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Noncommutative invariants

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REPÚBLICA ARGENTINA

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Foreword

These lectures propose an introduction to various problems about group actions by automorphisms on noncommutative algebras. The underlying noncommutativity deals with Poisson structures on polynomial algebras, their deformations into noncommutative associative algebras, some localized or completed versions, the associated Lie algebras. The typical objects are noncommutative polynomial algebras (Ore extensions), skewfields of fractions, noncommutative power series, and specially among them Weyl algebras, quantum spaces and tori, quantum groups. The typical results concern the finite generation of invariants, in continuity with Noether's and Hilbert's theorems in the classical theory. We try to provide a primer on some basic theorems and to give some evidence on many profounds links between the questions under consideration. It seems difficult to give within the framework of this course complete proofs of all general results (when they are known...); our choice is to illustrate the problems studied by concrete developments on the two-dimensional case, which it is rich enough to carry the whole interest of the situations, although being open to a direct approach. From this point of view, the following three diagrams can be seen as constituting parts of a guide through these lectures.

Picture 1: *action of* $\mathrm{SL}_2(\mathbb{C})$ *on symplectic Poisson two-dimensional polynomial algebras, associated deformations, and around*

$$
\begin{tikzcd}
F(\mathbb{C}) \arrow[r, "\sim"] & F(\mathbb{C})^{\Gamma} & & \text{$\Leftarrow$ modular action of finite $\Gamma \subset \mathrm{SL}_2(\mathbb{C})$} \\
\mathrm{Frac}\, A_1(\mathbb{C}) := D_1(\mathbb{C}) \arrow[u, hook, "\text{completion}"'] \arrow[r, "\sim"] & D_1(\mathbb{C})^G & & \\
A_1(\mathbb{C}) \arrow[u, hook, "\text{localization}"'] \arrow[r] & A_1(\mathbb{C})^G \arrow[u, hook, "\text{localization}"'] \arrow[r, dotted] & \mathrm{HH}_0(A_1(\mathbb{C})^G) & \\
S := \mathbb{C}[x,y] \arrow[u, squiggly, "\textit{deformation}"'] \arrow[r] \arrow[d, hook, "\text{diff extension}"'] & S^G \arrow[u, squiggly, "\textit{deformation}"'] \arrow[r, dotted] \arrow[d, hook, "\text{diff extension}"'] & \mathrm{HP}_0(S^G) \arrow[u, squiggly, "\textit{deformation}"'] & \text{$\Leftarrow$ linear action of finite $G \subset \mathrm{SL}_2(\mathbb{C})$} \\
\mathrm{Diff}\,(S) \simeq A_2(\mathbb{C}) \arrow[r] \arrow[d, hook, "\text{localization}"'] & A_2(\mathbb{C})^G \arrow[d, hook, "\text{localization}"'] & & \\
\mathcal{U}_{\mathrm{Pois}}(\mathrm{Frac}\, S) \simeq B_2(\mathbb{C}) \arrow[r, "\sim"] \arrow[d, hook, "\text{localization}"'] & B_2(\mathbb{C})^G \arrow[d, hook, "\text{localization}"'] & & \\
\mathrm{Frac}\, A_2(\mathbb{C}) := D_2(\mathbb{C}) \arrow[r, "\sim"] & D_2(\mathbb{C})^G & &
\end{tikzcd}
$$

Picture 2: *action of* $\mathrm{SL}_2(\mathbb{Z})$ *on multiplicative Poisson, two-dimensional polynomial algebras, quantum deformations, and around*

$$\begin{tikzcd}
F_q(\mathbb{C}) & & \\
\operatorname{Frac} T_q := D_1^q(\mathbb{C}) \arrow[u, hook, "\text{completion}"'] \arrow[r, "\sim"] & D_1^q(\mathbb{C})^G & \\
T_q := \mathbb{C}_q[x^{\pm 1}, y^{\pm 1}] \arrow[u, hook, "\text{localization}"'] \arrow[r] & T_q^G \arrow[u, hook, "\text{localization}"'] \arrow[r, dotted] & \mathrm{HH}_0(T_q^G) \\
T := \mathbb{C}[x^{\pm 1}, y^{\pm 1}] \arrow[u, squiggly, "\textit{deformation}"'] \arrow[r] & T^G \arrow[u, squiggly, "\textit{deformation}"'] \arrow[r, dotted] & \mathrm{HP}_0(T^G)
\end{tikzcd}$$

$\Leftarrow$ multiplicative action of finite $G \subset \mathrm{SL}_2(\mathbb{Z})$

Picture 3: *automorphisms of quantum and jordanian deformations of the plane, and around*

$$\begin{tikzcd}
T_q := \mathbb{C}_q[x^{\pm 1}, y^{\pm 1}] & S_q := \mathbb{C}_q[x,y] \arrow[l, hook, "\text{localiz.}"'] \arrow[r, hook] & \mathcal{O}_q(M_2) \arrow[l, dotted, bend left, "\text{coaction}"] \arrow[r, two heads] & \mathcal{O}_q(\mathrm{SL}_2) \\
T := \mathbb{C}[x^{\pm 1}, y^{\pm 1}] \arrow[u, squiggly, "\textit{deform.}"] & S := \mathbb{C}[x,y] \arrow[l, hook, "\text{localiz.}"'] \arrow[u, squiggly, "\textit{deform.}"] \arrow[d, squiggly, "\textit{deform.}"'] \arrow[r, hook] & \mathcal{O}(M_2) \arrow[u, squiggly, "\textit{deform.}"'] \arrow[d, squiggly, "\textit{deform.}"'] \arrow[l, dotted, bend left, "\text{coaction}"] \arrow[r, two heads] & \mathcal{O}(\mathrm{SL}_2) \arrow[u, squiggly, "\textit{deform.}"'] \arrow[d, squiggly, "\textit{deform.}"'] \\
& S^{\mathrm{J}} := \mathbb{C}^{\mathrm{J}}[x,y] \arrow[r, hook] & \mathcal{O}^{\mathrm{J}}(M_2) \arrow[l, dotted, bend left, "\text{coaction}"] \arrow[r, two heads] & \mathcal{O}^{\mathrm{J}}(\mathrm{SL}_2)
\end{tikzcd}$$

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1 Invariants of noncommutative polynomial rings

1.1 Invariants of noetherian rings under finite group actions

1.1.1 Noncommutative noetherian rings

Let R be a ring (non necessarily commutative). A left R-module M is said to be noetherian if M satisfies the ascending chain condition on left submodules, or equivalently if every left submodule of M is finitely generated. The ring R himself is a left noetherian ring if it is noetherian as left R-module. There is of course a similar definition for right modules, and a ring R is said to be noetherian if it is left noetherian and right noetherian (i.e. if every left ideal is finitely generated and every right ideal is finitely generated). It is classical (see for instance [8]) that: (i) *for any submodule N of a module M, we have: M noetherian if and only if N and M/N are noetherian* ; (ii) *any finite direct sum of noetherian modules is noetherian.*

Proof. We suppose that M is noetherian. Any ascending chain of submodules of N being an ascending chain of submodules of M, it is clear that N is noetherian. Let $C_1 \subset C_2 \subset \cdots \subset C_i \subset \cdots$ be an ascending chain of submodules of M/N. Any C_i is a quotient A_i/N where $A_1 \subset A_2 \subset \cdots \subset A_i \subset \cdots$ is an ascending chain of submodules of M. The noetherianity of M implies the existence of $n \geq 1$ such that $A_i = A_n$ for all $i \geq n$. Therefore $C_i = C_n$ for any $i \geq n$ and M/N is noetherian.

Suppose conversely that N and M/N are noetherian. Let $A_1 \subset A_2 \subset \cdots \subset A_i \subset \cdots$ be an ascending chain of submodules of M. From one hand, $(A_i \cap N)_{i\geq 0}$ is an ascending chain of submodules of N ; there exists $m \geq 1$ such that $A_i \cap N = A_m \cap N$ for any $i \geq m$. From the other hand, $((A_i + N)/N)_{i\geq 0}$ is an ascending chain of submodules of M/N ; there exists $p \geq 1$ such that $(A_i + N)/N = (A_m + N)/N$ for any $i \geq p$. Take $i \geq n := \max(m, p)$. We have $A_i \cap N = A_n \cap N$ and $(A_i + N)/N = (A_n + N)/N$. If $x \in A_i$, there exists $y \in A_n$ such that $x - y \in N$. Since $A_n \subset A_i$, it follows that $y \in A_i$, then $x - y \in (A_i \cap N) = (A_n \cap N)$. So $x - y \in A_n$ and finally $x \in A_n$. We conclude that $A_i \subset A_n$, i.e. $A_i = A_n$ and M is noetherian.

If N_1 and N_2 are noetherian left modules, the submodule $N = N_1 \oplus (0)$ in $M = N_1 \oplus N_2$ is noetherian by $N \simeq N_1$ and the module M/N is noetherian by $M/N \simeq N_2$. Therefore M is noetherian applying the previous property. The result follows by induction. $\square$

These properties imply in particular the following useful observation: *if R a left noetherian ring, then all finitely generated left R-modules are left noetherian.*

Let P be a finitely generated R-module, and $\{x_1 \ldots, x_n\}$ a generating family of P. Consider some free R-module M of rank n and $\{\xi_1, \ldots, \xi_n\}$ a R-basis of M. The R-module morphism $f : M \to P$ defined by $f(\xi_i) = x_i$ for any $1 \leq i \leq n$ is surjective. So $P \simeq M/\ker f$. Since $M \simeq \bigoplus_{1\leq i\leq n} R\xi_i$ and each $R\xi_i$ is a noetherian R-module because R is a noetherian ring, we deduce that M is a noetherian R-module. Then $M/\ker f$ is noetherian, i.e. P is noetherian. $\square$

1.1.2 Invariant ring and skew group ring

Let R be a ring and G a subgroup of the group $\operatorname{Aut} R$ of ring automorphisms of R. The invariant ring (or fixed ring) R^G is by definition the subring of R:

$$R^G = \{r \in R\,;\, g(r) = r \text{ for all } g \in G\}.$$

The skew group ring (or trivial crossed product) $R \star G$ is defined as the free left R-module with elements of G as a basis and with multiplication defined from relation:

$$\forall\, r, s \in R,\ \forall\, g, h \in G,\ (rg)(sh) = rg(s)gh.$$

In particular:

$$\forall\, r \in R,\ \forall\, g \in G,\ \ gr = g(r)g \ \text{ and } \ rg = gg^{-1}(r)$$

Every element of $R \star G$ as a unique expression as $\sum_{g\in G} r_g g$ with $r_g \in R$ for any $g \in G$ and $r_g = 0$ for all but finitely many g. It is clear that R is a subring of $R \star G$ (identifying r with $r1_G$), and that $R \star G$ is also a right R-module. In the particular case where G is finite, the left R-module $R \star G$ is finitely generated, then using the last observation of 1.1.1, we deduce immediately that:

$$\text{if } G \text{ is finite and } R \text{ is left noetherian, then } R \star G \text{ is left noetherian.} \tag{1}$$

Note that the noetherianity of $R \star G$ can be proved in the more general context where G is polycyclic by finite, see [13]. The skew group ring $R \star G$ is closely related to the invariant ring R^G, as shows for instance the following lemma (from [14]).

Lemma. *Let R be a ring, G a finite subgroup of $\operatorname{Aut} R$, and $S = R \star G$.*

(i) *The element $f = \sum_{g\in G} g$ of S satisfies $fS = fR$ and $Sf = Rf$.*

(ii) *If $|G|$ is invertible in R, the element $e = \frac{1}{|G|} f$ of S satisfies $e^2 = e$, $eS = eR$, and $eSe = eR^G \simeq R^G$.*

Proof. We have $fg = f = gf$ for all $g \in G$. For any $x = \sum_{g\in G} r_g g \in S$, we compute $fx = \sum_{g\in G} fr_g g = \sum_{g\in G} fgg^{-1}(r_g) = \sum_{g\in G} fg^{-1}(r_g) = f \sum_{g\in G} g^{-1}(r_g) \in fR$. We conclude that $fS \subseteq fR$; the converse is clear and so $sS = fR$. On the same way $xf = \sum_{g\in G} r_g gf = \sum_{g\in G} r_g f = (\sum_{g\in G} r_g) f$ implies $Sf \subseteq Rf$ and finally $Sf = Rf$.

It follows from point (i) that $e^2 = e$, $eS = eR$, $Se = Re$ and $eSe = eRe$. For $r \in R$, we compute:

$$\begin{aligned} ere &= \tfrac{e}{|G|} \textstyle\sum_{g\in G} rg = \tfrac{e}{|G|} \sum_{g\in G} gg^{-1}(r) = \tfrac{1}{|G|} \sum_{g\in G} egg^{-1}(r) \\ &= \tfrac{1}{|G|} \textstyle\sum_{g\in G} eg^{-1}(r) = \tfrac{e}{|G|} \sum_{g\in G} g^{-1}(r) = \tfrac{e}{|G|} \sum_{g\in G} g(r) = e\tau(r), \end{aligned}$$

where $\tau : R \to R^G$ is the trace map $r \mapsto \frac{1}{|G|} \sum_{g\in G} g(r)$. This proves that $eSe = e\tau(R)$. Since any $r \in R^G$ can be written $r = \tau(r)$, we have $R^G \subset \tau(R)$, so $R^G = \tau(R)$. Hence $eSe = eR^G$. Finally, because $er = re$ for any $r \in R^G$, the map $r \mapsto er$ defines a ring isomorphism $R^G \to eR^G$. $\square$

1.1.3 A finiteness theorem

The following theorem is due to S. Montgomery and L. W. Small (see [57]) and can be viewed as a noncommutative analogue of the classical Noether's theorem.

Theorem. *Let A be a commutative noetherian ring, R a non necessarily commutative ring such that A is a central subring of R and R is a finitely generated A-algebra, and G a finite group of A-algebra automorphisms of R such that $|G|$ is invertible in R. If R is left noetherian, then R^G is a finitely generated A-algebra.*

Proof. Let us introduce $S = R \star G$. As we have observed in 1.1.2, S is left noetherian. It is clear from the hypothesis that A is a central subring of S and that S is finitely generated as A-algebra (if $\{q_1, \dots, q_m\}$ generate R over A and $G = \{g_1, \dots, g_d\}$, then $\{q_1, \dots, q_m, g_1, \dots, g_d\}$ generate S over A).
As in 1.1.2, consider in S the element $e = \frac{1}{|G|}\sum_{g\in G} g$ which satisfies $e^2 = e$. In particular, eSe is a subring of S, eS is a left eSe-module, and SeS is a two-sided ideal of S. Observe firstly that eS is a finitely generated left eSe-module.

> Because S is left noetherian, SeS is finitely generated as a left ideal of S. Say that $SeS = \sum_i Sx_i$, and write $x_i = \sum_j v_{ij}ew_{ij}$ with $v_{ij} \in S$ and $w_{ij} \in S$ for all j. Choose $r \in S$. Then $er = eeer \in e(SeS)$, and so $er = e(\sum_i s_i x_i) = \sum es_i v_{ij} ew_{ij} = \sum es_i v_{ij} e^2 w_{ij}$. Thus the finite set $\{ew_{ij}\}$ generates eS as a left eSe-module.

Denote more briefly $eS = \sum_{i=1}^n eSex_i$ with $x_i \in S$, and take $t_1, t_2, \dots, t_m$ generators of S as a A-algebra. Now write $et_j = \sum_{i=1}^n ey_{ij}ex_i$ and $ex_kt_j = \sum_{i=1}^n ez_{ijk}ex_i$ with $y_{ij} \in S$ and $z_{ijk} \in S$ for all $1 \le j \le m$ and $1 \le k \le n$. Consider the finite set $E = \{ex_ie,\, ey_{ij}e,\, ez_{ijk}e\}_{1\le i,k\le n,\, 1\le j\le m}$. We compute:

$$et_1t_2e = (\sum_{i=1}^n ey_{i1}ex_i)t_2e = \sum_{i=1}^n ey_{i1}e(ex_it_2)e = \sum_{i=1}^n ey_{i1}e(\sum_{\ell=1}^n ez_{\ell 2i}ex_\ell)e = \sum_{i=1}^n ey_{i1}e(\sum_{\ell=1}^n ez_{\ell 2i}eex_\ell e),$$

and prove so inductively that any monomial $et_{j_1}t_{j_2}\dots t_{j_k}e$ with $1 \le j_1, j_2, \dots, j_k \le m$ can be expressed by a finite sum of products of elements of E. As any element of eSe is a linear combination of such monomials with coefficients in A, we conclude that E generates eSe as a A-algebra. By point (ii) of lemma 1.1.2, the proof is complete. $\square$

This theorem will apply in particular to the iterated Ore extensions (see further 1.3).

1.2 Invariants of simple rings under finite group actions

1.2.1 Simplicity of invariants

Definitions. Recall that a ring R is simple when (0) and R are the only two-sided ideals of R. An automorphism $g \in \operatorname{Aut} R$ is said to be inner if there exists $a \in R$ invertible in R such that $g(x) = axa^{-1}$ for all $x \in R$, and is said to be outer if it is not inner. A subgroup G of $\operatorname{Aut} R$ is outer when the identity map is the only inner automorphism in G.

We start with the following lemma (from [14]) about simplicity of crossed products.

Lemma. *Let R be a simple ring and G a finite outer subgroup of* $\operatorname{Aut} R$. *Then the ring $R \star G$ is simple.*

Proof. We denote $S = R \star G$. For any nonzero element $x = \sum_{g \in G} r_g g$ in S, define the length of x as the cardinal of the support $\{g \in G\,;\, r_g \neq 0\}$ of x. Let I be a two-sided nonzero ideal of $S = R \star G$ and ℓ be the minimal length of nonzero elements of I. Because I is a two-sided ideal and ℓ is minimal, it is clear that the set K of all elements $r \in R$ appearing as a coefficient in the decomposition of some element of I of length ℓ is a two-sided ideal of R. Since R is simple, we have $1 \in K$. So there exists in I some element with decomposition $1.g_0 + \sum_{g \in G, g \neq g_0} r_g.g$. Multiplying at the right by g_0^{-1}, we deduce that I contains an element $x = 1.1_G + \sum_{g \in G, g \neq 1_G} r_g.g$ of length ℓ.

If $x = 1.1_G$ (i.e. $\ell = 1$), then $I = S$ and we are done. Assume that $r_h \neq 0$ for some $h \in G, h \neq 1_G$. For any $r \in R$, the bracket $rx - xr = \sum_{g \in G, g \neq 1_G}(rr_g - r_g g(r))g$ lies in I and has shorter length than x. Since ℓ is minimal, it follows that $rx - xr = 0$. In particular: $rr_h - r_h h(r) = 0$ for all $r \in R$. Therefore $r_h R = R r_h$ is a two-sided ideal of R. The simplicity of R implies that $1 \in r_h R$, and so r_h is invertible in R. Hence $h(r) = r_h^{-1} r r_h$ for all $r \in R$, which says that h is an inner automorphism of R, which is impossible since G is outer and $h \neq 1_G$. $\square$

We need now a brief account on the notion of Morita equivalence. Two rings S and T are Morita equivalent when their categories of modules are equivalent. There exist several methods to characterize such an equivalence. None is obvious and we refer for instance to [1] or [13] for a serious presentation of this classical subject. In the limited frame of this notes, our basis will be the following concrete criterion (see [13], proposition 3.5.6): S and T are Morita equivalent if and only if T is a corner in some matrix algebra with entries in S, that is if and only if there exist an integer n and an idempotent element $e \in M_n(S)$ such that $T \simeq eM_n(S)e$ and $M_n(S)eM_n(S) = M_n(S)$.

Theorem. *Let R be a simple ring and G a finite outer subgroup of* $\operatorname{Aut} R$ *such that $|G|$ is invertible in R. Then:*

(i) *R^G and $R \star G$ are Morita equivalent,*

(ii) *the ring R^G is simple.*

Proof. We denote $S = R \star G$. By point (ii) of lemma 1.1.2, the element $e = \frac{1}{|G|}\sum_{g \in G} g$ of S satisfies $e^2 = e$ and we have a ring isomorphism $eSe \simeq R^G$. It is clear that SeS is a two-sided ideal of S. Thus $SeS = S$ since S is simple by the previous lemma. We just apply the above Morita equivalence criterion (with $n = 1$) to conclude that S and R^G are Morita equivalent.

Point (ii) can be deduced from the simplicity of S using the fact that simplicity is a Morita invariant. We give here a direct proof (which doesn't use Morita equivalence) of the simplicity of R^G. By point (iii) of lemma 1.1.2, it is equivalent to prove the simplicity of the subring eSe of S. Let I be a two-sided nonzero ideal of eSe. Denote by J the set of element $u \in S$ such that $eue \in I$. Thus $eJe = I$. Because $e^2 = e$ and I is right ideal of eSe, we have $euese = (eue)(ese) \in I$ for any $u \in J$, $s \in S$; then $ues \in J$. On the same way on the left $eseue = (ese)(eue) \in I$ implies $seu \in J$. Hence for any $s, s' \in S$ and $u \in J$, we deduce that $seues' = (seu)es'$ with $seu \in J$ from the second argument and then $seues' \in J$ from the first one. Therefore $sxs' \in J$ for any $x \in I = eJe$. In other words $SIS \subseteq J$. By simplicity of S (previous lemma), the two-sided nonzero ideal SIS of S equals to S. Finally $S \subseteq J$, thus $J = S$ and then $I = eSe$. We conclude that eSe is a simple ring and the proof is complete. $\square$

This theorem, from [14], is a fundamental argument in all homological studies of invariants of Weyl algebras (see further 2.2.2 and 2.3.1).

1.2.2 Central invariants

It is clear that the center $Z(R)$ of any ring R is stable under the action of any subgroup G of $\operatorname{Aut} R$, and that $Z(R)^G = Z(R) \cap R^G \subset Z(R^G)$. Our aim here is to prove (following [14]) that equality holds when R is simple and G outer. We need the following preliminary results.

LEMMA. *Let R be a simple ring and G a finite outer subgroup of $\operatorname{Aut} R$. Then we have a ring isomorphism between $Z(R)$ and the centralizer of R in $R \star G$.*

PROOF. We denote $S = R \star G$. Let $x = \sum_{g \in G} r_g g$ be an element of S such that $xr = rx$ for any $r \in R$. Since $xr - rx = \sum_{g \in G} (r_g g(r) - r r_g)\, g$, we deduce that any g in the support $\{g \in G\,;\, r_g \neq 0\}$ of x satisfies $r_g g(r) = r r_g$ for all $r \in R$. Hence $r_g R = R r_g$ is a two-sided ideal of R. It follows by simplicity of R that r_g is invertible in R and g is the inner isomorphism $r \mapsto r_g^{-1} r r_g$. Now the assumption on G implies $g = 1_G$. We conclude that $x = r_1 1_G$ with $r_1 \in Z(R)$. Up to the canonical embedding of R in S, we have proved that $\operatorname{Cent}_S(R) = Z(R)$. □

LEMMA. *Let R be a simple ring and G a finite outer subgroup of $\operatorname{Aut} R$. Then we have a ring isomorphism between $R \star G$ and $\operatorname{End}_{R^G} R$ for the canonical structure of right R^G-module of R.*

PROOF. With our usual notation $S = R \star G$, we define a left S-module structure on R by:

$$x \cdot r = \textstyle\sum_{g \in G} r_g g(r) \quad \text{for any } r \in R,\ x = \sum_{g \in G} r_g g \in S.$$

It is clear that $(y \cdot (x \cdot r)) = (yx) \cdot r$ for all $x, y \in S, r \in R$. For the particular element $f = \sum_{g \in G} g \in S$, we calculate $fx \cdot r$ for any $x \in S, r \in R$. Since $fS = fR$ (see lemma 1.1.2), the exists $r' \in R$ such that $fx = fr'$. Therefore $fx \cdot r = fr' \cdot r = f \cdot (r' \cdot r) = f \cdot (r'r) = \sum_{g \in G} g(r'r) = \tau(r'r)$ which is obviously an element of R^G. We have finally proved that $fx \cdot r \in R^G$ for all $x \in S, r \in R$.

We introduce for any $x \in S$ the map $\psi_x : R \to R,\ r \mapsto x \cdot r$. By an easy calculation, we have $\psi_x(ra) = \psi_x(r)a$ for all $r \in R, a \in R^G$. Therefore ψ_x is an endomorphism of R as a right R^G-module. The map $\psi : x \mapsto \psi_x$ is clearly a morphism of rings $S \to \operatorname{End}_{R^G} R$. By the lemma 1.2.1, S is simple, hence ψ is injective. In order to prove the surjectivity, we consider the two-sided ideal SfS ; using again the simplicity of S, we have $SfS = S$. In particular, $1_S = \sum_{i=1}^n u_i f v_i$ for some $u_1, \ldots, u_n, v_1, \ldots, v_n$ in S. Since $Sf = Rf$ (see lemma 1.1.2), any $u_i f$ can be written $w_i f$ with $w_i \in R$, and then $1_S = \sum_{i=1}^n w_i f v_i$ with $w_i \in R, v_i \in S$. We fix any $h \in \operatorname{End}_{R^G} R$ and associate $x = \sum_{i=1}^n h(w_i) f v_i \in S$. For $r \in R$, we compute $x \cdot r = \big(\sum_{i=1}^n h(w_i) f v_i\big) \cdot r = \sum_{i=1}^n h(w_i)(f v_i \cdot r)$. We know from the beginning of the proof that $f v_i \cdot r \in R^G$, and h is a right R^G-module endomorphism, hence:

$$\sum_{i=1}^n h(w_i)(f v_i \cdot r) = h\big(\sum_{i=1}^n w_i (f v_i \cdot r)\big) = h\big(\sum_{i=1}^n (w_i f v_i) \cdot r)\big) = h(1_S \cdot r) = h(r).$$

We have proved that $h = \psi_x$, therefore ψ is an isomorphism. □

THEOREM. *Let R be a simple ring and G a finite outer subgroup of $\operatorname{Aut} G$. Then $Z(R^G) = Z(R)^G$.*

Proof. For any $r \in R$, we denote by $\mu_r : R \to R$ the left multiplication $t \mapsto rt$ by r. It is clear that μ_r is right R^G-module endomorphism of R and the subset $\overline{R} := \{\mu_r \; ; \; r \in R\}$ of $\mathrm{End}_{R^G}R$ is a subring which is isomorphic to R via $\mu : r \mapsto \mu_r$. Moreover, by the definition of the isomorphism $\psi : S \to \mathrm{End}_{R^G}R$ in the previous lemma, the image ψ_r of an element $r \in R$ (identified canonically with $r1_G \in S$) is no more than μ_r. Hence $\psi(R) = \overline{R}$ and recalling the first lemma:

$$\mathrm{Cent}_{\mathrm{End}_{R^G}R}(\overline{R}) \simeq \mathrm{Cent}_S(R) = Z(R).$$

For any $r \in R$, we denote by $\nu_r : R \to R$ the right multiplication $t \mapsto tr$ by r. An easy calculation proves that, if $r \in \mathrm{Cent}_R(R^G)$, then ν_r is a right R^G-module endomorphism of R; since $\nu_r \circ \mu_s = \mu_s \circ \nu_r$ for all $r, s \in R$, it follows that ν restricts into a map

$$\nu : \mathrm{Cent}_R(R^G) \to \mathrm{Cent}_{\mathrm{End}_{R^G}R}(\overline{R}).$$

It is clearly a morphism of rings (with values in a commutative ring). The injectivity is obvious. For the surjectivity, we fix an element $h \in \mathrm{Cent}_{\mathrm{End}_{R^G}R}(\overline{R})$. By the previous lemma, there exists an element $x \in S$ such that $h = \psi_x \in \mathrm{End}_{R^G}R$, satisfying in particular $\psi_x \circ \mu_r = \mu_r \circ \psi_x$ for all $r \in R$. In other words, $x \cdot (rt) - r(x \cdot t) = 0$ for all $r, t \in R$. Using the development $x = \sum_{g \in G} r_g g$, it follows that $\sum_{g \in G} r_g g(rt) - \sum_{g \in G} r r_g g(t) = \sum_{g \in G} (r_g g(r) - r r_g)\, g(t) = 0$; denoting $y_r = \sum_{g \in G} (r_g g(r) - r r_g)\, g \in S$, we obtain $y_r \cdot t = 0$ for all $r, t \in R$, that is $\psi_{y_r} = 0$ in $\mathrm{End}_{R^G}R$, or equivalently $y_r = 0$ in S, for any $r \in R$. As seen previously in the first lemma, this implies that $x = r_1 1_G$ for some $r_1 \in Z(R)$. Therefore $h = \psi_x = \psi_{r_1 1_G} = \mu_{r_1}$. Since $r_1 \in Z(R)$, we have $\mu_{r_1} = \nu_{r_1}$ and the proof of the surjectivity of ν is complete.

So we have proved that $\mathrm{Cent}_R(R^G) \simeq \mathrm{Cent}_S(R)$. We conclude that $Z(R^G) = \mathrm{Cent}_R(R^G) \cap R^G = Z(R) \cap R^G = Z(R)^G$. $\square$

1.3 Invariants of Ore extensions under finite group actions

1.3.1 Iterated Ore extensions

Let A a non necessarily commutative ring. For any $\sigma \in \mathrm{Aut}\, A$, a σ-derivation of A is an additive map $\delta : A \to A$ such that $\delta(\alpha\beta) = \sigma(\alpha)\delta(\beta) + \delta(\alpha)\beta$ for all $\alpha, \beta \in A$.

For any automorphism σ of A and any σ-derivation δ of A, it is a technical elementary exercise to verify that there exists a ring R containing A as a subring and an element $x \in R$ such that R is a free left A-module with basis $\{x^n \,, n \geq 0\}$ and:

$$x\alpha = \sigma(\alpha)x + \delta(\alpha) \quad \text{for any} \quad \alpha \in A. \tag{2}$$

The ring R is called the Ore extension of A defined by σ and δ, and is denoted by $R = A[x\,; \sigma, \delta]$. Any element can be written uniquely as a finite sum $y = \sum_i \alpha_i x^i$ with $\alpha_i \in A$. The addition in R is the ordinary addition of polynomials, and the noncommutative multiplication in R is defined inductively from the commutation law (2). For $y \neq 0$, the nonnegative integer $n = \max\{i \,, \alpha_i \neq 0\}$ is called the degree of y and denoted by $\deg_x y$, and the corresponding α_n is the leading coefficient of y. By convention 0 has degree $-\infty$ and leading coefficient 0. It is clear that, if y, z are two non zero elements of R of respective degrees n, m and leading coefficients α, β, then yz has degree $n+m$ and leading coefficient $\alpha\sigma^n(\beta)$. We deduce in particular that:

if A is a domain, then $A[x\,;\,\sigma,\delta]$ is a domain.

In the particular case where $\delta = 0$, we simply denote $R = A[x\,;\,\sigma]$. The commutation relation becomes:

$$x\alpha = \sigma(\alpha)x \quad \text{for any} \quad \alpha \in A. \tag{3}$$

In the particular case where $\sigma = \mathrm{id}_A$, the map δ is an ordinary derivation of A and we simply denote $R = A[x\,;\,\delta]$. The commutation relation becomes:

$$x\alpha = \alpha x + \delta(\alpha) \quad \text{for any} \quad \alpha \in A. \tag{4}$$

When the coefficient ring A is a field, we have as in the commutative case an euclidian algorithm in $A[x\,;\,\sigma,\delta]$; the proofs of the following two results are straightforward adaptations of their commutative analogues and left to the reader (see for instance [3]).

Proposition. *Let $R = K[x\,;\,\sigma,\delta]$ where K is a non necessarily commutative field, σ is an automorphism of K, and δ is a σ-derivation of K. For any $a, b \in R$, with $b \neq 0$, there exist $q, r \in R$ unique such that $a = qb + r$ with $\deg_x r < \deg_x b$, and there exist $q', r' \in R$ unique such that $a = bq' + r'$ with $\deg_x r' < \deg_x b$.*

Corollary. *For K a non necessarily commutative field, all right ideal and all left ideals of $R = K[x\,;\,\sigma,\delta]$ are principal.*

Examples. Take $A = \mathbb{k}[y]$ the commutative polynomial ring in one variable over a commutative field $\mathbb{k}$.

(i) For $\delta = \partial_y$ the usual derivative, $\mathbb{k}[y][x\,;\,\partial_y]$ is the first Weyl algebra $A_1(\mathbb{k})$, with commutation law $xy - yx = 1$.

(ii) For $\delta = y\partial_y$, $\mathbb{k}[y][x\,;\,y\partial_y]$ is the enveloping algebra $U_1(\mathbb{k})$ of the non abelian two dimensional Lie algebra, with commutation law $xy - yx = y$. Note that $yx = (x-1)y$ and then $U_1(\mathbb{k})$ can also be viewed as $\mathbb{k}[x][y\,;\,\sigma]$ for σ the $\mathbb{k}$-automorphism of $\mathbb{k}[x]$ defined by $x \mapsto x - 1$.

(iii) For $\delta = y^2\partial_y$, $\mathbb{k}[y][x\,;\,y^2\partial_y]$ is the jordanian plane, with homogeneous commutation law $xy - yx = y^2$.

(iv) For σ the $\mathbb{k}$-automorphism of $\mathbb{k}[y]$ defined by $y \mapsto qy$ for some fixed scalar $q \in \mathbb{k}^\times$, $\mathbb{k}[y][x\,;\,\sigma]$ is the quantum plane, denoted by $\mathbb{k}_q[x, y]$, with commutation law $xy = qyx$.

(v) Consider again σ the $\mathbb{k}$-automorphism of $\mathbb{k}[y]$ defined by $y \mapsto qy$ for some fixed scalar $q \in \mathbb{k}^\times, q \neq 1$. The Jackson derivative is the additive map $\delta : \mathbb{k}[y] \to \mathbb{k}[y]$ defined by $\delta(f) = \frac{f(qy)-f(y)}{qy-y}$; it is a σ-derivation. The algebra $\mathbb{k}[y][x\,;\,\sigma,\delta]$ is then the first quantum Weyl algebra, denoted by A_1^q, with commutation law $xy - qyx = 1$.

Starting with a commutative field $\Bbbk$ and the commutative polynomial ring $R_1 = \Bbbk[x_1]$, and considering an automorphism σ_2 and a σ_2-derivation δ_2 of R_1, we can build the Ore extension $R_2 = R_1[x_2\,;\,\sigma_2, \delta_2]$. Taking an automorphism σ_3 and a σ_3-derivation δ_3 of R_2, we consider then $R_3 = R_2[x_3\,;\,\sigma_3, \delta_3]$. Iterating this process, we obtain a so called iterated Ore extension:

$$R_m = \Bbbk[x_1][x_2\,;\,\sigma_2, \delta_2][x_3\,;\,\sigma_3, \delta_3]\cdots[x_m\,;\,\sigma_m, \delta_m]. \tag{5}$$

It is clear from the construction that $\{x_1^{i_1}x_2^{i_2}\ldots x_m^{i_m}\}_{(i_1,i_2,\ldots,i_m)\in\mathbb{N}^m}$ is a left $\Bbbk$-basis of R_m, and that R_m is a domain. We give here some elementary examples (see also 2.1.1 below).

1. The Lie algebra $\mathfrak{sl}_2(\Bbbk)$ is $\Bbbk e \oplus \Bbbk f \oplus \Bbbk h$ with Lie brackets $[h, e] = 2e$, $[h, f] = -2f$ and $[e, f] = h$. By Poincaré-Birkhoff-Witt's theorem, its enveloping algebra $U(\mathfrak{sl}_2)$ admits $(h^i e^j f^k)_{i,j,k\in\mathbb{N}}$ as a left $\Bbbk$-basis. Then $U(\mathfrak{sl}_2) = \Bbbk[h][e\,;\,\sigma'][f\,;\,\sigma, \delta]$, where σ' is the $\Bbbk$-automorphism of $\Bbbk[h]$ defined by $h \mapsto h - 2$, σ is the $\Bbbk$-automorphism of $\Bbbk[h][e\,;\,\sigma']$ defined by $h \mapsto h + 2, e \mapsto e$, and δ is the σ-derivation of $\Bbbk[h][e\,;\,\sigma']$ defined by $\delta(h) = 0$ and $\delta(e) = -h$.

2. The Heisenberg Lie algebra $\mathfrak{sl}_3^+(\Bbbk)$ is $\Bbbk x\oplus\Bbbk y\oplus\Bbbk z$ with Lie brackets $[x, z] = [y, z] = 0$ and $[x, y] = z$. Then $U(\mathfrak{sl}_3^+) = \Bbbk[z][y][x\,;\,\delta]$ for $\delta = z\partial_y$. It can be proved much more generally that the enveloping algebra of any nilpotent Lie algebra of dimension n is an iterated Ore extension on n variables (with $\sigma_1 = \mathrm{id}$ for all i's in the formula 5).

3. Let $Q = (q_{ij})$ a $m \times m$ matrix with entries in $\Bbbk^\times$ such that $q_{ii} = 1$ and $q_{ij} = q_{ji}^{-1}$ for all i, j's. The quantum m-dimensional affine space parameterized by Q is the algebra $\Bbbk_Q[x_1, \ldots, x_m]$ generated over $\Bbbk$ by m generators $x_1, \ldots, x_m$ satisfying the commutation relations $x_ix_j = q_{ij}x_jx_i$. It is the iterated Ore extension:

$$\Bbbk_Q[x_1, \ldots, x_m] = \Bbbk[x_1][x_2\,;\,\sigma_2][x_3\,;\,\sigma_3]\cdots[x_m\,;\,\sigma_m]$$

with σ_i the $\Bbbk$-automorphism of $\Bbbk[x_1][x_2\,;\,\sigma_2,]\cdots[x_{i-1}\,;\,\sigma_{i-1}]$ defined by $\sigma_i(x_j) = q_{ij}x_j$ for any $1 \le j \le i - 1$.

1.3.2 Noetherianity and finiteness of invariants

The important following theorem (from [8]) can be viewed as a noncommutative version of Hilbert's basis theorem (see the historical note of [8] p. 20).

Theorem. *Let A a non necessarily commutative ring, σ an automorphism and δ a σ-derivation of A. If A is right (resp. left) noetherian, then $A[x\,;\,\sigma, \delta]$ is right (resp. left) noetherian.*

Proof. Assume that A is right noetherian. Let J be a non zero right ideal of $R = A[x\,;\,\sigma, \delta]$. We claim that the set L of leading coefficients of elements of J is a right ideal of A.

Take $\alpha, \beta \in L$. If $\alpha + \beta = 0$, we have $\alpha + \beta \in L$ obviously. So we assume $\alpha + \beta \ne 0$. Let $y, z \in J$ of respective degrees $m, n \in \mathbb{N}$ with respective leading

> coefficients α, β. In other words, $y = \alpha x^m + \cdots$ and $z = \beta x^n + \cdots$. If $n \geq m$, then $yx^{n-m} + z = (\alpha + \beta)x^n + \cdots$ lies in J, thus $\alpha + \beta \in L$. If $m > n$, then $y + zx^{m-n} = (\alpha + \beta)x^m + \cdots$ lies in J and $\alpha + \beta \in L$. Now take $\gamma \in A$ such that $\alpha\gamma \neq 0$. We have $y\sigma^{-m}(\gamma) = \alpha\gamma x^m + \cdots$. As $y\sigma^{-m}(\gamma) \in J$, it follows that $\alpha\gamma \in L$. We conclude that L is a right ideal of A.

A being right noetherian, introduce nonzero generators $\alpha_1, \ldots, \alpha_k$ of L as a right ideal of A. For any $1 \leq i \leq k$, let y_i be an element of J with leading coefficient α_i. Denote n_i the degree of y_i and $n = \max\{n_1, \ldots, n_k\}$. Each y_i can be replaced by $y_i x^{n-n_i}$. Hence there is no loss of generality in assuming that $y_1, \ldots, y_k$ all have the same degree n. Set N the left A-submodule of R generated by $1, x, x^2, \ldots, x^n$ (i.e. the set of elements of R whose degree is lower or equal than n). Using the commutation law $\alpha x = x\sigma^{-1}(\alpha) - \delta(\sigma^{-1}(\alpha))$ for any $\alpha \in A$, we observe that N is also the right A-submodule of R generated by $1, x, x^2, \ldots, x^n$. So N is a noetherian right A-module (any right module finitely generated over a right noetherian ring is right noetherian, see the last observation of 1.1.1). It follows that the right A-submodule $J \cap N$ of N is finitely generated, say generated by $z_1, \ldots, z_t$. Thus we have $J \cap N = z_1A + z_2A + \cdots + z_tA$. Set $I = y_1R + y_2R + \cdots + y_kR + z_1R + z_2R + \cdots + z_tR$. We will show that $J = I$.
The inclusion $I \subset J$ is trivial (all y_i and z_j are in the right ideal J of R). For the converse inclusion observe first that, A being a subring of R, we have: $J \cap N = z_1A + z_2A + \cdots + z_tA \subset z_1R + z_2R + \cdots + z_tR \subset I$. Thus I contains all elements of J with degree less than n. We will prove by induction on m that, for any integer $m \geq n$, we have: $\{p \in J\,;\, \deg_x p \leq m\} \subset I$.

> The assertion is right for $m = n$. Assume that it is satisfied up to a rank $m-1 \geq n$. Take $p \in J$ with degree m and leading coefficient α. We have $\alpha \in L$, then there exist $\beta_1, \ldots, \beta_k \in A$ such that $\alpha = \alpha_1\beta_1 + \cdots + \alpha_k\beta_k$. Set $q = \big[y_1\sigma^{-n}(\beta_1) + y_2\sigma^{-n}(\beta_2) + \cdots + y_k\sigma^{-n}(\beta_k)\big]x^{m-n}$, which lies in I by definition of I. Each y_i being of degree n and leading coefficient α_i, the degree of q is m and its leading coefficient is $\alpha_1\beta_1 + \cdots + \alpha_k\beta_k = \alpha$. It follows that $p - q$ is of degree less than m. We have $p \in J$ and $q \in I \subset J$, thus $p - q \in J$ and we can apply the induction assumption to deduce that $p - q \in I$, and then $p \in I$.

So we have proved that $J = I$. Since J was any right ideal of R and I is finitely generated as a right ideal of R, we conclude that R is right noetherian.
Now if A is left noetherian, the opposite ring A^{op} is right noetherian. It is easy to observe that $A[x\,;\,\sigma, \delta]^{\mathrm{op}}$ is isomorphic to $A^{\mathrm{op}}[x\,;\,\sigma^{-1}, -\delta\sigma^{-1}]$. Then the left noetherianity of R follows from the first part of the proof. □

Corollary. *Every iterated Ore extension over a commutative field $\Bbbk$ is a noetherian domain.*

Proof. We have seen in 1.3.1 that $A[x\,;\,\sigma, \delta]$ is a domain when A is a domain. We apply this argument and the previous theorem inductively starting from $\Bbbk$. □

From the previous corollary and theorem 1.1.3, we deduce immediately the following practical result:

Theorem. *Let R be an iterated Ore extension over a commutative field $\Bbbk$. Let G be a finite group of $\Bbbk$-automorphisms of R. We suppose that the order of G is prime with the characteristic of $\Bbbk$. Then R^G is a finitely generated $\Bbbk$-algebra.*

Additional result: *skew Laurent polynomials.* Let A be a ring and $\sigma \in \operatorname{Aut} A$. The ring $S = A[x^{\pm 1}\,;\,\sigma]$ is the set of finite sums $\sum_{i=m}^{p} \alpha_i x^i$ where $m \leq p$ in $\mathbb{Z}$ and $\alpha_i \in A$, with usual addition and noncommutative multiplication defined from relation (3) extended by $x^{-1}\alpha = \sigma^{-1}(\alpha)x^{-1}$ for any $\alpha \in A$.

Proposition. *If A is right(left) noetherian, then $A[x^{\pm 1}\,;\,\sigma]$ is right (left) netherian.*

Proof. It is clear that $R := A[x\,;\,\sigma]$ is a subring of $S := A[x^{\pm 1}\,;\,\sigma]$. Consider a right ideal I of S and denote $J = I \cap R$, which is a right ideal of R. Obviously $JS \subseteq I$. Any element $y \in I$ may be written as $y = \sum_{i=-n}^{n} \alpha_i x^i$ for some $n \geq 0$ and the α_i's in A. Then $yx^n \in J$ and so $y = yx^n x^{-n} \in JS$. Hence we have $I = JS$. The ring A being right noetherian, then so is R be previous theorem, whence J is a finitely generated right ideal of R, and consequently $I = JS$ is a finitely generated right ideal of S. We conclude that S is right noetherian. $\square$

2 Actions and invariants for Weyl algebras

2.1 Polynomial differential operator algebras

2.1.1 Weyl algebras

We fix an integer $n \geq 1$ and a commutative base field $\Bbbk$. Let $S = \Bbbk[q_1, q_2, \dots, q_n]$ be the commutative polynomial algebra in n variables. We denote by $\mathrm{End}_k S$ the $\Bbbk$-algebra of $\Bbbk$-linear endomorphisms of S. The canonical embedding $\mu : S \to \mathrm{End}_k S$ consisting in the identification of any polynomial f with the multiplication μ_f by f in S is a morphism of algebras. We consider in $\mathrm{End}_k S$ the k-vector space $\mathrm{Der}_k S$ consisting of the $\Bbbk$-derivations of S. It is a S-module with basis $(\partial_{q_1}, \partial_{q_2}, \dots, \partial_{q_n})$, where ∂_{q_i} is the usual derivative related to q_i. Then the algebra $\mathrm{Diff}\, S$ of differential operators on S is the subalgebra of $\mathrm{End}_k S$ generated by $\mu_{q_1}, \dots, \mu_{q_n}, \partial_{q_1}, \dots, \partial_{q_n}$. This algebra $\mathrm{Diff}\, S = \mathrm{Diff}\, \Bbbk[q_1, \dots, q_n]$ is called the n-th Weyl algebra over $\Bbbk$, and is denoted by $A_n(\Bbbk)$. For all $d \in \mathrm{Der}_k S$ and $f, h \in S$, the ordinary rule $d(fh) = d(f)h + fd(h)$ can be written $d\mu_f = \mu_f d + \mu_{d(f)}$ in $\mathrm{End}_k S$ or, up to the identification mentioned above, $df - fd = d(f)$. Denoting by p_i the derivative ∂_{q_i}, we obtain the following formal definition of $A_n(\Bbbk)$:

Definition. The Weyl algebra $A_n(\Bbbk)$ is the algebra generated over $\Bbbk$ by $2n$ generators $q_1, \dots, q_n$, $p_1, \dots, p_n$ with relations:

$$[p_i, q_i] = 1, \quad [p_i, q_j] = [p_i, p_j] = [q_i, q_j] = 0 \ \text{ for } \ i \neq j, \tag{6}$$

where $[\,.\,,\,.\,]$ is the canonical commutation bracket (i.e. $[a, b] = ab - ba$ for all $a, b \in A_n(\Bbbk)$). The monomials $(q_1^{i_1} \dots q_n^{i_n} p_1^{j_1} \dots p_n^{j_n})_{(i_1,\dots,i_n,j_1,\dots,j_n) \in \mathbb{N}^{2n}}$ are a $\Bbbk$-left basis of the algebra $A_n(\Bbbk)$, which can be viewed as the iterated Ore extensions:

$$A_n(\Bbbk) = A_{n-1}(\Bbbk)[q_n][p_n\,;\,\partial_{q_n}], \tag{7}$$

$$A_n(\Bbbk) = \Bbbk[q_1, q_2, \dots, q_n][p_1\,;\,\partial_{q_1}][p_2\,;\,\partial_{q_2}] \dots [p_n\,;\,\partial_{q_n}]. \tag{8}$$

It follows in particular that the invertible elements of $A_n(\Bbbk)$ are only the nonzero scalar in $\Bbbk^\times$, and so that any nontrivial automorphism of $A_n(\Bbbk)$ is outer.

