Uniqueness Condition for Auto-logistic model*

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Abstract

Auto-logistic model is widely used to describe binary images texture and spatial presence-absence data. Gibbs Sampler algorithm (like others) simulates this kind of model, but his performance depends on the model global properties. Under general conditions we have at least a global distribution that locally performs as the model specifies. We give a sufficient condition on the parameters, and this condition ensures the distribution is unique.

Keywords: Auto-logistic model - uniqueness - Dobrushin's condition - Gibbs measure - Simulation

1 Introduction

In Statistical Physics, Ising (in 1925) laid the foundation of the Random fields in his doctoral thesis ([9, 10]). He presented a ferromagnetic model where fix particles interact in a lattice, each one associated with a spin value +1 or -1. Besag applied this idea for the first time to image processing ([2]). Images statistical modelling is a valuable tool and there are a lot of interests in image processing at several knowledge areas. Ising and Besag models take into account the dependence between nearest pixels. These kinds of models are called Markovian Random Fields. By Hammersley-Clifford Theorem ([15]), they have Gibbs distribution. There are algorithms that simulate Gibbs distribution. Gibbs Sampler is the most popular one (see [5, 15]). It generates a Markov Chain of images converging to a realization of the subjacent model using its local dependence. The convergence holds if there is only one global distribution. Under general conditions we have existence, but uniqueness is not trivial (see [7]). There are a lot of works on this topic (see [1, 3, 12, 14]). The Dobrushin's condition theorem provides us with a sufficient condition to achieve it. To verify this condition is not easy and model dependent. The Auto-logistic one models the dependence of spatial binary data like binary images indicating presence-absence of something. These kind of data appear in several areas like biology and geoscience. We give a sufficient (but not necessary) condition that ensures uniqueness in the Auto-logistic model for 4 and 8 neighbours. This condition does not involve

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an external field and constrains interaction parameters to a bounded and convex region (in $\mathbb{R}^2$ or $\mathbb{R}^4$). We think this theorem could be extended to bigger neighbourhoods and to 3D lattices. The organization of this work is as follows. At section 2 we give some definitions we need. At section 3 we describe the model and condition bases. Then, at section 4 we present and demonstrate the theorem. Finally, we discuss about theorem condition at section 5.

2 Theoretical framework

Let the lattice $S \subseteq \mathbb{Z}^2$, not necessarily finite. We consider the following definitions as assumptions

$E = \{0,1\}$ and $\mathscr{E} = \mathcal{P}(E)$ is parts of E;

$x = \{x_s\}_{s \in S} \in E^S$ is a binary image;

$X = \{X_s\}_{s \in S}$ is the underlying stochastic process;

$V \subseteq S$; $x_V \doteq \{x_s\}_{s \in V} \in E^V$; $\mathscr{E}^V$ is the product σ-algebra;

$\mathscr{F}_V \doteq \left\{B \times E^{S \setminus V} / B \in \mathscr{E}^V\right\} \subseteq \mathscr{F} \doteq \mathscr{E}^S$;

$\mathscr{S} \doteq \left\{\Lambda \subset \mathbb{Z}^2 / 1 \leq \#(\Lambda) < \infty\right\}$;

$\Phi \doteq (\Phi_\Lambda)_{\Lambda \in \mathscr{S}}$ is a potential. That is $\Phi_\Lambda(x) = \phi_\Lambda(x_\Lambda)$ where ϕ_Λ is a real function $\mathscr{E}^\Lambda$-medible and there exists $H_\Lambda \doteq \sum_{\Delta \in \mathscr{S} \cap \Lambda} \Phi_\Delta$, it is the energy function;

$\gamma = (\gamma_\Lambda)_{\Lambda \in \mathscr{S}}$ is the Gibbsian specification (for Φ) that is

$\gamma_\Lambda(A|x) \doteq \frac{\sum_{\xi \in E^\Lambda} 1_A\left(\xi x_{S\setminus\Lambda}\right) exp\left(-H_\Lambda\left(\xi x_{S\setminus\Lambda}\right)\right)}{\sum_{\xi \in E^\Lambda} exp\left(-H_\Lambda\left(\xi x_{S\setminus\Lambda}\right)\right)}$, $A \in \mathscr{F}$;

$\mathscr{G}(\gamma) \doteq \{\mu / \mu(A \cap B) = \int_B \gamma_\Lambda(A|\,) d\mu \,\forall B \in \mathscr{F}_{S\setminus\Lambda}\}$ is the set of global Gibbsian probabilities μ in E^S such that $\gamma_\Lambda(A|x)$ is the probability of A with respect to μ conditional to $x_{S\setminus\Lambda}$.

