = 0,1$, ii) $(1+\phi^2(1,0)-\phi^2(0,1)-\phi^2(1,1))^2-4(\phi(1,0)+\phi(0,1)\phi(1,1))^2 > 0$, iii) $(1-\phi^2(0,1)) > |\phi(1,0)+\phi(0,1)\phi(1,1)|$. Then using a multinomial expansion for the polynomial $(1-\phi(1,0)z_1-\phi(0,1)z_2-\phi(1,1)z_1z_2)^{-1}$, $X(m,n)$ can be written as

$$X(m,n) = \sum_{k=0}^{\infty}\sum_{l=0}^{\infty}\sum_{r=0}^{\infty}\frac{(k+l+r)!}{k!\ l!\ r!}\phi^k(1,0)\phi^l(0,1)\phi^r(1,1)e(m-k-r,n-l-r). \quad (3)$$

(See also Tjostheim, 1978). In Section 2 we use (3) to define the RA estimator for models like (1).

The multinomial expansion used to derive (3) is not restrictive to models with three parameters. For a high dimensional parameter space the same expansion can be used.

The paper is organized by sections. Section 2 is concerned with the implementation of the RA estimator. Section 3 presents a simulation study to observe the sensitivity of the estimators under additive contamination. We compare the behavior of the RA and GM estimators with the M and LS estimators. Finally, in Section 4 we present some concluding remarks.

2 The RA and GM Estimators

Let X be a zero mean AR-2D process with $Var(e(m,n)) = \sigma^2$. Assume that X is observed on a strongly causal squared window $W_M = \{(k,l)\in S : 0\le k,l\le M\}$. (Guyon, 1995) Let us define $W_M \setminus T = \{(m,n)\in W_M : (m-1,n-1)\in W_M\}$, and $\boldsymbol{\phi}^T = (\phi(1,0),\phi(1,1),\phi(0,1))$. We define the residual of order (m,n) in $\boldsymbol{\phi}$ of X as follows,

$$r(m,n)(\boldsymbol{\phi}) = \begin{cases} -\sum_{(k,l)\in T'}\phi(k,l)X(m-k,n-l) & \text{if } (m,n)\in(W_m\setminus T) \\ 0 & \text{if otherwise} \end{cases}$$

where $T' = T\cup\{(0,0)\}$ and $\phi(0,0) = -1$. Then the RA estimator $\hat{\boldsymbol{\phi}}$ of $\boldsymbol{\phi}$ is defined by the following equations

$$\sum_{k=0}^{\infty}\sum_{l=0}^{\infty}\sum_{r=0}^{\infty}p_{\boldsymbol{\phi}}(k,l,r)\sum_{(m,n)\in(W_M\setminus T)}\eta\left(\frac{r(m,n)(\boldsymbol{\phi})}{\hat{\sigma}},\frac{r(m-i-k-r,n-j-l-r)(\boldsymbol{\phi})}{\hat{\sigma}}\right) = 0, \quad (4)$$

$$\sum_{(m,n)\in(W_M\setminus T)}\psi\left(\frac{r(m,n)(\boldsymbol{\phi})}{\hat{\sigma}}\right) = 0,$$

where $\hat{\sigma}$ is estimated independently by

$$\hat{\sigma} = Med(|r(m,n)(\tilde{\boldsymbol{\phi}})| : (m,n) \in (W_M \setminus T))/0.6745,$$

the constant 0.6745 is the median of the absolute value of a standard normal random variable,

$$p_{\boldsymbol{\phi}}(k,l,r) = \frac{(k+l+r)!}{k!\ l!\ r!}\phi(1,0)^k\phi(0,1)^l\phi(1,1)^r, \tag{5}$$

η is a continuous, bounded and odd function in two variables, ψ is a continuous, bounded and odd function, and $\tilde{\boldsymbol{\phi}}$ is the least square estimator of $\boldsymbol{\phi}$. Suitable proposals for η and ψ are discussed in Bustos and Yohai (1986).