Proposition. *If $\Bbbk$ is of characteristic zero, $A_n(\Bbbk)$ is a simple noetherian domain of center $\Bbbk$.*

Proof. By 1.3.2, $A_n(\Bbbk)$ is a noetherian domain independently of the characteristic. Let $a = \sum_{i,j} a_{i,j} q_n^i p_n^j$ be any element of $A_n(\Bbbk)$, with $a_{i,j} \in A_{n-1}(\Bbbk)$. We have:

$$[p_n, a] = \sum_{i,j} i\, a_{i,j}\, q_n^{i-1} p_n^j \qquad \text{and} \qquad [a, q_n] = \sum_{i,j} j\, a_{i,j}\, q_n^i p_n^{j-1} \tag{9}$$

If a is central in $A_n(\Bbbk)$, we have $[p_n, a] = [a, q_n] = 0$. Since $\Bbbk$ is of characteristic zero, we deduce from (9) that a reduces to $a_{0,0}$, and then $a \in A_{n-1}(\Bbbk)$. As a must be central in $A_{n-1}(\Bbbk)$, it follows by induction that $a \in \Bbbk$. Now consider a two-sided ideal I of $A_n(\Bbbk)$ and a a non zero element of I. We must have $aq_n \in I$ and $q_n a \in I$, thus $[a, q_n] \in I$. Similarly, $[p_n, a] \in I$.

Applying (8), we deduce after a finite number of steps that $a_{0,0} \in I$. We repeat the process with the element $a_{0,0}$ in $A_{n-1}(\Bbbk)$, and then inductively up to obtain $1 \in I$. This proves that the only two-sided ideals of $A_n(\Bbbk)$ are (0) and $A_n(\Bbbk)$. $\square$

PROPOSITION. *If $\Bbbk$ is of characteristic zero, then $A_n(\Bbbk)^G$ is a simple noetherian domain of center $\Bbbk$ for any finite subgroup G of* $\operatorname{Aut} A_n(\Bbbk)$.

PROOF. $A_n(\Bbbk)^G$ is simple by point (ii) of theorem 1.2.1 and noetherian by point (i) of theorem 1.2.1 and observation (1) of 1.1.2. Any nonzero central element of $A_n(\Bbbk)^G$ generates a two-sided principal ideal in $A_n(\Bbbk)^G$, so is invertible since $A_n(\Bbbk)^G$ is simple, and then belongs to $\Bbbk$. $\square$

2.1.2 Bernstein filtration

We refer for the results and proofs of this paragraph to [13] or [5]. Let us recall the following well known preliminary notions. Let R be a $\Bbbk$-algebra. We say that R is graded if there exists a sequence $(\mathcal{G}_i)_{i\geq 0}$ of $\Bbbk$-vector spaces satisfying the following two conditions:

$$\text{(i)} \quad R = \textstyle\bigoplus_{i\geq 0} \mathcal{G}_i \ ; \qquad\qquad \text{(ii)} \quad \mathcal{G}_i\mathcal{G}_j \subseteq \mathcal{G}_{i+j}.$$

Each $\mathcal{G}_i$ is called the homogeneous component of degree i of R. The must simple example of commutative graded $\Bbbk$-algebra is the commutative polynomial ring $\Bbbk[x_1, \dots, x_n]$ where the monomials $x_1^{k_1}x_2^{k_2}\dots x_n^{k_n}$ such that $k_1 + k_2 + \cdots + \cdots k_n = i$ form a $\Bbbk$-basis of the homogeneous component of degree i. Similarly the quantum space (see example (iv) and example 3 of 1.3.1) gives an easy noncommutative example.

A family $\mathcal{F} = (\mathcal{F}_i)_{i\geq 0}$ of $\Bbbk$-vector spaces of R is a filtration of R when the following three conditions are satisfied:

$$\text{(i)} \quad \mathcal{F}_0 \subseteq \mathcal{F}_1 \subseteq \mathcal{F}_2 \subseteq \cdots \subseteq R \ ; \qquad \text{(ii)} \quad R = \textstyle\bigcup_{i\geq 0} \mathcal{F}_i \ ; \qquad \text{(iii)} \quad \mathcal{F}_i\mathcal{F}_j \subseteq \mathcal{F}_{i+j}$$

In this case, we consider the $\Bbbk$-vector space $\operatorname{gr}_{\mathcal{F}}(R) := \bigoplus_{i\geq 0}(\mathcal{F}_i/\mathcal{F}_{i-1})$, with convention $\mathcal{F}_{-1} = 0$. In order to make it into a graded algebra, it is enough to define the product on the homogeneous component (then extend by linearity), and we do it by:

$$(x_n + \mathcal{F}_{n-1})(x_m + \mathcal{F}_{m-1}) = x_nx_m + \mathcal{F}_{n+m-1}, \ \text{ for any } n, m \geq 0, x_n \in \mathcal{F}_n, x_m \in \mathcal{F}_m.$$

A straightforward verification shows that $\operatorname{gr}_{\mathcal{F}}(R)$ is a graded $\Bbbk$-algebra whose homogeneous components are the $\mathcal{G}_i = \mathcal{F}_i/\mathcal{F}_{i-1}$. This is called the graded algebra of R associated to the filtration $\mathcal{F}$. Our first important application of this process is for Weyl algebras.

THEOREM. *For any nonnegative integer m, denote by $\mathcal{F}_m$ the $\Bbbk$-vector space generated in $A_n(\Bbbk)$ by monomials $q_1^{i_1}\dots q_n^{i_n}p_1^{j_1}\dots p_n^{j_n}$ such that $i_1 + \cdots + i_n + j_1 + \cdots + j_n \leq m$. Then:*

(i) *$F = (\mathcal{F}_m)_{m\in\mathbb{N}}$ is a filtration of $A_n(\Bbbk)$, called the Bernstein filtration.*

(ii) *The associated graded algebra $\operatorname{gr}_{\mathcal{F}}(A_n(\Bbbk))$ is the commutative polynomial algebra in $2n$ variables over $\Bbbk$.*

Proof. Point (i) is clear. For (ii), we consider the graded algebra $T := \mathrm{gr}_{\mathcal{F}}(A_n(\Bbbk)) = \bigoplus_{i\geq 0}\mathcal{F}_i/\mathcal{F}_{i-1}$ associated to the Bernstein filtration. For any $k\geq 0$, we introduce the canonical surjection $\pi_k : \mathcal{F}_k \to \mathcal{F}_k/\mathcal{F}_{k-1}$. By definition of the product in T, we have:

$$\pi_n(x_n)\pi_m(x_m) = \pi_{n+m}(x_nx_m) \text{ for any } n,m\geq 0, x_n\in\mathcal{F}_n, x_m\in\mathcal{F}_m.$$

In particular, for any monomial $u_k = q_1^{i_1}\dots q_n^{i_n}p_1^{j_1}\dots p_n^{j_n}$ with $i_1+\cdots+i_n+j_1+\cdots+j_n = k$, we have $\pi_k(u_k) = \pi_1(q_1)^{i_1}\dots\pi_1(q_n)^{i_n}\pi_1(p_1)^{j_1}\dots\pi_1(p_n)^{j_n}$. We define in T the $2n$-elements $t_i := \pi_1(q_i)$ and $t_{i+n} := \pi_1(p_i)$ for any $1\leq i\leq n$. Any element of $\mathcal{F}_k/\mathcal{F}_{k-1}$ can be written $\pi_k(x_k)$ for some $x_k\in\mathcal{F}_k$; there exists a monomial u_k of degree k as above such that $x_k = u_k + x_{k-1}$ with $x_{k-1}\in\mathcal{F}_{k-1}$, then $\pi_k(x_k) = \pi_k(u_k) = t_1^{i_1}\dots t_n^{i_n}t_{n+1}^{j_1}\dots t_{2n}^{j_n}$ with $i_1+\cdots+i_n+j_1+\cdots+j_n = k$. We conclude that T is generated by $t_1,\dots,t_{2n}$ as $\Bbbk$-algebra.
Since $[p_i,q_j] = [p_i,p_j] = [q_i,q_j] = 0$ in $A_n(\Bbbk)$ for all $1\leq i\neq j\leq n$, it is clear that $t_it_k = t_kt_i$ in T when $|k-i|\neq n$. Moreover, $p_iq_i = q_ip_i + 1$ implies $\pi_2(p_iq_i) = \pi_2(q_ip_i)$ then $t_i\,t_{i+n} = t_{i+n}\,t_i$ for any $1\leq i\leq n$. It follows that the $\Bbbk$-algebra T is commutative.
So we can consider the surjective morphism of rings $\phi : S := \Bbbk[z_1,\dots,z_{2n}] \to T$, where S is the commutative polynomial algebra in $2n$ variables over $\Bbbk$, defined by $\phi(z_i) = t_i$ for any $1\leq i\leq 2n$. In order to prove the injectivity, we consider $f\in S$ such that $\phi(f) = 0$. Because ϕ maps each z_i to the corresponding t_i, the degrees from S to T are respected by ϕ (i.e. ϕ is a graded morphism) and we can suppose without any restriction that f is homogeneous. We write $f = \sum\lambda_{i,j}\,z_1^{i_1}\dots z_n^{i_n}z_{n+1}^{j_1}\dots z_{2n}^{j_n}$ with $i_1+\cdots+i_n+j_1+\cdots+j_n = k$ and $\lambda_{i,j}\in\Bbbk$ for each monomial in the sum. Defining in $A_n(\Bbbk)$ the corresponding element $g = \sum\lambda_{i,j}\,q_1^{i_1}\dots q_n^{i_n}p_n^{j_1}\dots p_n^{j_n}$, we have:

$$\pi_k(g) = \sum\lambda_{i,j}\,t_1^{i_1}\dots t_n^{i_n}t_{n+1}^{j_1}\dots t_{2n}^{j_n} = \phi(f) = 0.$$

Thus $g\in\mathcal{F}_{k-1}$. But by definition g is a sum of monomials of total degree k, then all the coefficient $\lambda_{i,j}$ above are zero. We conclude that $f = 0$ and ϕ is injective as required. □

Although the Bernstein filtration and associated grading play a main role in many studies about the Weyl algebras (see in particular further the important proposition and theorem in 3.2.3), it could be sometimes usefull to consider other filtrations or graduations:

1. For any integer $r\geq 0$, define $\mathcal{C}_r$ to be the set of elements in $A_n(\Bbbk)$ which can be written as a finite sum $\sum_{j\in\mathbb{N}^n} f_j(q_1,\dots,q_n)\,p_1^{j_1}\dots p_n^{j_n}$ with $j_1+\cdots+j_n\leq r$, where $f_j\in\Bbbk[q_1,\dots,q_n]$. It is easy to prove that $(\mathcal{C}_r)_{r\geq 0}$ is a filtration of $A_n(\Bbbk)$. Note that in particular $\mathcal{C}_0 = \Bbbk[q_1,\dots,q_n]$ is an infinite dimensional $\Bbbk$-vector space (any $\mathcal{F}_i$ is finite dimensional in the case of the Bernstein filtration). This filtration "by the order of the differential operators" can be defined (unlike the Bernstein filtration) for other kinds of differential operator algebras.
2. We consider here the Weyl algebra $A_1(\Bbbk)$, with generators p,q and relation $[p,q] = 1$. For any integer $m\in\mathbb{Z}$, define V_m to be the set of elements in $A_1(\Bbbk)$ which can be written as a finite sum $\sum_{i,j\in\mathbb{N}} f_{i,j}\,p^iq^j$ with $i-j = m$, where $f_{i,j}\in\Bbbk$. In particular $V_0 = \Bbbk[pq]$ contains all monomials p^jq^j with $j\geq 0$ because of the formula:
$$p^jq^j = pq(pq+1)(pq+2)\dots(pq+j-1)$$
For $i\geq j$, we have $p^iq^j = p^{i-j}(p^jq^i)\in V_{i-j}$, and for $j\geq i$, we have $p^iq^j = (p^iq^i)q^{j-i}\in V_{-(j-i)}$. Hence
$$V_0 = \Bbbk[pq],\quad V_m = p^m\Bbbk[pq] \text{ and } V_{-m} = \Bbbk[pq]q^m \text{ for } m\geq 0.$$

Then $A_1(\Bbbk) = \bigoplus_{m\in\mathbb{Z}} V_m$ and, up to a natural extension of the definition, $\mathcal{V} = (V_m)_{m\in\mathbb{Z}}$ is a $\mathbb{Z}$-graduation of $A_1(\Bbbk)$ (see further the second comment at the end of 2.2.2 for an application of it).

2.2 Actions and invariants for A_1

2.2.1 A reminder on Kleinian surfaces

We consider the group $\mathrm{SL}_2(\mathbb{C})$ (briefly denoted by SL_2 if there is no doubt about the base field) and the trivial two dimensional representation $\rho : \mathrm{SL}_2 \to \mathrm{GL}(V)$ defined on a complex vector space $V = \mathbb{C}e_1 \oplus \mathbb{C}e_2$ by

$$\forall\ g = \left(\begin{smallmatrix}\alpha & \beta\\ \gamma & \delta\end{smallmatrix}\right) \in \mathrm{SL}_2,\quad g.e_1 = \alpha e_1 + \gamma e_2 \ \text{ and } \ g.e_2 = \beta e_1 + \delta e_2.$$

It defines on $\mathbb{C}[V] \simeq S(V^*) = \mathbb{C}[x, y]$ with $x = e_1^*$ and $y = e_2^*$ the canonical left action :

$$\forall\ g \in \mathrm{SL}_2,\ \forall\ f \in \mathbb{C}[V],\ \forall\ v \in V,\ (g.f)(v) = f(g^{-1}.v) = f(\rho(g^{-1})(v)). \tag{10}$$

which is equivalent to:

$$\forall\ g = \left(\begin{smallmatrix}\alpha & \beta\\ \gamma & \delta\end{smallmatrix}\right) \in \mathrm{SL}_2,\quad g.x = \delta x - \beta y \ \text{ and } \ g.y = -\gamma x + \alpha y \tag{11}$$

extended by algebra automorphism to any polynomial.
The description of the algebras $\mathbb{C}[x, y]^G$ for G a finite group of SL_2 is a classical topic in algebraic and geometric invariant theory. Let us recall that finite subgroups of SL_2 are classified up to conjugation in five types, two infinite families parameterized by the positive integers (the type A_{n-1} corresponding of the cyclic group of order n and the type D_n corresponding to the binary dihedral group of order $4n$) and three groups E_6, E_7, E_8 of respective orders 24, 48, 120. They can be explicitly described in the following way.
Let us denote $\zeta_n = \exp(2i\pi/n) \in \mathbb{C}$ for any integer $n \geq 1$ and consider in SL_2 the matrices:

$$\theta_n = \begin{pmatrix}\zeta_n & 0\\ 0 & \zeta_n^{-1}\end{pmatrix},\quad \mu = \left(\begin{smallmatrix}0 & i\\ i & 0\end{smallmatrix}\right),\quad \nu = \left(\begin{smallmatrix}0 & 1\\ -1 & 0\end{smallmatrix}\right),\quad \varphi = \begin{pmatrix}-\zeta_5^3 & 0\\ 0 & -\zeta_5^2\end{pmatrix},$$
$$\eta = \tfrac{1}{\sqrt{2}}\begin{pmatrix}\zeta_8^7 & \zeta_8^7\\ \zeta_8^5 & \zeta_8\end{pmatrix},\qquad \psi = \tfrac{1}{\zeta_5^2-\zeta_5^{-2}}\begin{pmatrix}\zeta_5+\zeta_5^{-1} & 1\\ 1 & -(\zeta_5+\zeta_5^{-1})\end{pmatrix}.$$

We define the following subgroups of SL_2:

- type A_{n-1} : the cyclic group C_n, of order n, generated by θ_n,
- type D_n : the binary dihedral group D_n, of order $4n$, generated by θ_{2n} and μ,
- type E_6 : the binary tetrahedral group T, of order 24, generated by θ_4, μ and η,
- type E_7 : the binary octahedral group O, of order 48, generated by θ_8, μ and η,
- type E_8 : the binary icosahedral group I, of order 120, generated by φ, ν and ψ.

Since any finite subgroup G of SL_2 is conjugate to a subgroup G' of these types (then $\mathbb{C}[x, y]^G \simeq \mathbb{C}[x, y]^{G'}$), we can suppose without restriction in the determination of the algebra of invariants $\mathbb{C}[x, y]^G$ for the natural action (11) that G is C_n, D_n, T, O or I. In each case, one can compute (see [17]) a system of three generators f_1, f_2, f_3 of the algebra of invariants $\mathbb{C}[x,y]^G$ for the natural action.

type	generators of $\mathbb{C}[x,y]^G$	equation of $\mathcal{F}$
A_{n-1}	$f_1 = xy, \quad f_2 = x^n, \quad f_3 = y^n$	$X^n - YZ = 0$
D_n	$f_1 = x^2y^2, \quad f_2 = x^{2n} + (-1)^n y^{2n},$ $f_3 = x^{2n+1}y - (-1)^n xy^{2n+1}$	$X^{n+1} + XY^2 + Z^2 = 0$
E_6	$f_1 = xy^5 - x^5y, \quad f_2 = x^8 + 14x^4y^4 + y^8,$ $f_3 = x^{12} - 33x^8y^4 - 33x^4y^8 + y^{12}$	$X^4 + Y^3 + Z^2 = 0$
E_7	$f_1 = x^8 + 14x^4y^4 + y^8, \; f_2 = x^{10}y^2 - 2x^6y^6 + x^2y^{10}$ $f_3 = x^{17}y - 34x^{13}y^5 + 34x^5y^{13} - xy^{17}$	$X^3Y + Y^3 + Z^2 = 0$
E_8	$f_1 = x^{11}y + 11x^6y^6 - xy^{11},$ $f_2 = x^{20} - 228x^{15}y^5 + 494x^{10}y^{10} + 228x^5y^{15} + y^{20},$ $f_3 = x^{30} + 522x^{25}y^5 - 10005x^{20}y^{10} - 10005x^{10}y^{20} - 522x^5y^{25} + y^{30}$	$X^5 + Y^3 + Z^2 = 0$

In all cases, the algebra $\mathbb{C}[x,y]^G = \mathbb{C}[f_1, f_2, f_3]$ appears as the factor of the polynomial algebra $\mathbb{C}[X,Y,Z]$ in three variables by the ideal generated by one irreducible polynomial F (of degree n, $n+1$, 4, 4, 5 respectively). The corresponding surfaces $\mathcal{F}$ of $\mathbb{C}^3$ are the Kleinian surfaces, which are the subject of many geometric, algebraic and homological studies. It is proved in [17] that, for G and G' two groups among the types $A_{n-1}, D_n, E_6, E_7, E_8$, the algebras $\mathbb{C}[x,y]^G$ and $\mathbb{C}[x,y]^{G'}$ are isomorphic if and only if $G = G'$.

2.2.2 Action of SL_2 on the Weyl algebra A_1

We consider now a analogue of the commutative context of 2.2.1 for the first Weyl algebra. Here we take $n = 1$ and $\Bbbk = \mathbb{C}$. We denote simply p for p_1 and q for q_1. Thus, $A_1(\mathbb{C})$ is the algebra generated over $\mathbb{C}$ by p, q with the only relation $[p,q] = 1$.

$$A_1(\mathbb{C}) = \mathbb{C}[q][p\,;\,\partial_q] = \mathbb{C}[p][q\,;\,-\partial_p]. \tag{12}$$

Any element of $\mathrm{SL}_2 = \mathrm{SL}_2(\mathbb{C})$ gives rise to a linear algebra automorphism on $A_1(\mathbb{C})$ defined by:

$$\forall\, g = \left(\begin{smallmatrix} \alpha & \beta \\ \gamma & \delta \end{smallmatrix}\right) \in \mathrm{SL}_2, \quad g(p) = \alpha p + \beta q \;\text{ and }\; g(q) = \gamma p + \delta q. \tag{13}$$

We start with some elementary examples of calculation of $A_1(\mathbb{C})^G$ for G an infinite subgroup of SL_2.

1. For $T = \{\left(\begin{smallmatrix} \alpha & 0 \\ 0 & \alpha^{-1} \end{smallmatrix}\right) \;;\; \alpha \in \mathbb{C}^\times\}$, we have $A_1(\mathbb{C})^T = \mathbb{C}[pq]$.

Proof. Choose $\delta \in \mathbb{C}^\times$ of infinite order and denote by g the automorphism $p \mapsto \delta p$ and $q \mapsto \delta^{-1}q$. For any monomial $\lambda_{i,j}p^iq^j$ with $\lambda_{i,j} \in \mathbb{C}$, we have $g(\lambda_{i,j}p^iq^j) = \lambda_{i,j}\delta^{i-j}p^iq^j$. Then a polynomial $\sum_{i,j}\lambda_{i,j}p^iq^j$ lies in $A_1(\mathbb{C})^T$ if and only if $\lambda_{i,j} = 0$ for $i \neq j$. □

2. For $U = \{\left(\begin{smallmatrix} 1 & \beta \\ 0 & 1 \end{smallmatrix}\right) \ ; \ \beta \in \mathbb{C}\}$, we have $A_1(\mathbb{C})^U = \mathbb{C}[q]$.

PROOF. Choose $\beta \in \mathbb{C}^\times$ and denote by g the automorphism $p \mapsto p + \beta q, q \mapsto q$. Any nonzero polynomial $f \in A_1(\mathbb{C})$ can be written $f = h_m(q)p^m + h_{m-1}(q)p^{m-1} + \cdots + h_0(q)$ with $h_i(q) \in \mathbb{C}[q]$, $h_m \neq 0$. Then $g(f) = h_m(q)(p + \beta q)^m + h_{m-1}(q)(p + \beta q)^{m-1} + \cdots + h_0(q)$. It follows from (12) that $(p+\beta q)^k = p^k + k\beta q p^{k-1} + \cdots$ for any $k \geq 1$. Therefore $g(f) = h_m(q)p^m + [h_{m-1}(q) + m\beta q h_m(q)]p^{m-1} + \cdots$. Supposing $g(f) = f$, we observe by a trivial identification that $m\beta q h_m(q) = 0$. We conclude that $f = h_0(q) \in \mathbb{C}[q]$. $\square$

3. We deduce in particular that $(A_1(\mathbb{C}))^{\mathrm{SL}_2} = \mathbb{C}$.

We consider now the more interesting case of finite subgroups of G. Denoting by $\iota : \mathrm{SL}_2 \hookrightarrow \operatorname{Aut} A_1(\mathbb{C})$ the canonical injection defined by (13), a subgroup of $\operatorname{Aut} A_1(\mathbb{C})$ is said to be linear admissible if it is the image by ι of one of the five types A_{n-1}, D_n, E_6, E_7, E_8 defined in 2.2.1. We can now formulate (from [27]:

THEOREM.

(i) *Any finite subgroup of* $\operatorname{Aut} A_1(\mathbb{C})$ *is conjugate to a linear admissible subgroup.*

(ii) *If G and G' are two linear admissible subgroups of* $\operatorname{Aut} A_1(\mathbb{C})$, *then* $A_1(\mathbb{C})^G \simeq A_1(\mathbb{C})^{G'}$ *if and only if $G = G'$.*

PROOF. It is not possible to give here a complete self contained proof of this theorem, which is based on many non trivial theorems from various papers. We indicate the structure of the main arguments and refer the interested reader to the original articles for further details.

First, we can naturally introduce two kinds of automorphisms of $A_1(\mathbb{C})$. The linear ones (preserving the vector space $\mathbb{C}p \oplus \mathbb{C}q$) correspond to the action (13) of SL_2. The triangular ones are of the form: $p \mapsto \alpha p + \beta, q \mapsto \alpha^{-1} q + f(p)$ with $\alpha \in \mathbb{C}^\times, \beta \in \mathbb{C}, f(p) \in \mathbb{C}[p]$, and form a subgroup denoted by J. It is proved in [40] that $\operatorname{Aut} A_1(\mathbb{C})$ is generated by the subgroups J and SL_2 (in fact the image L of SL_2 by the canonical injection ι). More precisely, it is shown in [19] that $\operatorname{Aut} A_1(\mathbb{C})$ is the amalgamated free products of L and J over their intersection. (i.e. if $g_i \in \mathrm{J} \setminus \mathrm{L}$ and $h_i \in \mathrm{L} \setminus \mathrm{J}$, then $g_1 h_1 g_2 h_2 \dots g_n h_n g_{n+1} \notin \mathrm{L}$). It follows by a theorem of Serre (see [15], théorème 8 p. 53) that any finite subgroup G of $\operatorname{Aut} A_1(\mathbb{C})$ is conjugate either to a subgroup of L or to a subgroup of J.

Suppose now that G is a finite subgroup of J. It acts on $\mathbb{C}p \oplus \mathbb{C}$ fixing $\mathbb{C}$. By semi-simplicity of G (see the lemma below), there exists $p' \in \mathbb{C}p \oplus \mathbb{C}$ such that G stabilizes $\mathbb{C}p'$ and $\mathbb{C}p \oplus \mathbb{C} = \mathbb{C}p' \oplus \mathbb{C}$. Then G acts on $\mathbb{C}q \oplus \mathbb{C}[p] = \mathbb{C}q \oplus \mathbb{C}[p']$ stabilizing $\mathbb{C}[p']$. Again by semi-simplicity of G, there exists $q' \in \mathbb{C}q \oplus \mathbb{C}[p']$ such that G stabilizes $\mathbb{C}q'$ and $\mathbb{C}q \oplus \mathbb{C}[p'] = \mathbb{C}q' \oplus \mathbb{C}[p']$. Denoting by h the triangular automorphism defined by $h(p) = p'$ and $h(q) = q'$, we conclude that $h^{-1}Gh$ acts diagonally on $\mathbb{C}p \oplus \mathbb{C}q$. In particular, G is conjugate to a subgroup of L.

Thus any finite subgroup of J is conjugate to a subgroup of linear automorphisms. Since the finite subgroups of SL_2 are classified up to conjugation in the five types A_n, D_n, E_6, E_7, E_8 (see 2.2.1), point (i) follows. The separation (ii), which cannot be obtained by the standard dimensional invariants, was first proved in [27] by an original method of "reduction modulo p". It can also be obtained from the argument of the second additional comment below. $\square$

In order to be complete, we recall in the following lemma the semi-simplicity argument used in the proof of the proposition.

LEMMA (MASCHKE). *Let $\rho : G \to \mathrm{GL}(V)$ a representation of a finite group G whose order doesn't divide the characteristic of the base field $\Bbbk$, with V a non necessarily finite dimensional vector space. Suppose that $V = W \oplus W_1$ with W and W_1 subspaces such that W is G-stable. Then there exists a G-stable subspace $W_2 \simeq W_1$ such that $V = W \oplus W_2$.*

PROOF. Denote by π the canonical projection $\pi : V \to W$ and define $f : V \to V$ by $f(v) = \frac{1}{|G|}\sum_{g\in G}\rho(g)(\pi(\rho(g)^{-1}(v)))$. Because W is G-stable, we have $f(v) \in W$ for all $v \in V$ and $f(w) = w$ for all $w \in W$. Then $\operatorname{Im} f = W$. An easy calculation shows that $f(\rho(h)(v)) = \rho(h)(f(v))$ for any $h \in G$ and $v \in V$. It follows that $\operatorname{Ker} f$ is G-stable. Then the lemma is proved with $W_2 = \operatorname{Ker} f$. $\square$

• *First additional comment: finite generation of $A_1(\mathbb{C})^G$.* By theorem 1.3.2, $A_1(\mathbb{C})^G$ is a finitely generated $\mathbb{C}$-algebra, and we can ask for explicit generators of $A_1(\mathbb{C})^G$ for any type of admissible G, similarly to the commutative case in 2.2.1.

EXAMPLE: consider the action $p \mapsto \zeta p$, $q \mapsto \zeta^{-1}q$ of the cyclic group C_n on $A_1(\mathbb{C})$, with ζ a primitive n-th root of unity in $\mathbb{C}$. Each monomial p^iq^j being an eigenvector for the action, it is clear that $A_1(\mathbb{C})^{C_n}$ is generated by invariants monomials. We recall now the calculations of the last example of 2.1.2: for $j \geq i$, we write $p^iq^j = (p^iq^i)q^{j-i}$ and observe that p^iq^i is invariant to deduce that $j - i = kn$ for some $k \geq 1$, and then $p^iq^j = (p^iq^i)q^{kn}$. Similarly, $p^iq^j = p^{kn}(p^jq^j)$ if $i > j$. We conclude with the formula $p^jq^j = pq(pq+1)(pq+2)\dots(pq+j-1)$ that $A_1(\mathbb{C})^{C_n}$ is generated by q^n, p^n and pq. This result is formally similar to the first case (type A_{n-1}) of 2.2.1, but we must of course take care that the generators don't commute here. More precisely we have: $pqp^n = p^n(pq-n)$, $q^npq = (pq-n)q^n$, and

$$p^nq^n - q^np^n = \prod_{i=1}^{n}(pq+i-1) - (-1)^n\prod_{i=1}^{n}(-pq+i).$$

We refer to [38] for calculation of generators for each of the five types of admissible G.

• *Second additional comment: dimension of the first Hochschild homology space of $A_1(\mathbb{C})^G$.* For any $\mathbb{C}$-algebra A, we can consider the $\mathbb{C}$-vector space $\mathrm{HH}_0(A) = A/[A,A]$ where $[A,A]$ denote the subspace generated by all brackets $[a,b] = ab-ba$ with $a,b \in A$. The paper [28] proves (using the Morita equivalence of $A_1(\mathbb{C})^G$ with $A_1(\mathbb{C}) \star G$ and a general result from [53] on Hochschild homology of crossed products) that $\dim_{\mathbb{C}} \mathrm{HH}_0(A_1(\mathbb{C})^G) = s(G) - 1$, where $s(G)$ is the number of conjugacy classes in G. Calculating case by case, it follows:

type	A_{n-1}	D_n	E_6	E_7	E_8
$\dim_{\mathbb{C}} \mathrm{HH}_0(A_1(\mathbb{C})^G)$	$n-1$	$n+2$	6	7	8

Example: *a direct elementary proof in the cyclic case.* The cyclic group C_n acts by $p \mapsto \zeta p,\ q \mapsto \zeta^{-1} q$ with ζ a primitive n-th root of unity. The invariant algebra $R_n := A_1(\mathbb{C})^{C_n}$ is generated by pq, p^n, q^n. This action respects the $\mathbb{Z}$-graduation $A_1(\mathbb{C}) = \bigoplus_{m\in\mathbb{Z}} V_n$ where $V_0 = \mathbb{C}[pq]$, $V_m = p^m V_0$ and $V_{-m} = V_0\, q^m$ for any if $m \geq 0$ (see example 2 at the end of 2.1.2). Hence R_n can be decomposed into:

$$R_n = \cdots \oplus V_0\, q^{2n} \oplus V_0\, q^n \oplus V_0 \oplus p^n V_0 \oplus p^{2n} V_0 \oplus \cdots$$

For integers $i \geq 0, k \geq 1$, we have $[p^{kn}(pq)^i\,,\, pq] = kn\, p^{kn}(pq)^i$ and $[pq\,,\,(pq)^i q^{kn}] = kn\,(pq)^i q^{kn}$, thus clearly:

$$(\cdots \oplus V_0\, q^{2n} \oplus V_0\, q^n) \oplus (p^n V_0 \oplus p^{2n} V_0 \oplus \cdots) \subseteq [R_n, R_n].$$

Therefore $\mathrm{HH}_0(R_n) \simeq V_0 \,/\, (\,[R_n, R_n] \cap V_0\,)$ and our aim is to identify $[R_n, R_n] \cap V_0$. Let $v(X) = (X-1)(X-2)\ldots(X-n) \in \mathbb{C}[X]$ and

$$L := \{ f(pq) - f(pq+n)\,;\, f(X) \in v(X)\mathbb{C}[X] \} \ \subseteq\ V_0.$$

We claim that $[R_n, R_n] \cap V_0 = L$. Observe first that a $\mathbb{C}$-basis of L is $(\ell_i)_{i\geq 0}$ where:

$$\ell_i = (pq)^i v(pq) - (pq+n)^i v(pq+n).$$

Since $q^n p^n = v(pq)$ and $p^n q^n = v(pq+n)$ (see for instance [40] p. 216), it follows that $\ell_i = (pq)^i q^n p^n - (pq+n)^i p^n q^n = (pq)^i q^n p^n - p^n (pq)^i q^n = [(pq)^i q^n, p^n]$. Hence $\ell_i \in ([R_n, R_n] \cap V_0)$ for any $i \geq 0$ and so $L \subseteq [R_n, R_n] \cap V_0$.

For the converse inclusion, note theta an arbitrary element of $[R_n, R_n] \cap V_0$ is a sum of commutators of the form:

$$C(f,g,k) = [f(pq)q^{kn}, p^{kn} g(pq)] \quad \text{with } k \geq 1 \text{ and } f, g \in \mathbb{C}[X].$$

We have: $C(f,g,k) = f(pq)q^{kn}p^{kn}g(pq) - p^{kn}g(pq)f(pq)q^{kn} = f(pq)g(pq)q^{kn}p^{kn} - g(pq+kn)f(pq+kn)p^{kn}q^{kn}$. By induction on k from the fundamental identities $q^n p^n = v(pq)$ and $p^n q^n = v(pq+n)$, one checks easily that $q^{kn}p^{kn} = w(pq)$ and $p^{kn}q^{kn} = w(pq+kn)$ for $w(X) = \prod_{i=0}^{k-1} v(X - in)$ which lies in the ideal $v(X)\mathbb{C}[X]$. Hence: $C(f,g,k) = f(pq)g(pq)w(pq) - g(pq+kn)f(pq+kn)w(pq+kn) \ \in\ L.$

Finally $\mathrm{HH}_0(R_n) = V_0/L$ with $V_0 = \mathbb{C}[pq]$ and L the subspace of V_0 with basis $(\ell_i)_{i\geq 0}$ such that $\deg_{pq} \ell_i = n + i - 1$. We conclude that a basis of V_0/L is $\{(pq)^j\}_{0\leq j\leq n-2}$ and the dimension is $n-1$.

• *Third additional comment: $A_1(\mathbb{C})^G$ as a deformation of the kleinian surfaces.* The linear action of the finite group G on the noncommutative algebra $A_1(\mathbb{C})$ induces canonically a linear action on the commutative graded algebra $S = \mathrm{gr}(A_1(\mathbb{C})) = \mathbb{C}[x,y]$ associated to the Bernstein filtration, which is the standard action considered in 2.2.1. We have then $\mathrm{gr}(A_1(\mathbb{C})^G) = S^G$ (see the last proposition of 2.1.1) and therefore the invariant algebra $A_1(\mathbb{C})^G$ can be seen as a noncommutative deformation of the algebra S^G of regular functions on the associated kleinian surface. This point of view will be developed further in 3.2.3.

2.3 Linear actions on A_n

2.3.1 Action of Sp_{2n} on the Weyl algebra A_n

An automorphism g of $A_n(\Bbbk)$ is linear if the $\Bbbk$-vector subspace $W = \Bbbk q_1 \oplus \cdots \oplus \Bbbk q_n \oplus \Bbbk p_1 \oplus \cdots \oplus \Bbbk p_n$ is stable under g. The restriction to W of the commutation bracket in $A_n(\Bbbk)$ defines an alternated bilinear form and relations (6) mean that $\mathcal{B} = (p_1, q_1, p_2, q_2 \ldots, p_n, q_n)$ is a symplectic basis of W. Then it is clear that the group of linear automorphisms of $A_n(\Bbbk)$ is isomorphic to the symplectic group $\mathrm{Sp}_{2n} = \mathrm{Sp}_{2n}(\Bbbk)$. The previous example 2.2.2 is just the case $n = 1$. For finite abelian groups of linear automorphisms and for $\Bbbk = \mathbb{C}$, the following result (from [24]) simplifies the situation in a way which is used as a key argument by many studies of this kind of actions (see [30], [29], [25], and further 5.4.2).

Proposition. *Any finite abelian subgroup of linear automorphisms of $A_n(\mathbb{C})$ is conjugated in Sp_{2n} to a subgroup of diagonal automorphisms.*

More precisely, with the above notations, for any finite abelian subgroup G of Sp_{2n}, there exist a symplectic basis $\mathcal{C} = (x_1, y_1, x_2, y_2, \ldots, x_n, y_n)$ of W and complex characters $\chi_1, \chi_2, \ldots, \chi_n$ of G such that:

$$g(x_j) = \chi_j(g)x_j \quad \text{and} \quad g(y_j) = \chi_j(g)^{-1}y_j, \quad \text{for all } g \in G.$$

Proof. By Schur's lemma and total reducibility (see below), there exists $\mathcal{U} = (u_1, u_2, \ldots, u_{2n})$ a basis of W and complex characters $\varphi_1, \varphi_2, \ldots, \varphi_n$ of G such that $g(u_j) = \varphi_j(g)u_j$ for any $1 \leq j \leq 2n$. Set $\omega_{i,j} = [u_i, u_j]$ for all $1 \leq i, j \leq 2n$. Up to permute the u_i's, one can suppose that $\omega_{1,2} \neq 0$. For any $3 \leq j \leq 2n$, let us define:

$$v_j = \omega_{1,2}u_j - \omega_{j,2}u_1 + \omega_{j,1}u_2.$$

Denote $x_1 = u_1$ and $y_1 = \omega_{1,2}^{-1}u_2$. Then $(x_1, y_1, v_3, v_4, \ldots, v_{2n})$ is a basis of W satisfying $[x_1, y_1] = 1$ and $[x_1, v_j] = [y_1, v_j] = 0$ for any $3 \leq j \leq 2n$. The action of G on this new basis can be described on the following way. It is clear that $g(x_1) = \varphi_1(g)x_1$ and $g(y_1) = \varphi_2(g)y_1$ for any $g \in G$. Since $\omega_{1,2} \neq 0$, we have $\varphi_2(g) = \varphi_1(g)^{-1}$. For $3 \leq j \leq 2n$, it follows from the definition of v_j that:

$$g(v_j) = \varphi_j(g)v_j + \omega_{j,2}\big(\varphi_j(g) - \varphi_1(g)\big)u_1 - \omega_{j,1}\big(\varphi_j(g) - \varphi_2(g)\big)u_2.$$

If $\omega_{j,2} \neq 0$, then $\varphi_j(g) = \varphi_2(g)^{-1} = \varphi_1(g)$. Similarly $\omega_{j,1} \neq 0$ implies $\varphi_j(g) = \varphi_2(g)$. Hence $g(v_j) = \varphi_j(g)v_j$ for any $3 \leq j \leq 2n$. Finally we conclude that the basis $(x_1, y_1, v_3, v_4, \ldots, v_{2n})$ of W satisfies $[x_1, y_1] = 1$ and $[x_1, v_j] = [y_1, v_j] = 0$ for any $3 \leq j \leq 2n$, and that G acts by:

$$g(x_1) = \varphi_1(g)x_1, \quad g(y_1) = \varphi_1(g)^{-1}y_1, \quad g(v_j) = \varphi_j(g)v_j \text{ for } 3 \leq j \leq 2n.$$

We repeat the process with the subspace generated by $v_3, \ldots, v_{2n}$. As W doesn't contain any totally isotropic subspace of dimension $\geq n + 1$, we can iterate this construction n times to obtain the basis $\mathcal{C}$ and the characters $\chi_1 = \varphi_1, \chi_2 = \varphi_3, \ldots, \chi_n = \varphi_{2n-1}$ of the proposition. □

In order to be complete, we recall in the following lemma two classical arguments on representation theory used at the beginning of the proof.

Lemma.

(i) (Total reducibility). *Let $\rho : G \to \mathrm{GL}(V)$ be a representation of a finite group G whose order doesn't divide the characteristic of $\Bbbk$, with V a finite dimensional vector space. Then $V = V_1 \oplus \cdots \oplus V_m$ with V_i G-stable and irreducible (i.e. V_i doesn't admit proper and non zero G-stable subspace) for any $1 \leq i \leq m$.*

(ii) (Schur's lemma). *If $\Bbbk$ is algebraically closed and G is abelian, then any finite dimensional irreducible representation of G is of dimension one.*

Proof. Because V is finite dimensional, (i) just follows from Maschke's lemma (see 2.2.2). For (ii), consider a finite dimensional irreducible representation $\rho : G \to \mathrm{GL}(V)$ of an abelian group G. Fix $s \in G$ and set $t = \rho(s)$. For any $g \in G$, $gs = sg$ implies $\rho(g)t = t\rho(g)$. Let $\lambda \in \Bbbk^*$ be a eigenvalue of t and denote $W = \{v \in V\,;\, t(v) = \lambda v\} \neq (0)$. For any $v \in W$, we have: $t(\rho(g)(v)) = \rho(g)(t(v)) = \rho(g)(\lambda v) = \lambda(\rho(g)(v))$ so $\rho(g)(v) \in W$. Hence W is G-stable and then $W = V$. We have proved: for all $s \in G$, there exists $\lambda \in \Bbbk^*$ such that $\rho(s) = \lambda\,\mathrm{id}_V$. In particular any one-dimensional subspace of V is G-stable. Since V is irreducible, we conclude that V is of dimension one. □

This proposition applies in particular to the subgroup generated by one automorphism of finite order. Under this form, it appears in [29] and [30] as an ingredient for the homological study of $A_n(\mathbb{C})^G$ when G is finite not necessarily abelian (another fundamental ingredient is the Morita equivalence between $A_n(\mathbb{C})^G$ and $A_n(\mathbb{C}) \star G$ by theorem 1.2.1, as $A_n(\mathbb{C})$ doesn't admit nontrivial inner automorphisms). We cannot develop here the elaborate proofs of these papers leading in particular to the following theorem, which describes very precisely the Hochschild (co)homology and Poincaré duality: *for any finite subgroup of linear automorphisms of $A_n(\mathbb{C})$, we have for all nonnegative integer j:*

$$\dim_{\mathbb{C}} \mathrm{HH}_j(A_n(\mathbb{C})^G) = \dim_{\mathbb{C}} \mathrm{HH}^{2n-j}(A_n(\mathbb{C})^G)) = a_j(G)$$

where $a_j(G)$ is the number of conjugacy classes of elements of G which admit the eigenvalue 1 with multiplicity j.

2.3.2 Finite triangular automorphism groups

Let g be an automorphism of $A_n(\Bbbk)$ and suppose that g is triangular with respect of the iterated Ore extension:

$$A_n(\Bbbk) = \Bbbk[q_1][p_1\,;\,\partial_{q_1}][q_2][p_2\,;\,\partial_{q_2}]\ldots[q_n][p_n\,;\,\partial_{q_n}]. \tag{14}$$

By straightforward calculations from relations (6), we can check that g stabilizes in fact any subalgebra $\Bbbk[q_i][p_i\,;\,\partial_{q_i}] \simeq A_1(\Bbbk)$, for $1 \leq i \leq n$, acting on the generators by:

$$g(q_i) = \alpha_i q_i + \gamma_i, \quad g(p_i) = \alpha_i^{-1} p_i + f_i(q_i), \quad \text{with } \alpha_i \in \Bbbk^\times,\ \gamma_i \in \Bbbk,\ f_i \in \Bbbk[q_i]. \tag{15}$$

So, similarly to the semisimplicity argument used in the particular case $n = 1$ in the proof of the main theorem of 2.2.2, we have:

LEMMA. *Any finite subgroup of triangular automorphisms of $A_n(\Bbbk)$ is conjugated in* $\mathrm{Aut}\,(A_n(\Bbbk))$ *to a finite abelian subgroup of diagonal automorphisms.*

PROOF. Let G be a finite subgroup of triangular automorphisms of $A_n(\Bbbk)$. In each subalgebra $\Bbbk[q_i][p_i\,;\,\partial_{q_i}]$, $1 \leq i \leq n$, consider the $\Bbbk$-vector spaces $F_i = \Bbbk \oplus \Bbbk q_i$ and $E_i = \Bbbk[q_i] \oplus \Bbbk p_i$. By (15), G acts on F_i fixing $\Bbbk$ and on E_i stabilizing $\Bbbk[q_i]$. By the semi-simplicity lemma 2.2.2, there exist $y_i \in F_i$ with $F_i = \Bbbk \oplus \Bbbk y_i$ and $x_i \in E_i$ with $E_i = \Bbbk[q_i] \oplus \Bbbk x_i$ such that $\Bbbk y_i$ and $\Bbbk x_i$ are G-stable. By construction, $y_i = \lambda_i q_i + \mu_i$ where $\lambda_i \in \Bbbk^\times$ and $\mu_i \in \Bbbk$. Up to multiply by a nonzero scalar, we can suppose that $x_i = \lambda_i^{-1} p_i + s_i(q_i)$ with $s_i \in \Bbbk[q_i]$. Let h be the triangular automorphism of $A_n(\Bbbk)$ defined by $h(q_i) = y_i$ and $h(p_i) = x_i$ for all $1 \leq i \leq n$. Then $h^{-1}Gh$ acts diagonally on the vectors of the basis $q_1, p_1, \ldots, q_n, p_n$. □

As seen by previous results, some favorable situations reduce to diagonal actions, i.e. actions of subgroups of the torus $(\Bbbk^\times)^n$ by $g(q_i) = \alpha_i q_i$ and $g(p_i) = \alpha_i^{-1} p_i$ with $\alpha_i \in \Bbbk^\times$. This is the most simple case of the following construction.