$\#\mathscr{G}(\gamma) \geq 1$ by Theorem 4.23 in [7].

3 Auto-logistic model and Dobrushin's condition

We consider the potential

$$\Phi_\Lambda(x) = \begin{cases} \beta_i x_t x_{t+v_i} & \Lambda = \{t, t+v_i\} \\ \beta_0 x_t & \Lambda = \{t\} \\ 0 & \text{otherwise} \end{cases},$$

where $t \in S$, $i = 1, .., g$ ($g = 2$, first order and $g = 4$, second order), $v_1 = (0,1)$, $v_2 = (1,0)$, $v_3 = (1,1)$, and $v_4 = (-1,1)$.

For $s \in S$ we define

$$\begin{aligned}
\gamma_s^0(B|x) &\doteq \gamma_{\{s\}}(B \times E^{S\setminus\{s\}}|x) \\
&= \frac{\sum_{\xi\in B} exp\left(-H_{\{s\}}\left(\xi x_{S\setminus\{s\}}\right)\right)}{\sum_{\xi\in E} exp\left(-H_{\{s\}}\left(\xi x_{S\setminus\{s\}}\right)\right)} \\
&= \frac{\sum_{\xi\in B} exp\left(-\sum_{\Lambda\in\{s\}\cap\mathscr{S}} \Phi_\Lambda\left(\xi x_{S\setminus\{s\}}\right)\right)}{\sum_{\xi\in E} exp\left(-\sum_{\Lambda\in\{s\}\cap\mathscr{S}} \Phi_\Lambda\left(\xi x_{S\setminus\{s\}}\right)\right)}, \; B \in \mathscr{E}.
\end{aligned}$$

Then, the local characteristic is

$$\pi_s(x) = \gamma_s^0(\{x_s\}|x) = \frac{e^{-x_s(\beta_0+\sum_{i=1}^{g}\beta_i(x_{s+v_i}+x_{s-v_i}))}}{e^{-(\beta_0+\sum_{i=1}^{g}\beta_i(x_{s+v_i}+x_{s-v_i}))}+1}.$$

We note that $\gamma_s^0(.|x)$ depends on $x_{\partial s}$, with

$\partial s = \{s \pm v_i\}_{i=1}^{g}$ neighbourhood of s.

For $g = 4$, $\partial s =$

$s - v_3$	$s - v_1$	$s + v_4$
$s - v_2$		$s + v_2$
$s - v_4$	$s + v_1$	$s + v_3$

.

β_0 is the external field parameter, if $\beta_0 = 0$ we say the model does not have an external field. $\beta = (\beta_0, \beta_1, \beta_2, \beta_3, \beta_4)$ is the parameter vector for the second order model (8 neighbours) and $\beta = (\beta_1, \beta_2, \beta_3, \beta_4)$ if does not have external field. $\beta = (\beta_0, \beta_1, \beta_2)$ is the parameter vector for the first order model (4 neighbors) and $\beta = (\beta_1, \beta_2)$ without an external field. It is easy to see that $g = 2$ is the particular case of $g = 4$ when $\beta_3 = \beta_4 = 0$, and Φ is translation-invariant (i.e. β does not depend on $t \in S$). It is remarkable that we can not consider the Auto-logistic regression model, since the external field depends on t ([8, 13]).

We define the Uniform Distance between the probabilities μ and $\tilde{\mu}$ by

$$d_U(\mu, \tilde{\mu}) \doteq \sup\{|\mu(B) - \tilde{\mu}(B)|\, B \in \mathscr{E}\}.$$

For s and t in S we define

$$\gamma_{s,t} \doteq \sup\left\{d_U(\gamma_s^0(\cdot|x), \gamma_s^0(\cdot|w))/x_{S\setminus t} = w_{S\setminus t}\right\}.$$

We note that $\gamma_{s,t} = 0$ if $t \notin \partial s$.