Note from (4) that the definition of the RA estimator involves infinite sums. However, for computing purposes we need to consider a finite version of equation (4). One way to accomplish this is to consider a finite support based in the fact that the residuals are non-null only for those values belonging to $W_m \setminus T$ (see Ojeda, Vallejos and Lucini, 2002). Here, we propose an alternative way to carry out the computational implementation of the RA estimator. Notice from (5) that if $k = l = r$, then $p_{\boldsymbol{\phi}}(k)$ is decreasing when k increases, because X is stationary. Hence we can find a value $k_0 = \min(k_s)$, where k_s is a solution of the inequality

$$p_{\boldsymbol{\phi}}(k) < \epsilon \tag{6}$$

with ϵ small. Stirling's formula can be used to get an approximated version of (6). To obtain the RA estimator of $\boldsymbol{\phi}$, instead of (4) we consider the following equation

$$\sum_{k=0}^{k_0}\sum_{l=0}^{k_0}\sum_{r=0}^{k_0} p_{\boldsymbol{\phi}}(k,l,r) \sum_{(m,n)\in(W_M\setminus T)} \eta\left(\frac{r(m,n)(\boldsymbol{\phi})}{\hat{\sigma}}, \frac{r(m-i-k-r, n-j-l-r)(\boldsymbol{\phi})}{\hat{\sigma}}\right) = 0. \tag{7}$$

Two types of contamination will be introduced, *innovation* and *additive* outliers (Martin, 1980). The following is a briefly description of those outliers.

Innovation Outliers: In this case the $e(n,m)$'s in (1) have a F contaminated Normal distribution noted as $F = (1-\alpha)N(0,\sigma^2) + \alpha\ G$, where G is an arbitrary distribution with variance $\tau^2 \geq \sigma^2$. This means that the innovations $e(m,n)$ comes from a $N(0,\sigma^2)$ with probability $1-\alpha$ and from an arbitrary distribution G having greater dispersion with probability α. The $e(n,m)$'s coming from G are considered outliers.

Additive Outliers: In this case , it is considered that the AR model is not perfectly observable due to a small fraction α (in practice $\alpha \leq 0.1$) of observations which are generated by the outlier process $\{\delta(i,j)\nu(i,j)\ \}$, where $\delta(i,j)$ is such that $P[\delta(i,j) = 1] = \alpha$ and $P[\delta(i,j) = 0] = 1-\alpha$, and the variables $\nu(i,j)$ have an arbitrary distribution H. α is similar to the innovation outliers case. Thus the observational model is

$$Y(i,j) = X(i,j) + \delta(i,j)\nu(i,j). \tag{8}$$

Therefore with probability $(1-\alpha)$ the AR model $X(i,j)$ itself is observed, and with probability α the observations $X(i,j)$ are corrupted by an error with distribution

H. Alternatively to the RA estimator of (8), the robust GM-estimator can be used, which is obtained by solving the equation

$$\sum_{ij} t_{ij}\psi\left[\frac{Y(i,j)-\boldsymbol{\phi}^T Z(i,j)}{{}_{ij}\sigma}\right] Z^T(i,j)=0, \tag{9}$$

where $Z(i,j)^T=[\,Y(i-1,j), Y(i-1,j-1), Y(i,j-1)]$, the influence function ψ is bounded and continuous, and t_{ij} and ${}_{ij}$ are weights corresponding to the respective $Z(i,j)$. There are several proposals for the choice of ψ as the robustness and the rate of convergence of the procedure that depend upon this function. Further details can be found in Martin (1980).

In the next section we will develop a simulation study to observe the performance of the RA estimator with respect to the M and GM estimators when the process is contaminated with additive contamination.

3 A simulation study

The simulations were performed for a grid of size 16×16 and for six sets of parameters

$$\begin{aligned}
(i)&\phi(1,0)=0.5, \phi(1,1)=0.1, \phi(0,1)=0.3,\\
(ii)&\phi(1,0)=0.3, \phi(1,1)=0.2, \phi(0,1)=0.2,\\
(iii)&\phi(1,0)=0.8, \phi(1,1)=0.7, \phi(0,1)=-0.6,\\
(iv)&\phi(1,0)=0.1,\ \phi(1,1)=0.2,\ \phi(0,1)=0.05,\\
(v)&\phi(1,0)=0.15,\ \phi(1,1)=0.07,\ \phi(0,1)=-0.1,\\
(vi)&\phi(1,0)=0.3, \phi(1,1)=0.2, \phi(0,1)=0.06.
\end{aligned}$$

For additive contamination we considered $\sigma^2=1$, $\tau^2=50$, and $\alpha=0.1$. Distribution H was supposed to be $N(0,\tau^2)$. Then, for each combination, 500 simulations of the data set were generated and the LS, RA, M, and GM estimators were computed. The value of k_0 is computed for each set of parameters from (6). In all cases we have considered $\epsilon=10^{-9}$. For example in cases (i) $k_0=14$, (ii) $k_0=18$, and (iii) $k_0=50$. The M estimates were computed as in Kashyap and Eom (1988). The GM estimates were obtained using the algorithm proposed by Allende, Galbiati and Vallejos (1998). The RA estimates were obtained from (7) by using the Newton-Raphson method. In this section we present only a brief summary of our results.