2.3.3 Dual action of GL_n on the Weyl algebra A_n

We consider here the case of a linear action on $A_n(\Bbbk)$ which extends an action on the polynomial functions by the following classical duality splitting process.

• We start with a vector space V of finite dimension n over $\Bbbk$, $(e_1, \ldots, e_n)$ a $\Bbbk$-basis of V, and $(x_1, \ldots, x_n)$ its dual basis in V^*, $S := \Bbbk[V] \simeq S(V^*) \simeq \Bbbk[x_1, \ldots, x_n]$. Let $\rho : G \to \mathrm{GL}(V)$ be a representation of some group G on V, with the corresponding left action:

$$\forall\; g \in G,\; \forall\; v \in V,\; g.v = \rho(g)(v), \tag{16}$$

extended canonically in an action by automorphisms on S by:

$$\forall\; g \in G,\; \forall\; f \in S,\; \forall\; v \in V,\; (g.f)(v) = f(g^{-1}.v) = f(\rho(g^{-1})(v)). \tag{17}$$

The restriction of this action to the subspace $V^* = \Bbbk x_1 \oplus \Bbbk x_2 \oplus \cdots \Bbbk x_n$ just corresponds to the dual representation of ρ [recall that $\rho^* : G \to \mathrm{GL}(V^*)$ is such that, for any $f \in V^*$, the linear form $\rho^*(g)(f)$ is given by $v \mapsto f(\rho(g^{-1})(v))$].

We put $W = V \oplus V^*$. Any element of W can be written uniquely $w = v + x$ with $v \in V$ and $f \in V^*$. Then we denote $w = (v, f)$. Combining the action (16) of G on V and the associated action (17) on V^*, we define the action:

$$\forall\; g \in G,\; \forall\; w = (v, f) \in W = V \oplus V^*,\; g.w = (g.v, g.f). \tag{18}$$

We define the following bilinear form $q : W \to \Bbbk$:

$$\forall\; (v, f) \in W,\; q(v, f) = f(v). \tag{19}$$

Considering the basis $(e_1, \ldots, e_n, x_1, \ldots, x_n)$ of W and its dual basis $(x_1, \ldots, x_n, \zeta_1, \ldots, \zeta_n)$ in W^*, we claim that

$$q = x_1\zeta_1 + \cdots + x_n\zeta_n \in \Bbbk[W]^G. \tag{20}$$

PROOF. From one hand, by definition of the x_i's and ζ_i's, we have $x_i(v,f) = x_i(v,0) = x_i(v)$ and $\zeta_i(v,f) = \zeta_i(0,f) = \zeta_i(f)$ for all $(v,f) \in W$. It follows that the polynomial function $q' = x_1\zeta_1 + \cdots + x_n\zeta_n$ is a bilinear form $W \to \Bbbk$. For any $1 \leq i,j \leq n$, we have: $q'(e_i,x_j) = \sum_{k=1}^{n} x_k(e_i,x_j)\zeta_k(e_i,x_j)$. Since $x_k(e_i,x_j) = \delta_{i,k}$ and $\zeta_k(e_i,x_j) = \delta_{j,k}$, we obtain $q'(e_i,x_j) = \delta_{i,j} = x_j(e_i) = q(e_i,x_j)$. Using the bilinearity of q and q', this proves that $q' = q$.

From the other hand, for any $g \in G$ and $(v,f) \in W$, we have $q(g.(v,f)) = (g.f)\big((g.v)\big) = f\big(\rho(g^{-1})(\rho(g)(v))\big) = f(v) = q(v,f)$. Therefore $g.q = q$ in $\Bbbk[W]$ for any $g \in G$. $\square$

• We start again with a vector space V of finite dimension n over $\Bbbk$, $(q_1,\ldots,q_n)$ a $\Bbbk$-basis of the dual V^*, $S := \Bbbk[V] \simeq S(V^*) \simeq \Bbbk[q_1,\ldots,q_n]$. As in 2.1.1, we denotes by $\mathrm{End}_k S$ the $\Bbbk$-algebra of $\Bbbk$-linear endomorphisms of S, $\mu : S \to \mathrm{End}_k S$ the canonical embedding defined by the multiplication, $\mathrm{Der}_k S$ the subspace of $\mathrm{End}_k S$ consisting of the $\Bbbk$-derivations of S, and $A_n(\Bbbk) = \mathrm{Diff}\, S$ the subalgebra of $\mathrm{End}_k S$ generated by $\mu_{q_1},\ldots,\mu_{q_n},\partial_{q_1},\ldots,\partial_{q_n}$. Let G be a subgroup of $\mathrm{GL}_n(\Bbbk)$ acting by linear automorphisms on V, via the natural representation $\rho : G \to \mathrm{GL}(V)$. By (17), this action extends canonically in an action by automorphisms on S whose restriction to the subspace $V^* = \Bbbk q_1 \oplus \Bbbk q_2 \oplus \cdots \Bbbk q_n$ just corresponds to the dual representation of ρ. Let us define the application:

$$G \times \mathrm{End}_k S \to \mathrm{End}_k S, \quad (g,\varphi) \mapsto g.\varphi := g\varphi g^{-1}. \tag{21}$$

For any $f \in S$, we have $g.\mu_f = \mu_{g(f)}$. So we obtain an action of G on $\mathrm{End}_k S$ which extends the action on S making covariant the morphism μ. We observe easily that the subspace $\mathrm{Der}_k S$ is stable under this action. We conclude that the restriction to $\mathrm{Diff}\, S$ of the action of G determines an action of G on the Weyl algebra. We claim that the restriction of this action to the vector space $U = \Bbbk\partial_{q_1} \oplus \Bbbk\partial_{q_2} \oplus \cdots \Bbbk\partial_{q_n}$ corresponds to the initial representation ρ.

PROOF. Denote by $(\beta_{i,j})$ the matrix of g^{-1} in the basis $(q_1,\ldots,q_n)$ of V^*. For all $1 \leq i,j \leq n$ and $g \in G$, we compute

$$(g.\partial_{q_i})(q_j) = g\partial_{q_i}g^{-1}(q_j) = g\partial_{q_i}\big(\sum_{m=1}^{n}\beta_{m,j}q_m\big) = \beta_{i,j} = \partial_{q_i}(g^{-1}(q_j)) = \partial_{q_i}(g^{-1}.q_j).$$

By (17), it follows that the action on U is dual to the action on V^*, which is itself dual of the initial action on V. $\square$

In other words, the so-defined action of G on $A_n(\Bbbk)$ is obtained from the linear action of G on S applying the duality splitting process exposed above. In particular, assertion (20) applies. We summarize this results in the following proposition, with the notation $p_i = \partial_{q_i}$.

PROPOSITION. *For any subgroup G of $\mathrm{GL}_n(\Bbbk)$, the action of G by linear automorphisms on $S = \Bbbk[q_1,\ldots,q_n]$ extends in an action by linear automorphisms on the Weyl algebra $A_n(\Bbbk)$ by:*

$$[g(p_i), q_j] = [p_i, g^{-1}(q_j)] \ \textit{ for all } g \in G,\ 1 \leq i,j \leq n, \tag{22}$$

or equivalently

$$g(p_i) = \sum_{j=1}^{n} \partial_{q_i}(g^{-1}(q_j))p_j \quad \textit{for all } g \in G,\ 1 \leq i \leq n. \tag{23}$$

In this action, the element $w = q_1p_1 + q_2p_2 + \cdots + q_np_n$ *lies in* $A_n(\mathbb{k})^G$.

• *First example: diagonal action.* The most simple situation (but interesting as we have seen before) is when G acts as a diagonal subgroup of $\mathrm{GL}_n(\mathbb{k})$, and then acts on $A_n(\mathbb{k})$ as a subgroup of the torus $(\mathbb{k}^\times)^n$ by:

$$g(q_i) = \alpha_i q_i, \quad g(p_i) = \alpha_i^{-1} p_i, \quad \text{with } g = (\alpha_1, \dots, \alpha_n) \in (\mathbb{k}^\times)^n. \tag{24}$$

If $G = (\mathbb{k}^\times)^n$, then $A_n(\mathbb{k})^G = \mathbb{k}[q_1p_1, q_2p_2, \dots, q_np_n]$.

PROOF. Any monomial $y = q_1^{j_1} \dots q_n^{j_n} p_1^{i_1} \dots p_n^{i_n}$ is an eigenvector under the action, and any element of $A_n(\mathbb{k})^G$ is a $\mathbb{k}$-linear combination of invariant monomials. If we choose $g = (\lambda_1, 1, \dots, 1)$ with λ_1 of infinite order in $\mathbb{k}^*$, the relation $g.y = y$ implies $i_1 = j_1$. Proceeding on the same way for all diagonal entries, we obtain $y = (q_1p_1)^{i_1}(q_2p_2)^{i_2} \dots (q_np_n)^{i_n}$. The result follows. □

If G is a finite subgroup of $(\mathbb{k}^\times)^n$ acting so, the invariant algebra $A_n(\mathbb{k})^G$ is finitely generated over $\mathbb{k}$ (by theorem 1.3.2). Since every monomial in the q_i's and p_i's is an eigenvector under the action of G, it's clear that we can find a finite family of $\mathbb{k}$-algebra generators of $A_n(\mathbb{k})^G$ constituted by invariant monomials. The case where $n = 1$ is detailed in the example of the first additional comment of 2.2.2. For $n > 1$, the determination of such a family becomes an arithmetical and combinatorial question depending on the mixing between the actions on the various copies of $A_1(\mathbb{k})$ in $A_n(\mathbb{k})$. We shall solve it completely at the level of the rational functions further in 5.4.2. For the moment, we only give the two following toy illustrations:

Example. For $G = \langle g \rangle$ the cyclic group of order 6 acting on $A_2(\mathbb{C})$ by:

$$g: \ p_1 \mapsto -p_1, \quad q_1 \mapsto -q_1, \quad p_2 \mapsto jp_2, \quad q_2 \mapsto j^2 q_2,$$

$A_2(\mathbb{C})^G$ is generated by $p_1^2, p_1q_1, q_1^2, p_2^3, p_2q_2, q_2^3$.

Example. For $G = \langle h \rangle$ the cyclic group of order 2 acting on $A_2(\mathbb{C})$ by:

$$h: \ p_1 \mapsto -p_1, \quad q_1 \mapsto -q_1, \quad p_2 \mapsto -p_2, \quad q_2 \mapsto -q_2,$$

$A_2(\mathbb{C})^G$ is generated by $p_1^2, p_1q_1, p_1p_2, p_1q_2, q_1^2, q_1p_2, q_1q_2, p_2^2, p_2q_2, q_2^2$.

• *Second example: differential operators over Kleinian surfaces.* We take $\mathbb{k} = \mathbb{C}$, $n = 2$, G a finite subgroup of SL_2 acting on $A_2(\mathbb{C})$ by:

$$\forall\ g = \left(\begin{smallmatrix} \alpha & \beta \\ \gamma & \delta \end{smallmatrix}\right) \in \mathrm{SL}_2, \quad \begin{cases} g(q_1) = \alpha q_1 + \beta q_2, & g(p_1) = \delta p_1 - \gamma p_2, \\ g(q_2) = \gamma q_1 + \delta q_2 & g(p_2) = -\beta p_1 + \alpha p_2. \end{cases} \tag{25}$$

This action is the extension, following the process described at the beginning of this paragraph, of the canonical action (11) on $\mathbb{C}[q_1, q_2]$ (don't mistake with (13) corresponding to the action on $A_1(\mathbb{C})$ described in 2.2.2). Applying theorem 5 from [52] (since G doesn't contain non trivial pseudo-reflections), we have $\mathrm{Diff}(S)^G = A_2(\mathbb{C})^G \simeq \mathrm{Diff}(S^G)$, the differential operator algebra over the Kleinian surface associated to G. As an application of the main results of part 5, we will prove further in 5.4.3 that $A_2(\mathbb{C})^G$ is rationally equivalent to $A_2(\mathbb{C})$.

• *Third example: dual action of the Weyl group on a Cartan subalgebra of a semi-simple complex Lie algebra.* Let $\mathfrak{g}$ a semi-simple Lie algebra of rank ℓ over $\mathbb{C}$ and $\mathfrak{h}$ a Cartan subalgebra. The Weyl group acts by linear automorphisms on $\mathbb{C}[\mathfrak{h}^*] \simeq S(\mathfrak{h})$, and then on $\mathrm{Diff}(\mathfrak{h}^*) \simeq A_\ell(\mathbb{C})$ following the process that we described above. The interested reader could find in [30] homological results and calculations concerning this action.

2.3.4 Non linear actions and polynomial automorphisms

Of course, the classical questions about invariants under subgroups of non necessarily linear automorphisms of a commutative polynomial algebra make sense for noncommutative polynomial algebras. It is not possible to give here a complete survey of the many papers devoted to the determination of such automorphism groups (see for instance the bibliographies of [19], [20], [21], [23], [40], [44],...). With the contents of the following sections in mind, we focus here on the iterated Ore extension in two variables over $\mathbb{C}$, for which we have a complete answer.

Classification lemma. *Let σ be a $\mathbb{C}$-automorphism of $\mathbb{C}[y]$ and δ a σ-derivation of $\mathbb{C}[y]$. Set $R = \mathbb{C}[y][x\,;\sigma,\delta]$. Up to $\mathbb{C}$-isomorphism, we have one and only one of the following five cases.*

(i) $R = \mathbb{C}[x, y]$ *is commutative;*

(ii) *there exists some $q \in \mathbb{C}^\times$, $q \neq 1$, such that $R = \mathbb{C}_q[x, y]$;*

(iii) *there exists some $q \in \mathbb{C}^\times$, $q \neq 1$, such that $R = A_1^q(\mathbb{C})$;*

(iv) *δ is an ordinary k-derivation such that $\delta(y) \notin \mathbb{C}$ and $R = \mathbb{C}[y][x\,;\delta]$;*

(v) $R = A_1(\mathbb{C})$.

Proof. There exists $q \in \mathbb{C}^\times$ and $s \in \mathbb{C}$ such that $\sigma(y) = qy + s$. If $q \neq 1$ we set $y' = y + s(q-1)^{-1}$ and obtain $R = \mathbb{C}[y'][x\,;\sigma,\delta]$ with $\sigma(y') = qy'$ and $\delta(y') = \delta(y) \in \mathbb{C}[y]$. In $\mathbb{C}[y']$ write $\delta(y') = \phi(y')(1-q)y' + r$ with $\phi(y') \in \mathbb{C}[y']$ and $r \in \mathbb{C}$. It follows that $x' = x - \phi(y')$ satisfies $x'y' - qy'x' = r$. Hence $R = \mathbb{C}[y'][x'\,;\sigma,\delta']$ with $\delta'(y') = r \in k$. If $r = 0$, then $R = \mathbb{C}_q[x', y']$. If $r \neq 0$, we set $x'' = r^{-1}x'$ and conclude that $R = A_1^q(\mathbb{C})$. Assume now that $q = 1$. If $s = 0$ then $\sigma = \mathrm{id}$ and $R = \mathbb{C}[y][x\,;\delta]$; we are in case (i) when $\delta = 0$, in case (v) when $\delta(y) \in \mathbb{C}^\times$, and in case (iv) when $\delta \neq 0$. If $s \neq 0$, we set first $y' = s^{-1}y$ to reduce to $R = k[y'][x\,;\sigma,\delta]$ with $\sigma(y') = y' + 1$

and $\delta(y') = s^{-1}\delta(y)$. Then we denote $x' = x + \delta(y')$, which satisfies $x'y' = (y' + 1)x'$, so that $R = \mathbb{C}[y'][x' ; \sigma]$ is the enveloping algebra $U_1(\mathbb{C})$ introduced in example (v) of 1.3.1. We write $U_1(\mathbb{C}) = \mathbb{C}[x'][y' ; -x'\partial_{x'}]$ and are then in case (iv). □

Main observation. *The group* Aut R *is explicitly known in each of the five cases above.*

• The description of the group Aut R for $R = \mathbb{C}[x, y]$ is a classical nontrivial problem. Its structure is very explicitly known. Papers by Jung, Van der Kulk, Rentschler, Makar-Limanov (see [18] for more complete references) led to prove that Aut R is generated by the subgroup $\mathrm{L}(R)$ of linear automorphisms (corresponding to the linear action of GL_2 on $\mathbb{C}x \oplus \mathbb{C}y$) and the subgroup $\mathrm{J}(R)$ of triangular automorphisms (of the form: $y \mapsto \alpha y + \beta, x \mapsto \lambda x + f$ with $\alpha, \lambda \in \mathbb{C}^\times, \beta \in \mathbb{C}, f \in \mathbb{C}[y]$). More precisely, Aut R is the amalgamated free product of $\mathrm{L}(R)$ and $\mathrm{J}(R)$ over their intersection and it follows from a theorem of Serre and a semisimplicity argument (already cited in the proof of theorem 2.2.2) that any finite subgroup of Aut R is conjugate to a subgroup of linear automorphisms.

• The automorphism group of $A_1(\mathbb{C})$ is also described from the works of [40] and [19] as a amalgamated free product of the subgroup of linear automorphisms and the subgroup of triangular ones (see above the proof of theorem in 2.2.2). Is structure is indeed as rich as in the commutative case.

• This is not the case for the quantum plane $\mathbb{C}_q[x, y]$ (with commutation rule $xy = qyx$, see example (iv) of 1.3.1), and for the quantum Weyl algebra $A_1^q(\mathbb{C})$ (with commutation rule $xy - qyx = 1$, see example (v) of 1.3.1). Will shall prove further in 4.1.1 that the automorphism group of the quantum plane $\mathbb{C}_q[x, y]$ is isomorphic to the torus $(\mathbb{C}^\times)^2$ acting by $(\alpha, \beta) : x \mapsto \alpha x,\ y \mapsto \beta y$. And the automorphism group of the quantum Weyl algebra $A_1^q(\mathbb{C})$, reduces to $(\mathbb{C}^\times)$ acting by $\alpha : x \mapsto \alpha x,\ y \mapsto \alpha^{-1}y$, see [21]. In both cases the automorphism group is very “small” which is an example of the general principle of “quantum rigidity”(see further 4.1.3).

• The following proposition (from [23]) solves the remaining case.

Proposition. *Suppose that δ is an ordinary derivation of $\mathbb{C}[y]$ satisfying $\delta(y) \notin \mathbb{C}$. Let p be the non constant polynomial in $\mathbb{C}[y]$ such that $\delta = p\partial_y$. Any automorphism of $R = \mathbb{C}[y][x ; \delta]$ is triangular, of the form:*

$$y \mapsto \alpha y + \beta, \quad x \mapsto \lambda x + f,$$

with $f \in \mathbb{C}[y]$, and $\alpha \in \mathbb{C}^\times$, $\lambda \in \mathbb{C}^\times$, $\beta \in C$ satisfying $p(\alpha y + \beta) = \alpha\lambda p(y)$.

Proof. For any $u \in \mathbb{C}[y]$, we have $xu = ux + p\partial_y(u)$, and then $xp = p.(x + \partial_y(p))$. Thus p is normal in R. It follows that the two-sided ideal I generated by the commutators $[r, s] = rs - sr$ with $r, s \in R$ is the principal ideal generated by $p = [x, y]$. For any automorphism $g \in \mathrm{Aut}\, R$, the element $g(p)$ generates I. So there exists $\varepsilon \in \mathbb{C}^\times$ such that $g(p) = \varepsilon p \in \mathbb{C}[y]$. As $\deg_x g(p) = n \deg_x g(y)$ where $n = \deg_x p \geq 1$ (by assumption), we deduce that $\deg_x g(y) = 0$, therefore $g(y) \in \mathbb{C}[y]$. Hence $g(\mathbb{C}[y]) \subset \mathbb{C}[y]$, and it's clear that there exists $\alpha \in \mathbb{C}^\times, \beta \in \mathbb{C}$ such that $g(y) = \alpha y + \beta$. Then, the surjectivity of g implies that $\deg_x(g(x)) = 1$. So there exist $\lambda \in \mathbb{C}^\times, f \in \mathbb{C}[y]$ such that $g(x) = \lambda x + f$. We have $p(\alpha y+\beta) = g(p) = [g(x), g(y)] = \alpha\lambda p(y)$. □

3 Deformation: Poisson structures on invariant algebras

3.1 Poisson invariant algebras

3.1.1 Basic notions on Poisson structures

We start with the following definition.

Definition. A commutative $\Bbbk$-algebra $\mathcal{A}$ is a Poisson algebra when there exists a bilinear antisymmetric map $\{\cdot,\cdot\}\ :\ \mathcal{A}\times\mathcal{A}\to\mathcal{A}$ satisfying the two conditions:

- Leibniz rule: $\{ab,c\}=a\{b,c\}+\{a,c\}b\quad$ for all $a,b,c\in\mathcal{A}$;

- Jacobi identity : $\{a,\{b,c\}\}+\{b,\{c,a\}\}+\{c,\{a,b\}\}=0\quad$ for all $a,b,c\in\mathcal{A}$.

Then the Poisson bracket $\{\cdot,\cdot\}$ defines a structure of Lie algebra on $\mathcal{A}$ and acts as a biderivation. It's clear that a Poisson bracket on a finitely generated algebra $\mathcal{A}$ is entirely determined by the values of $\{x_i,x_j\}$ for $i<j$ where $x_1,\dots,x_N$ generate $\mathcal{A}$.

Examples.

1. The commutative polynomial algebra in two variables $S=\Bbbk[x,y]$ is a Poisson algebra for the bracket defined on the generators by $\{x,y\}=1$, or equivalently for any $P,Q\in S$:
$$\{P,Q\}=\tfrac{\partial P}{\partial x}\tfrac{\partial Q}{\partial y}-\tfrac{\partial Q}{\partial x}\tfrac{\partial P}{\partial y}=P_1'Q_2'-Q_1'P_2'$$

2. More generally, $S=\Bbbk[x_1,\dots,x_n,y_1,\dots,y_n]$ is a Poisson algebra for the symplectic bracket defined on the generators by $\{x_i,y_j\}=\delta_{i,j}$ and $\{x_i,x_j\}=\{y_i,y_j\}=0$, or equivalently for any $P,Q\in S$:
$$\{P,Q\}=\sum_{i=1}^{n}\tfrac{\partial P}{\partial x_i}\tfrac{\partial Q}{\partial y_i}-\tfrac{\partial Q}{\partial x_i}\tfrac{\partial P}{\partial y_i}.$$

3. Let F be a fixed element of the polynomial algebra in three variables $S=\Bbbk[x,y,z]$; then there exists a Poisson bracket on S defined for any $P,Q\in S$ by :
$$\begin{aligned}\{P,Q\}&=\mathrm{Jac}(P,Q,F)\\&=(P_2'Q_3'-Q_2'P_3')F_1'+(P_3'Q_1'-Q_3'P_1')F_2'+(P_1'Q_2'-Q_1'P_2')F_3'.\end{aligned}$$
The brackets on the generators are then $\{x,y\}=F_3'$, $\{y,z\}=F_1'$, $\{z,x\}=F_2'$.

4. More generally, one can prove (see [34] for complete detailed calculations) that $S=\Bbbk[x_1,\dots,x_N]$ is a Poisson algebra for the Poisson bracket defined (when $N\geq 3$) for any $P,Q\in S$ by: $\{P,Q\}=\mathrm{Jac}(P,Q,F_1,\dots,F_{N-2})$, where $F_1,\dots,F_{N-2}$ are arbitrary chosen polynomials in S.

• *Poisson structure on quotient algebras.* An ideal I of a Poisson algebra $\mathcal{A}$ is a Poisson ideal when $\{a,x\}\in I$ for any $a\in\mathcal{A}, x\in I$; in this case, we also have $\{x,a\}\in I$ and the trivial observation $\{a,b\}-\{a',b'\}=\{a-a',b\}+\{a',b-b'\}$ for all $a,b\in\mathcal{A}$ allows to define on the algebra $\mathcal{A}/I$ the induced bracket $\{\overline{a},\overline{b}\}=\overline{\{a,b\}}$.

• *Poisson structure on localized algebras.* Let $\mathcal{S}$ be a multiplicative set containing 1 in a Poisson algebra $\mathcal{A}$. Then there exists exactly one Poisson bracket $\{\cdot,\cdot\}$ on the localization $\mathcal{S}^{-1}\mathcal{A}$ extending the bracket of $\mathcal{A}$. It is given by:

$$\{as^{-1},bt^{-1}\}=\{a,b\}s^{-1}t^{-1}-\{a,t\}bs^{-1}t^{-2}-\{s,b\}as^{-2}t^{-1}+\{s,t\}abs^{-2}t^{-2}$$

for any $a,b\in\mathcal{A}, s,t\in\mathcal{S}$. In particular, if $\mathcal{A}$ is a domain, the Poisson bracket on $\mathcal{A}$ extends canonically in a Poisson bracket on the field of fractions of $\mathcal{A}$.

• *Poisson structure on invariant algebras.* Let G be a group of algebra automorphisms of a Poisson algebra $\mathcal{A}$. An element $g\in G$ is said to be a Poisson automorphism when $g\{a,b\}=\{g(a),g(b)\}$. If any $g\in G$ is a Poisson automorphism, then the Poisson bracket of two elements of the invariant algebra $\mathcal{A}^G$ also lies in $\mathcal{A}^G$. We say that $\mathcal{A}^G$ is a Poisson subalgebra of $\mathcal{A}$.

3.1.2 Poisson structures on Kleinian surfaces

Main results of this paragraph come from [28] (see also [34]). Let G be a finite subgroup of $\mathrm{SL}_2(\mathbb{C})$ of one of the canonical types A_{n-1}, D_n, E_6, E_7, E_8 acting canonically by (11) on $S=\mathbb{C}[x,y]$.

From one hand, since $G\subset\mathrm{SL}_2$, each element of G is a Poisson automorphism of S for the symplectic bracket defined on S in example 1 of 3.1.1. Therefore S^G is a Poisson subagebra of S for the symplectic Poisson structure.

From the other hand, we have recalled in 2.2.1 that S^G is isomorphic to $\mathbb{C}[X,Y,Z]/(F)$ for some polynomial F irreducible in $\mathbb{C}[X,Y,Z]$ explicitly determined for each of the five types. Let us consider on $\mathbb{C}[X,Y,Z]$ the jacobian Poisson bracket associated to F, in the sense of example 3 of 3.1.1. For any polynomials $P\in\mathbb{C}[X,Y,Z]$ and $QF\in(F)$, we have $\{P,QF\}=\{P,Q\}F+\{P,F\}Q=\{P,Q\}F+\mathrm{Jac}(P,F,F)Q=\{P,Q\}F+0\in(F)$. Then (F) is a Poisson ideal and we can take the induced Poisson structure on $\mathbb{C}[X,Y,Z]/(F)$.

Proposition. *There exists a Poisson isomorphism between $\mathbb{C}[x,y]^G$ for the symplectic Poisson structure and $\mathbb{C}[X,Y,Z]/(F)$ for the jacobian Poisson structure associated to F.*

Proof. With the notations of 2.2.1, $\mathbb{C}[x,y]^G$ is generated by f_1,f_2,f_3 submitted to one relation $F(f_1,f_2,f_3)=0$ for suitable irreducible $F\in\mathbb{C}[X,Y,Z]$. The surjective morphism of algebras $\phi:\mathbb{C}[X,Y,Z]\to\mathbb{C}[x,y]^G$ defined by $X\mapsto f_1, Y\mapsto f_2, Z\mapsto f_3$ induces a surjective morphism $\Phi:\mathbb{C}[X,Y,Z]/(F)\to\mathbb{C}[x,y]^G$ because $\ker\phi\supset(F)$. From classical ringtheoretical results, the Krull dimension of $\mathbb{C}[x,y]^G$ is 2, and the irreducibility of F implies that $\mathbb{C}[X,Y,Z]/(F)$ is also of Krull dimension 2. We conclude that Φ is a algebra isomorphism. The strategy to deduce from Φ a Poisson isomorphism consists in the calculation of three constants $a_1,a_2,a_3\in\mathbb{Q}$ such that the polynomials $h_1=a_1f_1$, $h_2=a_2f_2$ and $h_3=a_3f_3$ in $\mathbb{C}[x,y]^G$ satisfy the relations:

$$\{h_1,h_2\}=F_3'(h_1,h_2,h_3),\quad \{h_2,h_3\}=F_1'(h_1,h_2,h_3),\quad \{h_3,h_1\}=F_2'(h_1,h_2,h_3)\qquad(\star)$$

with $F(h_1, h_2, h_3) = 0$, so that the isomorphism $\Psi : \mathbb{C}[X, Y, Z]/(F) \to \mathbb{C}[x, y]^G$ deduced from the map $X \mapsto h_1, Y \mapsto h_2, Z \mapsto h_3$ becomes a Poisson isomorphism.

The determination of a_1, a_2, a_3 is case by case. For instance, for G of type A_{n-1}, we have $f_1 = xy$, $f_2 = x^n$ and $f_3 = y^n$, with $F = X^n - YZ$ so $F_1' = nX^{n-1}$, $F_2' = -Z$ and $F_3' = -Y$. We compute $\{f_1, f_2\} = -nf_2$, $\{f_2, f_3\} = n^2 f_1^{n-1}$ and $\{f_3, f_1\} = -nf_3$. Setting $h_1 = a_1 f_1$, $h_2 = a_2 f_2$ and $h_3 = a_3 f_3$ and identifying in the above relations ($\star$), we obtain $a_1 = \frac{1}{n}$ and $a_2 a_3 = \frac{1}{n^n}$. Similar (but more complicated) calculations are detailed for each case in [28]. $\square$

We deduce from this proposition an interesting link between the Poisson algebraic structure of the algebra $\mathbb{C}[x, y]^G \simeq \mathbb{C}[X, Y, Z]/(F)$ and some geometrical invariant of the hypersurface $\mathcal{F}$ defined by F in the three dimensional affine space. There exists for any Poisson $\mathbb{C}$-algebra $\mathcal{A}$ a notion of Poisson homology ; the first term of it is just the $\mathbb{C}$-vector space:

$$\mathrm{HP}_0(\mathcal{A}) = \mathcal{A}/\{\mathcal{A}, \mathcal{A}\},$$

where $\{\mathcal{A}, \mathcal{A}\}$ is the subspace generated by all $\{a, b\}$ for $a, b \in \mathcal{A}$. Moreover, the Milnor number of the surface $\mathcal{F}$ is defined as the codimension of the jacobian ideal (i.e. the ideal generated by the derivative polynomials F_1', F_2', F_3'), that is:

$$\mu(\mathcal{F}) = \dim_{\mathbb{C}} \mathbb{C}[X, Y, Z]/(F_1', F_2', F_3').$$

Then we have:

Proposition. *For any finite subgroup of G, the Kleinian surface $\mathcal{F}$ associated to the polynomial $F \in \mathbb{C}[X, Y, Z]$ in the Poisson algebra isomorphism $\mathbb{C}[x, y]^G \simeq \mathbb{C}[X, Y, Z]/(F)$ satisfies the equality :*

$$\dim \mathrm{HP}_0(\mathbb{C}[x, y]^G) = \mu(\mathcal{F}).$$

Proof. Observe first that F is weighted homogeneous : referring to the description of each type of Kleinian surface in 2.2.1 and denoting the total degrees of the three generators $a := \deg f_1$, $b = \deg f_2$ and $c := \deg f_3$, there exists in each case an integer $d \geq 1$ (depending on a, b, c and F) such that:

$$F(\lambda^a X, \lambda^b Y, \lambda^c Z) = \lambda^d F(X, Y, Z), \quad \text{for any } \lambda \in \mathbb{C}. \tag{26}$$

where:

type	A_{n-1}	D_n	E_6	E_7	E_8
$a, b, c, \mathbf{d}$	$2, n, n, \mathbf{2n}$	$4, 2n, 2(n+1), \mathbf{4(n+1)}$	$6, 8, 12, \mathbf{24}$	$8, 12, 18, \mathbf{36}$	$12, 20, 30, \mathbf{60}$

It follows that :

$$a\, X F_1'(X, Y, Z) + b\, Y F_2'(X, Y, Z) + c\, Z F_3'(X, Y, Z) = d\, F(X, Y, Z) \tag{27}$$

Now denote $T := \mathbb{C}[X, Y, Z]$ with the jacobian Poisson bracket defined from

$$\{X, Y\} = F_3',\ \{Y, Z\} = F_1',\ \{Z, X\} = F_2', \tag{28}$$

and $I = (F_1', F_2', F_3')$. Relation (27) implies $(F) \subset I$. Hence $T/I \simeq \overline{T}/\overline{I}$ where $\overline{T} := T/(F) \simeq S^G$ from the previous proposition with notation $S = \mathbb{C}[x, y]$. If we prove that $\{T, T\} + (F) = I$, then $\{\overline{T}, \overline{T}\} = \overline{I}$, thus $\mathrm{HP}_0(S^G) = \overline{T}/\{\overline{T}, \overline{T}\} = \overline{T}/\overline{I} \simeq T/I$ and the proof will be complete.

Inclusion $\{T, T\} + (F) \subset I$ is clear from relations (28) and (27).

To prove the converse inclusion, it's sufficient to check that

$$X^mY^nZ^pF'_i \in \{T,T\}+(F) \quad \text{for any integers } m,n,p\geq 0,\ i=1,2,3. \tag{29}$$

Up to permutation of i, we can take $i=1$. In the case $m=0$, we have from (28) (see example 3 in 3.1.1) the identities $\{Y^{n+1}Z^p,Z\}=\mathrm{Jac}(Y^{n+1}Z^p,Z,F)=(n+1)Y^nZ^pF'_1$ and $\{Y,Y^nZ^{p+1}\}=\mathrm{Jac}(Y,Y^nZ^{p+1},F)=(p+1)Y^nZ^pF'_1$. It follows that:

$$Y^nZ^pF'_1=\tfrac{1}{n+1}\{Y^{n+1}Z^p,Z\}=\tfrac{1}{p+1}\{Y,Y^nZ^{p+1}\}\in\{S,S\}.$$

So we suppose now $m\geq 1$. Applying again the jacobian formula, we have:

$$\{Y,X^mY^nZ^{p+1}\}=(p+1)X^mY^nZ^pF'_1-mX^{m-1}Y^nZ^{p+1}F'_3, \tag{30}$$

$$\{X^mY^{n+1}Z^p,Z\}=(n+1)X^mY^nZ^pF'_1-mX^{m-1}Y^{n+1}Z^pF'_2. \tag{31}$$

From the other hand, Euler's identity (27) implies

$$d\,X^{m-1}Y^nZ\,F=aX^mY^nZ^pF'_1+bX^{m-1}Y^{n+1}Z^pF'_2+cX^{m-1}Y^nZ^{p+1}F'_3. \tag{32}$$

The three relations (30), (31), (32) can be interpreted as a linear system into the three variables $U=X^mY^nZ^pF'_1$, $V=X^{m-1}Y^{n+1}Z^pF'_2$ and $W=X^{m-1}Y^nZ^{p+1}F'_3$ whose determinant $\begin{vmatrix} p+1 & 0 & -m \\ n+1 & -m & 0 \\ a & b & c \end{vmatrix}=-m[c(p+1)+b(n+1)+am]$ doesn't vanish. Then each U,V,W appears as a linear combination of $\{Y,X^mY^nZ^{p+1}\}$, $\{X^mY^{n+1}Z^p,Z\}$ and $X^{m-1}Y^nZ\,F$, so as an element of $\{T,T\}+(F)$, which proves (29) and achieves the proof. □

Corollary. *The values of $\mu(\mathcal{F})$ by type of Kleinian surface are:*

type	A_{n-1}	D_n	E_6	E_7	E_8
$\dim_{\mathbb{C}}\mathrm{HP}_0(\mathbb{C}[x,y]^G)$	$n-1$	$n+2$	6	7	8

Proof. For the type A_{n-1}, we have $F=X^n+YZ$, then the ideal $I=(F'_1,F'_2,F'_3)$ is generated by Y,Z,X^{n-1} ; therefore a $\mathbb{C}$-basis of $\mathbb{C}[X,Y,Z]/I$ is $\{1,\overline{X},\overline{X}^2,\dots,\overline{X}^{n-2}\}$ whose cardinality is $n-1$. For the type D_n, $F=X^{n+1}+XY^2+Z^2$ satisfies $I=((n+1)X^n+Y^2,XY,Z)$; therefore a $\mathbb{C}$-basis of $\mathbb{C}[X,Y,Z]/I$ is $\{1,\overline{X};\overline{X}^2,\dots,\overline{X}^n,\overline{Y}\}$ whose cardinality is $n+2$. For the type E_6, $F=X^4+Y^3+Z^2$ then $I=(X^3,Y^2,Z)$ and a $\mathbb{C}$-basis of $\mathbb{C}[X,Y,Z)/I$ is $\{1,\overline{X},\overline{X}^2,\overline{Y},\overline{YX},\overline{YX}^2\}$ of cardinality 6. The cases E_7 and E_8 are similar with basis $\{1,\overline{X},\overline{Y},\overline{X}Y,\overline{X}^2,\overline{X}^3,\overline{X}^4\}$ and $\{1,\overline{X},\overline{Y},\overline{X}Y,\overline{X}^2,\overline{X}^3,\overline{X}^2Y,\overline{X}^3Y\}$ respectively. □

3.2 Deformations of Poisson algebras

3.2.1 General deformation process

We fix a non necessary commutative $\Bbbk$-algebra B. We suppose that there exists some element h of B which is central in B not invertible and not a zero divisor in B, such that $\mathcal{A}:=B/hB$ is a commutative $\Bbbk$-algebra.

$\mathcal{A}$ being commutative, any $u,v\in B$ satisfy $(u+hB)(v+hB)=(v+hB)(u+hB)$ and then $[u,v]:=uv-vu\in hB$. We denote by $\gamma(u,v)$ the element of B defined by $[u,v]=h\gamma(u,v)$. We set:

$$\{\overline{u},\overline{v}\}=\overline{\gamma(u,v)} \quad \text{for any } \overline{u},\overline{v}\in\mathcal{A}.$$

This is independent of the choice of u,v.

If $u' = u + hw$ with $w \in B$, we have $[u', v] = [u, v] + h[w, v]$ since h is central; thus $h\gamma(u', v) = h\gamma(u, v) + h^2\gamma(w, v)$, then $\gamma(u', v) = \gamma(u, v) + h\gamma(w, v)$ and so $\overline{\gamma(u', v)} = \overline{\gamma(u, v)}$. The result follows by antisymmetry.

This defines a Poisson bracket on $\mathcal{A}$.

Jacobi identity holds for $[\cdot, \cdot]$, thus for $\gamma(\cdot, \cdot)$ because h is central, and then for $\{\cdot, \cdot\}$. Using again the centrality of h, the Leibniz rule for $\{\cdot, \cdot\}$ follows from $[uv, w] = u[v, w] + [u, w]v$ for all $u, v, w \in B$.

Definitions. With the previous data and notation, we say that the noncommutative algebra B is a *quantization* of the Poisson algebra $\mathcal{A}$, and for any $\lambda \in \Bbbk$ such that the central element $h - \lambda$ of B is non invertible in B, the algebra $A_\lambda := B/(h - \lambda)B$ is a *deformation* of the Poisson algebra $\mathcal{A}$.

$$\begin{tikzcd} & B \arrow[dl] \arrow[dr] & \\ \mathcal{A} = B/hB \arrow[ur, dotted, bend left, "\text{quantization}"] \arrow[rr, squiggly, "\text{deformation}"'] & & A_\lambda = B/(h-\lambda)B \end{tikzcd}$$

Examples.

1. Let $\mathfrak{g}$ be a complex finite dimensional Lie algebra. Let B be the homogenized enveloping algebra $U_h(\mathfrak{g})$ of $\mathfrak{g}$, that is B is the $\mathbb{C}[h]$-algebra with generators a basis $\{x_1, \ldots, x_n\}$ of $\mathfrak{g}$ and relations: $x_i x_j - x_j x_i = h\,[x_i, x_j]$. It's clear that B is a quantization of the algebra $\mathcal{A} = \mathbb{C}[x_1, \ldots, x_n] \simeq S(\mathfrak{g}) \simeq \mathcal{O}(\mathfrak{g}^*)$ with the so-called Kirillov-Kostant Poisson bracket defined on the generators by $\{x_i, x_j\} = [x_i, x_j]_{\mathfrak{g}}$, and that the enveloping algebra $U(\mathfrak{g}) \simeq B/(h - 1)B$ is a deformation of $\mathcal{A}$.

$$\begin{tikzcd} & U_h(\mathfrak{g}) \arrow[dl] \arrow[dr] & \\ S(\mathfrak{g}) \simeq \mathcal{O}(\mathfrak{g}^*) \arrow[ur, dotted, bend left, "\text{quantization}"] \arrow[rr, squiggly, "\text{deformation}"'] & & U(\mathfrak{g}) \end{tikzcd}$$

2. In particular, let $\mathfrak{g}$ be the first Heisenberg Lie algebra $\mathfrak{h}_1 = \mathfrak{sl}_3^+$ and B be the homogenized enveloping algebra $U_h(\mathfrak{g})$, that is the $\mathbb{C}[h]$-algebra with generators a basis $\{x, y, z\}$ of $\mathfrak{g}$ with relations: $xy - yx = h\,[x, y] = hz$, $xz - zx = zy - yz = 0$. Then B is a quantization of the commutative algebra $\mathcal{A} = \mathbb{C}[x, y, z] \simeq S(\mathfrak{g})$ whose Poisson structure deduced from the brackets $\{x, y\} = z$ and $\{x, z\} = \{y, z\} = 0$ is of jacobian type (see example 3 of 3.1.1); the enveloping algebra $U(\mathfrak{g}) \simeq B/(h-1)B$ is a deformation of $\mathcal{A}$.

$$\begin{tikzcd}
& U_h(\mathfrak{sl}_3^+) \arrow[dl] \arrow[dr] & \\
\mathbb{C}[x,y,z] \arrow[ur, dotted, bend left, "\text{quantization}"] \arrow[rr, squiggly, "\text{deformation}"'] & & U(\mathfrak{sl}_3^+)
\end{tikzcd}$$

3. Let $B = U(\mathfrak{h}_n)$ be the enveloping algebra of the n-th Heisenberg algebra generated by $x_1, \ldots, x_n, y_1, \ldots, y_n, z$ with $[x_i, y_i] = z$ and $[x_i, y_j] = [x_i, x_j] = [y_i, y_j] = [x_i, z] = [y_i, z] = 0$ for $i \neq j$. This is a quantization of $\mathcal{A} = B/zB = \mathbb{C}[x_1, \ldots, x_n, y_1, \ldots, y_n]$ i.e. $\mathcal{A} \simeq \mathcal{O}(\mathbb{C}^{2n})$ with the Poisson symplectic structure (see second example in 3.1.1). Thus the n-th Weyl algebra $A_n(\mathbb{C}) = B/(z-1)B$ is a deformation of $\mathcal{A}$.

$$\begin{tikzcd}
& U(\mathfrak{h}_n) \arrow[dl] \arrow[dr] & \\
\mathcal{O}(\mathbb{C}^{2n})_{\text{sympl}} \arrow[ur, dotted, bend left, "\text{quantization}"] \arrow[rr, squiggly, "\text{deformation}"'] & & A_n(\mathbb{C})
\end{tikzcd}$$

3.2.2 Algebraic deformation process

We follow in 3.2.2 and 3.2.3 the results and writing of [34]. Let $\mathcal{A}$ be a commutative Poisson $\mathbb{k}$-algebra and R a non necessary commutative $\mathbb{k}$-algebra. By definition, we say that R is an algebraic deformation of $\mathcal{A}$ when there exists a filtration $\mathcal{F} := (\mathcal{F}_n)_{n\geq 0}$ of R with $\mathcal{F}_0 = \mathbb{k}$ [we also use the convention $\mathcal{F}_{-1} = (0)$] satisfying the following two conditions:

(i) $[\mathcal{F}_n, \mathcal{F}_m] \subseteq \mathcal{F}_{n+m-1}$ for all $n, m \geq 0$,

(ii) the associated graded space $\mathrm{gr}_{\mathcal{F}}(R) = \bigoplus_{n\geq 0} \mathcal{F}_n/\mathcal{F}_{n-1}$ is isomorphic to $\mathcal{A}$ as a Poisson algebra, for the product and Poisson bracket defined in $\mathrm{gr}_{\mathcal{F}}(R)$ from the product . and commutation bracket $[\,\cdot\,,\,\cdot\,]$ in R by:

$$(x_n + \mathcal{F}_{n-1})(x_m + \mathcal{F}_{m-1}) = x_n.x_m + \mathcal{F}_{n+m-1},$$
$$\{x_n + \mathcal{F}_{n-1}\,,\, x_m + \mathcal{F}_{m-1}\} = [x_n, x_m] + \mathcal{F}_{n+m-2}$$

Recall that by definition the Rees algebra of R related to the filtration $\mathcal{F}$ is the subalgebra $\mathrm{Rees}_{\mathcal{F}}(R) = \bigoplus_{n\geq 0} \mathcal{F}_n h^n$ in the noncommutative algebra $R[h]$ of polynomials in one central indeterminate h with coefficients in R.