Finally,

$$\alpha(\gamma) \doteq \sup_{s\in S}\left\{\sum_{t\in S}\gamma_{s,t}\right\}.$$

If $\alpha(\gamma) < 1$ then γ meets Dobrushin's condition and $\#\mathscr{G}(\gamma) = 1$ (Definition 8.6 and Theorem 8.7 in [7]).

4 Uniqueness Theorem

Theorem 4.1. *Let γ be the Gibbsian specification for Φ. Then*

$$2\sum_{l=1}^{g}\tanh(|\beta_l|/4) < 1 \Rightarrow \#\mathscr{G}(\gamma) = 1$$

Proof

To prove theorem we only need to check that $\alpha(\gamma) \leq 2\sum_{l=1}^{g}\tanh(|\beta_l|/4)$

Let $s \in S$, $t \in \partial s$, x and w in E^S such that $x_{S\setminus t} = w_{S\setminus t}$.

If $x_t = w_t$, then $x = w$ and $d_U(\gamma_s^0(\cdot|x), \gamma_s^0(\cdot|w)) = 0$.

If $x_t = 1 - w_t$, where $t = s + v_l$ or $t = s - v_l$, for $1 \leq l \leq g$. Without loss of generality, we assume that $t = s - v_l$. Then $x_r = w_r$, $r \neq s - v_l$ and $x_{s-v_l} = 1 - w_{s-v_l}$.

We note that

$$\begin{aligned} d_U(\gamma_s^0(\cdot|x), \gamma_s^0(\cdot|w)) &= \sup\left\{|\gamma_s^0(A|x) - \gamma_s^0(A|w)|/A \in \mathscr{E}\right\}, \\ &= \max\left\{|\gamma_s^0(A|x) - \gamma_s^0(A|w)|/A = \emptyset, E, \{0\}, \{1\}\right\}, \end{aligned}$$

and

$$\begin{aligned} |\gamma_s^0(\emptyset|x) - \gamma_s^0(\emptyset|w)| &= |0-0| = 0, \\ |\gamma_s^0(E|x) - \gamma_s^0(E|w)| &= |1-1| = 0, \\ |\gamma_s^0(\{1\}|x) - \gamma_s^0(\{1\}|w)| &= |\gamma_s^0(\{0\}|x) - \gamma_s^0(\{0\}|w)|, \end{aligned}$$

(because $\gamma_s^0(\{1\}|x) = 1 - \gamma_s^0(\{0\}|x)$), therefore

$$\begin{aligned} d_U(\gamma_s^0(\cdot|x), \gamma_s^0(\cdot|w)) &= \left|\gamma_s^0(\{1\}|x) - \gamma_s^0(\{1\}|w)\right|, \\ &= \left|\frac{e^{-(\beta_0+\sum_{i=1}^{g}\beta_i(x_{s+v_i}+x_{s-v_i}))}}{e^{-(\beta_0+\sum_{i=1}^{g}\beta_i(x_{s+v_i}+x_{s-v_i}))}+1}\right. \\ &\qquad \left. - \frac{e^{-(\beta_0+\sum_{i=1}^{g}\beta_i(w_{s+v_i}+w_{s-v_i}))}}{e^{-(\beta_0+\sum_{i=1}^{g}\beta_i(w_{s+v_i}+w_{s-v_i}))}+1}\right|, \\ &= \left|\frac{1}{1+e^{\beta_0+\beta_l(x_{s+v_l}+x_{s-v_l})+\sum_{i\neq l}\beta_i(x_{s+v_i}+x_{s-v_i})}}\right. \\ &\qquad \left. - \frac{1}{1+e^{\beta_0+\beta_l(w_{s+v_l}+w_{s-v_l})+\sum_{i\neq l}\beta_i(w_{s+v_i}+w_{s-v_i})}}\right|, \end{aligned}$$