Table 1 shows the results of a Monte Carlo study. In all cases the mean, standard deviation and the empirical mean squared error were computed. According to Vallejos and Garcia-Donato (2006)

$$MSE=\frac{1}{N}\sum_{i=1}^{N}(\hat{\phi}-\phi)^2. \tag{10}$$

where N is the number of simulation runs. Notice from Table 1 that in most of the cases, the smallest values of the MSE are for the GM and RA estimators. The bias

and variance of the LS and M estimators are strongly affected by the outliers. However, the GM and RA estimators are more resistant to the additive contamination. This patterns are visually confirmed in Figure 3. The performance of the estimators for the other cases not shown in Figure 3 is similar. In all cases the behavior of the GM and RA estimators are comparable and there is no clear patterns to prefer one of them.

The asymptotic normality of the GM estimator has not been studied. In the one-dimensional case, theoretical properties were studied by Bustos (1982), however its good robust behavior was reported by Allende, Galbiati and Vallejos (2001). The asymptotic properties of the RA estimator in the context of time series has been examined by Bustos and Yohai (1986). The consistency of the RA estimator for 2D autoregressive models has been shown by Ojeda (1999). The asymptotic normality of the RA estimator for the same models was examined by Bustos et, al. (2006). To illustrate the normality of the RA estimator we plot the estimated density function for case (ii), based on 500 simulations. The theoretical value is $\phi(0,1) = 0.2$. In Figure 4 we observe that there is no large departure from normality. In all cases the asymptotic normality was not rejected by the Anderson-Darling test at the level of significance $\alpha = 0.05$.

Case (vi) in the simulation study corresponds to a model for which the parameters satisfy $\phi(1,1) = \phi(1,0)\phi(0,1)$. These models are called doubly-geometric processes since its correlation function is the product of two geometric terms. Quenouille (1949) used its correlation structure for numerical comparison between certain planar sampling schemes. These processes are a special case of Moran's (1973) strongly Markovian process, and have also been given by Pichard (1978) as an example of unilateral Ising model. Martin (1979), (1990), studied the properties of the maximum likelihood estimators for the parameters of a doubly-geometric process. The standard deviation of the RA estimator is smaller than the standard deviation of the GM estimator. However, the GM estimator has smaller bias. In Figure 5 we show the boxplot for case (vi). The robust estimations performed well for $\phi(1,0)$ and $\phi(0,1)$. All estimations of $\phi(1,1)$ are strongly biased.

The performance of the GM and RA estimators were compared under several simulation studies that are not reported here. In all cases the behavior was similar. For window sizes of 32×32 and 64×64 we noted that the RA estimator behaves slightly better than the GM estimator and much better than the M and LS estimators. The same conclusion was reported by Ojeda, Lucini and Vallejos (2002). Thus we expect to have more precise RA estimations when the window size increases.

The main drawback of the RA estimator is that it requires more time to be computed than the GM estimator. This is because the system of equations (7) is highly nonlinear and depends on the dimension of the parameter space.