Theorem. *Under the above hypothesis, the algebra $B = \mathrm{Rees}_{\mathcal{F}}(R)$ satisfies the Poisson algebra isomorphism $B/hB \simeq \mathrm{gr}_{\mathcal{F}}(R) \simeq \mathcal{A}$ and the algebra isomorphism $B/(h-1)B \simeq R$. Thus B is a quantization of $\mathcal{A}$ and R is a deformation of $\mathcal{A}$.*

$$\begin{tikzcd}
& \mathrm{Rees}_{\mathcal{F}}(R) \arrow[dl] \arrow[dr] & \\
\mathrm{gr}_{\mathcal{F}}(R) \arrow[ur, dotted, "\text{quantization}"] \arrow[rr, squiggly, "\text{deformation}"'] & & R
\end{tikzcd}$$

Proof. In $B = \bigoplus_{n\geq 0} \mathcal{F}_n h^n \subset R[h]$, the element h is central, non zero divisor and non invertible. The linear map $\varphi : B \to \mathrm{gr}_{\mathcal{F}}(R)$ defined from $x_n h^n \mapsto x_n + \mathcal{F}_{n-1}$ is clearly surjective and a morphism of algebras following the definition (ii) of the product in $\mathrm{gr}_{\mathcal{F}}(R)$. We determine $\ker\varphi$. First we have $\varphi(h) = 1 + \mathcal{F}_0 = 0$ in $\mathrm{gr}_{\mathcal{F}}(R)$ thus $hB \subset \ker\varphi$. Now consider $f = x_0 + x_1 h + \cdots + x_n h^n \in B$ with $x_0, x_1, \ldots, x_n$ be elements of $\mathcal{F}_0, \mathcal{F}_1, \ldots, \mathcal{F}_n$ respectively such that $\varphi(f) = 0$. Then: $x_0 + \mathcal{F}_{-1} = 0,\ x_1 + \mathcal{F}_0 = 0, \ldots, x_n + \mathcal{F}_{n-1} = 0$, that means $x_0 = 0, x_1 \in \mathcal{F}_0, \ldots, x_n \in \mathcal{F}_{n-1}$. Thus $f = h(x_1 + x_2 h + \cdots + x_n h^{n-1})$. We conclude that $\ker\varphi = hB$ and $\widetilde{\varphi} : B/hB \to \mathrm{gr}_{\mathcal{F}}(R)$ is an isomorphism of algebras. By assumption $\mathrm{gr}_{\mathcal{F}}(R) \simeq \mathcal{A}$ so B/hB is commutative.
Moreover for $x_n \in \mathcal{F}_n, x_m \in \mathcal{F}_m$ we have $[x_n h^n, x_m h^m] = [x_n, x_m]h^{n+m} = h\gamma(x_n h^n, x_m h^m)$ with notation $\gamma(x_n h^n, x_m h^m) = [x_n, x_m]h^{n+m-1} = x_{n+m-1}h^{n+m-1}$, where $x_{n+m-1} := [x_n, x_m]$ lies on $\mathcal{F}_{n+m-1}$ because of the hypothesis (i) on the filtration. Thus the Poisson bracket defined on B/hB by general deformation process 3.2.1 is given by $\{x_n h^n + hB, x_m h^m + hB\} = [x_n, x_m]h^{n+m-1} + hB$ those image by $\widetilde{\varphi}$ is no more than $[x_n, x_m] + \mathcal{F}_{n+m-2}$ corresponding to the Poisson bracket defined by hypothesis (ii) in $\mathrm{gr}_{\mathcal{F}}(R)$. We conclude that $\widetilde{\varphi}$ is an Poisson isomorphism and B is a quantization of $\mathrm{gr}_{\mathcal{F}}(R)$.
In order to prove that R is a deformation of $\mathcal{A}$, we consider the linear map $\psi : B \to R$ defined from $x_n h^n \mapsto x_n$. It is clearly surjective and a morphism of algebras. We determine $\ker\psi$. First we have $\psi(h-1) = 1 - 1 = 0$ in R thus $(h-1)B \subset \ker\psi$. Now consider $f = x_0 + x_1 h + \cdots + x_n h^n \in B$ with $x_0, x_1, \ldots, x_n$ be elements of $\mathcal{F}_0, \mathcal{F}_1, \ldots, \mathcal{F}_n$ respectively such that $\psi(t) = 0$. Then: $x_0 + x_1 + \cdots + x_n = 0$, that means $x_n = -x_0 - x_1 - \cdots - x_{n-1}$. Therefore $f = x_0(1-h^n) + x_1 h(1-h^{n-1}) + \cdots + x_{n-1}h^{n-1}(1-h) \in (h-1)B$. We conclude that $\ker\psi = (h-1)B$ and $\widetilde{\psi} : B/(h-1)B \to R$ is an isomorphism of algebras. $\square$

Example. Let $A_n(\Bbbk)$ be the n-th Weyl algebra and $\mathcal{F}$ the Bernstein filtration defined in 2.1.2; it can be proved inductively from relations (6) on the generators that condition $[\mathcal{F}_n, \mathcal{F}_m] \subseteq \mathcal{F}_{n+m-1}$ is satisfied. Applying the algebraic deformation process, $A_n(\Bbbk)$ is a deformation of the Poisson algebra $\mathcal{A} = \mathrm{gr}_{\mathcal{F}}(A_n(\Bbbk))$. We have seen in 2.1.2 that $\mathcal{A} \simeq \mathcal{O}(\Bbbk^{2n}) \simeq \Bbbk[x_1, \ldots, x_n, y_1, \ldots, y_n]$ as an algebra. Morever by relation (ii) the corresponding Poisson bracket in $\mathcal{A}$ calculated on elements of $\mathcal{F}_1$ gives $\{x_i, x_j\} = [p_i, p_j] = 0$, $\{y_i, y_j\} = [q_i, q_j] = 0$ and $\{x_i, y_j\} = [p_i, q_j] = \delta_{i,j}$. Thus the Poisson structure on $\mathcal{A}$ is the symplectic one (see example 2 of 3.1.1).

$$\begin{tikzcd}
& \mathrm{Rees}_{\mathcal{F}}(A_n(\Bbbk)) \arrow[dl] \arrow[dr] & \\
\mathcal{O}(\Bbbk^{2n})_{\mathrm{sympl}} \arrow[ur, dotted, "\text{quantization}"] \arrow[rr, squiggly, "\text{deformation}"'] & & A_n(\Bbbk)
\end{tikzcd}$$

REMARK. The comparison of this diagram with the third example of 3.2.1 gives rise to the natural question asking for the equivalence or not of the two deformation processes, that means for the isomorphism or not of the enveloping algebra $U(\mathfrak{h}_n)$ with the Rees algebra of $A_n(\Bbbk)$ for the Bernstein filtration. We can prove that they are not isomorphic.

PROOF. We write it for A_1 for simplicity but the general case is similar. We take $R = A_1(\Bbbk)$ generated over $\Bbbk$ by p, q with $pq - qp = 1$. The denote by $\mathcal{F} = (\mathcal{F}_n)_{n\geq 0}$ the Bernstein filtration: a $\Bbbk$-basis of $\mathcal{F}_n$ being $(p^i q^j)_{i+j\leq n}$, we deduce that a $\Bbbk$-basis of the associated Rees algebra $B = \bigoplus_{n\geq 0} \mathcal{F}_n h^n \subset R[h]$ is $(p^i q^j h^n)_{i+j\leq n}$. Denoting $u := ph$ and $v := qh$, we can deduce that B is the $\Bbbk$-algebra generated over $\Bbbk$ by three generators u, v, h with relations $[u, h] = [v, h] = 0$ and $[u, v] = h^2$. The two-sided ideal I generated in B by the commutators is the ideal $h^2 B$ and then the abelianized algebra B/I contains the nonzero nilpotent element $\overline{h}$. At the opposite the enveloping algebra of the Heisenberg $\mathfrak{h}_1$ is the $\Bbbk$-algebra $H = U(\mathfrak{h}_1)$ generated over $\Bbbk$ by three generators x, y, z with relations $[x, z] = [y, z] = 0$ and $[x, y] = z$. The two-sided ideal J generated in H by the commutators is the ideal zH and then the abelianized algebra H/J is the polynomial algebra $\Bbbk[\overline{x}, \overline{y}]$. We conclude that $B \not\simeq H$ since their abelianized algebras are not isomorphic. $\square$

Indeed the algebraic deformation process presents useful specific properties in particular when we introduce some group action.

3.2.3 Deformations of invariant algebras

Let R be a non necessary commutative $\Bbbk$-algebra with a filtration satisfying conditions (i) and (ii) of 3.2.2. We consider the associated Poisson algebra $\mathrm{gr}_{\mathcal{F}}(R) = \bigoplus_{n\geq 0} \mathcal{F}_n/\mathcal{F}_{n-1}$. Now we suppose that some finite group G acts on R by automorphisms with respect of the filtration, i.e.

$$g.x_n \in \mathcal{F}_n \quad \text{for all } g \in G, x_n \in \mathcal{F}_n.$$

In this case, G acts naturally on $\mathcal{F}_n/\mathcal{F}_{n-1}$ by: $g.(x_n + \mathcal{F}_{n-1}) = g.x_n + \mathcal{F}_{n-1}$.

LEMMA. *We have the following isomorphism of vector spaces:* $(\mathcal{F}_n/\mathcal{F}_{n-1})^G \simeq \mathcal{F}_n^G/\mathcal{F}_{n-1}^G$.

PROOF. It's clear that $0 \to \mathcal{F}_{n-1} \hookrightarrow \mathcal{F}_n \to \mathcal{F}_n/\mathcal{F}_{n-1} \to 0$ is an exact sequence of G-modules, then the isomorphism $(\mathcal{F}_n/\mathcal{F}_{n-1})^G \simeq \mathcal{F}_n^G/\mathcal{F}_{n-1}^G$ just follows from the fact that $0 \to \mathcal{F}_{n-1}^G \hookrightarrow \mathcal{F}_n^G \to (\mathcal{F}_n/\mathcal{F}_{n-1})^G \to 0$ is an exact sequence, which is the direct application of the following general sublemma. $\square$

SUBLEMMA. *If* $0 \longrightarrow A \xrightarrow{\alpha} B \xrightarrow{\beta} C \longrightarrow 0$ *is an exact sequence of G-modules, then* $0 \longrightarrow A^G \xrightarrow{\alpha'} B^G \xrightarrow{\beta'} C^G \longrightarrow 0$ *is an exact sequence.*

PROOF. Since α and β are morphisms of G-modules, it's clear that $\alpha(A^G) \subset B^G$ and $\beta(B^G) \subset C^G$ and we can consider the restrictions α' and β'. The injectivity of

α' is trivial. It's clear that $\operatorname{Im}\alpha' \subset \operatorname{Im}\alpha \cap B^G$. Conversely, if $b \in B^G$ and $b = \alpha(a)$ for some $a \in A$, then for any $g \in G$ we have: $\alpha(a) = b = g \cdot b = g \cdot \alpha(a) = \alpha(g \cdot a)$ therefore $a = g \cdot a$ by injectivity of α,and so $a \in A^G$. We deduce that $\operatorname{Im}\alpha' = \operatorname{Im}\alpha \cap B^G = \operatorname{Ker}\beta \cap B^G = \operatorname{Ker}\beta'$. The last point is to check the surjectivity of β'. Let $c \in C^G$. By surjectivity of β there exists $b \in B$ such that $c = \beta(b)$. It is clear that $b' := \frac{1}{|G|}\sum_{g\in G} g \cdot b$ lies in B^G and we compute (using at the second equality the fact that β is a morphism of G-module):

$$\beta(b') = \frac{1}{|G|}\sum_{g\in G}\beta(g \cdot b) = \frac{1}{|G|}\sum_{g\in G} g \cdot \beta(b) = \frac{1}{|G|}\sum_{g\in G} g \cdot c = c.$$

We conclude that $c \in \beta(B^G)$, and then β' is surjective. □

Application. Consider the commutative Poisson algebra $S = \Bbbk[x_1, \dots, x_n, y_1, \dots, y_n] \simeq \mathcal{O}(\Bbbk^{2n})$ for the symplectic Poisson bracket (example 2 of 3.1.1) as the algebraic deformation of the Weyl algebra $A_n(\Bbbk)$ for the Bernstein filtration (see above in 3.2.2). Fix a finite subgroup G of the symplectic group $\mathrm{Sp}_{2n}(\Bbbk)$ acting by Poisson automorphisms on S and by automorphisms of noncommutative algebra on $A_n(\Bbbk)$. It's clear that any subspace $\mathcal{F}_n$ in the Bernstein filtration is stable under the action of G, then G acts on each $\mathcal{F}_m/\mathcal{F}_{m-1}$ and the Poisson isomorphism $S \simeq \mathrm{gr}_{\mathcal{F}}A_n(\Bbbk)$ is also a G-module isomorphism. Applying the previous lemma, we deduce:

$$S^G \simeq \left(\mathrm{gr}_{\mathcal{F}}A_n(\Bbbk)\right)^G \simeq \left(\textstyle\bigoplus_{m\geq 0}\mathcal{F}_m/\mathcal{F}_{m-1}\right)^G \simeq \textstyle\bigoplus_{m\geq 0}\left(\mathcal{F}_m/\mathcal{F}_{m-1}\right)^G \simeq \textstyle\bigoplus_{m\geq 0}\mathcal{F}_m^G/\mathcal{F}_{m-1}^G.$$

Therefore the filtration $\widetilde{\mathcal{F}} = (\mathcal{F}_m^G)_{m\geq 0} = (\mathcal{F}_m \cap A_n(\Bbbk)^G)_{m\geq 0}$ of the invariant algebra $A_n(\Bbbk)^G$ is such that $\mathrm{gr}_{\widetilde{\mathcal{F}}}(A_n(\Bbbk)^G) \simeq S^G$ as Poisson algebras. In other words, we have proved:

Proposition. *For any finite subgroup of G of linear automorphisms of $A_n(\Bbbk)$, the action of G induces an action on $\mathrm{gr}_{\mathcal{F}}(A_n(\Bbbk))$, the Bernstein filtration $\mathcal{F}$ induces a filtration $\widetilde{\mathcal{F}}$ of $A_n(\Bbbk)^G$, and we have:*

$$\mathrm{gr}_{\widetilde{\mathcal{F}}}(A_n(\Bbbk)^G) \simeq \mathrm{gr}_{\mathcal{F}}(A_n(\Bbbk))^G.$$

Condition (i) of 3.2.2 being obvious for $\widetilde{\mathcal{F}}$, we can conclude:

Theorem. *For any finite subgroup G of $\mathrm{Sp}_{2n}(\Bbbk)$ acting by Poisson automorphisms on $S = \Bbbk[x_1, \dots, x_n, y_1, \dots, y_n] \simeq \mathcal{O}(\Bbbk^{2n})$ and by automorphisms of noncommutative algebra on $A_n(\Bbbk)$, the invariant algebra $A_n(\Bbbk)^G$ is an algebraic deformation of the Poisson algebra S^G for the filtration induced by the Bernstein filtration.*

$$\begin{tikzcd}
\mathcal{O}(\Bbbk^{2n}) \arrow[r, squiggly, "\text{deformation}"'] & A_n(\Bbbk) \\
\mathcal{O}(\Bbbk^{2n})^G \arrow[u, hook] \arrow[r, squiggly, "\text{deformation}"'] & A_n(\Bbbk)^G \arrow[u, hook]
\end{tikzcd}$$

Remark. Suppose here that $n = 1$, $\Bbbk = \mathbb{C}$ and G a finite subgroup of SL_2. We consider in the affine space $V = \mathbb{C}^2$ the quotient variety $V|\,G$, so that $\mathcal{O}(V|\,G) \simeq \mathcal{O}(V)^G$. With our usual notations, $\mathcal{O}(V) = \mathbb{C}[x, y] = S$, the above deformation picture can be completed by homological considerations. Comparing the values of $\dim_{\mathbb{C}} \mathrm{HH}_0(A_1(\mathbb{C})^G)$ (see 2.2.2) and $\dim_{\mathbb{C}} \mathrm{HP}_0(S^G)$ (see 3.1.2), we just observe a vector space automorphism:

$$\mathrm{HH}_0(A_1(\mathbb{C})^G) \simeq \mathrm{HP}_0(S^G).$$

It can be interpreted as a deformation process at the level of the homological trace groups: considering the vector spaces $\mathcal{G}_m = \mathcal{F}_m^G / \left(\mathcal{F}_m^G \cap [A_1(\mathbb{C})^G, A_1(\mathbb{C})^G]\right)$, it is proved in [28] that $\mathrm{HP}_0(S^G) = \mathrm{gr}_{\mathcal{G}} \mathrm{HH}_0(A_1(\mathbb{C})^G) := \bigoplus_{m \geq 0} \mathcal{G}_m / \mathcal{G}_{m-1}$.

$$\begin{tikzcd}
\mathbb{C}[x,y] \arrow[r, rightsquigarrow, "\text{deformation}"'] & A_1(\mathbb{C}) \\
\mathbb{C}[x,y]^G \arrow[u, hook] \arrow[r, rightsquigarrow, "\text{deformation}"'] \arrow[d, dotted] & A_1(\mathbb{C})^G \arrow[u, hook] \arrow[d, dotted] \\
\mathrm{HP}_0(\mathbb{C}[x,y]^G) \arrow[r, rightsquigarrow, "\text{deformation}"'] & \mathrm{HH}_0(A_1(\mathbb{C})^G)
\end{tikzcd}$$

3.3 Lie structure on invariant algebras

3.3.1 Finiteness of the Lie structure on Poisson symplectic spaces

Preliminary. The results of section 3.3 come from [34]. We start with the basic situation where S is the commutative Poisson algebra $\mathbb{C}[x,y]$ with the symplectic Poisson bracket defined from $\{x,y\} = 1$. The generators x and y act on S by derivations:

$$\{\, x\,,\, \cdot\,\} = \partial_y \qquad \text{and} \qquad \{\, -y\,,\, \cdot\,\} = \partial_x. \tag{33}$$

We consider in the homogeneous component $S_2 = \mathbb{C}x^2 \oplus \mathbb{C}xy \oplus \mathbb{C}y^2$ of degree 2 in S the three elements: $e = \frac{1}{2}x^2$, $f = -\frac{1}{2}y^2$ and $h = -xy$, which act on S by Euler derivations:

$$\{\, e\,,\, \cdot\,\} = x\,\partial_y, \quad \{\, f\,,\, \cdot\,\} = y\,\partial_x \quad \text{and} \quad \{\, h\,,\, \cdot\,\} = x\,\partial_x - y\,\partial_y. \tag{34}$$

In particular: $[e,f] = h$, $[h,e] = 2e$ and $[h,f] = -2f$; hence $S_2 = \mathbb{C}e \oplus \mathbb{C}f \oplus \mathbb{C}h$ is a Lie subalgebra of $(S, \{\,\cdot\,,\,\cdot\,\})$, isomorphic to $\mathfrak{s}l_2(\mathbb{C})$. We introduce $V := \mathbb{C}x \oplus \mathbb{C}y$ and the subspace $F_2 = \mathbb{C} \oplus V \oplus S_2$ of elements of total degree ≤ 2 in S. It is clear that F_2 is a Lie subalgebra of S for the Lie structure defined by the Poisson bracket.

Proposition: *The Lie algebra $\mathbb{C}[x,y]$ for the symplectic bracket is finitely generated, and the Lie subalgebra F_2 is maximal.*

Proof. Let $\mathfrak{g}$ be a Lie subalgebra of S containing F_2 (i.e. $1, x, y, x^2, xy, y^2 \in \mathfrak{g}$) and any other element q of S of total degree ≥ 3 (ie $q \in \mathfrak{g}$ and $q \notin F_2$). Then $\{x,q\} \in \mathfrak{g}$ and $\{y,q\} \in \mathfrak{g}$; so applying many times the hamiltonian derivations (33), we obtain an element $p \in \mathfrak{g}$ of total degree 3. Let us denote:

$$p = \alpha x^3 + \beta x^2 y + \gamma x y^2 + \delta y^3 \ \text{ with } \alpha, \beta, \gamma, \delta \in \mathbb{C},\ (\alpha, \beta, \gamma, \delta) \neq (0,0,0,0).$$

Now the elements e, f, h lie in $\mathfrak{g}$ and then by (34): $\{e,p\} = \beta x^3 + 2\gamma x^2 y + 3\delta y^2 x \in \mathfrak{g}$, $\{e,\{e,p\}\} = 2\gamma x^3 + 6\delta x^2 y \in \mathfrak{g}$, and $\{e,\{e,\{e,p\}\}\} = 12\delta x^3 \in \mathfrak{g}$. We deduce that $x^3 \in \mathfrak{g}$. Applying to p the action of $\{f,\cdot\}$ we obtain similarly $y^3 \in \mathfrak{g}$. It follows that $x^2y = \frac{1}{3}\{f, x^3\}$ and $xy^2 = \frac{1}{3}\{e, y^3\}$ are also elements of $\mathfrak{g}$. Hence $\mathfrak{g}$ contains the homogeneous component S_3 of degree 3 in S. In particular $\{x^3, y^3\} = 9x^2y$ lies in $\mathfrak{g}$. By iterated application of $\{e,\cdot\}$ and $\{f,\cdot\}$, we conclude that $x^3y, x^4, xy^3, y^4 \in \mathfrak{g}$ and so $\mathfrak{g}$ contains the homogeneous component S_4. Suppose by induction that $S_n \in \mathfrak{g}$ for some $n \geq 3$. Then $x^2y^{n-1} = \frac{1}{3n}\{x^3, y^n\} \in \mathfrak{g}$ which implies by applications of $\{e,\cdot\}$ and $\{f,\cdot\}$ that S_{n+1}. We conclude that $\mathfrak{g} \supseteq \bigoplus_{n\geq 0} S_n = S$ and $S = \mathfrak{g}$ is generated by $1, x, y, x^2, xy, y^2$ and any element of degree ≥ 3 in S. □

This result can be extended to the dimension $2n$ and precised by a reduction to only two generators. This the purpose of the following theorem.

THEOREM. *The Lie algebra $\mathbb{C}[x_1,\dots,x_n,y_1,\dots,y_n]$ for the symplectic Poisson bracket is generated by the two elements:*

$$H = -\sum_{i=1}^{n} x_i y_i$$

$$T = x_1y_1x_2y_2\dots x_ny_n + \sum_{i=1}^{n} x_i + \sum_{i=1}^{n} y_i + \sum_{i=1}^{n}(x_i^{i+2} + y_i^{i+2})$$

PROOF. The adjoint action of H is given by $\{H,\cdot\} = \sum_{i=1}(x_i\partial_{x_i} - y_i\partial_{y_i})$. The element T appears as a sum of eigenvectors for this action:

$$\begin{aligned}
&X := \sum_{i=1}^{n} x_i \text{ and } Y := \sum_{i=1}^{n} y_i \quad \text{satisfy} \quad \{H,X\} = X \ \text{ and } \ \{H,Y\} = -Y,\\
&Z := x_1y_1x_2y_2\dots x_ny_n \qquad \text{satisfies} \quad \{H,Z\} = 0,\\
&X_i := x_i^{i+2} \text{ and } Y_i := y_i^{i+2} \quad \text{satisfy} \quad \{H,X_i\} = (i+2)X_i \ \text{ and } \ \{H,Y_i\} = -(i+2)Y_i.
\end{aligned}$$

The iterated adjoint action of H on T produces a linear $(2n+3)\times(2n+3)$ system:

$$\begin{array}{rcccccccc}
T = X & + & Y & + & \sum_{i=1}^{n} X_i & + & \sum_{i=1}^{n} Y_i & + & Z\\
\{H,T\} = X & + & (-1)Y & + & \sum_{i=1}^{n}(i+2)X_i & + & \sum_{i=1}^{n}(-i-2)Y_i & + & 0\\
\{\{H,T\}\} = X & + & (-1)^2Y & + & \sum_{i=1}^{n}(i+2)^2X_i & + & \sum_{i=1}^{n}(-i-2)^2Y_i & + & 0\\
\dots & & & & \dots & & & & \\
(\operatorname{ad} H)^{2n+2}(T) = X & + & (-1)^{2n+2}Y & + & \sum_{i=1}^{n}(i+2)^{2n+2}X_i & + & \sum_{i=1}^{n}(-i-2)^{2n+2}Y_i & + & 0
\end{array}$$

whose determinant is a nonzero Vandermonde determinant. Hence the system is invertible and any element of the family $\mathcal{M} := \{X, Y, X_1,\dots,X_n,Y_1,\dots,Y_n,Z\}$ can be expressed as a linear combination of the brackets $(\operatorname{ad} H)^j(T)$ for $0 \leq j \leq 2n+2$. Thus each vector of $\mathcal{M}$ lies in the Lie subalgebra $\mathfrak{g}$ of $S := \mathbb{C}[x_1,\dots,x_n,y_1,\dots,y_n]$ generated by H and T.

Moreover, $(\operatorname{ad} X)$ acts as $\sum_{i=1}^{n}\partial_{y_i}$ and $(\operatorname{ad}(-Y))$ as $\sum_{i=1}^{n}\partial_{x_i}$ then we have for $1 \leq i \leq n$:

$$(\operatorname{ad}(-Y))^i(X_i) = \tfrac{1}{2}(i+2)!\,x_i^2 \quad \text{and} \quad (\mathrm{adX}\,(Y))^i(Y_i) = \tfrac{1}{2}(i+2)!\,y_i^2,$$

which imply that $x_1^2,\dots,x_n^2,y_1^2,\dots,y_n^2 \in \mathfrak{g}$. Applying again $(\operatorname{ad}(-Y))$ and $(\operatorname{ad} X)$, we deduce that $\mathfrak{g}$ contains $x_1,\dots,x_n,y_1,\dots,y_n$. Hence $\mathfrak{g}$ contains the homogeneous component of degree one $S_1 = V = \mathbb{C}x_1 \oplus\dots\oplus \mathbb{C}x_n \oplus \mathbb{C}y_1 \oplus\dots\oplus \mathbb{C}y_n$. By iteration of appropriate $(\operatorname{ad} x_i)$ and $(\operatorname{ad} y_i)$ acting on the product Z, it follows that any monomial with factors x_i and y_i appearing only with exponent one lies on $\mathfrak{g}$.

In particular, $\mathfrak{g}$ contains all monomials x_ix_j, y_iy_j for $1 \leq i \neq j \leq n$ and x_iy_j for $1 \leq i, j \leq n$, in addition of the squares x_i^2 and y_i^2. Thus, $\mathfrak{g}$ contains the homogeneous component of degree two:

$$S_2 = \bigoplus_{1\leq i\leq n} \mathbb{C}x_i^2 \oplus \bigoplus_{1\leq i\leq n} \mathbb{C}y_i^2 \oplus \bigoplus_{1\leq i,j\leq n} \mathbb{C}x_iy_j \oplus \bigoplus_{1\leq i\neq j\leq n} \mathbb{C}x_ix_j \oplus \bigoplus_{1\leq i\neq j\leq n} \mathbb{C}y_iy_j$$

For degree three, $\mathfrak{g}$ contains similarly all monomials $x_ix_jx_k$ and $y_iy_jy_k$ for pairwise distinct i, j, k, and all monomials $x_ix_jy_k$ and $x_ky_iy_j$ for $i \neq j$. From the relations

$$(\mathrm{ad}(-Y))^{i-3}(X_i) = \tfrac{1}{6}(i+2)!\, x_i^3 \quad \text{and} \quad (\mathrm{ad}X)^{i-3}(Y_i) = \tfrac{1}{6}(i+2)!\, y_i^3$$

we deduce that $x_i^3, y_i^3 \in \mathfrak{g}$ for any $1 \leq i \leq n$. The calculation of $\{x_ix_j, y_j^3\}$ implies $x_iy_j^2 \in \mathfrak{g}$ (for $i \neq j$ or for $i = j$). Finally it follows from $\{x_iy_j, x_j^3\} = -3x_ix_j^2$ that $x_ix_j^2 \in \mathfrak{g}$ for $i \neq j$. Similarly $x_i^2y_j \in \mathfrak{g}$ and $y_iy_j^2 \in \mathfrak{g}$. We conclude that the homogeneous component S_3 of degree 3 is a subspace of $\mathfrak{g}$.

Suppose now by induction that $\mathfrak{g}$ contains the homogeneous component S_{m-1} of S for some $m \geq 4$. Take any monomial $f = x_1^{a_1} \cdots x_n^{a_n} y_1^{b_1} \cdots y_n^{b_n} \in S_m$. If all exponents are ≤ 1, then f can be obtained from $Z \in \mathfrak{g}$ by application of appropriate $\mathrm{ad}(x_i)$ and $\mathrm{ad}(y_j)$ and therefore $f \in \mathfrak{g}$. If at least one exponent is ≥ 2, we can suppose without any restriction that $a_1 \geq 2$, and then $f = \frac{1}{3}\{x_1^3, h\} = x_1^2\partial_{y_1}(h)$ with $h = x_1^{a_1-2} \cdots x_n^{a_n} y_1^{b_1+1} \cdots y_n^{b_n} \in S_{m-1}$. The result follows by induction. □

Remark. It follows from the description of the homogeneous component S_2 detailed in the proof above that $\dim_{\mathbb{C}} S_2 = n(2n+1)$. It is also obvious in this direct sum that the Poisson bracket of two generators lies in S_2. Hence S_2 is a Lie subalgebra of S. It can be proved by a straightforward verification that S_2 is isomorphic to the symplectic Lie algebra $\mathfrak{sp}_{2n}(\mathbb{C})$.

3.3.2 Finiteness of the Lie structure on Kleinian surfaces

The algebra $S = \mathbb{C}[x_1, \ldots, x_n, y_1, \ldots, y_n] = \mathcal{O}(V)$ of regular functions on the symplectic space V of dimension $2n$ is finitely generated as a Lie algebra for the Poisson structure. It is then a natural question (in the continuity of Noether's theorem about the ring structure or its noncommutative analogues) to ask whether the invariant Poisson subalgebra $S^G \simeq \mathcal{O}(V|G)$ under the action of a finite subgroup of Sp_{2n} (see theorem 3.2.3) is also finitely generated as a Lie subalgebra, and whether it is also the case for its deformation $A_n(\mathbb{C})^G$. We give a positive answer in the case of Kleinian surfaces (i.e. $n = 1$).

We fix G a finite subgroup of $\mathrm{SL}_2(\mathbb{C})$ acting linearly by automorphisms on $S = \mathbb{C}[x, y]$; we consider the Poisson isomorphism $S^G \simeq \mathbb{C}[X, Y, Z]/(F)$ proved in 3.1.2, where F is an irreducible element of $\mathbb{C}[X, Y, Z]$ defining the corresponding Kleinian surface, see 2.2.1.

Denoting by a, b, c the total degrees in S of the three homogeneous generators f_1, f_2, f_3 of S^G given in 2.2.1, we define the weight of a monomial by $w(X^iY^jZ^k) = ai + bj + ck$ and the weight of any polynomial in $\mathbb{C}[X, Y, Z]$ as the maximum of the weights of its monomials. In particular the weight of each monomial appearing in the polynomial F is the same, as observed previously in (26), and we have denoted it by d. The integer d takes values $2n, 4n+4, 24, 36, 60$ for G of type $A_{n-1}, D_n, E_6, E_7, E_8$ respectively, and we can note that $d = 2\max(a, b, c)$.

We have given in the last corollary in 3.1.2 a basis $(\overline{P}_1, \dots, \overline{P}_m)$ of $\mathbb{C}[X,Y,Z]/(F'_1, F'_2, F'_3)$. The corresponding monomials $P_1, \dots, P_m \in \mathbb{C}[X,Y,Z]$ satisfy :

$$w(P_i) \leq d-4 \quad \text{for any} \quad 1 \leq i \leq m, \tag{35}$$

as observed in the following table:

type	a,b,c	$F(X,Y,Z)$	d	$P_1, \dots, P_m$	$\max w(P_i)$
A_{n-1}	$2,n,n$	$X^n + YZ$	$2n$	$1, X, X^2, \dots, X^{n-2}$	$2n-4$
D_n	$4, 2n, 2n+2$	$X^{n+1} + XY^2 + Z^2$	$4n+4$	$1, X, X^2, \dots, X^n, Y$	$4n$
E_6	$6, 8, 12$	$X^4 + Y^3 + Z^2$	24	$1, X, X^2, Y, YX, YX^2$	20
E_7	$8, 12, 18$	$X^3Y + Y^3 + Z^2$	36	$1, X, X^2, X^3, X^4, Y, XY$	32
E_8	$12, 20, 30$	$X^5 + Y^3 + Z^2$	60	$1, Y, X, XY, X^2, X^3, X^2Y, X^3Y$	56

Proposition. *The Lie algebra $\mathbb{C}[x,y]^G \simeq \mathbb{C}[X,Y,Z]/(F)$ for the Poisson bracket is finitely generated (by its elements of degree $\leq d$), for any finite subgroup G of SL_2.*

Proof. We denote by $\mathfrak{g}$ the Lie subalgebra of $H := \mathbb{C}[X,Y,Z]/(F)$ generated by the elements $P+(F)$ for $P \in \mathbb{C}[X,Y,Z]$ such that $w(P) \leq d$. Our goal is to prove that, for any $P \in \mathbb{C}[X,Y,Z]$, we have $P+(F) \in \mathfrak{g}$. We proceed by induction on $w(P)$. It is obvious when $w(P) \leq d$. Suppose that there exists some integer $e > d$ such that $P+(F) \in \mathfrak{g}$ for all polynomials P of weight $w(P) < e$.

Now we fix some $P \in \mathbb{C}[X,Y,Z]$ with $w(P) = e$. Considering $\mathbb{C}[X,Y,Z]/(F'_1, F'_2, F'_3)$, we can write with the notations above:

$$P = \alpha_1 P_1 + \dots + \alpha_m P_m + Q_1 F'_1 + Q_2 F'_2 + Q_3 F'_3,$$

with $\alpha_1, \alpha_2, \dots, \alpha_m \in \mathbb{C}$ and $Q_1, Q_2, Q_3 \in \mathbb{C}[X,Y,Z]$. For $i = 1,2,3$, denote by Q_i^j the homogeneous part of weight j in Q_i. Thus

$$P = \alpha_1 P_1 + \dots + \alpha_m P_m + \sum_j Q_1^j F'_1 + \sum_j Q_2^j F'_2 + \sum_j Q_3^j F'_3,$$

From different previous observations, we have $w(F'_1) = d-a, w(F'_2) = d-b, w(F'_3) = d-c$, $w(\alpha_1 P_1 + \dots + \alpha_m P_m) \leq d-4 < e$, hence $w(P) = e$ implies that

$$\sum_{j > e-d+a} Q_1^j F'_1 + \sum_{j > e-d+b} Q_2^j F'_2 + \sum_{j > e-d+c} Q_3^j F'_3 = 0,$$

and then

$$P = \alpha_1 P_1 + \dots + \alpha_m P_m + \sum_{j \leq e-d+a} Q_1^j F'_1 + \sum_{j \leq e-d+b} Q_2^j F'_2 + \sum_{j \leq e-d+c} Q_3^j F'_3.$$

From the other hand, it follows from the induction hypothesis that:

$$\alpha_1 P_1 + \dots + \alpha_m P_m + \sum_{j < e-d+a} Q_1^j F'_1 + \sum_{j < e-d+b} Q_2^j F'_2 + \sum_{j < e-d+c} Q_3^j F'_3 \ + \ (F) \ \in \ \mathfrak{g}.$$

To sum up, it is sufficient to prove that $Q_1^{e-d+a} F'_1 + (F)$, $Q_2^{e-d+b} F'_2 + (F)$ and $Q_3^{e-d+c} F'_3 + (F)$ are elements of $\in \mathfrak{g}$. By linearity and symmetry up to permutation of Q_1, Q_2 and Q_3, it is finally enough to prove that:

$$\text{if } Q = X^m Y^n Z^p F'_1 \ \text{ with } am + bn + cp = e-d+a, \ \text{ then } \ Q + (F) \in \mathfrak{g} \tag{36}$$

• First case: $m,n,p>0$. We have:

$$\{Y^2, X^mY^{n-1}Z^{p+1}\} = 2(p+1)X^mY^nZ^pF_1' - 2mX^{m-1}Y^nZ^{p+1}F_3',$$
$$\{X^mY^{n+1}Z^{p-1}, Z^2\} = 2(n+1)X^mY^nZ^pF_1' - 2mX^{m-1}Y^{n+1}Z^pF_3',$$

and from Euler identity (27)

$$dX^{m-1}Y^nZ^pF = aX^mY^nZ^pF_1' + bX^{m-1}Y^{n+1}Z^pF_2' + cX^{m-1}Y^nZ^{p+1}F_3'.$$

These three relations form a linear system with three variables $X^mY^nZ^pF_1'$, $X^{m-1}Y^{n+1}Z^pF_2'$ and $X^{m-1}Y^nZ^{p+1}F_3'$. Its determinant is $\begin{vmatrix} 2(p+1) & 0 & -2m \\ 2(n+1) & 0 & -2m \\ a & b & c \end{vmatrix} = -4m(ma+(n+1)b+(p+1)c) < 0$. Hence $Q = X^mY^nZ^pF_1'$ is a linear combination of $X^{m-1}Y^nZ^pF$, $\{Y^2, X^mY^{n-1}Z^{p+1}\}$ and $\{X^mY^{n+1}Z^{p-1}, Z^2\}$. Therefore we only have to check that $\{Y^2, X^mY^{n-1}Z^{p+1}\} + (F) \in \mathfrak{g}$ and $\{X^mY^{n+1}Z^{p-1}, Z^2\} + (F) \in \mathfrak{g}$.

> Since $w(Y^2) = 2n \leq d$, we have $Y^2 + (F) \in \mathfrak{g}$. Moreover $w(X^mY^{n-1}Z^{p+1}) = ma+nb+pc+c-b < ma+nb+pc+d-a = w(X^mY^nZ^pF_1') = e$ and then by induction hypothesis, $X^mY^{n-1}Z^{p+1}+(F) \in \mathfrak{g}$. We deduce that $\{Y^2, X^mY^{n-1}Z^{p+1}\}+(F) \in \mathfrak{g}$. The calculations for $\{X^mY^{n+1}Z^{p-1}, Z^2\} + (F) \in \mathfrak{g}$ are quite similar.

We conclude that $Q + (F) \in \mathfrak{g}$ in this case.

• Second case: $m>0$ and $p=0$ (the case $m>0$ and $n=0$ is similar). We have:

$$\{X^{m-1}Y^{n+1}, XZ\} = (n+1)X^mY^nF_1' - (m-1)X^{m-1}Y^{n+1}F_2' - (n+1)X^{m-1}Y^nZF_3',$$
$$\{X^mY^n, YZ\} = nX^mY^nF_1' - mX^{m-1}Y^{n+1}F_2' + mX^{m-1}Y^nZF_3',$$

and from Euler identity (27)

$$dX^{m-1}Y^nF = aX^mY^nF_1' + bX^{m-1}Y^{n+1}F_2' + cX^{m-1}Y^nZF_3'.$$

These three relations form a linear system with three variables $X^mY^nF_1'$, $X^{m-1}Y^{n+1}ZF_2'$ and $X^{m-1}Y^nZF_3'$. Its determinant is $\begin{vmatrix} n+1 & -(m-1) & -(n+1) \\ m & -n & n \\ a & b & c \end{vmatrix} = -(m+n)(ma+(n+1)b+c) < 0$.

Hence $Q = X^mY^nF_1'$ is a linear combination of the elements $X^{m-1}Y^nF$, $\{X^{m-1}Y^{n+1}, XZ\}$ and $\{X^mY^n, YZ\}$. Similarly to the previous case, we conclude that $Q + (F) \in \mathfrak{g}$.

• Third case: $m=0$. Then $w(F_1')$ is lower than d and $F_1' + (F) \in \mathfrak{g}$. Hence we suppos $n>0$ (the case $p>0$ is similar). We have: $Q = Y^nZ^pF_1' = \frac{1}{2(p+1)}\{Y^2, Y^{n-1}Z^{p+1}\}$, and the induction hypothesis implies that $Q + (F) \in \mathfrak{g}$.

In conclusion, assertion (36) us proved in all cases and the proof is complete. □

Remark. Let $\mathcal{U} = (u_1, \ldots, u_m)$ be a family of elements of S^G generating S^G as a Lie subalgebra of S. Denote by V the vector space generated by $\mathcal{U}$. Then we have: $S^G = V + \{V,V\} + \{V,\{V,V\}\} + \cdots$ and then $S^G \subset V + \{S^G, S^G\}$. Therefore the classes $\overline{u}_1, \ldots, \overline{u}_m$ modulo $\{S^G, S^G\}$ generate the subspace $S^G/\{S^G, S^G\}$; thus its dimension is at least m. We conclude that the cardinality of a generating family of S^G as Lie subalgebra of S for the Poisson bracket is always greater than $\mu = \dim_{\mathbb{C}} \mathrm{HP}_0(S^G)$. So comparing the values of d and μ given in the two tables in 3.1.2, the following proposition is a real improvement of the previous result.

THEOREM. *For any finite subgroup G of SL_2, the Lie algebra $\mathbb{C}[x,y]^G$ for the Poisson bracket is generated by a (minimal) family whose cardinality is 2 in the trivial case where G is cyclic of order 2, and exactly* $\dim_{\mathbb{C}} \mathrm{HP}_0(\mathbb{C}[x,y]^G)$ *in all other cases.*

PROOF. We only outline the general method and illustrate it by complete calculations in the cyclic case; the four other cases are developed in [34]. We denote $S = \mathbb{C}[x,y]$.

• One find in any case of G a finite-dimensional subspace V_0 of S^G and an integer $N \geq 1$, such that the iterated adjoint action of V_0 on the subspace finite-dimensional V_1 of S generated by the elements of degree $\leq N$ generates S as a Lie algebra. In other terms, $S = \sum_{k\geq 1} V_k$ where $V_{k+1} = \{V_0, V_k\}$ for any $k \geq 1$. The Reynolds operator $\rho_G := \frac{1}{|G|}\sum_{g\in G} g$ is a projection $S \to S^G$. Since each g is a Poisson automorphism, we have

$$\rho_G(V_{k+1}) = \rho_G(\{V_0, V_k\}) = \{\rho_G(V_0), \rho_G(V_k)\} = \{V_0, \rho_G(V_k)\}.$$

Hence $S^G = \rho_G(S)$ is generated as a Lie algebra by the subspace V_0 and the subspace $W_1 := \rho_G(V_1)$ of invariants of degree $\leq N$. Finally one reduces case by case the number of generators to obtain the minimal value $\mu = \dim_{\mathbb{C}} \mathrm{HP}_0(S^G)$.

• We suppose now that is G cyclic of order $n \geq 3$. As an associative algebra, S^G is generated by x^n, y^n, xy. We take $V_0 := \mathbb{C}x^2y^2 \oplus \mathbb{C}x^{n+1}y \oplus \mathbb{C}xy^{n+1} \subset S^G$ and $N := 2(n-1)$. Hence V_1 is the subspace of S generated by monomials x^py^q of degree $p+q \leq 2(n-1)$. We have:

$$\begin{aligned}
&\{x^2y^2, x^{p-1}x^{q-1}\} = 2(q-p)x^py^q \quad \text{for all } p,q \geq 1,\\
&\{x^{n+1}y, x^{p-n}y^q\} = [(n+1)q - p + n]x^py^q \quad \text{for all } p \geq n,\ q \geq 1,\\
&\{xy^{n+1}, x^py^{q-n}\} = [q - n - (n+1)p]x^py^q \quad \text{for all } p \geq 1,\ q \geq n.
\end{aligned}$$

By straightforward induction on the total degree $p+q$, we check from these relations that any monomial x^py^q, $p,q \geq 1$, lies in the vector space $\sum_{k\geq 1} V_k$ with $V_{k+1} = \{V_0, V_k\}$ for $k \geq 1$. We conclude that $S = \sum_{k\geq 1} V_k$ as announced. Hence by the general above argument, S^G is generated as a Lie algebra by V_0 and W_1, where V_0 is generated by $(x^2y^2, xy^{n+1}, x^{n+1}y)$ and W_1 is generated by: $\{(xy)^i\}_{0\leq i\leq n-1} \ \cup\ \{(xy)^ix^n\}_{0\leq i\leq \frac{n}{2}-1} \ \cup\ \{(xy)^jy^n\}_{0\leq j\leq \frac{n}{2}-1}$.

Let us denote by $\mathfrak{L}$ the Lie subalgebra of S^G generated by the $n-1$ invariant elements: $1 + x^n + y^n, xy, (xy)^2, \ldots, (xy)^{n-2}$. We claim that $S^G = \mathfrak{L}$. It's enough to check that any generator of V_0 and W_1 above lies in $\mathfrak{L}$.

Firstly: $\{1 + x^n + y^n, xy\} = nx^n - ny^n$ hence $x^n - y^n \in \mathfrak{L}$,
then: $\{xy, x^n - y^n\} = -nx^n - ny^n$ hence $x^n + y^n \in \mathfrak{L}$,
thus $\mathfrak{L}$ contains $1, x^n, y^n$,
and $\{x^n, y^n\} = n^2x^{n-1}y^{n-1}$ implies $(xy)^{n-1} \in \mathfrak{L}$.