$$
\begin{aligned}
&= \left| \frac{1}{1+e^{\beta_0+\beta_l(x_{s+v_l}+x_{s-v_l})+\sum_{i\neq l}\beta_i(x_{s+v_i}+x_{s-v_i})}} \right. \\
&\quad \left. - \frac{1}{1+e^{\beta_0+\beta_l(x_{s+v_l}+(1-x_{s-v_l}))+\sum_{i\neq l}\beta_i(x_{s+v_i}+x_{s-v_i})}} \right|, \\
&= \left| \frac{1}{1+e^{\beta_l x_{s-v_l}}e^{\theta}} - \frac{1}{1+e^{\beta_l(1-x_{s-v_l})}e^{\theta}} \right|^{1} \\
&= \left| \frac{e^{\beta_l(1-x_{s-v_l})} - e^{\beta_l x_{s-v_l}}}{e^{-\theta}+e^{\beta_l(x_{s-v_l})}+e^{\beta_l(1-x_{s-v_l})}+e^{\beta_l}e^{\theta}} \right|, \\
&= \frac{|1-e^{\beta_l}|}{e^{\beta_l}e^{\theta}+e^{-\theta}+e^{\beta_l}e^{0}+e^{-0}}, \\
&\le \frac{|1-e^{\beta_l}|}{e^{\beta_l}e^{-\beta_l/2}+e^{-(-\beta_l/2)}+e^{\beta_l}+1}, \\
&= \frac{|1-e^{\beta_l}|}{(1+e^{\beta_l/2})^2} = \frac{(1+e^{\beta_l/2})|1-e^{\beta_l/2}|}{(1+e^{\beta_l/2})^2} \\
&= \frac{|1-e^{\beta_l/2}|}{1+e^{\beta_l/2}} = \frac{e^{|\beta_l|/2}-1}{e^{|\beta_l|/2}+1} = \tanh(|\beta_l|/4)
\end{aligned}
$$

(since $e^{\beta_l}e^{-\beta_l/2}+e^{\beta_l/2} \le e^{\beta_l}e^{z}+e^{-z}$, $z \in \mathbb{R}$).

Therefore $\gamma_{s,s-v_l} \le \tanh(|\beta_l|/4)$ and $\sum_{t\in\partial s}\gamma_{s,t} \le \sum_{l=1}^{g} 2\tanh(|\beta_l|/4)$, $\forall s \in S$, then

$$\alpha(\gamma) = \sup_{s\in S}\left\{\sum_{t\in\partial s}\gamma_{s,t}\right\} \le 2\sum_{l=1}^{g}\tanh(|\beta_l|/4)$$

■

Remark 4.1. *The counterpart is false ($\not\Leftarrow$)*

Proof

If we identify 0 with -1, first order Auto-logistic model with $\beta_0/2 = \beta_1 = \beta_2$ is the (-1)-normalized Ising model for $\beta_1/4$ and without an external field (see Example 3.3.33 in [4]). The $\beta_1 = \beta_2 = 1.6$ case does not reach our theorem condition. But there is uniqueness because $\beta_1/4 < \beta_c = (\log(1+\sqrt{2}))/2 = 0.4402$ (Ising critical parameter, see page 100 of [7] and example in [6]).

5 Discussion

Theorem 4.1 provides us with a region for interaction parameters. We called it Uniqueness region and we can see its graphic in figure 1 for the first order model.

These parameters constrains ensure uniqueness but limit models diversity. There are a lot of textures, like the one in figure 2, which can not be characterized for the Auto-logistic model if parameters must lie in Uniqueness region. Images in figure 2 came from an Auto-logistic model with $\beta = (20, -20, -20, 10, 10)$. Image in figure 2(a) was generated with 500 iterations of Gibbs Sampler and image in figure 2(b) was generated with 8000 iterations of the same algorithm.

[1] $\theta = \beta_0 + \beta_l(x_{s+v_l}) + \sum_{i\neq l}\beta_i(x_{s+v_i}+x_{s-v_i})$

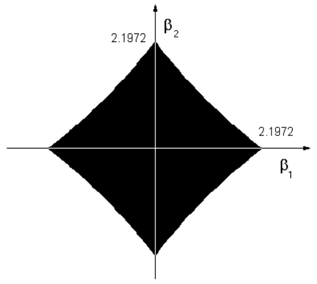
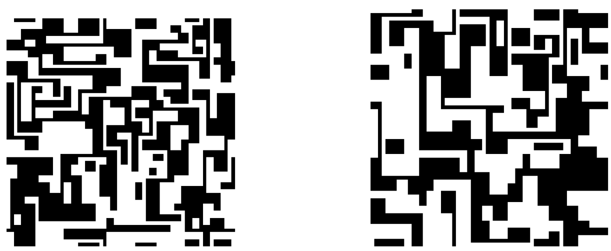