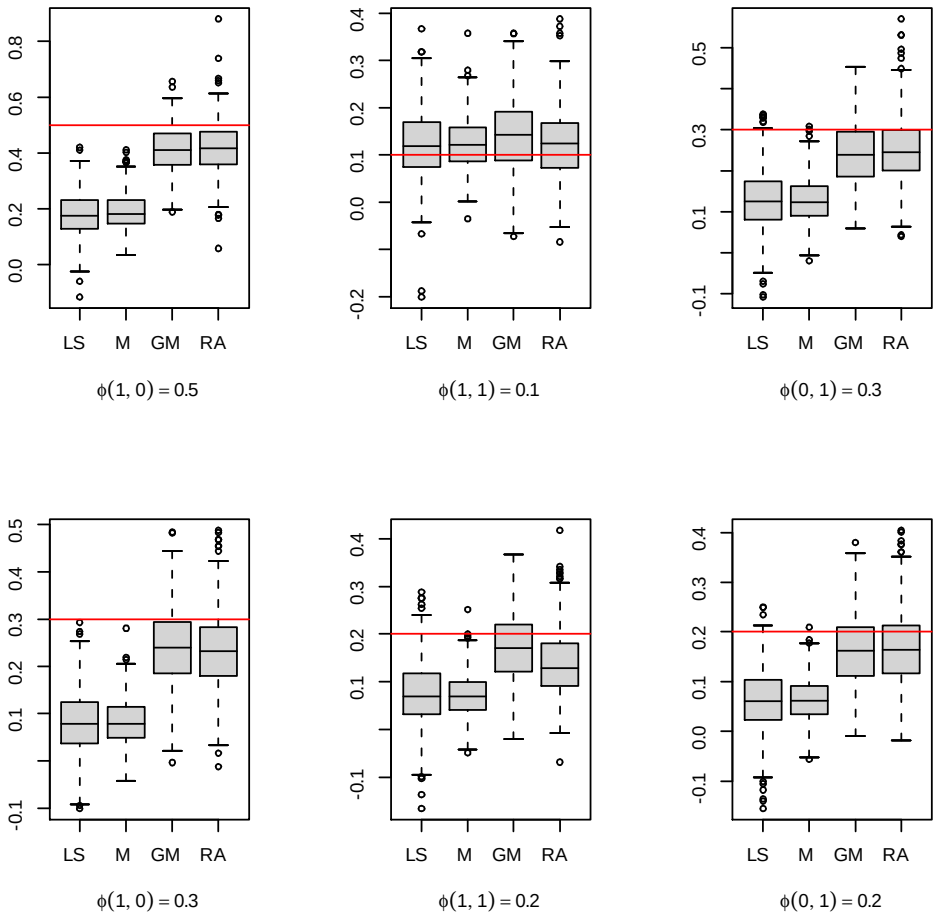

Figure 3: LS, M, GM, and RA estimators for cases (i) and (ii).

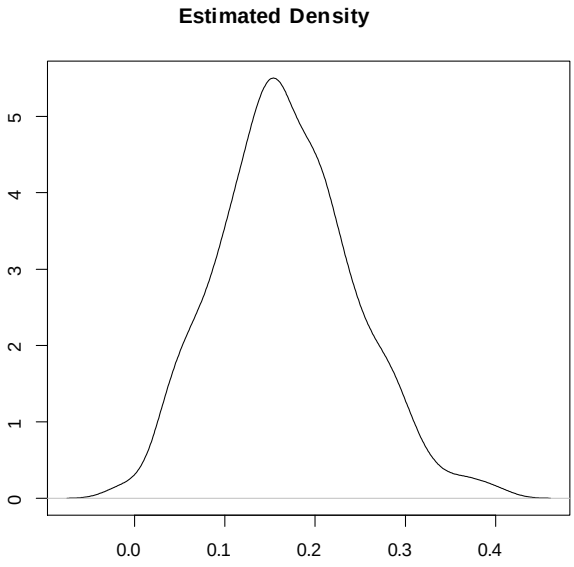

Figure 4: Case (ii), $\phi(0,1) = 0.2$

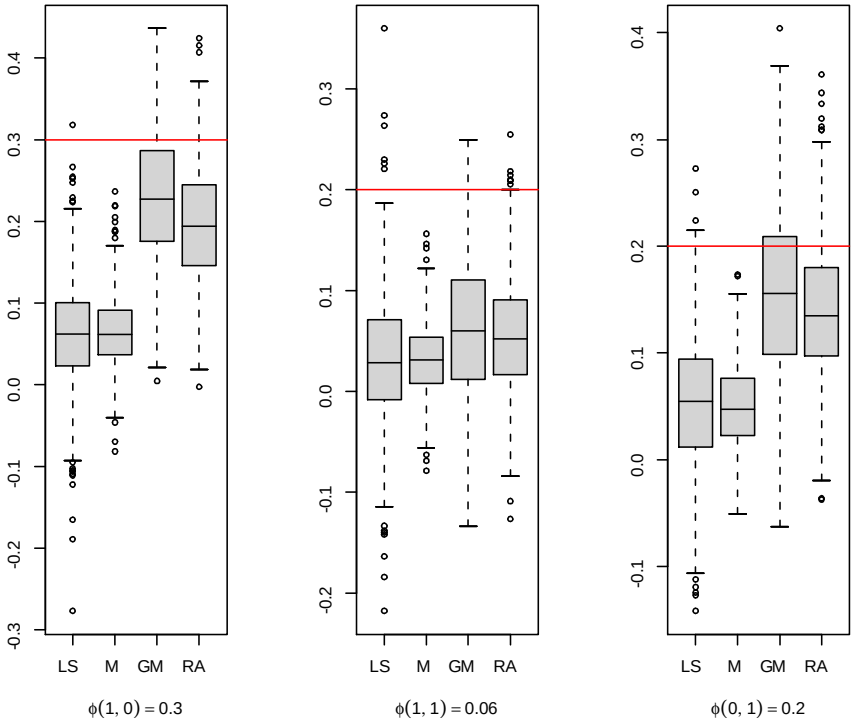

Figure 5: LS, M, GM, and RA estimators for case (vi)