Now, $\mathfrak{L}$ contains $(xy)^ix^n = \frac{1}{n(i+1)}\{x^n, (xy)^{i+1}\}$ and similarly $(xy)^iy^n$ for any $0 \leq i \leq n-2$. We conclude that $S^G = \mathfrak{L}$ is generated by $n-1$ elements, with $n - 1 = \dim_{\mathbb{C}}\mathrm{HP}_0(S^G)$. The proof is complete in this case.

• In the particular case $n = 2$, we have $\dim_{\mathbb{C}}\mathrm{HP}_0(S^G) = 1$ and one invariant can of course not generate the nonzero Lie algebra S^G. In this case, one proves by direct calculations using the same kind of arguments as above that S^G is generated by the two elements $(1 + xy)$ and $x^2 + y^2 + (xy)^2$. $\square$

REMARK. We give as an illustration the minimal generating families calculated (following the method exposed at the beginning of previous proof but with highly nontrivial computations) in [34]. An interesting observation is that this Lie algebra generators are powers of the homogeneous algebra generators described in 2.2.1

type	algebra generators of $\mathbb{C}[x,y]^G$	Lie algebra generators of $\mathbb{C}[x,y]^G$	μ
D_n	$f_1 = x^2y^2, \quad f_2 = x^{2n} + (-1)^n y^{2n},$ $f_3 = x^{2n+1}y - (-1)^n xy^{2n+1}$	$1,\ f_1,\ f_1^2, \dots, f_1^{n-2}, f_2$	n
E_6	$f_1 = xy^5 - x^5y, \quad f_2 = x^8 + 14x^4y^4 + y^8,$ $f_3 = x^{12} - 33x^8y^4 - 33x^4y^8 + y^{12}$	$1,\ f_1,\ f_1^2,\ f_2,\ f_1f_2,\ f_1^2f_2$	6
E_7	$f_1 = x^8 + 14x^4y^4 + y^8,\ f_2 = x^{10}y^2 - 2x^6y^6 + x^2y^{10}$ $f_3 = x^{17}y - 34x^{13}y^5 + 34x^5y^{13} - xy^{17}$	$1,\ f_1,\ f_1^2,\ f_2,\ f_1f_2,\ f_2^2,\ f_1f_2^2$	7
E_8	$f_1 = x^{11}y + 11x^6y^6 - xy^{11},$ $f_2 = x^{20} - 228x^{15}y^5 + 494x^{10}y^{10} + 228x^5y^{15} + y^{20},$ $f_3 = x^{30} + 522x^{25}y^5 - 10005x^{20}y^{10}$ $-10005x^{10}y^{20} - 522x^5y^{25} + y^{30}$	$1,\ f_1,\ f_1^2,\ f_1^3,\ f_2,\ f_1f_2,\ f_1^2f_2,\ f_1^3f_2$	8

3.3.3 Lie structures on deformations

We start with the following lemma (from [34]) related to the general context of the algebraic deformations studied in 3.2.2.

Lemma. *Let $\mathcal{A}$ be a commutative Poisson algebra and R a noncommutative algebra with a filtration $\mathcal{F}$ such that R is the algebraic deformation of $\mathcal{A}$ associated to $\mathcal{F}$. Suppose that we have a finite family $a_0, a_1, \dots, a_p$ generating $\mathcal{A}$ as a Lie algebra for the Poisson bracket such that each a_i is homogeneous (i.e. $a_i \in \mathcal{F}_{m_i}/\mathcal{F}_{m_i-1}$ for some m_i) ; for any $0 \le i \le p$, let us choose $b_i \in \mathcal{F}_{m_i}$ such that $a_i = b_i + \mathcal{F}_{m_i-1}$. Then $b_0, b_1, \dots, b_p$ generate R as a Lie algebra for the commutator bracket.*

Proof. We denote by $\mathfrak{b}$ the Lie subalgebra of R (for the commutator bracket) generated by $b_0, b_1, \dots, b_p$. Our goal is to prove that $R = \mathfrak{b}$.

First step. By assumption, $b_0, b_1, \dots, b_p$ generate $\mathrm{gr}_{\mathcal{F}}(R) = \bigoplus_{m\ge 0} \mathcal{F}_m/\mathcal{F}_{m-1}$ as a Lie algebra for the Poisson bracket. It means that $\mathrm{gr}_{\mathcal{F}}(R) = \sum_{n\ge 1} \mathfrak{L}_n(V)$ where V is the vector subspace of $\mathrm{gr}_{\mathcal{F}}(R)$ generated by the elements $b_0, b_1, \dots, b_p$, and the $\mathfrak{L}_n(V)$ are defined inductively by $\mathfrak{L}_1(V) = V$ and $\mathfrak{L}_{n+1}(V) = \{V, \mathfrak{L}_n(V)\}$. Any element of $V = \mathfrak{L}_1(V)$ is a linear combination of elements $a_i = b_i + \mathcal{F}_{m_i-1}$ with $b_i \in \mathcal{F}_{m_i} \cap \mathfrak{b}$. Suppose by induction that for some fixed $n \ge 1$, $\mathfrak{L}_n(V)$ is a linear subspace generated by elements of the form $x_m + \mathcal{F}_{m-1}$ with $x_m \in \mathcal{F}_m \cap \mathfrak{b}$. Take then $x \in \mathfrak{L}_{n+1}(V)$; it is a finite sum:

$$\begin{aligned} x &= \textstyle\sum \{a_i, y_i\} && \text{with } y_i \in \mathfrak{L}_n(V) \\ &= \textstyle\sum \{b_i + \mathcal{F}_{m_i-1}\,,\, x_m + \mathcal{F}_{m-1}\} && \text{with } x_m \in \mathcal{F}_m \cap \mathfrak{b} \\ &= \textstyle\sum ([b_i, x_m] + \mathcal{F}_{m_i+m-2}) && \text{with } [b_i, x_m] \in \mathcal{F}_{m_i+m-1} \cap \mathfrak{b}. \end{aligned}$$

We conclude that $\mathrm{gr}_{\mathcal{F}}(R)$ is generated as a vector subspace by the elements of the form $x_m + \mathcal{F}_{m-1}$ with $x_m \in \mathcal{F}_m \cap \mathfrak{b}$.

Second step. We prove now that $\mathcal{F}_m \subseteq \mathfrak{b}$ for any $m \ge 0$. We have $\mathcal{F}_{-1} = (0) \subseteq \mathfrak{b}$. Suppose that $\mathcal{F}_m \subseteq \mathfrak{b}$ for some $m \ge -1$. Take $x \in \mathcal{F}_{m+1}$. Then the first step implies $x + \mathcal{F}_m = x_{m+1} + \mathcal{F}_m$

with $x_{m+1} \in \mathfrak{b}$. Hence $x \in \mathfrak{b} + \mathcal{F}_m \subset \mathfrak{b}$ by induction hypothesis. We conclude that $R \subseteq \mathfrak{b}$ and the proof is complete. $\square$

COROLLARY. *Let $\mathcal{A}$ be a commutative Poisson algebra and R a noncommutative algebra with a filtration $\mathcal{F}$ such that R is the algebraic deformation of $\mathcal{A}$ associated to $\mathcal{F}$. If $\mathcal{A}$ is finitely generated as a Lie algebra for the Poisson bracket, then R is finitely generated as a Lie algebra for the commutator bracket.*

PROOF. If $a_0, a_1, \dots, a_p$ is a generating family of $\mathrm{gr}_{\mathcal{F}}(R) \simeq \mathcal{A}$, the family composed by all homogeneous components of all elements a_i is still a finite family of generators ; now the previous lemma applies. $\square$

A direct application of this deformation results concerns the Lie structure defined from the commutator bracket in the invariant for the Weyl algebra (in the context of 3.2.3)

THEOREM. *For any finite subgroup G of the Weyl algebra $A_1(\mathbb{C})$, the invariant algebra $A_1(\mathbb{C})^G$ is finitely generated as a Lie algebra for the commutator bracket.*

PROOF. It follows directly from the previous corollary, theorem 3.2.3, and assertion (i) of theorem 2.2.2. $\square$

REMARK. *Explicit generators for the Lie algebra $A_1(\mathbb{C})^G$.* In cases D_n, E_6, E_7, E_8, the generators of the Lie algebra $\mathbb{C}[x, y]^G$ given at the end of 3.3.2 are homogeneous; thus the previous lemma produces directly generators of the Lie algebra $A_1(\mathbb{C})^G$ (replacing x by p and y by q).

In the case A_{n-1} with $n \geq 3$, we have proved that $\mathbb{C}[x, y]^G$ is generated by the $n-1$ elements $1 + x^n + y^n, xy, (xy)^2, \dots, (xy)^{n-2}$. Thus $\mathbb{C}[x, y]^G$ is generated by the n homogeneous elements $1, x^n + y^n, xy, (xy)^2, \dots, (xy)^{n-2}$. Hence applying the lemma, $A_1(\mathbb{C})^G$ is generated as a Lie algebra by the n elements $1, p^n + q^n, pq, (pq)^2, \dots, (pq)^{n-2}$. Moreover, we have:

$$[pq, 1 + p^n + q^n] = n(-p^n + q^n) \quad \text{and} \quad [pq, -p^n + q^n] = n(p^n + q^n).$$

Hence the Lie algebra generated by $1+p^n+q^n, pq, (pq)^2, \dots, (pq)^{n-2}$ contains p^n+q^n and 1 ; therefore $A_1(\mathbb{C})^G$ is generated by $1 + p^n + q^n, pq, (pq)^2, \dots, (pq)^{n-2}$.

In the case A_1, we have proved that $\mathbb{C}[x, y]^G$ is generated by the two elements $1 + xy$ and $x^2 + y^2 + (xy)^2$. Thus $\mathbb{C}[x, y]^G$ is generated by the four homogeneous elements $1, xy, x^2 + y^2, (xy)^2$. Hence applying the lemma, $A_1(\mathbb{C})^G$ is generated as a Lie algebra by the four elements $1, pq, p^2 + q^2, (pq)^2$. Moreover, we have:

$$[1 + pq, p^2 + q^2 + p^2q^2] = -2p^2 + 2q^2,$$
$$[1 + pq, -p^2 + q^2] = 2p^2 + 2q^2, \quad \text{and} \quad [p^2, q^2] = 4pq - 2.$$

Hence the Lie algebra generated by $1 + pq, p^2 + q^2 + (pq)^2$ contains $p^2, q^2, 1, pq, p^2q^2$ which generate $A_1(\mathbb{C})^G$. Finally $A_1(\mathbb{C})^G$ is generated by $1 + pq$ and $p^2 + q^2 + (pq)^2$.

4 Quantization: automorphisms and invariants for quantum algebras

4.1 Quantum deformations and their automorphisms

4.1.1 Quantum deformations of the plane

We fix $\Bbbk$ a commutative base field. We recall that for any $q \in \Bbbk^\times$ the quantum plane $\Bbbk_q[x,y]$ is the algebra generated over $\Bbbk$ by two elements x and y with relation

$$xy = qyx. \tag{37}$$

Proposition (the quantum plane). *Suppose that q is not a root of one.*

(i) *$\Bbbk_q[x,y]$ is a noncommutative noetherian domain of center $\Bbbk$.*

(ii) *The $\Bbbk$-algebra $\Bbbk_q[x,y]$ is not simple.*

(iii) *The $\Bbbk$-automorphism group of $\Bbbk_q[x,y]$ reduces to the 2-dimensional torus $(\Bbbk^\times)^2$ acting by $x \mapsto \alpha x$ and $y \mapsto \beta y$ for any $(\alpha,\beta) \in (\Bbbk^\times)^2$.*

(iv) *$\Bbbk_q[x,y]$ is a deformation of the commutative Poisson algebra $\Bbbk[x,y]$ related to the Poisson bracket defined from $\{x,y\} = xy$.*

Proof. Noetherianity in point (i) follows from example (iv) in 1.3.1 and last corollary of 1.3.2; because q is not a root of one, it is straightforward to observe that the centralizer of y reduces to $\Bbbk[y]$, and then that an element of $\Bbbk[y]$ commutes with x only when it is a constant. The non simplicity of $\Bbbk_q[x,y]$ in (ii) follows from the fact that any non trivial monomial is normal (generates a two-sided ideal).
Assertion (iii) first appeared in [20], as a particular case of more general results. We give here a short independent proof. Let z be a normal element of $\Bbbk_q[x,y]$. We have in particular $zy = uz$ and $zx = vz$ for some $u,v \in \Bbbk_q[x,y]$. Considering $\deg_x$ in the first equality, we have $u \in \Bbbk[y]$. Denoting $z = \sum_m f_m(y)x^m$, relation $zy = uz$ implies $\sum_m f_m(y)(q^m y - u(y))x^m = 0$; since q is not a root of one, there exists one nonnegative integer i such that $z = f_i(y)x^i$. From the second equality $f_i(y)x^{i+1} = vz$, it is easy to deduce that $z = \alpha y^j x^i$ for some nonnegative integer j and some $\alpha \in \Bbbk$. This proves that the normal elements of $\Bbbk_q[x,y]$ are the monomials. Now let g be a $\Bbbk$-automorphism of $\Bbbk_q[x,y]$. It preserves the set of nonzero normal elements. Hence we have $g(x) = \alpha y^j x^i$ and $g(y) = \beta y^k x^h$ with $\alpha,\beta \in \Bbbk^\times$ and j,i,k,h nonnegative integers; because q is not a root of one, the relation $g(x)g(y) = qg(y)g(x)$ implies that $ik - hj = 1$. Writing similar formulas for g^{-1} and identifying the exponents in $g^{-1}(g(x)) = x$ and $g^{-1}(g(y)) = y$, we obtain easily $j = h = 0$ and $i = k = 1$.
For the last point (iv), let us introduce the noncommutative algebra B generated by three variables h,x,y with commutation relations $xh = hx$, $yh = hy$ and $xy - yx = hxy$. It is clear that $\mathcal{A} := B/hB$ is the commutative algebra $\Bbbk[\overline{x},\overline{y}]$. Moreover $uv - vu = h\gamma(u,v)$ with $\gamma(u,v) \in B$ for all $u,v \in B$. Thus by the method described in 3.2.1, $\{\overline{u},\overline{v}\} = \overline{\gamma(u,v)}$ defines a Poisson bracket in $\mathcal{A}$. In particular $\{\overline{x},\overline{y}\} = \overline{xy}$. For any $\lambda \in \Bbbk$, $\lambda \neq 0$, $\lambda \neq 1$, the quotient

$B/(h-\lambda)B$ is a deformation of $\mathcal{A}$. This deformation is generated by two elements X and Y with relation $YX=(1-\lambda)XY$, that is the quantum plane $\Bbbk_q[X,Y]$ for $q=(1-\lambda)^{-1}$. $\square$

We define the jordanian plane $\Bbbk^{\mathrm{J}}[x,y]$ as the algebra generated over $\Bbbk$ by x and y with relation $xy-yx=y^2$.

PROPOSITION (the jordanian plane). *Suppose that $\Bbbk$ is of characteristic zero.*

(i) $\Bbbk^{\mathrm{J}}[x,y]$ *is a noncommutative noetherian domain of center* $\Bbbk$.

(ii) *The $\Bbbk$-algebra $\Bbbk^{\mathrm{J}}[x,y]$ is not simple.*

(iii) *The $\Bbbk$-automorphism group of $\Bbbk^{\mathrm{J}}[x,y]$ reduces to the semi-direct product of $\Bbbk^\times$ by the additive group $\Bbbk[y]$, acting by*

$$x\mapsto \alpha x+f(y) \quad\text{and}\quad y\mapsto \alpha y \quad\text{for any } \alpha\in\Bbbk^\times,\, f\in\Bbbk[y].$$

(iv) $\Bbbk^{\mathrm{J}}[x,y]$ *is a deformation of the commutative Poisson algebra $\Bbbk[x,y]$ related to the Poisson bracket defined from $\{x,y\}=y^2$.*

PROOF. Noetherianity in point (i) follows from example (iii) in 1.3.1 and last corollary of 1.3.2; the determination of the center follows from easy calculations on the centralizers of x and y using the assumption $\operatorname{char}\Bbbk=0$. The non simplicity of $\Bbbk^{\mathrm{J}}[x,y]$ in (ii) is clear since y is normal. Point (iii) is a direct application of the last proposition of 2.3.4. The proof of (iv) is quite similar to the previous proposition considering here the algebra B generated by h,x,y with $hx=xh$, $hy=yh$ and $xy-yx=hy^2$. $\square$

> REMARK. We consider the quantum plane $\Bbbk_q[x,y]$ with $q\in\Bbbk^\times$. If $q=1$, it is just the commutative plane $\Bbbk[x,y]$. If $q\neq 1$, we set $\lambda=\frac{1}{1-q}$; the algebra $\Bbbk_q[x,y]$ is also generated by $x'=x+\lambda y$ and $y'=y$. We compute $x'y'-qy'x'=xy+\lambda y^2-qyx-\lambda qy^2=(1-q)\lambda y^2=(y')^2$. In other words, *the algebra generated over $\Bbbk$ by x and y with $xy-qyx=y^2$ is the quantum plane in the case $q\neq 1$ and the jordanian plane in the case $q=1$.* This property is sometimes called the contraction principle from the quantic case to the jordanian case.

We concentrate now on the quantum case (we will return to the jordanian situation further in 4.1.3). The quantum plane admitting non trivial quotient cannot constitute a quantum analogue of the Weyl algebra. Moreover, its automorphism group is to small to provide an interesting invariant theory for finite subgroups. Therefore we introduce the localization $\Bbbk_q[x^{\pm1},y^{\pm1}]$ of $\Bbbk_q[x,y]$ with respect of the multiplicative set generated by x and y; this is the algebra of Laurent polynomials, with $\Bbbk$-basis $(x^iy^j)_{i,j\in\mathbb{Z}}$ and commutation law $xy=qyx$ extended in $x^{-1}y=q^{-1}yx^{-1}$, $xy^{-1}=q^{-1}y^{-1}x$, or $x^{-1}y^{-1}=qy^{-1}x^{-1}$.

PROPOSITION (the quantum torus). *Suppose that q is not a root of one.*

(i) $\Bbbk_q[x^{\pm1},y^{\pm1}]$ *is a noncommutative noetherian domain of center* $\Bbbk$.

(ii) *The* $\mathbb{k}$*-algebra* $\mathbb{k}_q[x^{\pm 1}, y^{\pm 1}]$ *is simple.*

(iii) *The* $\mathbb{k}$*-automorphism group of* $\mathbb{k}_q[x^{\pm 1}, y^{\pm 1}]$ *is isomorphic to the semi-direct product of the 2-dimensional torus* $(\mathbb{k}^\times)^2$ *by* $\mathrm{SL}_2(\mathbb{Z})$ *acting by*

$$x \mapsto \alpha y^c x^a \quad \text{and} \quad y \mapsto \beta y^d x^b \tag{38}$$

for any $(\alpha, \beta) \in (\mathbb{k}^\times)^2$ *and* $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z})$.

(iv) $\mathbb{k}_q[x^{\pm 1}, y^{\pm 1}]$ *is a deformation of the commutative Poisson algebra* $\mathbb{k}[x^{\pm 1}, y^{\pm 1}]$ *related to the Poisson bracket defined by localization from* $\{x, y\} = xy$.

Proof. Points (i) and (iv) are clear by localization ; see last comment of 1.3.2, and 3.1.1 (or further the proof of theorem 4.2.2). The proof following of (ii) is a multiplicative adaptation of the argument for the Weyl algebra (see 2.1.1). For any nonzero element $s \in \mathbb{k}_q[x^{\pm 1}, y^{\pm 1}]$, denote by $\ell(s)$ the length of s (i.e. the number of monomials with nonzero coefficients in the decomposition of s related to the $\mathbb{k}$-basis $(x^i y^j)_{i,j \in \mathbb{Z}}$). Let I be a nonzero two-sided ideal of $\mathbb{k}_q[x^{\pm 1}, y^{\pm 1}]$. Let s be a nonzero element of I whose length is the minimum of the lengths of nonzero elements of I. Choose $(a, b) \in \mathbb{Z}^2$ in the support of s; we have $s = \alpha x^a y^b + s'$ where $\alpha \in \mathbb{k}^\times$ and $s' \in \mathbb{k}_q[x^{\pm 1}, y^{\pm 1}]$. Then $u := \alpha^{-1} s y^{-b} x^{-a} = 1 + s''$ where $s'' = \alpha^{-1} s' y^{-b} x^{-a}$. Because $s \in I$ and I is an ideal, we have $u \in I$. Hence $u - xux^{-1} \in I$. In other words $u - xux^{-1} = s'' - xs''x^{-1} \in I$. Since multiplying a monomial by x on the left and x^{-1} consists in multiplying it by a nonzero constant, is is clear that $\ell(s'' - xs''x^{-1}) \leq \ell(s'') = \ell(s') < \ell(s)$. By minimality of $\ell(s)$, we deduce that $u - xux^{-1} = 0$. We prove similarly that $u = yuy^{-1}$. Finally $u \in I$, $u \neq 0$, lies in the center $\mathbb{k}$ of $\mathbb{k}_q[x^{\pm 1}, y^{\pm 1}]$. So $I = \mathbb{k}_q[x^{\pm 1}, y^{\pm 1}]$ and point (ii) is proved.

To prove (iii) let us consider θ an automorphism of $\mathbb{k}_q[x^{\pm 1}, y^{\pm 1}]$. It preserves the group U of invertible elements. In particular $\theta(x) \in U$ and $\theta(y) \in U$. It is easy to prove that U is the set of nonzero monomials. Thus there exist $\alpha, \beta \in \mathbb{k}^\times$ and $a, b, c, d \in \mathbb{Z}$ such that $\theta(x) = \alpha y^c x^a$ and $\theta(y) = \beta y^d x^b$. By identification of coefficients in the identity $\theta(x)\theta(y) = q\theta(y)\theta(x)$ we deduce $q^{ad-bc} = 1$ and the assumption q not a root of one implies $ad - bc = 1$. Conversely any $a, b, c, d \in \mathbb{Z}$ with $ad - bc = 1$ define an automorphism $\theta : x \mapsto y^c x^a, y \mapsto y^d c^b$ with $\theta^{-1} : x \mapsto q^m y^{-c} x^d, y \mapsto q^n y^a c^{-b}$ where the exponents m, n depend on a, b, c, d (see further precisions in 4.2.1); point (iii) follows by computing the composition of such automorphisms and diagonal automorphisms $x \mapsto \alpha x, y \mapsto \beta y$ with $\alpha, \beta \in \mathbb{k}^*$. □

In conclusion, as well as the Weyl algebra is a simple noncommutative deformation of the symplectic plane, the quantum torus is a simple noncommutative deformation of the plane for its so called multiplicative Poisson structure.

Comment. We restrict here to dimension two but n-dimensional versions of the quantum plane and the quantum torus (see [2]) are of course the object of many studies. Related to the above properties, we refer to the papers [54], [59] and [61].

4.1.2 Induced Lie structures

• We consider the commutative Poisson algebra $S = \mathbb{C}[x, y]$ for the multiplicative Poisson bracket, defined in point (iv) of the first proposition of 4.1.1. That is:

$$\{x, y\} = xy, \tag{39}$$

or more generally

$$\{x^a y^b, x^c y^d\} = (ad - bc) x^{a+c} y^{b+d} \quad \text{for all } a, b, c, d \in \mathbb{N}. \tag{40}$$

We claim (from [34]) that $\mathrm{HP}_0(S)$ is not finite dimensional.

> Proof. The Poisson bracket $\{f, g\}$ of two polynomials $f, g \in S$ is a linear combination with coefficients in $\mathbb{C}$ of terms $h = \{x^a y^b, x^c y^d\}$ with $a, b, c, d \in \mathbb{N}$. If $a + c = 0$, then $a = c = 0$ thus $ad - bc = 0$, and therefore $h = 0$. With the same argument for $b + d$ we deduce that $\{f, g\}$ is a linear combination of monomials $x^u y^v$ where the integers $u = a + c$ and $v = b + d$ are ≥ 1. Such a monomial can be obtained as $x^u y^v = \frac{1}{v}\{x, x^{u-1} y^v\}$. So we have: $S = \mathbb{C} \oplus (\bigoplus_{u \geq 1} \mathbb{C} x^u) \oplus (\bigoplus_{v \geq 1} \mathbb{C} y^v) \oplus \{S, S\}$, and then $S/\{S, S\}$ is not finite dimensional. $\square$

In particular, $\mathbb{C}[x, y]$ is not finitely generated as a Lie algebra for the bracket $\{\cdot, \cdot\}$ under consideration. Then we work in the following on the localized form.

• We consider the commutative Poisson algebra $T = \mathbb{C}[x^{\pm 1}, y^{\pm 1}]$ for the multiplicative Poisson bracket, defined in point (iv) of the third proposition of 4.1.1 by localization of the previous one. That is:

$$xy = \{x, y\}, \quad x^{-1} y^{-1} = \{x^{-1}, y^{-1}\}, \quad x^{-1} y = -\{x^{-1}, y\}, \quad xy^{-1} = -\{x, y^{-1}\}, \tag{41}$$

or more generally

$$\{x^a y^b, x^c y^d\} = (ad - bc) x^{a+c} y^{b+d} \quad \text{for all } a, b, c, d \in \mathbb{Z}. \tag{42}$$

PROPOSITION. *For the multiplicative Poisson bracket on the commutative algebra $T = \mathbb{C}[x^{\pm 1}, y^{\pm 1}]$, the following holds:*

(i) $\dim_{\mathbb{C}} \mathrm{HP}_0(T) = 1$

(ii) *T is finitely generated as a Lie algebra.*

(iii) *The only Lie ideals of T are $\mathbb{C}$, T and the vector space T^+ generated by the non constant monomials. In particular $T/\mathbb{C}$ is a simple Lie algebra.*

PROOF. We follow [35]. For any $a, b \in \mathbb{Z}$, we have $\{x^{a+b} y^{b-a}, x^{-b} y^a\} = (a^2 + b^2) x^a y^b$. Thus any non constant monomial is an element of $\{T, T\}$. Moreover $a + c = b + d = 0$ implies $ad - bc = 0$ and it follows then from (42) that the constant term of any bracket $\{f, g\}$ with $f, g \in T$ is necessarily zero. In conclusion $T = \mathbb{C} \oplus \{T, T\}$ and point (i) is proved. Now denote by $\mathfrak{g}$ the Lie subalgebra of T generated by the five elements $1, x, x^{-1}, y, y^{-1}$. We claim that any monomial $x^a y^b$ with $a, b \in \mathbb{Z}$, $a \neq 0$, $b \neq 0$ lies in $\mathfrak{g}$.

> We proceed by induction on $|a| + |b|$. The case $|a| + |b| = 2$ follows from the identities (41). Now suppose $|a| + |b| > 2$. Then $|a| \geq 2$ or $|b| \geq 2$. Up to permute, we can suppose $|a| \geq 2$. If $a \geq 2$, then $x^a y^b = -\frac{1}{b}\{x^{a-1} y^b, x\}$ with $x \in \mathfrak{g}$ by definition of $\mathfrak{g}$ and $x^{a-1} y^b \in \mathfrak{g}$ by induction hypothesis. If $a \leq -2$, then $x^a y^b = \frac{1}{b}\{x^{a+1} y^b, x^{-1}\}$ with $x^{-1} \in \mathfrak{g}$ by definition of $\mathfrak{g}$ and $x^{a+1} y^b \in \mathfrak{g}$ by induction hypothesis.

Moreover $x^a = -\frac{1}{a}\{x^a y, y^{-1}\}$ for $a \neq 0$ with $x^a y \in \mathfrak{g}$ from the previous step, and then $x^a \in \mathfrak{g}$. Similarly $y^b = \frac{1}{b}\{xy^b, x^{-1}\} \in \mathfrak{g}$ for $b \neq 0$. In conclusion $\mathfrak{g}$ contains all monomials and finally $\mathfrak{g} = T$. In order to proved (iii), let us introduce a Lie ideal I of T non reduced to $\mathbb{C}$. We choose in I an element u of minimal length s among the non constant elements of I. We denote it $u := \alpha_1 x^{a_1} y_{b_1} + \cdots + \alpha_s x^{a_s} y^{b_s}$ where the α_k's are in $\mathbb{C}$. Suppose that $s \geq 2$. For any $(i,j) \in \mathbb{Z}^2$, $\{x^i y^j, u\} \in I$. Observe that $\{x^i y^j, u\} = \sum_{k=1}^{s} \alpha_k (ia_k - jb_k) x^{i+a_k} y^{j+b_k}$ is of length $\leq s$. If there exists some $(i,j) \in \mathbb{Z}^2$, $(i,j) \neq (0,0)$, such that all (a_k, b_k) are proportional to (i,j) [i.e. there exists $\lambda_1, \ldots, \lambda_s \in \mathbb{Q}$ such that $(a_k, b_k) = \lambda(i,j)$ for any $1 \leq k \leq s$], then we consider in I the non constant element $\{x^j y^{-i}, u\} = \sum_{k=1}^{s} 2a_k \lambda_k ij\, \alpha_k x^{j+\lambda_k i} y^{-i+\lambda_k j}$ with non proportional exponents $(j + \lambda_k i, -i + \lambda_k j)$. So, up to this change of monomial, we can suppose without any restriction that at least two pairs of exponents, for instance (a_1, b_1) and (a_2, b_2), are non proportional in $\mathbb{Q}$. Hence $\{x^{a_1} y^{b_1}, u\}$ is a non constant element of I whose length is strictly lower than s. Contradiction. It follows that we necessarily have $s = 1$. So I contains a monomial $u = x^a y^b$ with $(a,b) \neq (0,0)$. Without lost of generality we can assume $a \neq 0$. Then $-ay = \{x^{-a} y^{1-b}, u\} \in I$. Thus I contains y and then $\{xy^{-1}, y\} = x$. Similarly $x^{-1} \in I$ and $y^{-1} \in I$. If $1 \in I$, then $I = \mathfrak{g} = T$ by point (ii). Otherwise $I = T^+$. □

• We consider now the quantum deformation $\mathbb{C}_q[x^{\pm 1}, y^{\pm 1}]$. We suppose that q is not a root of one in $\mathbb{C}$. The Lie structure under consideration is the commutator's one. Hence the bracket of two monomials is given by

$$[x, y] = (q-1)yx = (1-q^{-1})xy, \tag{43}$$

or more generally

$$[x^a y^b, x^c y^d] = (q^{-bc} - q^{-ad}) x^{a+c} y^{b+d} \quad \text{for all } a, b, c, d \in \mathbb{Z}. \tag{44}$$

The similarity between the original multiplicative Poisson bracket on $T = \mathbb{C}[x^{\pm 1}, y^{\pm 1}]$ and the deformed commutator bracket on the quantum deformation $\mathbb{C}_q[x^{\pm 1}, y^{\pm 1}]$ appears in particular in the main observation that $(q^{-bc} - q^{-ad}) = 0$ in (44) if and only if $ad - bc = 0$ in (42). So it is not surprising to obtain a quite parallel result (see [34] and [35]):

Proposition. *Suppose that q is not a root of one. For the quantum torus $T_q = \mathbb{C}_q[x^{\pm 1}, y^{\pm 1}]$, the following holds:*

(i) $\dim_{\mathbb{C}} \mathrm{HH}_0(T_q) = 1$

(ii) *T_q is finitely generated as a Lie algebra for the commutator bracket.*

(iii) *The only Lie ideals of T_q are $\mathbb{C}$, T_q and the vector space T_q^+ generated by the non constant monomials.*

Proof. It suffices to copy out mutatis mutandis the previous demonstration □

Remark. Let I be a nonzero two-sided ideal of the associative algebra T_q. It is a fortiori a Lie ideal and we can apply point (iii) above. If $I = \mathbb{C}$, then $I = T_q$. If $I = T_q^+$, then $x \in I$ and by the definition of an ideal $1 = xx^{-1} \in I$, thus $I = T_q$. So we find again the simplicity of the associative algebra T_q proved in (ii) of the third proposition of 4.1.1.

4.1.3 Rigidity of quantum groups

Up to isomorphism, the search of Hopf algebras with the same representation theory as SL_2 leads to two noncommutative deformations of the Hopf algbra $\mathcal{O}(\mathrm{SL}_2)$ of regular functions on SL_2. There are obtained from the corresponding bialgebras of 2×2 matrices applying the Faddeev-Reshetihin-Takhtajan construction to one of the Hecke symmetries:

$$\text{quantum: } R_q = \begin{pmatrix} q^{-1} & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & q^{-1}-q & 0 \\ 0 & 0 & 0 & q^{-1} \end{pmatrix} \qquad \text{jordanian: } R^{\mathrm{J}} = \begin{pmatrix} 1 & -1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \tag{45}$$

where $q \in \Bbbk^\times$. In both cases the matrix $R = R_q$ or $R = R^{\mathrm{J}}$ viewed as an endomorphism of $V \otimes V$ for a 2-dimensional $\Bbbk$-vector space V satisfies in the group of linear automorphisms of $V \otimes V \otimes V$ the Yang-Baxter relation:

$$(R \otimes \mathrm{id}_V)(\mathrm{id}_V \otimes R)(R \otimes \mathrm{id}_V) = (\mathrm{id}_V \otimes R)(R \otimes \mathrm{id}_V)(\mathrm{id}_V \otimes R),$$

and the Hecke condition in $\mathrm{End}(V \otimes V)$:

$$(R_q - q^{-1}\,\mathrm{id}_{V\otimes V})(R_q + q^{-1}\,\mathrm{id}_{V\otimes V}) = 0 \quad \text{or} \quad (R^{\mathrm{J}} - \mathrm{id}_{V\otimes V})(R^{\mathrm{J}} + \mathrm{id}_{V\otimes V}) = 0.$$

Then we consider the algebra A generated by four generators a, b, c, d with relations:

$$R \times [\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \otimes \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)] = [\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \otimes \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)] \times R \ \in \ \mathrm{End}(V \otimes V) \tag{46}$$

Comment. A is also a bialgebra for the coproduct $\Delta : A \to A \otimes A$ and counit $\varepsilon : A \to \Bbbk$ defined from the matrix product:

$$\begin{pmatrix} \Delta(a) & \Delta(b) \\ \Delta(c) & \Delta(d) \end{pmatrix} = \begin{pmatrix} a\otimes a + b\otimes c & a\otimes b + b\otimes d \\ c\otimes a + d\otimes c & c\otimes b + d\otimes d \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \dot{\otimes} \begin{pmatrix} a & b \\ c & d \end{pmatrix} \ \text{ and } \ \begin{pmatrix} \varepsilon(a) & \varepsilon(b) \\ \varepsilon(c) & \varepsilon(d) \end{pmatrix} = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right).$$

It follows from relations (46) that the assignments $\delta(x) = a \otimes x + b \otimes y$ and $\delta(y) = c \otimes x + d \otimes y$ satisfy $\delta(x)\delta(y) = q\delta(y)\delta(x)$ if $xy = qyx$ and $R = R_q$, and $\delta(x)\delta(y) - \delta(y)\delta(x) = \delta(y)^2$ if $xy - yx = y^2$ and $R = R^{\mathrm{J}}$. Then δ defines a coaction $P \to A \otimes P$ where P is the quantum plane $\Bbbk_q[x, y]$ if $R = R_q$, and P is the jordanian plane $\Bbbk^{\mathrm{J}}[x, y]$ if $R = R^{\mathrm{J}}$. In both cases, there exists an analogue z of the determinant which is central in A and we define the corresponding analogue $A' = A/(z-1)A$ of $\mathcal{O}(\mathrm{SL}_2)$; the ideal $(z - 1)A$ is a coideal and the bialgebra A' is a (non commutative and non cocommutative) Hopf algebra where the antipode $S : A' \to A'$ is respectively given by $\begin{pmatrix} S(a) & S(b) \\ S(c) & S(d) \end{pmatrix} = \begin{pmatrix} d & -q^{-1}b \\ -qc & a \end{pmatrix}$ or $\begin{pmatrix} c+d & -a+c-b+d \\ -c & a-c \end{pmatrix}$.

Definition. *For $q \in \Bbbk^\times$, the algebra $\mathcal{O}_q(M_2)$ of quantum 2×2 matrices is the algebra generated over $\Bbbk$ by four generators a, b, c, d with relations:*

$$\begin{cases} ab = qba, & bd = qdb, & ac = qca, \\ cd = qdc, & cb = bc, & ad - da = (q - q^{-1})bc. \end{cases} \tag{47}$$

The quantum determinant is the central element $z_q = ad - qbc = da - qcb$ and the algebra $\mathcal{O}_q(M_2)/(z_q - 1)\mathcal{O}_q(M_2)$ is the quantum deformation of $\mathcal{O}(\mathrm{SL}_2)$, denoted by $\mathcal{O}_q(\mathrm{SL}_2)$.

REMARK. In the same way than in point (iv) of the first proposition of 4.1.1 we can observe that $\mathcal{O}_q(\mathrm{SL}_2)$ is a deformation of the commutative Poisson algebra $\mathcal{O}(\mathrm{SL}_2)$ for the standard Poisson bracket defined from $\{a,b\}=ab$, $\{a,c\}=ac$, $\{b,c\}=0$, $\{b,d\}=bd$, $\{c,d\}=cd$ and $\{a,d\}=2bc$; see III.5.5 in [2] for more details.

DEFINITION. *The algebra $\mathcal{O}^{\mathrm{J}}(M_2)$ of jordanian 2×2 matrices is the algebra generated over $\Bbbk$ by four generators a,b,c,d with relations:*

$$\left\{\begin{array}{lll} [a,c]=c^2, & [d,c]=c^2, & [a,d]=dc-ac, \\ [a,b]=ad-bc+ac-a^2, & [d,b]=ad-bc+ac-d^2, & [b,c]=dc+ac-c^2. \end{array}\right. \tag{48}$$

The jordanian determinant is the central element $z^{\mathrm{J}}=ad-bc+ac$ and the algebra $\mathcal{O}^{\mathrm{J}}(M_2)/(z^{\mathrm{J}}-1)\mathcal{O}^{\mathrm{J}}(M_2)$ is the jordanian deformation of $\mathcal{O}(\mathrm{SL}_2)$, denoted by $\mathcal{O}^{\mathrm{J}}(\mathrm{SL}_2)$.

THEOREM. *For q not a root of one, the automorphism group of $\mathcal{O}_q(M_2)$ reduces to the semi-direct product $(\Bbbk^\times)^3\rtimes\langle\tau\rangle$ where τ is the involution of $\mathcal{O}_q(M_2)$ defined by:*

$$\tau\ :\ a\mapsto a,\quad b\mapsto c,\quad c\mapsto b,\quad d\mapsto d \tag{49}$$

and the 3-dimensional torus $(\Bbbk^\times)^3$ acts by:

$$\mu_{\alpha,\beta,\gamma}\ :\ a\mapsto\alpha a,\quad b\mapsto\beta b,\quad c\mapsto\gamma c,\quad d\mapsto\alpha^{-1}\beta\gamma d \tag{50}$$

PROOF. This theorem is proved in [20] using the structure of derivations to deduce the automorphisms. We give here an alternative direct demonstration.

Step 1. Denote by A the algebra $\mathcal{O}_q(M_2)$ and U the noncommutative Laurent polynomial algebra $U=\Bbbk[b,c,z][a^{\pm1}\,;\,\sigma]$ where σ is defined by $\sigma(b)=qb$, $\sigma(c)=qc$ and $\sigma(z)=z$. It follows from relations (47) and relation $d=(z+qcb)a^{-1}$ that A can be embedded in U. Using the canonical form of any element $t\in A$ as a finite development $\sum_{i\in\mathbb{Z}}f_i(b,c,z)a^i$ in U with very simple relations $ab=qba$, $ac=qca$ and $az=za$, it is easy to see that t commutes with b and c if and only if $t\in\Bbbk[b,c,z]$, and then t also commute with a if and only if $t\in\Bbbk[z]$. We prove so that the center $Z(A)$ of A reduces to $\Bbbk[z]$. By similar calculations, we verify that the set of normal elements of A is $N(A)=\bigcup_{m\geq0}N_m(A)$ where $N_m(A)=\bigoplus_{i,j,k\geq0,i+j=m}\Bbbk\, b^ic^jz^k$.

Step 2. Let σ be an automorphism of A. Using the natural grading of A, we can note: $\sigma(a)=a_0+a_1+a^+$ with $a_0\in\Bbbk$, $a_1\in\Bbbk a\oplus\Bbbk b\oplus\Bbbk c\oplus\Bbbk d$, and $\deg a^+\geq 2$. With similar notations for $\sigma(b)$, $\sigma(c)$ and $\sigma(d)$, we apply σ to the first four relations of (47) and deduce obviously by identifications that $a_0=b_0=c_0=d_0$ and $a_1b_1=qb_1a_1$, $a_1c_1=qc_1a_1$, $b_1d_1=qd_1b_1$ and $c_1d_1=qd_1c_1$. Writing each a_1,b_1,c_1,d_1 in the basis a,b,c,d, these four q-commutations imply $a_1=\alpha a$, $b_1=\beta b+\gamma'c$, $c_1=\beta'b+\gamma c$ and $d_1=\delta d$ with $\alpha,\beta,\beta',\gamma,\gamma',\delta\in\Bbbk$. Moreover the restriction of σ to $Z(A)=\Bbbk[z]$ is an automorphism of $\Bbbk[z]$. Hence there exists $\lambda,\mu\in\Bbbk$, $\lambda\neq0$ such that $\sigma(z)=\lambda z+\mu$. Therefore $a_1d_1-qb_1c_1=\lambda(ad-qbc)+\mu$. It follows that $\mu=0$, $\alpha\delta=\lambda$, $\gamma\gamma'=\beta\beta'=0$ and $\beta\gamma+\beta'\gamma'=\lambda$. We conclude that, up to compose σ by τ and the automorphism $\mu_{\alpha,\beta,\gamma}$ described by (49) and (50), we can suppose without any restriction in the following that $a_1=a$, $b_1=b$, $c_1=c$ and $d_1=d$. In other words, $\sigma(a)=a+a^+$, $\sigma(b)=b+b^+$, $\sigma(c)=c+c^+$ and $\sigma(d)=d+d^+$, with $\sigma(z)=z$.

Step 3. The element b is normal in A, then $\sigma(b)$ is normal in A; because $\sigma(b)=b+b^+$ and $b\in N_1(A)$, the rest b^+ must be an element of the component $N_1(A)$ of $N(A)$. In particular

$ab^+ = qb^+a$. Developing the relation $\sigma(a)\sigma(b) = q\sigma(b)\sigma(a)$ into $ab + a^+b + ab^+ + a^+b^+ = qba + qb^+a + qa^+b + qa^+b^+$, the simplification by $ab = qba$ and $ab^+ = qb^+a$ together with the degree conditions $\deg b < \deg b^+$ imply $a^+b = qba^+$. Thus the development of a^+ in U reduces to $a^+ = f(b,c,z)a$ with $f \in \Bbbk[b,c,z]$. Applying the same argument for the automorphism σ^{-1} with $\sigma^{-1}(a) = a + g(b,c,z)a$, we obtain in the subalgebra $\Bbbk[b,c,z][a\,;\,\sigma]$ of U the identity $a = \sigma(\sigma^{-1}(a)) = (1+\sigma(g))(1+f)a$. Hence $f = g = 0$ and then $\sigma(a) = a$. Similarly $\sigma(d) = d$.

Step 4. Since $\sigma(ad) = ad$ and $\sigma(z) = z$, we have $\sigma(bc) = bc$. The element $b^+ \in N_1(A)$ is of the form $b^+ = f(z)b + g(z)c$ with $f, g \in \Bbbk[z]$. Similarly $c^+ = h(z)b + \ell(z)c$. The identification $(b+b^+)(c+c^+) = bc$ in $\Bbbk[z][b,c]$ gives then $g = h = 0$. Hence $\sigma(b) = b+f(z)b$ and $\sigma(c) = c+\ell(z)$ and the argument $b = \sigma(\sigma^{-1}(b))$ as above for a implies $f = 0$. Finally $\sigma(b) = b$, and similarly $\sigma(c) = c$. We conclude $\sigma = \mathrm{id}_A$ and the proof is complete. $\square$

We can deduce from this theorem (see also [20] and [33]) that:

- the group of algebra automorphisms of $\mathcal{O}_q(\mathrm{SL}_2)$ reduces to the semi-direct product $(\Bbbk^\times)^2 \rtimes \langle\tau\rangle$ where τ is the involution defined by: $a \mapsto a,\ b \mapsto c,\ c \mapsto b,\ d \mapsto d$ and the 2-dimensional torus $(\Bbbk^\times)^2$ acts by $a \mapsto \alpha a,\ b \mapsto \beta b,\ c \mapsto \beta^{-1}c,\ d \mapsto \alpha^{-1}d$.
- the Hopf algebra automorphisms of $\mathcal{O}_q(\mathrm{SL}_2)$ are the $a \mapsto a,\ b \mapsto \beta b,\ c \mapsto \beta^{-1}c,\ d \mapsto d$.