Figure 1: Uniqueness region



(a) 500 iterations of Gibbs Sampler (b) 8000 iterations of Gibbs Sampler

Figure 2: Images of size 64×64, from an Autologistic model with $\beta = (20, -20, -20, 10, 10)$

(a) 500 iterations of Gibbs Sampler (b) 8000 iterations of Gibbs Sampler

Figure 3: Images of size 64×64, from an Autologistic model with $\hat{\beta}_{NR} = (0.66, -0.66, -0.66, 0.33, 0.32)$

We estimate β of image in figure 2(b) maximizing Pseudo-likelihood function within Uniqueness region. We get $\hat{\beta}_{NR} = (0.66, -0.66, -0.66, 0.33, 0.32)$ using

(a) 500 iterations of Gibbs Sampler (b) 8000 Gibbs iterations

Figure 4: Images of size 64×64, from an Autologistic model with $\hat{\beta}_{SA} = (0.82, -0.95, -0.69, -0.07, 0.23)$

Newton-Raphson method (see [11]) and $\hat{\beta}_{SA} = (0.82, -0.95, -0.69, -0.07, 0.23)$ using Simulated Annealing (see [5, 15]). Images in figure 3 were generated with Gibbs Sampler and $\hat{\beta}_{NR}$. Images in figure 4 were generated with Gibbs Sampler and $\hat{\beta}_{NR}$. The difference between images in figure 2 and images in figures 3 and 4 is remarkable. This shows us the constrained model limits. However Uniqueness region avoids the phenomenon known in Statistical Physics as Phase transition.

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References

[1] S. Albeverioa, Y. Kondratieva, M. Röcknera, and T. Tsikalenkoa. Uniqueness of gibbs states on loop lattices. *Comptes Rendus de l'Académie des Sciences - Series I - Mathematics*, 324(12):1401–1406, 1997.

[2] J. E. Besag. Spatialo interaction and the statistical analysis of lattice systems (with discussion). *Jr. R. Stat. Soc. B*, 2:192–236, 1974.

[3] V. Betz and J. Lorinczi. Uniqueness of gibbs measures relative to brownian motion. *Annales de l'Institut Henri Poincare (B) Probability and Statistics*, 39:877 – 889, 2003.

[4] O.H. Bustos and A. Guerrero. *Breve introducción a la matemática de la estadística espacial.* Ensaios Matemáticos, Volume 20, 2011 , Sociedade Brasileira de Matemática, 2011.

[5] O.H. Bustos and S.M Ojeda. *Campos Aleatorios Markovianos en Procesamiento de Imágenes*. Trabajos de Matemática, Serie B 25/94, Facultad de Matemática, Astronomía y Física, Universidad Nacional de Córdoba, 1994.

[6] F. Comets. Detecting phase transition for gibbs measures. *The Annals of Applied Probability*, 7, No. 2:545–563, 1997.

[7] H. O. Georgii. *Gibbs Measure and phase transitions*. De Gruyter, Berlin, 1988.

[8] J. Hughes, M. Haran, and P. Caragea. Autologistic models for binary data on a lattice. *Environmetrics*, 22:857–871, 2011.

[9] E. Ising. Beitrag zur theorie des ferro- und paramagnetismus. In *Dissertation, Mathematish-Naturewissenschaftliche Fakultät der Universität Hamburg*, 1924.

[10] E. Ising. Beitrag zur theorie des ferromagnetismus. *Z. Phys*, 31:253–258, 1997.

[11] J.-O. Johansson. Parameter-estimation in the auto-binomial model using the coding- and pseudo-likelihood method approached with simulated annealing and numerical optimization. *Pattern Recognition Letters*, 22:1233–1246, 2001.

[12] D. Kepa and Y. Kozitsky. Uniqueness of gibbs states of a quantum system on graphs. *Reports on Mathematical Physics*, 59:281–288, 2007.

[13] Z. Wang and Y. Zheng. Analysis of binary data via a centered spatial-temporal autologistic regression model. *Environ Ecol Stat*, 20:37–57, 2013.

[14] D. Weitz. Combinatorial criteria for uniqueness of gibbs measures. *Random Structures and Algorithms*, 27(4):445–475, 2005.

[15] G. Winkler. *Image Analysis, Random Fields and Dynamic Monte Carlo Methods - A Mathematical Introduction*. Springer-Verlag, Berlin, 1995.