Window Size 16×16

	$\hat{\phi}(1,0)$	s.d.$(\hat{\phi}(1,0))$	MSE	$\hat{\phi}(1,1)$	s.d$(\hat{\phi}(1,1))$	MSE	$\hat{\phi}(0,1)$	s.d.$(\hat{\phi}(0,1))$	MSE
					case (i)				
LS	0.1784	0.0787	0.1095	0.1214	0.0707	0.0052	0.1286	0.0746	0.0349
M	0.1893	0.0665	0.1009	0.1236	0.0509	0.0031	0.1294	0.0559	0.0321
GM	0.4087	0.0825	0.0151	0.1427	0.0795	0.0081	0.2391	0.0770	0.0096
RA	0.4189	0.0905	0.0147	0.1241	0.0698	0.0054	0.2519	0.0800	0.0087
					case (ii)				
LS	0.0805	0.0666	0.0525	0.0738	0.0677	0.0204	0.0613	0.0641	0.0233
M	0.0827	0.0475	0.0494	0.0731	0.0445	0.0180	0.0636	0.0430	0.0204
GM	0.2396	0.0792	0.0098	0.1715	0.0768	0.0066	0.1619	0.0738	0.0068
RA	0.2333	0.0787	0.0106	0.1378	0.0706	0.0088	0.1679	0.0749	0.0066
					case (iii)				
LS	0.3656	0.1125	0.2013	0.1278	0.0821	0.3340	-0.1238	0.0770	0.2326
M	0.4110	0.1136	0.1641	0.1339	0.0736	0.3258	-0.1216	0.0643	0.2329
GM	0.6825	0.0812	0.0203	0.4529	0.1007	0.0711	-0.3831	0.0990	0.0567
RA	0.6647	0.1192	0.0324	0.4161	0.1525	0.1037	-0.3481	0.1206	0.0779
					case (iv)				
LS	0.0174	0.0675	0.0113	0.0369	0.0652	0.0308	0.0194	0.0649	0.0051
M	0.0195	0.0371	0.0078	0.0379	0.0344	0.0274	0.0161	0.0339	0.0022
GM	0.0758	0.0723	0.0058	0.1474	0.0736	0.0081	0.0421	0.0744	0.0055
RA	0.0644	0.0608	0.0049	0.1099	0.0631	0.0120	0.0473	0.0569	0.0032
					case (v)				
LS	0.0262	0.0640	0.0194	0.0095	0.0627	0.0075	-0.0203	0.0649	0.0105
M	0.0275	0.0367	0.0163	0.0097	0.0351	0.0048	-0.0189	0.0345	0.0077
GM	0.1067	0.0764	0.0077	0.0484	0.0738	0.0059	-0.07437	0.0744	0.0061
RA	0.0782	0.0577	0.0084	0.03232	0.0547	0.0044	-0.0509	0.0545	0.0053
					case (vi)				
LS	0.0612	0.0707	0.0620	0.0314	0.0652	0.0050	0.0529	0.0671	0.0261
M	0.0650	0.0438	0.0571	0.0319	0.0352	0.0020	0.0503	0.0394	0.0239
GM	0.2301	0.0786	0.0110	0.0615	0.0727	0.0052	0.1546	0.0790	0.0083
RA	0.1964	0.0721	0.0159	0.0564	0.0590	0.0034	0.1394	0.0641	0.0077

Table 1: LS, M, GM, and RA estimators for cases $(i) - (vi)$.

4 Conclusions

In this paper the performance of the GM and RA estimators for first order spatial autoregressive models was examined. Using Monte Carlo simulation we observed that the performance of the RA and GM estimators are highly superior than the M and LS estimators. The behavior of the GM and RA estimators is comparable under additive contamination. All estimators underestimate the true value. The asymptotic behavior of the GM has not been studied, but the computational implementation is simpler than the RA estimator and for small window size has the best performance. Robust estimators for the doubly-geometric process need further research.

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