Remark. The corresponding result for the jordanian matrices is proved in the paper [44]. The proof is somewhat more difficult and clearly too long to be developed here. The description is the following. We suppose $\Bbbk$ of characteristic zero and consider the algebra $A = \mathcal{O}^{\mathrm{J}}(M_2)$ with generators a, b, c, d and relations (48):

1. there exists $\tau \in \mathrm{Aut}\,A$ such that: $\tau(a) = d,\ \tau(b) = b,\ \tau(c) = c,\ \tau(d) = a$, and $T = \{\mathrm{id}_A, \tau\}$ is a subgroup of order 2 of $\mathrm{Aut}\,A$;
2. for all $\alpha \in \Bbbk^\times$, there exists $\sigma_\alpha \in \mathrm{Aut}\,A$ such that:
$$\sigma_\alpha(a) = \alpha a, \quad \sigma_\alpha(b) = \alpha b, \quad \sigma_\alpha(c) = \alpha c, \quad \sigma_\alpha(d) = \alpha d,$$
and $H = \{\sigma_\alpha\,;\,\alpha \in \Bbbk^\times\}$ is a subgroup of $\mathrm{Aut}\,A$ isomorphic to $\Bbbk^\times$;
3. for all $q(z) \in \Bbbk[z]$, there exists $\eta_q \in \mathrm{Aut}\,A$ such that:
$$\eta_q(a) = a, \quad \eta_q(b) = b + q(z)a, \quad \eta_q(c) = c, \quad \eta_q(d) = d + q(z)c,$$
and $G_1 = \{\eta_q\,;\,q(z) \in \Bbbk[z]\}$ is a subgroup of $\mathrm{Aut}\,A$ isomorphic to the additive group $\Bbbk[z]$;
4. for all $p(z,u,c) \in \Bbbk[z,u,c]$, where $u = d - a$, there exists $\xi_p \in \mathrm{Aut}\,A$ such that:
$$\xi_p(a) = a + p(z,u,c)c, \quad \xi_p(c) = c, \quad \xi_p(d) = d + p(z,u,c)c,$$
$$\xi_p(b) = b + ap(z,u,c) + p(z,u,c)d + p(z,u,c)^2c,$$
and $G_2 = \{\xi_p\,;\,p(z,u,c) \in \Bbbk[z,u,c]\}$ is a subgroup of $\mathrm{Aut}\,A$ isomorphic to the additive group $\Bbbk[z,u,c]$.

Then the main theorem asserts that:
$$\mathrm{Aut}\,\mathcal{O}^{\mathrm{J}}(M_2) = [(G_2 \rtimes G_1) \rtimes H] \rtimes T,$$
and a corollary proves that:
$$\mathrm{Aut}\,\mathcal{O}^{\mathrm{J}}(\mathrm{SL}_2) = [(G_2' \rtimes G_1') \rtimes H'] \rtimes T.$$
where $H' = \{\sigma_\alpha \in H\,;\,\alpha = \pm 1\}$ of order 2, $G_1' = \{\eta_q \in G_1\,;\,q \in \Bbbk\} \simeq \Bbbk$ and $G_2' = \{\xi_p \in G_2\,;\,p(u,c) \in \Bbbk[u,c]\} \simeq \Bbbk[u,c]$

Comment. Let us recall that the structure of the automorphism group of a commutative algebra of polynomials in n indeterminates is known only for $n \leq 2$ (see 2.3.4). For $n \geq 3$, it is the subject of many problems, studies and open questions connected with profound topics (tameness conjecture, Dixmier conjecture, Jacobian conjecture,... see [18]) with recent fundamental progresses (by Alexei Belov-Kanel and Maxim Kontsevich, Ivan P. Shestakov and Ualbai U. Umirbaev). For instance the structure of the automorphism group of $\mathbb{C}[x, y, z, t] \simeq \mathcal{O}(M_2)$ or $\mathcal{O}(SL_2)$ is still unknown. The same problem for the quantum algebras $\mathcal{O}_q(M_2)$ and $\mathcal{O}_q(SL_2)$ turns out to be trivial with a very "small" group of automorphisms. The fact that the quantization leads to a more rigid situation is a well known phenomenon, observed in many other cases: quantum spaces, quantum groups, quantum enveloping algebras (see for instance references in [20], [21], [32], [46], [61],...). The jordanian deformation gives rise to a very different picture; in some sense, it is intermediate between the extremely rich commutative situation and the very rigid quantum case (see [44]).

4.2 Multiplicative invariants

4.2.1 Actions for multiplicative Poisson structures and deformations

The action of the group $\mathrm{SL}_2(\mathbb{Z})$ on the Poisson algebra $T = \mathbb{C}[x^{\pm 1}, y^{\pm}]$ for the multiplicative bracket defined by (41) and the corresponding "multiplicative invariant theory"' (see [12]) is deformed into an action by automorphisms on the quantum torus $T_q = \mathbb{C}_q[x^{\pm 1}, y^{\pm}]$ previously encountered in point (iii) of the third proposition of 4.1.1 and detailed in the following. It is useful to introduce $\hat{q}$ a square root of q^{-1}. Relation (37) rewrites into

$$yx = \hat{q}^{\,2} xy \tag{51}$$

We start with the description of the actions.

Proposition. *We suppose that q is not a root of one.*

(i) *The group $\mathrm{SL}_2(\mathbb{Z})$ acts by Poisson automorphisms on the commutative algebra $T = \mathbb{C}[x^{\pm 1}, y^{\pm}]$ for the multiplicative bracket. The action is defined by:*

$$g.x = x^a y^c \quad \text{and} \quad g.y = x^b y^d \quad \text{for } g = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z}), \tag{52}$$

or more generally for any $m, n \in \mathbb{Z}$,

$$g.(x^m y^n) = x^{am+bn} y^{cm+dn} \tag{53}$$

(ii) *The group $\mathrm{SL}_2(\mathbb{Z})$ acts by algebra automorphisms on the quantum torus $T_q = \mathbb{C}_q[x^{\pm 1}, y^{\pm}]$. The action is defined by:*

$$g.x = \hat{q}^{\,ac} x^a y^c \quad \text{and} \quad g.y = \hat{q}^{\,bd} x^b y^d \quad \text{for } g = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z}), \tag{54}$$

or more generally for any $m, n \in \mathbb{Z}$,

$$g.(x^m y^n) = \hat{q}^{\,(am+bn)(cm+dn)-mn} x^{am+bn} y^{cm+dn} \tag{55}$$

Proof. From one hand $g'.(g.x) = g'.(x^ay^c) = (x^{a'}y^{c'})^a(x^{b'}y^{d'})^c = x^{a'a+b'c}y^{c'a+d'c} = (g'g).x$. Similarly $g'.(g.y) = (g'g).y$. From the other hand it follows from (42) that $\{g.x, g.y\} = \{x^ay^c, x^by^d\} = (ad-bc)x^{a+b}y^{c+d} = g.(xy) = g.\{x,y\}$ thus any $g \in \mathrm{SL}_2(\mathbb{Z})$ defines a Poisson automorphism. The proof of (i) is complete. For the quantum case, we have:

$$\begin{aligned}(g.y)(g.x) &= \hat{q}^{\,bd}x^by^d\hat{q}^{\,ac}x^ay^c = \hat{q}^{\,bd+ac+2ad}x^{a+b}y^{c+d} = \hat{q}^{\,ac+bd+2+2bc}x^{a+b}y^{c+d}\\ &= \hat{q}^{\,2}\hat{q}^{\,ac+bd+2bc}x^{a+b}y^{c+d} = \hat{q}^{\,2}\hat{q}^{\,ac}x^ay^c\hat{q}^{\,bd}x^by^d = \hat{q}^{\,2}(g.x)(g.y).\end{aligned}$$

Moreover: $\quad g'.(g.x) = g'.(\hat{q}^{\,ac}x^ay^c) = \hat{q}^{\,ac+a'c'a^2+2b'c'ac+b'd'c^2}x^{a'a+b'c}y^{c'a+d'c}$,

and $\qquad (g'g).x = \hat{q}^{\,(a'a+b'c)(c'a+d'c)}x^{a'a+b'c}y^{c'a+d'c}$;

the exponents of $\hat{q}$ are similar because $a'd' = 1+b'c'$, hence $g'.(g.x) = (g'g).x$. On the same way $g'.(g.y) = (g'g).y$. □

4.2.2 Invariants for multiplicative Poisson stuctures and deformations

Just like the classification of finite subgroups of $\mathrm{SL}_2(\mathbb{C})$ is the starting point for the invariant theory of symplectic Poisson commutative algebra $\mathbb{C}[x,y]$ (see 2.2.1) and through deformation of the Weyl algebra $A_1(\mathbb{C})$ (see 2.2.2 and 3.2.3), on the same way the multiplicative invariant theory deals with the invariants of $\mathbb{C}[x^{\pm1}, y^{\pm1}]$ under the action of classified finite subgroups of $\mathrm{SL}_2(\mathbb{Z})$ and can be extended to the quantum torus by deformation process; the following theorem (from [34]) is a multiplicative analogue of 3.2.3.

Theorem. *For any finite subgroup G of $\mathrm{SL}_2(\mathbb{Z})$ acting by Poisson automorphisms on the commutative algebra $T = \mathbb{C}[x^{\pm1}, y^{\pm1}]$ and by algebra automorphisms on the quantum torus $T_q = \mathbb{C}_q[x^{\pm1}, y^{\pm}]$, with q not a root of one, the noncommutative invariant algebra T_q^G is a deformation of the commutative Poisson invariant algebra T^G.*

$$\begin{tikzcd} \mathbb{C}[x^{\pm1}, y^{\pm1}] \arrow[r, rightsquigarrow, "\text{deformation}"'] & \mathbb{C}_q[x^{\pm1}, y^{\pm1}] \\ \mathbb{C}[x^{\pm1}, y^{\pm1}]^G \arrow[u, hook] \arrow[r, rightsquigarrow, "\text{deformation}"'] & \mathbb{C}_q[x^{\pm1}, y^{\pm1}]^G \arrow[u, hook] \end{tikzcd}$$

Proof. Let B be the noncommutative Laurent polynomial algebra generated by three generators x, y, z and their inverses $x^{\pm1}, y^{\pm1}, z^{\pm1}$ with relations $zx = xz$, $zy = yz$ and $yx = z^2xy$. The element $h := 2(1-z)$ is central and non invertible in B. It is clear that $\mathcal{A} := B/hB$ is isomorphic to the commutative algebra $T = \mathbb{C}[x^{\pm1}, y^{\pm1}]$. We calculate in B the commutator:

$$xy - yx = (1-z^2)xy = 2(1-z)\tfrac{1}{2}(1+z)xy = h\gamma(x,y) \quad \text{with notation } \gamma(x,y) := \tfrac{1}{2}(1+z)xy.$$

In the algebra B/hB, we have $\overline{\gamma(x,y)} = \frac{1}{2}(1+\overline{z})\,\overline{xy} = \overline{xy}$. Hence the algebra isomorphism $\mathcal{A} \simeq T$ is a Poisson isomorphism for the multiplicative Poisson bracket on T; then B is a quantization of $\mathcal{A}$ is the sense of 3.2.1 and $B/(h-\lambda)B$ is a deformation of $\mathcal{A}$ for any $\lambda \in \mathbb{C}$ such that $h-\lambda$ is not invertible in B. In particular for $\lambda = 2(1-\hat{q}\,)$ where $\hat{q}$ is a square root of q^{-1}, the deformed algebra $B/(h-\lambda)B$ is isomorphic to the quantum torus T_q.
By calculations formally similar to the proof of point (ii) in the previous proposition, we easily observe that any subgroup G of $\mathrm{SL}_2(\mathbb{Z})$ acts by automorphisms on B by:

$$g.x = z^{ac}x^ay^c, \quad g.y = z^{bd}x^by^d, \quad g.z = z \quad \text{for } g = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z}).$$

The ideal hB being stable under this action, the action induces an action on $B/hB = \mathcal{A}$ and $B^G/hB^G \simeq \mathcal{A}^G$ by application of the sublemma of 3.2.3. The Poisson isomorphism $\mathcal{A} \simeq T$ being clearly equivariant, it follows that $\mathcal{A}^G \simeq T^G$ and we finally $B^G/hB^G \simeq T^G$ as Poisson algebras. Similarly $B^G/(h-\lambda)B^G \simeq T_q^G$ as associative algebras where $\lambda = 2(1-\hat{q}\,)$. We conclude that B^G is a quantization and T_q^G is a deformation of T^G. In other words, the invariants of the deformed algebra constitute a deformation of the initial invariant algebra. □

Remark. We mention here some general way to obtain invariant elements in T (or T_q mutatis mutandis). Let G be a finite subgroup of $\mathrm{SL}_2(\mathbb{Z})$. Consider the canonical linear action of G on the lattice $\mathbb{Z}^2$, i.e. :

$$g.(m,n) = (am+bn, cm+dn) \quad \text{for } (m,n) \in \mathbb{Z}^2,\ g = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z}). \tag{56}$$

Relation (53) can be rewritten $g.(x^m y^n) = x^i y^j$ where $(i,j) = g.(m,n)$. We introduce the Reynolds operator $\rho_G : T \to T^G$ defined by $f \mapsto \rho_G(f) = \frac{1}{|G|}\sum_{g\in G} g.f$. For any $(m,n) \in \mathbb{Z}^2$, we consider

$$R_{m,n} := \rho_G(x^m y^n) = \frac{1}{|G|} \sum_{(i,j)\in G.(m,n)} x^i y^j \ \in\ T^G, \tag{57}$$

where $G.(m,n)$ is the G-orbit of (m,n). By surjectivity of ρ_G, the $(R_{m,n})_{(m,n)\in\mathbb{Z}^2}$ generate T^G as a $\mathbb{C}$-vector space. Then $\mathbb{Z}^2$ being the disjoint union of its G-orbits, a $\mathbb{C}$-basis of T^G is $(R_{m,n})_{(m,n)\in\Omega}$ where Ω is a set of representatives of the G-orbits. This basis (or its adapted quantum form, see [34] p. 97) provides candidates for generators of T^G as associative algebra or Lie algebra for the Poisson bracket.

Take for instance $G = \{I_2, -I_2\}$, then $R_{m,n} = R_{-m,-n}$ for any $(m,n) \in \mathbb{Z}^2$; we choose $\Omega = (\mathbb{N}^* \times \mathbb{Z}) \cup (\{0\} \times \mathbb{N})$. It's an exercise to check (by induction from identities verified by the $R_{m,n}$'s) that any $R_{m,n}$ such that $(m,n) \in \Omega$ lies in the subalgebra of T^G generated over $\mathbb{C}$ by $\xi_1 := 2R_{1,0} = x + x^{-1}$, $\xi_2 := 2R_{0,1} = y + y^{-1}$ and $\theta := 2R_{1,1}xy + x^{-1}y^{-1}$; hence T^G is generated as a $\mathbb{C}$-algebra by ξ_1, ξ_2, θ. Similarly, T^G is generated as a Lie algebra for the bracket (42) by the five elements $R_{0,0} = 1$, ξ_1, ξ_2, θ and $R_{2,0} = \frac{1}{2}(x^2 + x^{-2})$.

Comment. Previous results open the way for a wide program of systematic study of multiplicative/quantum invariants in parallel with the more classical symplectic/Weyl theory. This program is greatly initiated in [34]. We couldn't think of developing it here with details and proofs but it seems interesting to give some brief overview about the obtained results.

• 1. The classification up to conjugation of finite subgroups of $\mathrm{GL}_2(\mathbb{Z})$ is well known; the description of the twelve types (classically denoted $\mathcal{G}_1$ to $\mathcal{G}_{12}$) can be found in [12]. In particular the finite subgroups of $\mathrm{SL}_2(\mathbb{Z})$ correspond to the four (all cyclic) cases:

$$\mathcal{G}_7 = \langle \mathrm{x} \rangle \simeq \mathrm{C}_6, \quad \mathcal{G}_8 = \langle \mathrm{ds} \rangle \simeq C_4, \quad \mathcal{G}_9 = \langle \mathrm{x}^2 \rangle \simeq \mathrm{C}_3, \quad \mathcal{G}_{10} = \langle \mathrm{x}^3 \rangle \simeq \mathrm{C}_2,$$

where $\mathrm{x} = \left(\begin{smallmatrix} 1 & -1 \\ 1 & 0 \end{smallmatrix}\right)$, $\mathrm{d} = \left(\begin{smallmatrix} -1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$ and $\mathrm{s} = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$ are the three basic matrices used in the description of any finite subgroup of $\mathrm{GL}_2(\mathbb{Z})$. Explicitly:

$$\mathrm{x} = \left(\begin{smallmatrix} 1 & -1 \\ 1 & 0 \end{smallmatrix}\right), \quad \mathrm{ds} = \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right), \quad \mathrm{x}^2 = \left(\begin{smallmatrix} 0 & -1 \\ 1 & -1 \end{smallmatrix}\right), \quad \mathrm{x}^3 = \left(\begin{smallmatrix} -1 & 0 \\ 0 & -1 \end{smallmatrix}\right).$$

Remark. It is easy to verify that the property for any finite subgroup G of $\mathrm{SL}_2(\mathbb{Z})$ to be conjugated in $\mathrm{GL}_2(\mathbb{Z})$ to a $\mathcal{G}_i$ ($i = 7, 8, 9, 10$) implies that G is conjugated to $\mathcal{G}_i$ in $\mathrm{SL}_2(\mathbb{Z})$ (see [34] p. 73). Denoting by h an element in $\mathrm{SL}_2(\mathbb{Z})$ such that $G = h^{-1}\mathcal{G}_i h$, the assignment $P \mapsto h.P$ defines a Poisson isomorphism $\mathbb{C}[x^{\pm 1}, y^{\pm 1}]^G \to \mathbb{C}[x^{\pm 1}, y^{\pm 1}]^{\mathcal{G}_i}$. In conclusion, in the study of invariants of $\mathbb{C}[x^{\pm 1}, y^{\pm 1}]$ under the action of a finite subgroup G of $\mathrm{SL}_2(\mathbb{Z})$ we can suppose without restriction thet G is one of the $\mathcal{G}_7, \mathcal{G}_8, \mathcal{G}_9, \mathcal{G}_{10}$.

• 2. Just like the Kleinian surfaces for the case of finite groups of $\mathrm{SL}_2(\mathbb{C})$ acting on $\mathbb{C}[x, y]$, the invariant subalgebra $\mathbb{C}[x^{\pm 1}, y^{\pm 1}]^G$ for each type of finite subgroup of $\mathrm{SL}_2(\mathbb{Z})$ acting by (52) and (53) is generated (as an associative algebra) by three elements with one relation. From [12], we have:

G	generators of $\mathbb{C}[x^{\pm 1}, y^{\pm 1}]^G$ and relation
$\mathcal{G}_{10} \simeq C_2$	$\xi_1 = x + x^{-1}, \quad \xi_2 = y + y^{-1}, \quad \theta = xy + x^{-1}y^{-1}$ $\theta\xi_1\xi_2 = \theta^2 + \xi_1^2 + \xi_2^2 - 4$
$\mathcal{G}_9 \simeq C_3$	$\eta_+ = x + y + x^{-1}y^{-1}, \quad \eta_- = x^{-1} + y^{-1} + xy, \quad \varphi = xy^2 + x^{-2}y^{-1} + xy^{-1} + 6$ $\varphi\eta_+\eta_- = \eta_+^3 + \eta_-^3 + \varphi^2 - 9\varphi + 27$
$\mathcal{G}_8 \simeq C_4$	$\sigma_1 = \xi_1 + \xi_2, \quad \sigma_2 = \xi_1\xi_2, \quad \rho = xy^2 + x^{-1}y^{-2} + x^2y^{-1} + x^{-2}y + 3\sigma_1$ $\rho^2 = \rho\sigma_1(\sigma_2 + 4) + 4\sigma_1^2\sigma_2 - \sigma_1^4 - \sigma_2(\sigma_2 + 4)^2$
$\mathcal{G}_7 \simeq C_6$	$\tau_1 = \eta_+ + \eta_-, \quad \tau_2 = \eta_+\eta_-, \quad \sigma = \eta_+\varphi + \eta_-(x^{-1}y^{-2} + x^2y + x^{-1}y + 6)$ $\sigma^2 = \tau_1(\tau_2 + 9)\sigma - \tau_2(\tau_2 + 9)^2 + (\tau_1^2 - 4\tau_2)(3\tau_1\tau_2 - \tau_1^3 - 27)$

The surfaces in the 3-dimensional affine space corresponding to the algebraic relation between the three generators in each case are studied in [34] (in particular the type of the isolated singularities are determined).

• 3. The next step concerns the finiteness of the Lie structures on invariants. From one hand $\mathbb{C}[x^{\pm 1}, y^{\pm 1}]^G$ is finitely generated as a Lie algebra for the multiplicative Poisson bracket (42); this is a multiplicative analogue of the first proposition of 3.3.2. From the other hand the same is true after quantum deformation,i.e. $\mathbb{C}_q[x^{\pm 1}, y^{\pm 1}]^G$ is finitely generated as a Lie algebra for the commutator bracket; this is a multiplicative analogue of theorem 3.3.3. Moreover the cardinality of a generating family of the Lie algebra $(\mathbb{C}[x^{\pm 1}, y^{\pm 1}]^G, \{\cdot, \cdot\})$ and a generating family of the Lie algebra $(\mathbb{C}_q[x^{\pm 1}, y^{\pm 1}]^G, [\cdot, \cdot])$ calculated in [34] are the same (by type: 5 for $\mathcal{G}_{10}$, 7 for $\mathcal{G}_9$, 8 for $\mathcal{G}_8$, 9 for $\mathcal{G}_7$).

• 4. The multiplicative analogue of the last remark of 3.2.3 leads to compare the dimensions of $\mathrm{HH}_0(\mathbb{C}_q[x^{\pm 1}, y^{\pm 1}]^G)$ and $\mathrm{HP}_0(\mathbb{C}[x^{\pm 1}, y^{\pm 1}]^G)$.
- The answer is complete for G of type $\mathcal{G}_{10}$. In this case:

$$\dim_{\mathbb{C}} \mathrm{HH}_0(\mathbb{C}_q[x^{\pm 1}, y^{\pm 1}]^G) = \dim_{\mathbb{C}} \mathrm{HP}_0(\mathbb{C}[x^{\pm 1}, y^{\pm 1}]^G) = 5.$$

The last equality also proves (see theorem 3.3.2 for the symplectic analogue) that the family of five Lie algebra generators cited above is of minimal cardinality.
- In the other three cases, the determination of $\dim_{\mathbb{C}} \mathrm{HH}_0(\mathbb{C}_q[x^{\pm 1}, y^{\pm 1}]^G)$ can be found in [34]; the dimension is 7 for $\mathcal{G}_9$, 8 for $\mathcal{G}_8$ and 9 for $\mathcal{G}_7$.

5 Localization: actions on noncommutative rational functions

5.1 Commutative rational invariants

Preliminary remark: *extension of an action to the field of fractions.* Let S be a commutative ring. Assume that S is a domain and consider $F = \operatorname{Frac} S$ the field of fractions of S. Any automorphism of S extends into an automorphism of F and it's obvious that, for any subgroup G of $\operatorname{Aut} S$, we have $\operatorname{Frac} S^G \subseteq F^G$. For finite G, the converse is true:

Proposition. *If G is a finite subgroup of automorphisms of a commutative domain S with field of fractions F, then we have:* $\operatorname{Frac} S^G = F^G$.

Proof. For any $x \in F^G$, there exist $a, b \in S$, $b \neq 0$, such that $x = \frac{a}{b}$. Define $b' = \prod_{g \in G, g \neq \mathrm{id}_S} g(b)$. Then $bb' \in S^G$ and $x = \frac{ab'}{bb'}$, with $ab' = x(bb') \in F^G \cap S = S^G$. □

This applies in particular to a polynomial algebra $S = \Bbbk[x_1, \ldots, x_n]$ and its field of rational functions $F = \Bbbk(x_1, \ldots, x_n)$, and we formulate in this case the following problem about the structure of F^G.

5.1.1 Noether's problem

Let $\Bbbk$ be commutative field of characteristic zero. Let G be a finite subgroup of $\mathrm{GL}_n(\Bbbk)$ acting canonically by linear automorphisms on $S = \Bbbk[x_1, \ldots, x_n]$, and then on $F = \operatorname{Frac} S = \Bbbk(x_1, \ldots, x_n)$. We consider the subfield $F^G = \operatorname{Frac} S^G$ of F.

> Remark 1. It's well known (by Artin's lemma, see for instance [11] page 194) that $[F : F^G] = |G|$, and then $\operatorname{trdeg}_{\Bbbk} F^G = \operatorname{trdeg}_{\Bbbk} F = n$.

> Remark 2. We know from classical invariant theory that S^G is finitely generated (say by m elements) as a $\Bbbk$-algebra. Thus F^G is finitely generated (say by p elements) as a field extension of $\Bbbk$, with $p \leq m$. We can have $p < m$; example: $S = \Bbbk(x, y)$ and $G = \langle g \rangle$ for $g : x \mapsto -x, y \mapsto -y$, then $S^G = \Bbbk[x^2, y^2, xy] = \Bbbk[X, Y, Z]/(Z^2 - XY)$ and $F^G = \Bbbk(xy, x^{-1}y)$.

> Remark 3. Suppose that S^G is not only finitely generated, but isomorphic to a polynomial algebra $\Bbbk[y_1, \ldots, y_m]$, with $y_1, \ldots, y_m$ algebraically independent over $\Bbbk$. Then we have $F^G = \Bbbk(y_1, \ldots, y_m)$. Thus $m = n$ by remark 1.

Now we can consider the main question:

Problem (Noether's problem) : is F^G a purely transcendental extension of $\Bbbk$?

An abundant literature has been devoted (and is still devoted) to this question and it's out of the question to give here a comprehensive presentation of it. We just point out the following facts.

• The answer is positive if S^G is a polynomial algebra. By remark 3, we have then $S^G = \mathbb{k}[x_1, \dots, x_n]$ and $F^G = \mathbb{k}(x_1, \dots, x_n)$. This is in particular the case when G is the symmetric group S_n acting by permutation of the x_j's (see for instance [10] p. 3), or more generally when Shephard-Todd and Chevalley theorem applies.

• The answer is positive if $n = 1$. This is an obvious consequence of Lüroth's theorem (see [9] p. 520): if $F = \mathbb{k}(x)$ is a purely transcendental extension of degree 1 of $\mathbb{k}$, then for any intermediate subfield $\mathbb{k} \subsetneq L \subset F$, there exists some $v \in F$ transcendental over $\mathbb{k}$ such that $F = \mathbb{k}(v)$.

• The answer is positive if $n = 2$. This is an obvious consequence of Castelnuovo's theorem (see [9] p. 523): if $F = \mathbb{k}(x, y)$ is a purely transcendental extension of degree 2 of $\mathbb{k}$, then for any intermediate subfield $\mathbb{k} \subsetneq L \subset F$ such that $[F : L] < +\infty$, there exists some $v, w \in F$ such that $F = \mathbb{k}(v, w)$ is purely transcendental of degree 2.

• The answer is positive for all $n \geq 1$ when G is abelian and $\mathbb{k}$ is algebraically closed. This is a classical theorem by E. Fischer (1915), see [37] for a proof, or corollary 2 in 5.1.2 below.

Among other cases of positive results, we can cite the cases where G is any subgroup of S_n for $1 \leq n \leq 4$, the case where $G = A_5$ for $n = 5$ by Sheperd-Barron or Maeda (see [50] and [55]), the case where G is the cyclic group of order n in S_n for $1 \leq n \leq 7$ and $n = 11$. The first counterexamples (Swan 1969, Lenstra 1974) were for $\mathbb{k} = \mathbb{Q}$ (and G the cyclic group of order n in S_n for $n = 47$ and $n = 8$ respectively). D. Saltmann produced in 1984 the first counter-example for $\mathbb{k}$ algebraically closed (see [50], [63], [64]).

5.1.2 Miyata's theorem

The following result concerns invariants under actions on rational functions resulting from an action on polynomials.

THEOREM (T. MIYATA). *Let K be a commutative field, $S = K[x]$ the commutative ring of polynomials in one variable over K, and $F = K(x)$ the field of fractions of S. Let G be a group of ring automorphisms of S such that $g(K) \subseteq K$ for any $g \in G$.*

(i) *if $S^G \subseteq K$, then $F^G = S^G = K^G$.*

(ii) *if $S^G \not\subset K$, then for any $u \in S^G, u \notin K$ of degree $m = \min\{\deg_x y\,;\, y \in S^G, y \notin K\}$ we have $S^G = K^G[u]$ and $F^G = K^G(u)$.*

We don't give a proof of this theorem here, because we will prove it further (see 5.3) in the more general context of Ore extensions; for a self-contained proof on the commutative case, we refer the reader to [50] or [56]. Observe that the group G is not necessarily finite.

COROLLARY 1 (W. BURNSIDE). *The answer to Noether's problem is positive if $n = 3$.*

PROOF. Let G be a finite subgroup of $\mathrm{GL}_3(\mathbb{k})$ acting linearly on $S = \mathbb{k}[x, y, z]$. We introduce in $F = \mathbb{k}(x, y, z)$ the subalgebra $S_1 = \mathbb{k}(\frac{y}{x}, \frac{z}{x})[x]$, which satisfies $\mathrm{Frac}\, S_1 = F$. Let $g \in G$. We have:

$$g(x) = \alpha x + \beta y + \gamma z, \quad g(y) = \alpha' x + \beta' y + \gamma' z, \quad g(z) = \alpha'' x + \beta'' y + \gamma'' z.$$

Thus:

$$g(\tfrac{y}{x}) = \frac{\alpha' + \beta' \frac{y}{x} + \gamma' \frac{z}{x}}{\alpha + \beta \frac{y}{x} + \gamma \frac{z}{x}} \quad \text{and} \quad g(\tfrac{z}{x}) = \frac{\alpha'' + \beta'' \frac{y}{x} + \gamma'' \frac{z}{x}}{\alpha + \beta \frac{y}{x} + \gamma \frac{z}{x}}.$$

It follows that the subfield $K = \Bbbk(\frac{z}{x}, \frac{y}{x})$ is stable under the action of G, and we can apply the theorem to the algebra $S_1 = K[x]$. The finiteness of G implies that $[F : F^G]$ is finite and so $S_1^G \not\subset K$. Thus we are in the second case of the theorem. There exists $u \in S_1^G$ of minimal degree ≥ 1 such that $S_1^G = K^G[u]$ and $F^G = K^G(u)$. By Castelnuovo's theorem (see in 5.1.1 above), $K^G = \Bbbk(v; w)$ is purely transcendental of degree two, and then $F^G = \Bbbk(v, w)(u) = \Bbbk(u, v, w)$. $\square$

Of course, we can prove similarly that the answer to Noether's problem is positive if $n = 2$ using Lüroth's theorem instead of Castelnuovo's theorem.

Corollary 2 (E. Fischer). *If $\Bbbk$ is algebraically closed, the answer to Noether's problem is positive for G abelian.*

Proof. Here we assume that G is a finite abelian subgroup of $\mathrm{GL}_n(\Bbbk)$. By total reducibility and Schur's lemma (see 2.3.1) we can suppose up to conjugation that there exist complex characters $\chi_1, \ldots, \chi_n$ of G such that $g(x_j) = \chi_j(g) x_j$ for all $1 \leq j \leq n$ and all $g \in G$. In particular, G acts on $S_1 = \Bbbk(x_2, \ldots, x_n)[x_1]$ stabilizing $K_1 = \Bbbk(x_2, \ldots, x_n)$; thus $\Bbbk(x_1, \ldots, x_n)^G = K_1^G(u_1)$ for some $u_1 \in S_1^G$. We apply then Miyata's theorem inductively to conclude. $\square$

Another application due to E. B. Vinberg concerns the rational finite dimensional representations of solvable connected linear algebraic groups and uses Lie-Kolchin theorem about triangulability of such representations in order to apply inductively Miyata's theorem (see [68] for more details).

5.2 Noncommutative rational functions

5.2.1 Skewfields of fractions for noncommutative noetherian domains

Let A be a ring (non necessarily commutative). Assume that A is a domain; then the set $S = \{a \in A\,; a \neq 0\}$ is multiplicative. We say that S is a (left and right) Ore set if it satisfies the two properties:

$$[\forall\ (a, s) \in A \times S,\ \exists\ (b, t) \in A \times S,\ at = sb]$$
$$\text{and}\quad [\forall\ (a, s) \in A \times S,\ \exists\ (b', t') \in A \times S,\ t'a = b's].$$

In this case, we define an equivalence on $A \times S$ by $(a, s) \sim (b, t)$ if there exist $c, d \in A$ such that $ac = bd$ and $sc = td$. The factor set $D = (A \times S)/\sim$ is canonically equipped with a structure of skewfield (or noncommutative division ring), which is the smallest skewfield containing A. We name D the skewfield of fractions of A, denoted by $\operatorname{Frac} A$. Concretely, we have:

$$\forall\ q \in \operatorname{Frac} A, \quad [\exists\ (a, s) \in A \times S,\ q = as^{-1}] \ \text{ and } \ [\exists\ (b, t) \in A \times S,\ q = t^{-1}b], \qquad (58)$$

and more generally:

$$\forall\ q_1, \ldots, q_k \in \operatorname{Frac} A, \begin{cases} \exists\ a_1, \ldots, a_k, b_1, \ldots b_k \in A, \\ \exists\ s, t \in S, \end{cases} \quad q_i = a_i s^{-1} = t^{-1} b_i \text{ for } 1 \leq i \leq k. \quad (59)$$

We refer the reader to [7], [8], [13] for more details on this standard construction. An important point is that noetherianity is a sufficient condition for A to admit such a skewfield of fractions.

Lemma. *Any noetherian domain admits a skewfield of fractions.*

Proof. Let $(a, s) \in A \times S$, $a \neq 0$, where S is the set of nonzero elements of A. For any integer $n \geq 0$, denote by I_n the left ideal generated by $a, as, as^2, \dots, as^n$. We have $I_n \subseteq I_{n+1}$ for all $n \geq 0$. Since A is noetherian, there exists some $m \geq 0$ such that $I_m = I_{m+1}$. In particular, $as^{m+1} = c_0 a + c_1 as + \cdots + c_m as^m$ for some $c_0, c_1, \dots, c_m \in A$. Denote by k the smallest index such that $c_k \neq 0$. Because A is a domain, we can simplify by s^k and write $as^{m+1-k} = c_k a + c_{k+1} as + \cdots + c_m as^{m-k}$. With $t' = c_k \in S$ and $b' = as^{m-k} - c_{k+1} a - \cdots - c_m as^{m-k-1}$, we conclude that $t'a = b's$. So S is a left Ore set; the proof is similar on the right. □

Remark 1. Many results which are very simple for commutative fields of fractions become more difficult for skewfields. This is the case for instance of the following noncommutative analogue of the preliminary proposition of 5.1:

let R be a domain satisfying the left and right Ore conditions, let F be the skewfield of fractions of R, let G be a finite subgroup of automorphisms of R such that $|G|$ is invertible in R, then R^G satisfies the left and right Ore conditions and we have $\operatorname{Frac} R^G = F^G$.

> Sketch of the proof. We start with a preliminary observation. Let I and J be two nonzero left ideals of R. Take $a \in I, a \neq 0, s \in J, s \neq 0$. Since R satisfies the left Ore condition, there exist b', t' nonzero in R such that $t'a = b's$. This element is nonzero (since R is a domain) and lies in $I \cap J$. By induction, we prove similarly that: the intersection of any family of nonzero left ideals of R is a nonzero left ideal of R.
>
> Now fix a nonzero element $x \in F^G$. By (58), there exist nonzero elements $b, t \in R$ such that $x = t^{-1}b$. It's clear that $I = \bigcap_{g \in G} g(Rt)$ is a left ideal of R which is stable under the action of G. Then we can apply Bergman's and Isaacs' theorem (see corollary 1.5 in [14] or original paper [36] for a proof of this nontrivial result) to deduce that I contains a nontrivial fixed point. In other words, there exists a nonzero element v in $R^G \cap I$. In particular $v \in Rt$ can be written $v = dt$ for some nonzero $d \in R$, and so $x = t^{-1}b = t^{-1}d^{-1}db = v^{-1}db$. Since $x \in F^G$ and $v \in R^G$, we have $db = vx \in F^G \cap R = R^G$. Denoting $u = db$, we have proved that: any nonzero $x \in F^G$ can be written $x = v^{-1}u$ with v and u nonzero elements of R^G.
>
> Finally, let a, s be two nonzero elements of R^G. Then $x = st^{-1} \in F^G$. By the second step, there exist $u, v \in R^G$ such that $st^{-1} = v^{-1}u$, and then $vs = ut$. This proves that R^G satisfies the left Ore condition. The proof is similar on the right. Therefore R^G admits a skewfield of fractions and the equality $\operatorname{Frac} R^G = F^G$ is clear from the second step of the proof. □

Remark 2. There exists a noncommutative analogue of Galois theory. We cannot develop it here, but just mention the following version of Artin's lemma (see remark 1 of 5.1.1): *Let D be a skewfield and G a finite group of automorphisms of D. Then $[D : D^G] \leq |G|$. If moreover G doesn't contain any non trivial inner automorphism, then $[D : D^G] = |G|$.*

> We refer the reader to [4] (theorem 3.3.7) or [14] (lemma 2.18). □

5.2.2 Noncommutative rational functions

Let A a ring, σ an automorphism of A, δ a σ-derivation of A, and $R = A[x\,;\,\sigma,\delta]$ the associated Ore extension. We have seen in 1.3.1 that R is a domain when A is a domain, and in 1.3.2 that R is noetherian when A is noetherian. So we conclude by the lemma of 5.2.1 that, if A is a noetherian domain, then the Ore extension $R = A[x\,;\,\sigma,\delta]$ admits a skewfield of fractions. Denoting $K = \operatorname{Frac} A$, it's easy to check that σ and δ extend uniquely into an automorphism and a σ-derivation of K, and we can then consider the Ore extension $S = K[x\,;\,\sigma,\delta]$. It follows from (59) that any polynomial $f \in S$ can be written $f = gs^{-1} = t^{-1}h$ with s,t nonzero in A and $g,h \in R$. We deduce that $\operatorname{Frac} R = \operatorname{Frac} S$. This skewfield is denoted by $K(x\,;\,\sigma,\delta)$.

$$\begin{array}{l} \text{If} \qquad \operatorname{Frac} A = K, \quad R = A[x\,;\,\sigma,\delta], \quad S = K[x\,;\,\sigma,\delta], \\ \text{then: } D = \operatorname{Frac} R = \operatorname{Frac} S = K(x\,;\,\sigma,\delta). \end{array} \tag{60}$$

In the case of an iterated Ore extension (5) over a base field $\Bbbk$, we have by induction: if $R_m = \Bbbk[x_1][x_2\,;\,\sigma_2,\delta_2]\cdots[x_m\,;\,\sigma_m,\delta_m]$, then $\operatorname{Frac} R_m = \Bbbk(x_1)(x_2\,;\,\sigma_2,\delta_2)\cdots(x_m\,;\,\sigma_m,\delta_m)$. We simply denote $D = K(x\,;\,\sigma)$ when $\delta = 0$ and $D = K(x\,;\,\delta)$ when $\sigma = \mathrm{id}_A$.

Remark. It's useful in many circumstances to observe (see proposition 8.7.1 of [3]) that $K(x\,;\,\sigma,\delta)$ can be embedded into the skewfield $F = K(\!(x^{-1}\,;\,\sigma^{-1},-\delta\sigma^{-1})\!)$ whose elements are the Laurent series $\sum_{j\geq m}\alpha_j x^{-j}$ with $m \in \mathbb{Z}$ and $\alpha_j \in K$, with the commutation law:

$$x^{-1}\alpha = \sum_{n\geq 1}\sigma^{-1}(-\delta\sigma^{-1})^{n-1}(\alpha)x^{-n} = \sigma^{-1}(\alpha)x^{-1} - x^{-1}\delta\sigma^{-1}(\alpha)x^{-1} \quad \text{for all } \alpha \in K.$$

Indeed, multiplying on the left and the right by x, we obtain the commutation law of $S = K[x\,;\,\sigma,\delta]$; then S appears as a subring of F, and so D is a subfield of F. In particular, for $\delta = 0$, we denote $F = K(\!(x^{-1}\,;\,\sigma^{-1})\!)$ and just have: $x^{-1}\alpha = \sigma^{-1}(\alpha)x^{-1}$. If $\sigma = \mathrm{id}_K$, then $F = K(\!(x^{-1}\,;\,-\delta)\!)$ is a pseudo-differential operator skewfield, with commutation law:

$$x^{-1}\alpha = \alpha x^{-1} - \delta(\alpha)x^{-2} + \cdots + (-1)^n\delta^n(\alpha)x^{-n-1} + \cdots = \alpha x^{-1} - x^{-1}\delta(\alpha)x^{-1}.$$

It follows from the embedding of D into $K(\!(x^{-1}\,;\,\sigma^{-1},-\delta\sigma^{-1})\!)$ that D is canonically equipped with the discrete valuation $v_{x^{-1}}$, or more simply v, satisfying $v(s) = -\deg s$ for all $s \in S$.

Lemma. *Let K be a skewfield, with center $Z(K)$.*

(i) *Let σ be an automorphism of K. Assume that, for all $n \geq 1$, the automorphism σ^n is not inner. Then the center $Z(D)$ of $D = K(x\,;\,\sigma)$ is the subfield $Z(K) \cap K^\sigma$, where $K^\sigma = \{a \in K\,;\,\sigma(a) = a\}$.*

(ii) *Let δ be a derivation of K. Assume that K is of characteristic zero and δ is not inner. Then the center $Z(D)$ of $D = K(x\,;\,\delta)$ is $Z(K) \cap K_\delta$, where $K_\delta = \{a \in K\,;\,\delta(a) = 0\}$.*

Proof. In the embedding of $D = K(x\,;\sigma)$ in $F = K(\!(x^{-1}\,;\sigma^{-1})\!)$, any element $f \in D$ can be written $f = \sum_{j\geq m} \alpha_j x^{-j}$ with $m \in \mathbb{Z}$ and $\alpha_j \in K$ for all $j \geq m$. Assume that f is central. Then $xf = fx$ and $\alpha f = f\alpha$ for any $\alpha \in K$. This is equivalent to $\alpha_j \in K^\sigma$ and $\alpha\alpha_j = \alpha_j\sigma^{-j}(\alpha)$ for all $j \geq m$. Since σ^j is not inner, we necessarily have $\alpha_j = 0$ for $j \neq 0$. This achieve the proof of (i). Under the assumptions of point (ii), let us consider now an element $f \in D = K(x\,;\delta) \subseteq F = K(\!(x^{-1}\,;-\delta)\!)$. From the relation $\alpha f = f\alpha$ for any $\alpha \in K$, we deduce using the fact that δ is not inner that $f \in K$, and so $f \in Z(K)$. Then $f \in K_\delta$ follows from the relation $fx = xf$. $\square$

5.2.3 Weyl skewfields

We fix a commutative base field $\Bbbk$.

• We consider firstly as in (12) the first Weyl algebra $A_1(\Bbbk) = \Bbbk[q][p\,;\partial_q] = \Bbbk[p][q\,;-\partial_p]$. Its skewfield of fractions is named the first Weyl skewfield, classically denoted by $D_1(\Bbbk)$:

$$D_1(\Bbbk) = \operatorname{Frac} A_1(\Bbbk) = \Bbbk(q)(p\,;\partial_q) = \Bbbk(p)(q\,;-\partial_p). \tag{61}$$

It would be useful in many circumstances to give another presentation of $D_1(\Bbbk)$. Set $w = pq$; it follows from relation $pq - qp = 1$ that $wq = qw + q$ and $pw = (w+1)p$. Thus the subalgebra of $A_1(\Bbbk)$ generated by q and w, and the subalgebra of $A_1(\Bbbk)$ generated by p and w are both isomorphic to the enveloping algebra $U_1(\Bbbk)$ defined in example (ii) of 1.3.1. It's clear that $\operatorname{Frac} A_1(\Bbbk) = \operatorname{Frac} U_1(\Bbbk)$. We conclude:

$$D_1(\Bbbk) = \Bbbk(q)(w\,;d), \quad \text{with } d = q\partial_q \text{ the Euler derivation in } \Bbbk(q), \tag{62}$$

$$D_1(\Bbbk) = \Bbbk(w)(p\,;\sigma), \quad \text{with } \sigma \in \operatorname{Aut}\Bbbk(w) \text{ defined by } \sigma(w) = w+1. \tag{63}$$

Applying the last lemma in 5.2.2, we obtain:

$$\text{if } \Bbbk \text{ is of characteristic zero, then } Z(D_1(\Bbbk)) = \Bbbk. \tag{64}$$

The situation where $\Bbbk$ is of characteristic $\ell > 0$ is quite different, and out of our main interest here, since $D_1(\Bbbk)$ is then of finite dimension ℓ^2 over its center $\Bbbk(p^\ell, q^\ell)$.

• We defined similarly the n-th Weyl skewfield $D_n(\Bbbk) = \operatorname{Frac} A_n(\Bbbk)$. Using (8,14), we write:

$$D_n(\Bbbk) = \Bbbk(q_1, q_2, \ldots, q_n)(p_1\,;\partial_{q_1})(p_2\,;\partial_{q_2})\ldots(p_n\,;\partial_{q_n}), \tag{65}$$

$$D_n(\Bbbk) = \Bbbk(q_1)(p_1\,;\partial_{q_1})(q_2)(p_2\,;\partial_{q_2})\ldots(q_n)(p_n\,;\partial_{q_n}). \tag{66}$$

Reasoning as above on the products $w_i = p_iq_i$ for all $1 \leq i \leq n$, which satisfy the relations

$$p_iw_i - w_ip_i = p_i, \qquad w_iq_i - q_iw_i = q_i, \qquad [p_i, w_j] = [q_i, w_j] = [w_i, w_j] = 0 \text{ si } j \neq i, \tag{67}$$

we obtain the alternative presentations:

$$D_n(\Bbbk) = \Bbbk(q_1, q_2, \ldots, q_n)(w_1\,;d_1)(w_2\,;d_2)\ldots(w_n\,;d_n), \tag{68}$$

with d_i the Euler derivative $d_i = q_i\partial_{q_i}$ for all $1 \leq i \leq n$, and:

$$D_n(\Bbbk) = \Bbbk(w_1, w_2, \dots, w_n)(p_1\,;\,\sigma_1)(p_2\,;\,\sigma_2)\cdots(p_n\,;\,\sigma_n), \tag{69}$$

where each automorphism σ_i is defined on $\Bbbk(w_1, w_2, \dots, w_n)$ by $\sigma_i(w_j) = w_j + \delta_{i,j}$, and fixes the p_j's for $j < i$.

• If we replace $\Bbbk$ by a purely transcendental extension $K = \Bbbk(z_1, z_2, \dots, z_t)$ of degree t of $\Bbbk$, the skewfield $D_n(K)$ is denoted by $\mathcal{D}_{n,t}(\Bbbk)$. By convention, we set $\mathcal{D}_{0,t}(\Bbbk) = K$. To sum up:

$$\mathcal{D}_{n,t}(\Bbbk) = D_n(\Bbbk(z_1, \dots, z_t)) \quad \text{for all} \quad t \geq 0, n \geq 0. \tag{70}$$

One can prove using inductively the last lemma of 5.2.2 (see also [47] or [25]) that:

$$\text{if } \Bbbk \text{ is of characteristic zero, then } Z(\mathcal{D}_{n,t}(\Bbbk)) = \Bbbk(z_1, \dots, z_t). \tag{71}$$

Comment. The skewfields $\mathcal{D}_{n,t}$ play a fundamental role in Lie theory and are in the center of an important conjecture (the Gelfand-Kirillov conjecture) on rational equivalence of enveloping algebras: for "many" classes of algebraic Lie algebras $\mathfrak{g}$ the skewfield of fractions of the enveloping algebra $U(\mathfrak{g})$ is isomorphic to a Weyl skewfield $\mathcal{D}_{n,t}(\Bbbk)$ (see [47], I.2.11 of [2], [31], [25], [26], [60], and 5.4 below).

• Finally, for any $q \in \Bbbk^\times$, the skewfield of fractions the quantum plane $\Bbbk_q[x, y]$ defined in example (iv) of 1.3.1 is sometimes called the first quantum Weyl skewfield, denoted by:

$$D_1^q(\Bbbk) = \operatorname{Frac}\Bbbk_q[x, y] = \Bbbk_q(x, y) = \Bbbk(y)(x\,;\,\sigma) \text{ where } \sigma \in \operatorname{Aut}\Bbbk(y) \text{ with } \sigma(y) = qy. \tag{72}$$

These skewfields (or more generally their n-dimensional versions as in the last example of 1.3.1) play for the quantum algebras a role similar to the one of Weyl skewfields in classical Lie theory (see II.10.4 of [2], [25], [60]). It follows from last lemma in 5.2.2, that:

$$\text{if } q \text{ is not a root of one in } \Bbbk, \text{ then } Z(D_1^q(\Bbbk)) = \Bbbk. \tag{73}$$

The situation where q is of finite order $\ell > 0$ on $\Bbbk^\times$ is quite different, and out of our main interest here, since $D_1^q(\Bbbk)$ is then of finite dimension ℓ^2 over its center $\Bbbk(p^\ell, q^\ell)$.

Let us recall that the first quantum Weyl algebra (see example (v) of 1.3.1) is the algebra $A_1^q(\Bbbk)$ generated by x and y with commutation law $xy - qyx = 1$. We observe that the element $z = xy - yx = (q-1)yx + 1$ satisfies the relation $zy = qyz$. Since $x = (q-1)^{-1}y^{-1}(z-1)$, $\operatorname{Frac}A_1^q(\Bbbk)$ is equal to the subfield generated by z and y, which is clearly isomorphic to $D_1^q(\Bbbk)$. Thus we have proved that:

$$\operatorname{Frac}A_1^q(\Bbbk) \simeq D_1^q(\Bbbk). \tag{74}$$

5.3 Noncommutative rational invariants

5.3.1 Noncommutative analogue of Miyata's theorem

We can now formulate for Ore extensions an analogue of theorem 5.1.2. We start with a technical lemma.

LEMMA. *Let K be a non necessarily commutative field, σ an automorphism and δ a σ-derivation of K. We consider the Ore extension $S = K[x\,;\,\sigma,\delta]$. Take $u \in S$ such that $\deg_x(u) \geq 1$.*

(i) For any non necessarily commutative subfield L of K, the family $\mathcal{U} = \{u^i\,;\,i \in \mathbb{N}\}$ is right and left free over L.

(ii) If the left free L-module T generated by $\mathcal{U}$ is a subring of S, then there exist an ring endomorphism σ' and a σ'-derivation δ' of L such that $T = L[u\,;\,\sigma',\delta']$. If moreover T is equal to the right free L-module T' generated by $\mathcal{U}$, then σ' is an automorphism de L.

(iii) In the particular case where K is commutative, then σ' is the restriction of σ^m to L, with $m = \deg_x(u)$.

PROOF. Point (i) is straightforward considering the term of highest degree in a left L-linear sum of a finite number of elements of $\mathcal{U}$. Consider now $\alpha \in L \subseteq T$. We have $\deg_x(u\alpha) = \deg_x u$ and $u\alpha \in T$; thus there exist uniquely determined $\alpha_0, \alpha_1 \in L$ such that $u\alpha = \alpha_0 + \alpha_1 u$. So we define two $L \to L$ maps $\sigma' : \alpha \longmapsto \alpha_1$ and $\delta' : \alpha \longmapsto \alpha_0$ satisfying $u\alpha = \sigma'(\alpha)u + \delta'(\alpha)$ for all $\alpha \in L$. Denoting $u = \lambda_m x^m + \cdots + \lambda_1 x + \lambda_0$ with $m \geq 1$, $\lambda_i \in K$ for any $0 \leq i \leq m$ and $\lambda_m \neq 0$, then $\lambda_m \sigma^m(\alpha) = \sigma'(\alpha)\lambda_m$ for all $\alpha \in L$. We deduce that σ' is a ring endomorphism of L, and prove also point (iii). The associativity and distributivity in the ring T imply that δ' is a σ'-derivation. When $T' = T$, there exists for all $\beta \in L$ two elements β_1 and β_0 in L such that $\beta u = u\beta_1 + \beta_0 = \sigma'(\beta_1)u + \delta'(\beta_1) + \beta_0$. Thus $\beta = \sigma'(\beta_1)$ and σ' is surjective. $\square$

THEOREM ([23]). *Let K be a non necessarily commutative field, σ an automorphism and δ a σ-derivation of K. We consider the Ore extension $S = K[x,\,;\,\sigma,\delta]$ and its skewfield of fractions $D = \operatorname{Frac} S = K(x\,;\,\sigma,\delta)$. Let G be a group of ring automorphisms of S such that $g(K) \subseteq K$ for any $g \in G$.*

(i) *if $S^G \subseteq K$, then $D^G = S^G = K^G$.*

(ii) *if $S^G \not\subseteq K$, then for any $u \in S^G, u \notin K$ of degree $m = \min\{\deg_x y\,;\,y \in S^G, y \notin K\}$, there exist an automorphism σ' and a σ'-derivation δ' of K^G such that $S^G = K^G[u\,;\,\sigma',\delta']$ and $D^G = \operatorname{Frac}(S^G) = K^G(u\,;\,\sigma',\delta')$.*

PROOF. We simply denote here deg for $\deg_x$. Take $g \in G$ and $n = \deg g(x)$; the assumption $g(K) \subseteq K$ implies $\deg g(s) \in n\mathbb{N} \cup \{-\infty\}$ for all $s \in S$ and so $n = 1$ since g is surjective. We deduce:

$$\deg g(s) = \deg s \text{ for all } g \in G \text{ and } s \in S. \qquad (*)$$

If $S^G \subset K$, then $S^G = K^G$. If $S^G \not\subseteq K$, let us choose in $\{s \in S^G\,;\, \deg s \geq 1\}$ an element u of minimal degree m. In order to apply the previous lemma for $L = K^G$, we check that the free left K^G-module T generated by the powers of u is equal to the subring S^G of S. The inclusion $T \subseteq S^G$ is clear. For the converse, let us fix $s \in S^G$. By the proposition in 1.3.1, there exist q_1 and r_1 unique in S such that $s = q_1u + r_1$ and $\deg r_1 < \deg u$. For any $g \in G$, we have then: $s = g(s) = g(q_1)g(u) + g(r_1) = g(q_1)u + g(r_1)$. Since $\deg g(r_1) = \deg r_1 < \deg u$ by (*), it follows from the uniqueness of q_1 and r_1 that $g(q_1) = q_1$ and $g(r_1) = r_1$. So $r_1 \in S^G$; since $\deg r_1 < \deg u$ and $\deg u$ is minimal, we deduce that $r_1 \in K^G$. Moreover, $q_1 \in S^G$, and $\deg q_1 < \deg s$ because $\deg u \geq 1$. To sum up, we obtain $s = q_1u + r_1$ with $r_1 \in K^G$ and $q_1 \in S^G$ such that $\deg q_1 < \deg s$. We decompose similarly q_1 into $q_1 = q_2u + r_2$ with $r_2 \in K^G$ and $q_2 \in S^G$ such that $\deg q_2 < \deg q_1$. We obtain $s = q_2u^2 + r_2u + r_1$. By iteration, it follows that $s \in T$. The same process using the right euclidian division in S proves that S^G is also the right free L-module T' generated by the powers of u. Then we deduce from point (ii) of the previous lemma that there exist an automorphism σ' of K^G and a σ'-derivation δ' of de K^G such that $S^G = K^G[u\,;\,\sigma', \delta']$.

In both cases (i) and (ii), the inclusion $\mathrm{Frac}\,(S^G) \subseteq D^G$ is clear. For the converse (which follows from remark 1 of 5.2.1 in the particular case where G is finite), we have to prove that:

$$\text{for any } a \text{ and } b \text{ non-zero in } S,\ ab^{-1} \in D^G \text{ implies } ab^{-1} \in \mathrm{Frac}\,(S^G). \qquad (**)$$

We proceed by induction on $\deg a + \deg b$. If $\deg a + \deg b = 0$, then $a \in K$, $b \in K$. Thus $ab^{-1} \in D^G$ is equivalent to $ab^{-1} \in K^G \subseteq S^G$; the result follows. Assume now that (**) is satisfied for all (a, b) such that $\deg a + \deg b \leq n$, for some fixed integer $n \geq 0$. Suppose that a and b are non-zero in S with $ab^{-1} \in D^G$ and $\deg a + \deg b = n + 1$. Up to replace ab^{-1} by its inverse, we can without any restriction suppose that $\deg b \leq \deg a$. By the proposition of 1.3.1, there exist $q, r \in S$ uniquely determined such that:

$$a = qb + r \quad \text{ with } \deg r < \deg b \leq \deg a. \qquad (***)$$

For all $g \in G$, we have $g(ab^{-1}) = ab^{-1}$ and we can so introduce the element $c = a^{-1}g(a) = b^{-1}g(b)$ in D. Denoting by val the discrete valuation $v_{x^{-1}}$ in D (see the remark in 5.2.2), it follows from (*) that $\mathrm{val}\, c = 0$. Applying g to (***), we have $g(a) = g(q)g(b) + g(r)$; in other words, $qbc + rc = ac = g(q)bc + g(r)$, or equivalently: $(g(q) - q)bc = rc - g(r)$. The valuation of the left member is $\mathrm{val}\,(g(q) - q) + \mathrm{val}\, b$. For the right member, we have $\mathrm{val}\, g(r) = -\deg g(r) = -\deg r = \mathrm{val}\, r = \mathrm{val}\, rc$, thus $\mathrm{val}\,(rc - g(r)) \geq \mathrm{val}\, r$. Since $g(q) - q$, b and r are in S, we conclude that: $\deg(g(q) - q) + \deg(b) \leq \deg(r)$. The inequality $\deg b \leq \deg r$ being incompatible with (***), it follows that $g(q) = q$, and then $g(r) = rc$. Therefore we have $g(rb^{-1}) = rc(bc)^{-1} = rb^{-1}$. So we have proved that $ab^{-1} = (qb + r)b^{-1} = q + rb^{-1}$ with $q \in S^G$ and $rb^{-1} \in D^G$ such that $\deg(r) + \deg(b) < 2\deg(b) \leq \deg(a) + \deg(b) = n + 1$. If $r = 0$, then $ab^{-1} = q \in S^G$. If not, we apply the inductive assumption to rb^{-1}: there exist r_1 and b_1 non zero in S^G such that que $rb^{-1} = r_1b_1^{-1}$, and so $ab^{-1} = (qb_1 + r_1)b_1^{-1} \in \mathrm{Frac}\,(S^G)$. □

5.3.2 Rational invariants of the first Weyl algebra

We consider here the action of finite subgroups of automorphisms of the Weyl algebra $A_1(\mathbb{C})$ on its skewfield of fractions $D_1(\mathbb{C})$. We know from theorem 2.2.2 that the algebras $A_1(\mathbb{C})^G$ and $A_1(\mathbb{C})^{G'}$ are not isomorphic when the finite subgroups G and G' are not isomorphic. However, these algebras are always rationally equivalent, as proved by the following theorem from [23].

Theorem. *For any finite subgroup G of* $\operatorname{Aut} A_1(\mathbb{C})$, *we have:* $D_1(\mathbb{C})^G \simeq D_1(\mathbb{C})$.

Proof. With the notations of 2.2.2 and 5.2.3, we have $R = A_1(\mathbb{C})$ generated by p and q with $pq - qp = 1$ and $D = D_1(\mathbb{C}) = \operatorname{Frac} R$. The element $w = pq$ of R satisfies $p^m w - wp^m = mp^m$ for all $m \geq 1$. The field of fractions of the subalgebra U_m of R generated by p^m and w is $Q_m = \mathbb{C}(w)(p^m\,;\,\sigma^m)$, where σ is the $\mathbb{C}$-automorphism of $\mathbb{C}(w)$ defined by $\sigma(w) = w + 1$. In particular, $Q_1 = \mathbb{C}(w)(p\,;\,\sigma) = D$. It's clear $Q_m \simeq D$ for all $m \geq 1$. Let us define $v = p^{-1}q$, which satisfies $wv - vw = 2v$. Since $wv^{-1} = p^2$, we have $Q_2 = \mathbb{C}(w)(p^2\,;\,\sigma^2) = \mathbb{C}(v)(w\,;\,2v\partial_v)$. We denote by S the subalgebra $\mathbb{C}(v)[w\,;\,2v\partial_v]$.
Let G be a finite subgroup of $\operatorname{Aut} R$. From theorem 2.2.2, we can suppose without any restriction that G is linear admissible. In the cyclic case of order n, the group G is generated by the automorphism $g_n : p \mapsto \zeta_n p, q \mapsto \zeta_n^{-1} q$ for ζ_n a primitive n-th root of one. Then we have: $g_n(w) = w$, therefore $D^G = D^{g_n} = Q_1^{g_n} = Q_n \simeq D$. Assume now that we are in one of the cases D_n, E_6, E_7, E_8. Thus G necessarily contains the involution $e : p \mapsto -p, q \mapsto -q$ (because $\mu^2 = \nu^2 = \theta_2$ with the notations of 2.2.1), which satisfies $D^e = Q_2$. Let g be any element of G. By (13), there exist $\alpha, \beta, \gamma, \delta \in \mathbb{C}$ satisfying $\alpha\delta - \beta\gamma = 1$ such that $g(p) = \alpha p + \beta q$ and $g(q) = \gamma p + \delta q$. Thus $g(p) = p(\alpha + \beta v)$ and $g(q) = p(\gamma + \delta v)$, and so:

$$g(v) = \frac{\gamma + \delta v}{\alpha + \beta v} \in \mathbb{C}(v). \qquad (\dagger)$$

Moreover, $g(w) = \alpha\gamma p^2 + \beta\delta q^2 + \alpha\delta pq + \beta\gamma qp$. From relations $qp = pq - 1$, $p^2 = wv^{-1} = v^{-1}w - 2v^{-1}$ and $q^2 = v + vw = wv - v$, it follows that:

$$g(w) = \Big(\frac{\beta\delta v^2 + (\alpha\delta + \beta\gamma)v + \alpha\gamma}{v}\Big)w + \Big(\frac{\beta\delta v^2 - \beta\gamma v - 2\alpha\gamma}{v}\Big). \qquad (\ddagger)$$

We deduce from ($\dagger$) and ($\ddagger$) that the restrictions to the algebra $S = \mathbb{C}(v)[w\,;\,2v\partial_v]$ of the extensions to D of the elements of G determine a subgroup $G' \simeq G/(e)$ of $\operatorname{Aut} S$. Since $e \in G$ and $D^e = Q_2 = \operatorname{Frac} S$, we have $D^G = Q_2^{G'}$.
Assertion ($\dagger$) allows to apply theorem 5.3.1 for $K = \mathbb{C}(v)$, $d = 2v\partial_v$ and $S = K[w\,;\,d]$. By remark 2 of 5.2.1, we have: $[Q_2 : Q_2^{G'}] \leq |G'| < +\infty$, therefore $S^{G'} \not\subseteq K$. From the theorem of 5.3.1 and point (iii) of the lemma of 5.3.1, there exists $u \in S^{G'}$ of positive degree (related to w) and d' a derivation of $\mathbb{C}(v)^{G'}$ such that $S^{G'} = \mathbb{C}(v)^{G'}[u\,;\,d']$ and $Q_2^{G'} = \mathbb{C}(v)^{G'}(u\,;\,d')$. By Lüroth theorem (see 5.1.1), $\mathbb{C}(v)^{G'}$ is a purely transcendental extension $\mathbb{C}(z)$ de $\mathbb{C}$. If d' vanishes on $\mathbb{C}(z)$, then the subfield $Q_2^{G'}$ of Q_2 would be $\mathbb{C}(z, u)$ with transcendence degree > 1 over $\mathbb{C}$, which is impossible since $Q_2 \simeq D_1(\mathbb{C})$ (it's a well known but not trivial result that $D_1(\mathbb{C})$ doesn't contain commutative subfield of transcendence degree > 1 ; see [13], corollary 6.6.18). Therefore $d'(z) \neq 0$; defining $t = d'(z)^{-1}u$, we obtain $Q_2^{G'} = \mathbb{C}(z)(t\,;\,\partial_z) \simeq D_1(\mathbb{C})$. $\square$

Example 1. In the case where $G = C_n$ is cyclic of order n, we have seen in the proof that $D^G = Q_n$ is generated by $w = pq$ and p^n; then a pair (p_n, q_n) of generators of $D_1(\mathbb{C})^{C_n}$ satisfying $[p_n, q_n] = 1$ is $p_n = p^n$ et $q_n = (np^n)^{-1}pq$.

Example 2. In the case where $G = D_n$ is binary dihedral of order $4n$ (see 2.2.1), the interested reader could find in [23] the calculation of the following pair (p_n, q_n) of generators of $D_1(\mathbb{C})^{D_n}$ satisfying $[p_n, q_n] = 1$:

$$p_n = \frac{1}{16n}\big((p^{-1}q)^{-n} - (p^{-1}q)^n\big)\Big(\frac{(p^{-1}q)^n - 1}{(p^{-1}q)^n + 1}\Big)^2(2pq - 1), \qquad q_n = \Big(\frac{(p^{-1}q)^n + 1}{(p^{-1}q)^n - 1}\Big)^2.$$

5.3.3 Rational invariants of polynomial functions in two variables

We consider $R = \mathbb{C}[y][x\,;\,\sigma,\delta]$, for σ a $\mathbb{C}$-automorphism and δ a σ-derivation of $\mathbb{C}[y]$.

LEMMA. *If $R = \mathbb{C}[y][x\,;\,\delta]$, with δ an ordinary derivation of $\mathbb{C}[y]$ such that $\delta(y) \notin \mathbb{C}$, then* $\operatorname{Frac} R^G \simeq D_1(\mathbb{C})$ *for any finite subgroup of* $\operatorname{Aut} R$.

PROOF. Let us denote $K = \mathbb{C}(y)$ and $D = \operatorname{Frac} R = \mathbb{C}(y)(x\,;\,\delta)$. Replacing x by $x' = \delta(y)^{-1}x$, we have $D = \mathbb{C}(y)(x'\,;\,\partial_y)$,and so $D \simeq D_1(\mathbb{C})$. Since $\delta(y) \notin \mathbb{C}$, the proposition of 2.3.4 implies that any $g \in \operatorname{Aut} R$ satisfies $g(K) \subseteq K$ for $K = \mathbb{C}(y)$, and the restriction of g to $S = \mathbb{C}(y)[x\,;\,\delta]$ of the extension to $D = \operatorname{Frac} S$ determines an automorphism of S. For G a finite subgroup of $\operatorname{Aut} R$ we can apply the theorem of 5.3.1 and point (iii) of the lemma of 5.3.1: there exist $u \in S^G$ of positive degree and δ' a derivation of $\mathbb{C}(y)^G$ such that $S^G = \mathbb{C}(y)^G[u\,;\,\delta']$ and $D^G = \mathbb{C}(y)^G(u\,;\,\delta')$. Then we achieve the proof as in the proof of the previous theorem. □

LEMMA. *If R is the quantum plane $\mathbb{C}_q[x,y]$ for $q \in \mathbb{C}^\times$ not a root of one, then* $\operatorname{Frac} R^G \simeq D_1^{q'}(\mathbb{C})$ *with $q' = q^{|G|}$ for any finite subgroup G of* $\operatorname{Aut} R$.

PROOF. Let G a finite group of $\operatorname{Aut} R$ where $R = \mathbb{C}_q[x,y]$. By point (iii) of the first proposition of 4.1.1, there exists for any $g \in G$ a pair $(\alpha_g,\beta_g) \in \mathbb{C}^\times \times \mathbb{C}^\times$ such that $g(y) = \alpha_g y$ and $g(x) = \beta_g x$. Denote by m and m' the orders of the cyclic groups $\{\alpha_g\,;\,g \in G\}$ and $\{\beta_g\,;\,g \in G\}$ of $\mathbb{C}^\times$ respectively. In particular, $\mathbb{C}(y)^G = \mathbb{C}(y^m)$. We can apply the theorem of 5.3.1 to the extension $S = \mathbb{C}(y)[x\,;\,\sigma]$ of $R = \mathbb{C}[y][x\,;\,\sigma]$, where $\sigma(y) = qy$. We have $S^G \neq \mathbb{C}(y)^G$ because $x^{m'} \in S^G$. Let n be the minimal degree related to x of the elements of S^G of positive degree. For any $u \in S^G$ of degree n, there exist σ' and δ' such that $S^G = \mathbb{C}(y^m)[u\,;\,\sigma',\delta']$. By assertion (iii) of the lemma of 5.3.1, the automorphism σ' of $\mathbb{C}(y^m)$ is the restriction of σ^n to $\mathbb{C}(y^m)$. We show firstly that we can choose u monomial. We develop $u = a_n(y)x^n + \cdots + a_1(y)x + a_0(y)$ with $n \geq 1$, $a_i(y) \in \mathbb{C}(y)$ for all $0 \leq i \leq n$ and $a_n(y) \neq 0$. Denote by $p \in \mathbb{Z}$ the valuation (related to y) of $a_n(y)$ in the extension $\mathbb{C}(\!(y)\!)$ of $\mathbb{C}(y)$. The action of G being diagonal on $\mathbb{C}x \oplus \mathbb{C}y$, the monomial $v = y^p x^n$ lies in S^G. So we obtain $S^G = \mathbb{C}(y^m)[v\,;\,\sigma^n]$ and $D^G = \mathbb{C}(y^m)(v\,;\,\sigma^n) \simeq D_1^{q'}$ for $q' = q^{mn}$. We have to check that $mn = |G|$. Let $g \in G$ determining an inner automorphism of $D = \operatorname{Frac} R = \operatorname{Frac} S$; there exists non-zero $r \in D$ such that $g(s) = rsr^{-1}$ of all $s \in D$. Denoting by d the order of g in G, we have then r^d central in D, and so $r^d \in \mathbb{C}$ by (73). Embedding $D = \mathbb{C}(y)(x\,;\,\sigma)$ in $\mathbb{C}(y)(\!(x^{-1}\,;\,\sigma^{-1})\!)$, see remark in 5.2.2, we deduce that $r \in \mathbb{C}$ and so $g = id_R$. We have proved that any nontrivial automorphism in G is outer. Applying remark 2 of 5.2.1, it follows that $[D : D^G] = |G|$. We have:

$$D^G = \mathbb{C}(y^m)(y^p x^n\,;\,\sigma^n) \;\subseteq\; L = \mathbb{C}(y)(y^p x^n\,;\,\sigma^n) = \mathbb{C}(y)(x^n\,;\,\sigma^n) \subseteq\; D = \mathbb{C}(y)(x\,;\,\sigma).$$

Thus $[D : L] = n$ and $[L : D^G] = m$. We conclude $|G| = [D : D^G] = mn$. □

LEMMA. *If R is the quantum Weyl algebra $A_1^q(\mathbb{C})$ for $q \in \mathbb{C}^\times$ not a root of one, then* $\operatorname{Frac} R^G \simeq D_1^{q'}(\mathbb{C})$ *with $q' = q^{|G|}$ for any finite subgroup of* $\operatorname{Aut} R$.

PROOF. The proof is easier than in the case of the quantum plane and left to the reader as an exercise (use assertion (74) and the description of $\operatorname{Aut} A_1^q(\mathbb{C})$ recalled in 2.3.4, from [21]) ; see proposition 3.5 of [23] for details. □

We are now in position to summarize in the following theorem the results on rational invariants for Ore extensions in two variables.

THEOREM. *Let $R = \mathbb{C}[y][x\,;\sigma,\delta]$ with σ an automorphism and δ a σ-derivation of $\mathbb{C}[y]$. Let $D = \operatorname{Frac} R$ with center $\mathbb{C}$. Then we are in one of the two following cases:*

(i) $D \simeq D_1(\mathbb{C})$, and $D^G \simeq D_1(\mathbb{C})$ for any finite subgroup G of $\operatorname{Aut} R$;

(ii) there exists $q \in \mathbb{C}^\times$ not a root of one such that $D \simeq D_1^q(\mathbb{C})$, and $D^G \simeq D_1^{q^{|G|}}(\mathbb{C})$ for any finite subgroup G of $\operatorname{Aut} R$.

PROOF. We just combine the classification lemma 2.3.4 with the assertions (64) and (73) on the centers, the main theorem of 5.3.2, and the three previous lemmas. □

REMARK. It could be relevant to underline here that previous results only concern actions on $\operatorname{Frac} R$ which extend actions on R. The question of determining D^G for other types of subgroups G of $\operatorname{Aut} D$ is another problem, and the structure of the groups $\operatorname{Aut} D_1(\mathbb{C})$ and $\operatorname{Aut} D_1^q(\mathbb{C})$ remains unknown (see [22]). In particular, we can define a notion of rational triangular automorphism related to one of the presentations (61) or (63) of the Weyl skewfield $D_1(\mathbb{C})$; the three following results are proved in [24].

1. The automorphisms of $D_1(\mathbb{C}) = \mathbb{C}(q)(p\,;\partial_q)$ which stabilizing $\mathbb{C}(q)$ are of the form:
$$\theta:\ q \mapsto \theta(q) = \tfrac{\alpha q+\beta}{\gamma q+\delta},\quad p \mapsto \theta(p) = \tfrac{1}{\partial_q(\theta(q))}p + f(q),$$
for $\left(\begin{smallmatrix}\alpha & \beta\\ \gamma & \delta\end{smallmatrix}\right) \in \mathrm{GL}_2(\mathbb{C})$ and $f(q) \in \mathbb{C}(q)$.

2. The automorphisms of $D_1(\mathbb{C}) = \mathbb{C}(pq)(p\,;\sigma)$ stabilizing $\mathbb{C}(pq)$ are of the form:
$$\theta:\ pq \mapsto \theta(pq) = pq + \alpha,\quad p \mapsto \theta(p) = f(pq)p,$$
for $\alpha \in \mathbb{C}$ and $f(pq) \in \mathbb{C}(pq)$, or are the product of such an automorphism by the involution $pq \mapsto -pq,\ p \mapsto p^{-1}$.

3. For any finite subgroup of $\operatorname{Aut} D_1(\mathbb{C})$ stabilizing one of the three subfields $\mathbb{C}(p)$, $\mathbb{C}(q)$ or $\mathbb{C}(pq)$, we have $D_1(\mathbb{C})^G \simeq D_1(\mathbb{C})$.

5.4 Noncommutative Noether's problem

5.4.1 Rational invariants and the Gelfand-Kirillov conjecture

Let $\Bbbk$ be a field of characteristic zero. We have seen in 2.3.3 that any representation of dimension n of a group G gives rise to an action of G on the commutative polynomial algebra $S = \Bbbk[q_1,\ldots,q_n]$, which extends canonically into an action by automorphisms on the Weyl algebra $A_n(\Bbbk)$ defined from relations (22) or (23), and then to the Weyl skewfield $D_n(\Bbbk)$. Following the philosophy of the Gelfand-Kirillov problem (see above 5.2.3) by considering the Weyl skewfields $\mathcal{D}_{n,t}(\Bbbk)$ as significant classical noncommutative analogues of the purely transcendental extensions of $\mathbb{k}$, the following question appears of a relevant noncommutative formulation of Noether's problem.

Question: do we have $D_n(\mathbb{k})^G \simeq \mathcal{D}_{m,t}(\mathbb{k})$ for some nonnegative integers m and t ?

By somewhat specialized considerations on various noncommutative versions of the transcendence degree (which cannot be developed here), we can give the following two precisions (see [26] for the proofs):

1. if we have a positive answer to the above question, then m and t satisfy $2m+t \leq 2n$;
2. if we have a positive answer to the above question for a finite group G, then $m = n$ and $t = 0$, and so $D_n(\mathbb{k})^G \simeq D_n(\mathbb{k})$.

5.4.2 Rational invariants under linear actions of finite abelian groups

The main result (form [26]) is the following.

THEOREM. *For a representation of a group G (non necessarily finite) which is a direct summand of n representations of dimension one, there exists a unique integer $0 \leq s \leq n$ such that $D_n(\mathbb{k})^G \simeq \mathcal{D}_{n-s,s}(\mathbb{k})$.*

PROOF. By (71), the integer s is no more than the transcendence degree over $\mathbb{k}$ of the center of $\mathcal{D}_{n-s,s}(\mathbb{k})$ and so is unique. Now we proceed by induction on n to prove the existence of s.

1) Assume that $n = 1$. Then G acts on $A_1(\mathbb{k}) = \mathbb{k}[q_1][p_1\,;\,\partial_{q_1}]$ by automorphisms of the form:

$$g(q_1) = \chi_1(g)q_1, \quad g(p_1) = \chi_1(g)^{-1}p_1, \qquad \text{for all } g \in G$$

where χ_1 is a character $G \to \mathbb{k}^\times$. The element $w_1 = p_1q_1$ is invariant under G. We define in $D_1(\mathbb{k}) = \mathbb{k}(w_1)(p_1,\ \sigma_1)$, see (63), the subalgebra $S_1 = \mathbb{k}(w_1)[p_1,\ \sigma_1]$. We have $\operatorname{Frac} S_1 = D_1(\mathbb{k})$. Any $g \in G$ fixes w_1 and acts on p_1 by $g(p_1) = \chi_1(g)p_1$. We can apply the theorem of 5.3.1. If $S_1^G \subseteq \mathbb{k}(w_1)$, then $D_1(\mathbb{k})^G = S_1^G = \mathbb{k}(w_1)^G = \mathbb{k}(w_1)$; we deduce that in this case $D_1(\mathbb{k})^G \simeq \mathcal{D}_{1-s,s}(\mathbb{k})$ with $s = 1$. If $S_1^G \not\subseteq \mathbb{k}(w_1)$, then S_1^G is an Ore extension $\mathbb{k}(w_1)[u\,;\,\sigma', \delta']$ for some automorphism σ' and some σ'-derivation δ' of $\mathbb{k}(w_1)$, and some polynomial u in the variable p_1 with coefficients in $\mathbb{k}(w_1)$ such that $g(u) = u$ for all $g \in G$ and of minimal degree. Because of the form of the action of G on p_1, we can choose without any restriction $u = p_1^a$ for an integer $a \geq 1$, and so $\sigma' = \sigma_1^a$ and $\delta' = 0$. To sum up, $D_1(\mathbb{k})^G = \operatorname{Frac} S_1^G = \mathbb{k}(w_1)(p_1^a\,;\,\sigma_1^a)$. This skewfield is also generated by $x = p_1^a$ and $y = a^{-1}w_1p_1^{-a}$ which satisfy $xy - yx = 1$. We conclude that $D_1(\mathbb{k})^G \simeq D_1(\mathbb{k}) = \mathcal{D}_{1-s,s}(\mathbb{k})$ for $s = 0$.

2) Now suppose that the theorem is true for any direct summand of $n-1$ representations of dimension one of any group over any field of characteristic zero. Let us consider a direct summand of n representations of dimension one of a group G over $\mathbb{k}$. Then G acts on $A_n(\mathbb{k})$ by automorphisms of the form:

$$g(q_i) = \chi_i(g)q_i, \quad g(p_i) = \chi_i(g)^{-1}p_i, \qquad \text{for all } g \in G \text{ and } 1 \leq i \leq n,$$

where $\chi_1, \chi_2, \ldots, \chi_n$ are characters $G \to \mathbb{k}^\times$. Thus, recalling the notation $w_i = p_iq_i$, we have:

$$g(w_i) = w_i, \qquad \text{for any } g \in G \text{ and any } 1 \leq i \leq n.$$

In $D_n(\Bbbk) = \Bbbk(w_1, w_2, \ldots, w_n)(p_1 ; \sigma_1)(p_2 ; \sigma_2) \cdots (p_{n-1} ; \sigma_{n-1})(p_n ; \sigma_n)$, see (69), let us consider the subfields:

$$\begin{array}{rcl} L &=& \Bbbk(w_n), \\ K &=& \Bbbk(w_1, w_2, \ldots, w_n)(p_1 ; \sigma_1)(p_2 ; \sigma_2) \cdots (p_{n-1} ; \sigma_{n-1}) \\ &=& \Bbbk(w_n)(w_1, w_2, \ldots, w_{n-1})(p_1 ; \sigma_1)(p_2 ; \sigma_2) \cdots (p_{n-1} ; \sigma_{n-1}) \\ &\simeq& D_{n-1}(L), \end{array}$$

and the subalgebra $S_n = K[p_n ; \sigma_n]$ which satisfies $\operatorname{Frac} S_n = D_n(\Bbbk)$. Applying the induction hypothesis to the restriction of the action of G by L-automorphisms on $A_{n-1}(L)$, there exists an integer $0 \leq s \leq n-1$ such that: $D_{n-1}(L)^G \simeq \mathcal{D}_{n-1-s,s}(L) \simeq \mathcal{D}_{n-(s+1),s+1}(\Bbbk)$. Since K is stable under the action of G, we can apply the theorem of 5.3.1 to the ring $S_n = K[p_n ; \sigma_n]$. Two cases are possible.

First case: $S_n^G = K^G$. Then we obtain:

$$D_n(\Bbbk)^G = \operatorname{Frac}(S_n^G) = K^G \simeq D_{n-1}(L)^G \simeq \mathcal{D}_{n-(s+1),s+1}(\Bbbk).$$

Second case: there exists a polynomial $u \in S_n$ with $\deg_{p_n} u \geq 1$ such that $g(u) = u$ for all $g \in G$. Choosing u such that $\deg_{p_n} u$ is minimal, there exist an automorphism σ' and a σ'-derivation δ' of K^G such that $S_n^G = K^G[u ; \sigma', \delta']$ and $D_n(\Bbbk)^G = \operatorname{Frac} S_n^G = K^G(u ; \sigma', \delta')$.
Let us develop $u = f_m p_n^m + \cdots + f_1 p_n + f_0$ with $m \geq 1$ and $f_i \in K^G$ for all $0 \leq i \leq m$. In view of the action of G on p_n, it's clear that the monomial $f_m p_n^m$ is then invariant under G. Using the embedding in skewfield of Laurent series (see 5.2.2), we can develop f_m in:

$$\overline{K} = \Bbbk(w_1, w_2, \ldots, w_n)((p_1^{-1} ; \sigma_1^{-1}))((p_2^{-1} ; \sigma_2^{-1})) \cdots ((p_{n-1}^{-1} ; \sigma_{n-1}^{-1})).$$

The action of G extends to $\overline{K}$ acting diagonally on the p_i's and fixing w_i's. Therefore we can choose without any restriction a monomial u:

$$u = p_1^{a_1} \ldots p_n^{a_n} \text{ with } (a_1, \ldots, a_n) \in \mathbb{Z}^n, \text{ and } a_n \geq 1.$$

For any $1 \leq j \leq n$, we have $uw_j = (w_j + a_j)u$. Let us introduce the elements:

$$w_1' = w_1 - a_n^{-1} a_1 w_n, \quad w_2' = a_n w_2 - a_n^{-1} a_2 w_n, \quad \ldots, \quad w_{n-1}' = a_n w_{n-1} - a_n^{-1} a_{n-1} w_n.$$

We obtain: $w_j' u = u w_j'$ for any $1 \leq j \leq n-1$. Since $\sigma_i(w_j') = w_j' + \delta_{i,j}$ pour $1 \leq i, j \leq n-1$, the skewfield $F_{n-1} = \Bbbk(w_1', w_2', \ldots, w_{n-1}')(p_1 ; \sigma_1)(p_2 ; \sigma_2) \cdots (p_{n-1} ; \sigma_{n-1})$ is isomorphic to $D_{n-1}(\Bbbk)$. More precisely, F_{n-1} is the skewfield of fractions of the algebra

$$\Bbbk[q_1', \ldots, q_{n-1}'][p_1 ; \partial_{q_1'}] \ldots [p_{n-1} ; \partial_{q_{n-1}'}],$$

where $q_i' = w_i p_i^{-1}$ for any $1 \leq i \leq n-1$. This algebra is isomorphic to the Weyl algebra $A_{n-1}(\Bbbk)$. Applying the induction hypothesis, there exists $0 \leq s \leq n-1$ such that $F_{n-1}^G \simeq \mathcal{D}_{n-1-s,s}(\Bbbk)$. It's clear by definition of the w_j''s that $\Bbbk(w_n)(w_1', w_2', \ldots, w_{n-1}') = \Bbbk(w_n)(w_1, w_2, \ldots, w_{n-1})$; since w_n commutes with all the elements of F_{n-1}, we deduce that $K = F_{n-1}(w_n)$. The algebra $S_n^G = K^G[u ; \sigma', \delta']$ can then be written $S_n^G = F_{n-1}^G(w_n)[u ; \sigma', \delta']$. The generator u commutes with w_j' for any $0 \leq j \leq n-1$ as we have seen above, commutes with all the p_i's by definition, and satisfies with w_n the relation $uw_n = (w_n + a_n)u$. Therefore the change of variables $w_n' = a_n^{-1} w_n$ implies: $S_n^G = F_{n-1}^G(w_n')[u ; \sigma'']$, with σ'' which is the identity map on F_{n-1}^G and satisfies $\sigma''(w_n') = w_n' + 1$. It follows that: $\operatorname{Frac} S_n^G \simeq D_1(F_{n-1}^G) \simeq D_1(\mathcal{D}_{n-1-s,s}(\Bbbk)) \simeq \mathcal{D}_{n-s,s}(\Bbbk)$. □

Corollary (Application to finite abelian groups). *We suppose here that $\Bbbk$ is algebraically closed. Then, for any finite dimensional representation of a finite abelian group G, we have $D_n(\Bbbk)^G \simeq D_n(\Bbbk)$.*

Proof. By Schur's lemma and total reducibility, any finite representation of G is a direct summand of one dimensional representations (see 2.3.1). Then the result follows from the previous theorem and remark 2 of 5.4.1. $\square$

This result already appears in [24]. The following corollary proves in particular that for non necessarily finite groups G, all possible values of s can be obtained in the previous theorem.

Corollary (Application to the canonical action of subgroups of a torus). *For an integer $n \geq 1$, let $\mathbb{T}_n$ be the torus $(\Bbbk^\times)^n$ acting canonically on the vector space $\Bbbk^n$. Then:*

(i) for any subgroup G of $\mathbb{T}_n$, there exists a unique integer $0 \leq s \leq n$ such that $D_n(\Bbbk)^G \simeq \mathcal{D}_{n-s,s}(\Bbbk)$;

(ii) for any integer $0 \leq s \leq n$ there exists at least one subgroup G of $\mathbb{T}_n$ such that $D_n(\Bbbk)^G \simeq \mathcal{D}_{n-s,s}(\Bbbk)$;

(iii) in particular $s = n$ if $G = \mathbb{T}_n$, and $s = 0$ if G is finite.

Proof. Point (i) is just the application of the previous theorem. For (ii), let us fix an integer $0 \leq s \leq n$ and consider in $\mathbb{T}_n$ the subgroup:

$$G = \{\mathrm{Diag}\,(\alpha_1, \dots, \alpha_s, 1, \dots, 1) \ ; \ (\alpha_1, \dots, \alpha_s) \in (\Bbbk^*)^s\} \simeq \mathbb{T}_s,$$

acting by automorphisms on $A_n(\Bbbk)$:

$$\begin{array}{lll} q_i \mapsto \alpha_i q_i, & p_i \mapsto \alpha_i^{-1} p_i, & \text{pour tout } \ 1 \leq i \leq s, \\ q_i \mapsto q_i, & p_i \mapsto p_i, & \text{pour tout } \ s+1 \leq i \leq n. \end{array}$$

In the skewfield $D_n(\Bbbk) = \Bbbk(w_1, w_2, \dots, w_n)(p_1\,;\sigma_1)(p_2\,;\sigma_2)\cdots(p_n\,;\sigma_n)$, we introduce the subfield $K = \Bbbk(w_1, w_2, \dots, w_n)(p_{s+1}\,;\sigma_{s+1})(p_{s+2}\,;\sigma_{s+2})\cdots(p_n\,;\sigma_n)$. Then the subalgebra $S = K[p_1\,;\sigma_1]\cdots[p_s\,;\sigma_s]$ satisfies $\mathrm{Frac}\, S = D_n(\Bbbk)$. It's clear that K is invariant under the action of G. If $S^G \not\subset K$, we can find in particular in S^G a monomial:

$$u = v p_1^{d_1} p_2^{d_2} \cdots p_s^{d_s}, \quad v \in K,\ v \neq 0,\ d_1, \dots, d_s \in \mathbb{N},\ (d_1, \dots, d_s) \neq (0, \dots, 0),$$

then $\alpha_1^{d_1}\alpha_2^{d_2}\cdots\alpha_s^{d_s} = 1$ for all $(\alpha_1, \alpha_2, \dots, \alpha_s) \in (\Bbbk^*)^s$, and so a contradiction. We conclude with theorem 5.3.1 that $(\mathrm{Frac}\, S)^G = S^G = K^G$, and so $D_n(\Bbbk)^G = K$. It's clear that $K \simeq \mathcal{D}_{n-s,s}(\Bbbk)$; this achieves the proof of point (ii). Point (iii) follows then from the previous corollary. $\square$

The actions of tori $\mathbb{T}_n$ on the Weyl algebras have been studied in particular in [58].

5.4.3 Rational invariants for differential operators on Kleinian surfaces

Another situation where it's possible to give a positive answer to the question of 5.4.1 is the case of a 2-dimensional representation. Using the main theorem 5.3.1 as a key argument, one can the prove (by technical developments which cannot be detailed her; see [26] for a complete proof) the following general result.

THEOREM ([26]).

(i) *For any 2-dimensional representation of a group G, there exist two nonnegative integers m, t with $1 \leq m + t \leq 2$ such that $D_2(\Bbbk)^G \simeq \mathcal{D}_{m,t}(\Bbbk)$.*

(ii) *In particular, for any 2-dimensional representation of a finite group G, we have $D_2(\Bbbk)^G \simeq D_2(\Bbbk)$.*

As an application, let us consider again the canonical linear action of a finite subgroup G of $\mathrm{SL}_2(\mathbb{C})$ on $S = \mathbb{C}[x, y] = \mathbb{C}[V]$ for $V = \mathbb{C}^2$. The corresponding invariant algebra S^G is one of the Kleinian surfaces studied in 2.2.1. This action extends to the rational functions field $K = \operatorname{Frac} S = \mathbb{C}(x, y)$ and it follows from Castelnuovo or Burnside theorems (see 5.1.1 and 5.1.2) that $K^G \simeq K$. Considering the first Weyl algebra $A_1(\mathbb{C})$ as a noncommutative deformation of $\mathbb{C}[x, y]$, we have studied in 2.2.2 the action of G on $A_1(\mathbb{C})$ and the associated deformation $A_1(\mathbb{C})^G$ of the Kleinian surface S^G. The extension of the action to $\operatorname{Frac} A_1(\mathbb{C}) = D_1(\mathbb{C})$ has been considered in 5.3.2, and we have proved that $D_1(\mathbb{C})^G \simeq D_1(\mathbb{C})$. From another point of view, we can apply to the action of G on S the duality extension process described in 2.3.3 in order to obtain an action on $A_2(\mathbb{C})$. As explained in second example 2.3.3, the invariant algebra $A_2(\mathbb{C})^G = (\operatorname{Diff} S)^G$ is then isomorphic to $\operatorname{Diff}(S^G)$; in other words the invariants of differential operators on S are isomorphic to the differential operators on the Kleinian surface S^G (by theorem 5 of [52]). Of course the action extends to $D_2(\mathbb{C}) = \operatorname{Frac} A_2(\mathbb{C})$ and the following corollary follows then from point (ii) of the previous theorem (see also further the end of 5.5.2).

COROLLARY. *Let G be a finite subgroup of SL_2; for the action on $A_2(\mathbb{C}) = \operatorname{Diff} S$ canonically deduced from the natural action of G on $S = \mathbb{C}[x, y]$, we have $D_2(\mathbb{C})^G \simeq D_2(\mathbb{C})$.*

The method used in [26] to prove this result allows to compute explicitly, according to each type of G in the classification of 2.2.1, some generators P_1, P_2, Q_1, Q_2 of $D_2(\mathbb{C})^G$ satisfying canonical relations $[P_1, Q_1] = [P_2, Q_2] = 1$ and $[P_i, P_j] = [Q_i, Q_j] = [P_i, Q_j] = 0$ for $i \neq j$. For instance, denoting $A_2(\mathbb{C}) = \mathbb{C}[q_1, q_2][p_1\,;\partial_{q_1}][p_1\,;\partial_{q_2}]$, a solution for the type A_{n-1} is:

$n = 2p$	$n = 2p + 1$
$Q_1 = q_1^p q_2^{-p}$	$Q_1 = q_1^{2p+1} q_2^{-2p-1}$
$Q_2 = q_1 q_2$	$Q_2 = q_1^{p+1} q_2^{-p}$
$P_1 = \frac{1}{2p}\, q_1^{1-p} q_2^p p_1 - \frac{1}{2p}\, q_1^{-p} q_2^{p+1} p_2$	$P_1 = -\frac{p}{2p+1}\, q_1^{-2p} q_2^{2p+1} p_1 - \frac{p+1}{2p+1}\, q_1^{-2p-1} q_2^{2p+2} p_2$
$P_2 = \frac{1}{2}(q_2^{-1} p_1 + q_1^{-1} p_2)$	$P_2 = q_1^{-p} q_2^p p_1 + q_1^{-p-1} q_2^{p+1} p_2$

5.5 Poisson structure on invariants and localization

5.5.1 Poisson analogue of Noether's problem

Formulation of the main question. We come back to the commutative situation of 3.1.1 where $\mathcal{A}$ is a commutative Poisson algebra over a base field $\Bbbk$. Suppose that $\mathcal{A}$ is a domain and consider its field of fractions $F = \operatorname{Frac}\mathcal{A}$. The Poisson bracket extends canonically to F (see 3.1.1 with $\mathcal{S} = \mathcal{A} \setminus \{0\}$). Take now a finite group G of Poisson automorphisms of $\mathcal{A}$. We know (see 3.1.1) that in this case $\mathcal{A}^G$ is a Poisson subalgebra. From the preliminary proposition of 5.1, the field of fractions of $\mathcal{A}^G$ is $\operatorname{Frac}(\mathcal{A}^G) = (\operatorname{Frac}\mathcal{A})^G = F^G$. At the intersection of problems 5.1.1 and 5.4.1, we formulate (see [34]) the following question:

Problem (Poisson-Noether's problem) : is there a field isomorphism between F^G and F which is a Poisson isomorphism ?

> Comment: *a Poisson version of Gelfand-Kirillov problem.* For any field K and any integer $n \geq 1$, denote by $F_n(K)$ the field of fractions of the symplectic Poisson algebra of dimension $2n$ over K (see example 2 in 3.1.1). Let $\mathfrak{g}$ be an algebraic Lie algebra over a base field $\Bbbk$, $S(\mathfrak{g})$ the symmetric algebra, $L(\mathfrak{g})$ the field of fractions of $S(\mathfrak{g})$. Similarly to the Gelfand-Kirillov problem (see 5.2.3), we can ask:
>
> *Question:* do we have $L(\mathfrak{g}) \simeq F_m(\Bbbk(z_1, \dots, z_t))$ for some nonegative integers m, t ?
>
> Here $L(\mathfrak{g}) \simeq F_m(\Bbbk(z_1, \dots, z_t))$ means that the Poisson center of $L(\mathfrak{g})$ is purely transcendental of degree t over $\Bbbk$ and $L(\mathfrak{g})$ is isomorphic to the field of fractions of a symplectic Poisson algebra of dimension $2m$ over this Poisson center. The original geometric motivations of this problem arise from [67]. Recent algebraic results on it can be found in [48] and [66].

The most natural question is then the following:

Problem (symplectic Poisson-Noether's problem) : for a finite subgroup G of the symplectic group $\mathrm{Sp}_{2n}(\mathbb{C})$ acting by the linear canonical Poisson action on the symplectic polynomial algebra of dimension $2n$, do we have a Poisson isomorphism $F_n(\mathbb{C})^G \simeq F_n$? [or more generally for any G, a Poisson isomorphism $F_n(\mathbb{C})^G \simeq F_m(\mathbb{C}(z_1, \dots, z_t))$ for some nonegative integers m, t such that $2n \geq 2m + t$].

Examples in the case of the symplectic plane. We take for $\mathcal{A}$ the algebra $\mathbb{C}[x, y]$ with the symplectic Poisson bracket defined from $\{x, y\} = 1$. Thus $F = \operatorname{Frac}\mathcal{A} = \mathbb{C}(x, y)$. We introduce $w := xy \in \mathcal{A}$, an the subfields $Q_m = \mathbb{C}(w, x^m)$ of F for all $m \geq 1$. In particular $Q_1 = \mathbb{C}(w, x) = \mathbb{C}(x, y) = F$. The Q_m are stable for the Poisson bracket since

$$\{x^m, w\} = m x^m \quad \text{for any } m \geq 1. \tag{75}$$

Hence the element $z_m := \frac{1}{m} x^{-m} w = \frac{1}{m} y x^{1-m}$ satisfies $\{z_m, w\} = -m z_m$ and we deduce:

$$Q_m = \mathbb{C}(z_m, x^m), \quad \text{with } \{z_m, x^m\} = 1 \text{ for any } m \geq 1. \tag{76}$$

So each Q_m is isomorphic to $\mathbb{C}(x,y)$ as a field and as a Poisson algebra. We also need the element $v := x^{-1}y = 2z_2 \in F$; because $wv^{-1} = x^2$, we have

$$Q_2 = \mathbb{C}(w,x^2) = \mathbb{C}(v,w), \quad \text{with } \{w,v\} = 2v \text{ for any } m \geq 1. \tag{77}$$

• Example. Let G be the cyclic subgroup of order n in SL_2 generated by the automorphism g_n acting on $\mathbb{C}[x,y]$ by $g_n(x) = \zeta_n x$ and $g_n(y) = \zeta_n^{-1}y$ for ζ_n a n-th primitive root of one. Then $g_n(w) = w$. The algebra $S := \mathbb{C}(w)[x]$ is such that $\operatorname{Frac} S = F$ and g_n acts on S fixing w and multiplying x by ζ_n. Thus it is clear that $S^G = \mathbb{C}(w)[x^n]$ and it follows directly from (commutative) theorem 5.1.2 that $F^G = \mathbb{C}(w,x^n) = Q_n$. Finally we have proved that:

$$\mathbb{C}(x,y)^G = \mathbb{C}(p_n,q_n) \quad \text{with } p_n = \tfrac{1}{n}yx^{1-n} \text{ and } q_n = x^n \text{ satisfying } \{p_n,q_n\} = 1,$$

• Example. Let G be the binary dihedral subgroup of order $4n$ in SL_2 generated by the automorphism g_{2n} acting on $\mathbb{C}[x,y]$ by $g_{2n}(x) = \zeta_{2n}x$ and $g_{2n}(y) = \zeta_{2n}^{-1}y$ for ζ_{2n} a $2n$-th primitive root of one, and the automorphism μ define by $\mu(x) = iy$ and $\mu(y) = ix$ (see 2.2.1). We have $F^G = (F^{g_{2n}})^\mu = Q_{2n}^\mu$. Since $x^2 = wv^{-1}$, we have $x^{2n} = w^n v^{-n}$; thus $Q_{2n} = \mathbb{C}(w,x^{2n}) = \mathbb{C}(w,v^n)$, with $\{w,v^n\} = 2nv^n$. The action of μ on Q_{2n} is given by $\mu(w) = -w$ and $\mu(v^n) = v^{-n}$. The element $s_n := \frac{1}{2n}(v^{-n} - v^n)w$ satisfies $\mu(s_n) = s_n$ and $Q_{2n} = \mathbb{C}(s_n,v^n)$, with $\{s_n,v^n\} = 1 - v^{2n}$. By a last change of variable $t_n := (v^n+1)(v^n-1)^{-1}$, we deduce $\mathbb{C}(v^n) = \mathbb{C}(t_n)$ by Lüroth's theorem, and the action of μ reduces to $\mu(t_n) = -t_n$. Because $\mu(s_n) = s_n$, we have $Q_{2n}^\mu = \mathbb{C}(s_n,t_n)^\mu = \mathbb{C}(s_n,t_n^2)$. We compute:

$$\begin{aligned}\{s_n,t_n\} &= \big(\{s_n,v^n\}(v^n-1) - (v^n+1)\{s_n,v^n\}\big)(v^n-1)^{-2} \\ &= -2(1-v^{2n})(1-v^n)^{-2} = 2t_n,\end{aligned}$$

and then $\{s_n,t_n^2\} = 2t_n\{s_n,t_n\} = 4t_n^2$. It follows that $Q_{2n}^\mu = \mathbb{C}(s_n,t_n^2)$, with $\{s_n,t_n^2\} = 4t_n^2$. Denoting finally $p_n := (2t_n)^{-2}s_n$ and $q_n := t_n^2$, we conclude that $Q_{2n}^\mu = \mathbb{C}(p_n,q_n)$, with $\{p_n,q_n\} = 1$. We have proved that:

$$\mathbb{C}(x,y)^G = \mathbb{C}(p_n,q_n) \quad \text{with } p_n \text{ and } q_n \text{ satisfying } \{p_n,q_n\} = 1 \text{ defined by :}$$
$$p_n = \frac{1}{8n}\big((x^{-1}y)^{-n} - (x^{-1}y)^n\big)\Big(\frac{(x^{-1}y)^n - 1}{(x^{-1}y)^n + 1}\Big)^2 xy, \quad \text{and} \quad q_n = \Big(\frac{(x^{-1}y)^n + 1}{(x^{-1}y)^n - 1}\Big)^2.$$

Hence the answer to the Poisson-Noether's problem is positive in both cases. More generally, we have (from [34]):

Proposition. *Let $\mathcal{A}$ be the Poisson algebra $\mathbb{C}[x,y]$ for the symplectic bracket. Let $F = \mathbb{C}(x,y)$ be its field of fractions. For any finite subgroup G of SL_2 acting linearly on $\mathcal{A}$, there exist two elements p and q in F^G such that $F^G = \mathbb{C}(p,q)$ and $\{p,q\} = 1$.*

Therefore the assignment $x \mapsto p$ and $y \mapsto q$ defines an field isomorphism from F to F^G which is also a Poisson isomorphism.

Proof. The proof is somewhat formally similar to the noncommutative case in 5.3.2. Let G be any finite subgroup of SL_2. The cyclic case being solved in the first above example, we can suppose that the type of G is D_n, E_6, E_7 or E_8. Then G contains the involution e defined by $e(x)=-x$ and $e(y)=-y$, with $F^e=Q_2$ with notation (77). Take any $g\in G$. There exist $\alpha,\beta,\gamma,\varepsilon\in\mathbb{C}$ with $\alpha\varepsilon-\beta\gamma=1$ such that $g(x)=\alpha x+\beta y$ and $g(y)=\gamma x+\varepsilon y$. Recall that $w:=xy$ and $v:=x^{-1}y$. Since $g(x)=x(\alpha+\beta v)$ and $g(y)=x(\gamma+\varepsilon v)$, we obtain

$$g(v)=\tfrac{\gamma+\varepsilon v}{\alpha+\beta v}\in k(v). \qquad (\dagger)$$

Moreover, $g(w)=\alpha\gamma x^2+\beta\varepsilon y^2+\alpha\varepsilon xy+\beta\gamma yx$ and then

$$g(w)=\left(\tfrac{\beta\varepsilon v^2+(\alpha\varepsilon+\beta\gamma)v+\alpha\gamma}{v}\right)w. \qquad (\ddagger)$$

It follows from ($\dagger$) and ($\ddagger$) that the restrictions to the algebra $S=\mathbb{C}(v)[w]$ of the extensions to F of the elements of G determine a subgroup $G'\simeq G/(e)$ of $\mathrm{Aut}_{\mathbb{C}}S$. Because $e\in G$ and $F^e=Q_2=\mathrm{Frac}\,S$, we deduce that $F^G=Q_2^{G'}$.

Denoting $K=\mathbb{C}(v)$, assertion ($\dagger$) allows to apply theorem 5.1.2 with $S=K[w]$ and $Q_2=\mathrm{Frac}\,S=K(w)=\mathbb{C}(v,w)$. Since $S^{G'}\not\subseteq K$ because $[Q_2:Q_2^{G'}]=|G'|<+\infty$, there exists $u\in S^{G'}$ of degree $w\geq 1$ minimal among the degrees of all elements $S^{G'}\setminus K^{G'}$ such that $S^{G'}=K^{G'}[u]$ and $Q_2^{G'}=K^{G'}(u)$. Denote $u=a_m(v)w^m+a_{m-1}(v)w^{m-1}+\cdots+a_1(v)w+a_0(v)$, with $a_i(v)\in K$ for any $0\leq i\leq m$ and $a_m(v)\neq 0$. For any $h(v)\in K$ we have $\{a_i(v),h(v)\}=0$ thus $\{a_i(v)w^i,h(v)\}=a_i(v)\{w^i,h(v)\}$. Since $\{w,v\}=2v$ implies $\{w,h(v)\}=2v\partial_v(h(v))$, it follows that $\{w^i,h(v)\}=2v\partial_v(h(v))w^{i-1}$. Finally:

$$\{u,h(v)\}=2mva_m(v)\partial_v(h(v))w^{m-1}+\cdots \quad \text{for any } h(v)\in K$$

In particular, if $h(v)\in K^{G'}$, then $\{u,h(v)\}\in S^{G'}$ because $u\in S^{G'}$ and the elements of G' are Poisson automorphisms of S. By minimality of the degree m of u among degrees (related to w) of elements in $S^{G'}\setminus K^{G'}$, it is impossible that $m-1\geq 1$ when $\partial_v(h(v))\neq 0$. So we have proved:

$$\text{if } h(v)\in K^{G'} \text{ with } h(v)\notin\mathbb{C}, \text{ then } \{u,h(v)\}\in K.$$

By Lüroth's theorem, $\mathbb{C}(v)^{G'}$ is a purely transcendental extension $\mathbb{C}(z)$ of $\mathbb{C}$. Since $z\in K$ and $z\notin\mathbb{C}$, it follows from previous calculations that $m=1$ and $\{u,z\}=2va_1(v)\partial_v(z(v))\neq 0$. We introduce $t:=\{u,z\}^{-1}u$ in order to obtain $Q_2^{G'}=\mathbb{C}(z,t)$ with $\{t,z\}=1$, and the proof is complete. □

Remark. Another example of positive answer to the symplectic form of Poisson-Noether's problem in higher dimension can be found in [34] and concerns the action of the Weyl group B_2 of rank two on the symplectic polynomial algebra in four generators. The author also gives a Poisson analogue of Miyata's theorem, and the following interesting example related to the general (non symplectic) formulation of the Poisson-Noether's problem

Example for the multiplicative Poisson structure. We return here to the situation studied in 4.2 where a finite subgroup G of $\mathrm{SL}_2(\mathbb{Z})$ acts by Poisson automorphisms defined from (52) and (53) on the commutative Poisson algebra $T=\mathbb{C}[x^{\pm1},y^{\pm1}]$ for the multiplicative Poisson bracket $\{x,y\}=xy$, see (39) or (40). As an illustration of the corresponding Poisson-Noether's problem that in the particular case where G is the group $\mathcal{G}_{10}$ of order two (see 4.2.2), it is proved in [34] that:

Claim. *There exists a Poisson isomorphism $F^G \simeq F$ where $F = \mathbb{C}(x, y)$, for the multiplicative Poisson bracket $\{x, y\} = xy$.*

Proof. Here G is just $\{I_2, e\}$ where $e := -I_2$ acting by (52) and (53), that is $e.x = x^{-1}$ and $e.y = y^{-1}$. It is known that T^G is generated by $\xi_1 = x + x^{-1}$, $\xi_2 = y+y^{-1}$ and $\theta = xy+x^{-1}y^{-1}$, submitted to the relation $\theta\xi_1\xi_2-\theta^2-\xi_2^2-\xi_2^2+4 = 0$.

Step 1. In $F^G = \mathbb{C}(\xi_1, \xi_2, \theta)$, this algebraic dependence relation rewrites into:

$$(2\theta - \xi_1\xi_2)^2 = \xi_1^2\xi_2^2 - 4(\xi_1^2 + \xi_2^2 - 4) \Leftrightarrow (2\theta - \xi_1\xi_2)^2 = (\xi_1^2 - 4)(\xi_2^2 - 4)$$
$$\Leftrightarrow \Big(\frac{2\theta - \xi_1\xi_2}{\xi_2 - 2}\Big)^2 = (\xi_1^2 - 4)\,\frac{\xi_2 + 2}{\xi_2 - 2}.$$

Let us introduce $\eta := \dfrac{2\theta - \xi_1\xi_2}{\xi_2 - 2} \in F^G$ and $\alpha := \dfrac{\eta^2}{\xi_1^2 - 4} = \dfrac{\xi_2 + 2}{\xi_2 - 2} \in \mathbb{C}(\eta, \xi_1)$.

We have: $\xi_2 = \dfrac{2(\alpha + 1)}{\alpha - 1} \in \mathbb{C}(\eta, \xi_1)$ and then $\theta = \frac{1}{2}(\eta(\xi_2 - 2) + \xi_1\xi_2) \in \mathbb{C}(\eta, \xi_1)$.

We conclude that $F^G = \mathbb{C}(\eta, \xi_1)$.

Step 2. Concerning the Poisson structure, we start from:

$$\{\xi_1, \xi_2\} = 2\theta - \xi_1\xi_2,\ \{\xi_2, \theta\} = 2\xi_1 - \theta\xi_2 \text{ and } \{\theta, \xi_1\} = 2\xi_2 - \theta\xi_1.$$

Thus:
$$\begin{aligned}\{\eta, \xi_1\} = \{\tfrac{2\theta-\xi_1\xi_2}{\xi_2-2}\,,\, \xi_1\} &= \tfrac{-2\theta+\xi_1\xi_2}{(\xi_2-2)^2}\,\{\xi_2 - 2\,,\, \xi_1\} + \tfrac{1}{\xi_2-2}\{2\theta - \xi_1\xi_2\,,\, \xi_1\}\\ &= \big(\tfrac{2\theta-\xi_1\xi_2}{\xi_2-2}\big)^2 + \tfrac{1}{\xi_2-2}\big(2(2\xi_2 - \theta\xi_1) + \xi_1(2\theta - \xi_1\xi_2)\big)\\ &= \eta^2 + \tfrac{\xi_2}{\xi_2-2}(4 - \xi_1^2) = \eta^2 + \tfrac{1}{2}(\alpha + 1)(4 - \xi_1^2)\\ &= \tfrac{1}{2}(\eta^2 - \xi_1^2 + 4).\end{aligned}$$

Hence: $\{\eta\,,\, \eta^2 - \xi_1^2 + 4\} = -\xi_1(\eta^2 - \xi_1^2 + 4)$ and $\{\xi_1\,,\, \eta^2 - \xi_1^2 + 4\} = -\eta(\eta^2 - \xi_1^2 + 4)$.

Therefore: $\{\eta + \xi_1\,,\, \eta^2 - \xi_1^2 + 4\} = -(\eta + \xi_1)(\eta^2 - \xi_1^2 + 4)$ and then:

$$\{\tfrac{1}{\eta+\xi_1}\,,\, \eta^2 - \xi_1^2 + 4\} = \tfrac{1}{\eta+\xi_1}(\eta^2 - \xi_1^2 + 4).$$

Step 3. Conclusion: we define $p := \frac{1}{\eta+\xi_1}$ and $q := \eta^2 - \xi_1^2 + 4 = (\eta + \xi_1)(\eta - \xi_1) + 4$. From the first step, we have $F^G = \mathbb{C}(\eta+\xi_1, \eta-\xi_1) = \mathbb{C}(p, q)$. From the second step $\{p, q\} = pq$. □

5.5.2 Invariants of symplectic Poisson enveloping algebras

Introduction. Let $\mathcal{A}$ be a commutative Poisson algebra over a base field $\Bbbk$. For any $a \in \mathcal{A}$, the derivation $\{a, \cdot\}$ of $\mathcal{A}$ is called the hamiltonian derivation associated to a. We denote it by σ_a. From Jacobi identity we deduce that $\sigma_a\sigma_b - \sigma_b\sigma_a = \sigma_{\{a,b\}}$ for any $a, b \in \mathcal{A}$, then the space $\mathrm{Der}_{\mathrm{Ham}}(\mathcal{A})$ of hamiltonian derivations of $\mathcal{A}$ is a Lie subalgebra of $\mathrm{Der}\mathcal{A}$. The notion of Poisson enveloping algebra $\mathcal{U}_{\mathrm{Pois}}(\mathcal{A})$ defined in [51] can be easily described in the particular case where the Poisson structure on $\mathcal{A}$ is the symplectic one.

We fix V a $\mathbb{C}$-vector space of dimension n. Let $(q_1, \dots, q_n)$ be a basis of V, $(q_{n+1}, \dots, q_{2n})$ its dual basis, and $\{\cdot, \cdot\}$ the symplectic Poisson bracket defined on $V \oplus V^*$, and then on $\mathcal{O}(V \oplus V^*) := S = \mathbb{C}[q_1, \dots, q_n, q_{n+1}, \dots, q_{2n}]$ by

$$\{q_i, q_j\} = \delta_{n+i,j} \quad \text{for any } 1 \leq i \leq n,\ n+1 \leq j \leq 2n. \tag{78}$$

• For any Poisson subalgebra S' of S, the Poisson enveloping algebra $\mathcal{U}_{\mathrm{Pois}}(S')$ is defined as the subalgebra of $\mathrm{End}_{\Bbbk}(S')$ generated by the multiplications μ_a by all elements $a \in S'$ and the hamiltonian derivation σ_a for all $s \in S'$. It is clear in particular for $S' = S$ that $\mathcal{U}_{\mathrm{Pois}}(S) \subseteq \mathrm{Diff}\, S = A_{2n}(\mathbb{C})$ the Weyl algebra

$$A_{2n}(\mathbb{C}) = \mathbb{C}[q_1, \dots, q_{2n}][p_1\,;\,\partial_1] \dots [p_{2n}\,;\,\partial_{2n}].$$

Denoting by $\sigma : S \to A_{2n}(\mathbb{C})$ the map $a \mapsto \sigma_a$, and up to the usual identifications $\mu_{q_i} = q_i$ and $\partial_{q_i} = p_i$ (see 2.3.3), it follows from (78) that:

$$\sigma(a) = \sum_{i=1}^{n} (\partial_i(a) p_{n+i} - \partial_{n+i}(a) p_i) \quad \text{for any } a \in S, \tag{79}$$

In particular $\sigma(q_i) = p_{n+i}$ and $\sigma(q_{n+i}) = -p_i$ for any $1 \leq i \leq n$. Thus any p_j $(1 \leq j \leq 2n)$ acts as an hamiltonian derivation on S, and therefore:

$$\mathcal{U}_{\mathrm{Pois}}(S) = A_{2n}(\mathbb{C}) \tag{80}$$

• Consider $K = \mathrm{Frac}\, S = \mathbb{C}(q_1, \dots, q_{2n})$, and the algebra $B_{2n}(\mathbb{C})$ of differential operators with rational coefficients:

$$B_{2n}(\mathbb{C}) = K[p_1\,;\,\partial_1] \dots [p_{2n}\,;\,\partial_{2n}] = \mathbb{C}(q_1, \dots, q_{2n})[p_1\,;\,\partial_1] \dots [p_{2n}\,;\,\partial_{2n}].$$

Both algebras $A_{2n}(\mathbb{C}) \subset B_{2n}(\mathbb{C})$ have the same skewfield of fractions which is the Weyl skewfield

$$D_{2n}(\mathbb{C}) = K(p_1\,;\,\partial_1) \dots (p_{2n}\,;\,\partial_{2n}) = \mathbb{C}(q_1, \dots, q_{2n})(p_1\,;\,\partial_1) \dots (p_{2n}\,;\,\partial_{2n}).$$

For the Poisson structure on K extending the bracket in S, we can also extend the map $\sigma : K \to \mathrm{Der} K$ defined by $c \mapsto \sigma_c$. For any $c = a^{-1}b$ with $a, b \in S, a \neq 0$, we have $\sigma(c) = a^{-1}\sigma(b) - a^{-2}b\sigma(a)$. Therefore the subalgebra of $\mathrm{End} K$ generated by K and $\sigma(K)$ is the same that the subalgebra generated by K and $\sigma(S)$. This last algebra being generated by K and the p_j's $(1 \leq j \leq 2n)$, we conclude that

$$\mathcal{U}_{\mathrm{Pois}}(K) = B_{2n}(\mathbb{C}). \tag{81}$$

• Let G be a finite subgroup of the symplectic group $\mathrm{Sp}(V \oplus V^*) \simeq \mathrm{Sp}_{2n}(\mathbb{C})$, acting by Poisson automorphisms on S. The invariant algebra is a Poisson subalgebra of S. Then we can consider the enveloping Poisson algebra

$$\mathcal{V} := \mathcal{U}_{\mathrm{Pois}}(S^G).$$

The action of G extends canonically to $K = \operatorname{Frac} S$. Since G is finite, we have $K^G = \operatorname{Frac}(S^G)$ and we can introduce:

$$\mathcal{W} := \mathcal{U}_{\mathrm{Pois}}(K^G).$$

This action also extends canonically (see 2.3.3) into an action by automorphisms on $\operatorname{Diff} S = A_{2n}(\mathbb{C})$, and then on $B_{2n}(\mathbb{C})$ and $D_{2n}(\mathbb{C})$. We have the following inclusions:

$$(1)\quad \mathcal{V} \subset \mathcal{W}, \qquad (2)\quad \mathcal{V} \subset A_{2n}(\mathbb{C})^G, \qquad (3)\quad \mathcal{W} \subset B_{2n}(\mathbb{C})^G,$$

or in other words:

$$(1)\ \mathcal{U}_{\mathrm{Pois}}(S^G) \subset \mathcal{U}_{\mathrm{Pois}}(K^G), \quad (2)\ \mathcal{U}_{\mathrm{Pois}}(S^G) \subset \mathcal{U}_{\mathrm{Pois}}(S)^G, \quad (3)\ \mathcal{U}_{\mathrm{Pois}}(K^G) \subset \mathcal{U}_{\mathrm{Pois}}(K)^G.$$

> Proof. Assertion (1) is clear. Take $a \in S^G$ and consider the hamiltonian derivation $\sigma_a \in A_{2n}(\mathbb{C})$. For any $g \in G$, we apply (21) to calculate $g.\sigma_a = g\sigma_a g^{-1}$. Then for any $x \in S$, we have $(g.\sigma_a)(x) = g(\sigma_a(g^{-1}(x))) = g(\{a, g^{-1}(x)\}) = \{g(a), x\} = \{a, x\}$. Thus $g.\sigma_a = \sigma_a$ for all $g \in G$ and $a \in S^G$. We conclude that $\sigma(S^G) \subset A_{2n}^G$; this is enough to prove (2). The proof of (3) is similar. □

We are now in position to summarize in the following theorem the main results concerning this kind of invariant algebras. Another complementary result lies in [51] which proves that $\mathcal{U}_{\mathrm{Pois}}(S^G)$ and $\mathcal{U}_{\mathrm{Pois}}(S)^G$ are not Morita equivalent. We emphasize here in particular the quite different picture between the original algebras and their localized versions.

Theorem. *Let G be a nontrivial finite subgroup of $\mathrm{Sp}_{2n}(\mathbb{C})$. If G is abelian, or for any G when $n = 1$, we have:*

(i) $\mathcal{U}_{\mathrm{Pois}}(S^G) \neq \mathcal{U}_{\mathrm{Pois}}(S)^G = A_{2n}(\mathbb{C})^G \not\simeq A_{2n}(\mathbb{C})$.

(ii) $\mathcal{U}_{\mathrm{Pois}}(K^G) = \mathcal{U}_{\mathrm{Pois}}(K)^G = B_{2n}(\mathbb{C})^G \simeq B_{2n}(\mathbb{C})$.

• Proof of assertion (i). We denote $N = 2n$ and consider on $A_N(\mathbb{C})$ the $\mathbb{Z}$-graduation extending the natural graduation on S by giving degree 1 to each q_i and degree -1 to each p_i.

$$A_N(\mathbb{C}) = \bigoplus_{j\in\mathbb{Z}} T_j \qquad \text{and} \qquad S = \bigoplus_{j\in\mathbb{N}} S_j$$

where T_j is spanned by monomials $q_1^{a_1} \dots q_N^{a_N} p_1^{b_1} \dots p_N^{b_N}$ such that $a_1 + \cdots + a_N - b_1 - \cdots - b_N = j$, and S_j by monomials $q_1^{a_1} \dots q_N^{a_N}$ such that $a_1 + \cdots + a_N = j$. We know by Noether's theorem that the subalgebra S^G is finitely generated. We claim that more precisely S^G is here generated by homogeneous elements of degree ≥ 2.

> This is clear from 2.2.1 when $n = 1$ (and so $G \subset SL_2$). In the case where G is abelian, we can suppose by total reducibility argument (exactly as in 2.3.1) that (up to conjugation) any automorphism $g \in G$ acts on the symplectic basis (78) of S by $g(q_i) = \chi_i(g)q_i$ and $g(q_{i+n} = \chi_i(g)^{-1}q_{i+n}$ for some complex characters $\chi_1, \dots, \chi_n$ of G, and the result follows.

Let s be any element in S^G. It decomposes into $s = s_0 + s_2 + s_3 + \cdots + s_k$ with $s_0 \in \mathbb{C}$ and $s_j \in S_j$ for any $2 \le j \le k$. With the usual notations $\partial_i = \partial_{q_i}$, it follows that $\partial_i(s_0) = 0$ and $\deg \partial_i(s_j) \ge 1$ for any $1 \le i \le 2n$. Hence for any $1 \le i \le n$, we have $\deg \partial_i(s) \ge 1$, $\deg \partial_{n+i}(s) \ge 1$ and $\deg p_{n+i} = \deg p_i = -1$. We conclude with relation (79) that $\sigma(s) \in \bigoplus_{j\ge 1} T_j$ for any $s \in S^G$. Since $S^G \subset S \subset \bigoplus_{j\ge 0} T_j$ and $\mathcal{U}_{\mathrm{Pois}}(S^G)$ is generated by S^G and $\sigma(S^G)$, we conclude that:

$$\mathcal{U}_{\mathrm{Pois}}(S^G) \subset \textstyle\bigoplus_{j\ge 0} T_j.$$

The group G acts linearly on $W = \oplus_{1\le i\le N} \mathbb{C} q_i$ but also on $W' = \oplus_{1\le i\le N} \mathbb{C} p_i$ [this follows from the definition of the extension of the action studied in 2.3.3, see in particular identities (21), (22) and (23)]. Thus, applying the same argument as above to the action of G on $S' = \mathcal{O}(W') = \mathbb{C}[p_1, \dots, p_N]$, we know that S'^G is generated by homogeneous elements of degree ≥ 2 into the p_i's. Such an elements lies in $A_N(\mathbb{C})^G$ but not in $\bigoplus_{j\ge 0} T_j$. We have proved that the inclusion $\mathcal{U}_{\mathrm{Pois}}(S^G) \subset A_{2n}(\mathbb{C})^G$ is not an equality. For the non isomorphism of $A_{2n}(\mathbb{C})^G$ with $A_{2n}(\mathbb{C})$, see references at the end of section 2.3.1 □

• Proof of assertion (ii) in the abelian case. We fix a finite abelian subgroup G of $\mathrm{Sp}(V \oplus V^*)$. By total reducibility (and as in 2.3.1, see above), we can suppose up to change the symplectic basis that G acts on $V \oplus V^*$ and then on $S = \mathcal{O}(V \oplus V^*) = \mathbb{C}[q_1, \dots, q_{2n}]$ by

$$g(q_j) = \phi_j(g) q_j \quad \text{and} \quad g(q_{j+n}) = \phi_j(g)^{-1} q_{j+n} \quad \text{for any } g \in G,\ 1 \le j \le n,$$

where $\phi_1, \dots, \phi_n$ are complex characters of G. Following (21), (22) and (23) defining the extension of the action to $A_{2n}(2\mathbb{C})$, and therefore to $B_{2n}(\mathbb{C})$ and $D_{2n}(\mathbb{C})$, we obtain

$$g(p_j) = \phi_j(g)^{-1} p_j \quad \text{and} \quad g(p_{j+n}) = \phi_j(g) p_{j+n} \quad \text{for any } g \in G,\ 1 \le j \le n.$$

The elements $w_1, \dots, w_{2n} \in A_{2n}(\mathbb{C})$ defined by $w_j := q_j p_j$ for any $1 \le j \le 2n$ satisfy the relations

$$[w_j, q_i] = \delta_{i,j} q_i \quad \text{and} \quad [w_j, w_i] = 0 \ \text{ for all } 1 \le i, j \le 2n.$$

Then it is clear that

$$B_{2n}(\mathbb{C}) = K[w_1\,;\, d_1] \dots [w_{2n}\,;\, d_{2n}] = \mathbb{C}(q_1, \dots, q_{2n})[w_1\,;\, d_1] \dots [w_{2n}\,;\, d_{2n}],$$

where d_j denotes the Euler derivative $d_j = q_j \partial_{q_j}$. By construction, all w_j's are G-invariants. Hence K^G is stable for each d_j because if $a \in K^G$, then $d_j(a) = w_j a - a w_j$ with $w_j \in A_{2n}(\mathbb{C})^G$. Moreover the monomials into the w_j's being a basis of $B_{2n}(\mathbb{C})$ over K, they are also a basis of $B_{2n}(\mathbb{C})^G$ over K^G. To sum up, we have:

$$B_{2n}(\mathbb{C})^G = K^G[w_1\,;\, d_1] \dots [w_{2n}\,;\, d_{2n}].$$

The first step consists in the determination of the commutative invariant field K^G. Let us introduce in S the elements $y_1, \dots, y_n$ defined by $y_j := q_j q_{j+n}$ for any $1 \le j \le n$. Denoting $F = \mathbb{C}(q_1, \dots, q_n) \subset K$, we have $K = F(q_{n+1}, \dots, q_{2n}) = F(y_1, \dots, y_n)$. The y_j's being G-invariants by construction it follows that $K^G = F^G(y_1, \dots, y_n)$. Now observe that G acts diagonally on the generators $q_1, \dots, q_n$ of F; thus we can apply theorem 1 of [37] (see also corollary 2 in 5.1.2) to describe F^G as the purely transcendental extension

$$F^G = \mathbb{C}(z_1, \dots, z_n), \quad \text{with } z_j = q_1^{m_{j,1}} q_2^{m_{j,2}} \cdots q_j^{m_{j,j}} \text{ for any } 1 \le j \le n$$

where the $m_{j,i}$'s (for $1 \le i \le j \le n$) are $\frac{n(n+1)}{2}$ nonnegative integer such that $m_{j,j} \neq 0$ for any $1 \le j \le n$. We conclude that

$$K^G = \mathbb{C}(y_1, \dots, y_n, z_1, \dots, z_n).$$

The second step consists in the determination of $\mathcal{W} = \mathcal{U}_{\mathrm{Pois}}(K^G)$. Recall that $\mathcal{W}$ is the subalgebra of $B_{2n}(\mathbb{C})^G$ generated over $\mathbb{C}$ by K^G and $\sigma(S^G)$. Since $y_1, \dots, y_n, z_1, \dots, z_n$ are elements of S^G, it is clear that $\mathcal{W}$ contains the subalgebra $\mathcal{W}'$ generated over $\mathbb{C}$ by K^G and the set $E = \{\sigma(y_j)\,,\, \sigma(z_j) \mid 1 \le j \le n\}$. We calculate $\sigma(y_j)$ for any $1 \le j \le n$:

$$\sigma(y_j) = \sigma(q_j q_{j+n}) = q_j \sigma(q_{j+n}) + q_{j+n}\sigma(q_j) = -q_j p_j + q_{j+n} p_{j+n} = -w_j + w_{j+n}.$$

To compute $\sigma(z_j)$, observe that $\sigma(q_i^{m_{j,i}}) = m_{j,i} q_i^{m_{j,i}-1} p_{n+i}$. We deduce:

$$\sigma\Big(\prod_{i=1}^{j} q_i^{m_{j,i}}\Big) = \sum_{i=1}^{j} m_{j,i} q_i^{m_{j,i}-1} \Big[\prod_{k=1, k\neq i}^{j} q_k^{m_{j,k}}\Big] p_{n+i} = \sum_{i=1}^{j} m_{j,i} z_j q_i^{-1} p_{n+i} = \sum_{i=1}^{j} m_{j,i} z_j q_i^{-1} q_{n+i}^{-1} w_{n+i}.$$

Hence the n elements $\sigma(z_1), \sigma(z_2), \dots, \sigma(z_n)$ in E are given by the linear system

$$\begin{pmatrix} \sigma(z_1) \\ \sigma(z_2) \\ \vdots \\ \sigma(z_n) \end{pmatrix} = R \begin{pmatrix} w_{n+1} \\ w_{n+2} \\ \vdots \\ w_{2n} \end{pmatrix},$$

where R is the $n \times n$ triangular matrix with entries in K^Gwhose general entry (on the j-th row and i-th column) is $r_{j,i} = m_{j,i} z_j q_i^{-1} q_{n+i}^{-1}$ when $i \le j$, and zero if $i > j$. Its determinant is

$$\det R = \Big(\prod_{j=1}^{n} z_j\Big)\Big(\prod_{i=1}^{2n} q_i^{-1}\Big)\Big(\prod_{j=1}^{n} m_{j,j}\Big) \neq 0.$$

We deduce that the elements $w_{n+1}, \dots, w_{2n}$ of $B_{2n}(\mathbb{C})^G$ can be expressed as linear combinations with coefficients in K^G of the elements $\sigma(z_1), \sigma(z_2), \dots, \sigma(z_n)$ de E. Thus $w_{n+1}, \dots, w_{2n} \in \mathcal{W}'$. Since $w_j = w_{n+j} - \sigma(y_j)$ for any $1 \le j \le n$, we finally conclude that $w_j \in \mathcal{W}'$ for all $1 \le j \le 2n$. Hence $\mathcal{W}'$ contains K^G and all w_j for $1 \le j \le 2n$, then $\mathcal{W}' \supseteq B_{2n}(\mathbb{C})^G$. Since $\mathcal{W}' \subseteq \mathcal{W} \subseteq B_{2n}(\mathbb{C})^G$, the three algebras are equal.

The third step consists in proving the isomorphism $B_{2n}(\mathbb{C})^G \simeq B_{2n}(\mathbb{C})$. We start from the description:

$$B_{2n}(\mathbb{C})^G = \mathbb{C}(z_1, \dots, z_n, y_1, \dots, y_n,)[w_1\,;\, d_1] \dots [w_{2n}\,;\, d_{2n}],$$

where the generators $z_j = q_1^{m_{j,1}} q_2^{m_{j,2}} \cdots q_j^{m_{j,j}}$ and $y_j = q_j q_{j+n}$ (for $1 \le j \le n$) satisfy the following commutation relations (for $1 \le k \le 2n$)

$$[w_k, z_j] = \begin{cases} m_{j,k} z_j & \text{if } 1 \le k \le j\,, \\ 0 & \text{otherwise,} \end{cases} \qquad [w_k, y_j] = \begin{cases} y_j & \text{if } k = j \text{ or } k = n + j\,, \\ 0 & \text{otherwise.} \end{cases}$$

We realize the linear change of variables $t_i := \sum_{k=i}^{n} \alpha_{i,k}(w_k - w_{k+n})$ for any $1 \leq i \leq n$ where $(\alpha_{i,i}, \alpha_{i,i+1}, \ldots, \alpha_{i,n})$ is the unique solution in $\mathbb{C}$ of the system of $n-i+1$ following equations:

$$\begin{cases} \alpha_{i,i} m_{i,i} = 1, \\ \sum_{k=i}^{i+h} \alpha_{i,k} m_{i+h,k} = 0 \quad \text{pour } 1 \leq h \leq n-i. \end{cases}$$

By construction we have for all $1 \leq i, j \leq n$ the relations $[t_i, t_j] = [t_i, y_j] = 0$ and $[t_i, z_j] = \delta_{i,j} z_j$. By the new change of notations:

$$\begin{array}{ll} v_j := z_j \ \text{ if } 1 \leq j \leq n, & \text{and} \quad v_j := y_{n-j} \ \text{ if } n+1 \leq j \leq 2n, \\ u_i := t_i \ \text{ if } 1 \leq i \leq n & \text{and} \quad u_i := w_{n-i} \ \text{ if } n+1 \leq i \leq 2n, \end{array}$$

we obtain $[u_i, u_j] = [v_i, v_j] = 0$ and $[u_i, v_j] = \delta_{i,j} v_j$ for all $1 \leq i, j \leq 2n$. Therefore

$$B_{2n}(\mathbb{C})^G = \mathbb{C}(v_1, \ldots, v_{2n},)[u_1\,;\,D_1]\ldots[u_{2n}\,;\,D_{2n}],$$

where D_i denotes for any $1 \leq i \leq 2n$ the Euler derivative $D_i = v_i \partial_{v_i}$. Now it is enough to replace each generator u_i by $v_i^{-1} u_i$ to conclude $B_{2n}(\mathbb{C})^G \simeq B_{2n}(\mathbb{C})$. □

Observation. This last result provides an alternative proof of the first corollary in 5.4.2. More precisely, it can be viewed as an intermediate situation between the non-isomorphism $A_{2n}(\mathbb{C})^G \not\simeq A_{2n}(\mathbb{C})$ and the isomorphism $D_{2n}(\mathbb{C})^G \not\simeq D_{2n}(\mathbb{C})$, proving that the localization only by the functions (i.e. the elements of S) is sufficient to obtain the isomorphism.

Remark. The missing case to achieve the proof of the theorem concerns assertion (ii) in the particular situation where $n = 1$. Then G is a (non necessarily abelian) subgroup of $\mathrm{SL}_2(\mathbb{C})$, acting by the canonical linear action on $S = \mathbb{C}[q_1, q_2]$, extended by duality to the Weyl algebra $A_2(\mathbb{C})$, and then to the localizations $B_2(\mathbb{C})$ and $D_2(\mathbb{C})$. The proof is too technical to take place here integrally; thus we just outline the main argumentation and refer to [45] for a complete detailed writing.

• Sketch of the proof of assertion (ii) in the SL_2 case. Here $S = \mathbb{C}[q_1, q_2]$ and $K = \mathbb{C}(q_1, q_2)$. We know (see 2.2.1) that the subalgebra S^G is generated by three homogeneous polynomials f_1, f_2, f_3 into the variables q_1, q_2. Referring to (79), we denote:

$$h_i = \sigma(f_i) = \partial_1(f_i)p_2 - \partial_2(f_i)p_1 \qquad \text{for } i = 1, 2, 3.$$

Thus the algebra $\mathcal{W} = \mathcal{U}_{\mathrm{Pois}}(K^G)$, which is defined as the subalgebra of $B_2(\mathbb{C})^G$ generated by K^G and $\sigma(S^G)$, is equivalently generated by K^G and $\{h_1, h_2, h_3\}$. Moreover applying the operator $\sigma = \partial_1 p_2 - \partial_2 p_1$ to the algebraic equation $F(f_1, f_2, f_3) = 0$ of the Kleinian surface $\mathcal{F}$, we deduce for the h_1, h_2, h_3 a linear relation with nonzero left coefficients in K^G. This relation allows to express h_3 as a linear combination of h_1 and h_2 with coefficients in K^G. Explicitly :

G of type A_{n-1}	$f_1^n - f_2 f_3 = 0$	$n f_1^{n-1} h_1 - f_3 h_2 - f_2 h_3 = 0$
G of type D_n	$f_1^{n+1} + f_1 f_2^2 + f_3^2 = 0$	$((n+1)f_1^n + f_2^2)h_1 + 2f_1 f_2 h_2 + 2f_3 h_3 = 0$
G of type E_6	$f_1^4 + f_2^3 + f_3^2 = 0$	$4f_1^3 h_1 + 3f_2^2 h_2 + 2f_3 h_3 = 0$
G of type E_7	$f_1^3 f_2 + f_2^3 + f_3^2 = 0$	$3f_1^2 f_2 h_1 + (f_1^3 + 3f_2^2)h_2 + 2f_3 h_3 = 0$
G of type E_8	$f_1^5 + f_2^3 + f_3^2 = 0$	$5f_1^4 h_1 + 3f_2^2 h_2 + 2f_3 h_3 = 0$

We deduce that $\mathcal{W}$ is the subalgebra of $B_2(\mathbb{C})^G$ generated by K^G and the elements h_1, h_2. For each of the five cases, let us define $y := \frac{1}{d_1} f_1 h_1$ with $d_1 = \deg f_1$. It is clear that $y \in B_2(\mathbb{C})^G$. An important (but technical) step in the proof consists then in proving by direct calculations in connection with the Casimir element $w := q_1p_1 + q_2p_2$ (see the proposition in 2.3.3) that $B_2(\mathbb{C})^G$ can be described as a iterated Ore extension over K^G :

$$B_2(\mathbb{C})^G = K^G[y\,;\,D][w\,;\,D'] \tag{82}$$

where the derivations D and D' traduce the intended commutation relations between the generators. Now $h_2 = s_1 y + s_2 w$ is a linear combination of y and w whose coefficients in K^G can be explicitly calculated: $s_1 = -[\partial_1(g_2)q_1 + \partial_2(g_2)q_2]$ and $s_2 = -\frac{1}{d_1} {g_1}^{-1}[d_1 g_1 \partial_1(g_2) - d_2 g_2 \partial_1(g_1)]q_2^{-1}$.

G of type A_{n-1}	$d_1 = 2$	$d_2 = n$	$h_1 = -2g_1 y$	$h_2 = -ng_2 y - \frac{n}{2} g_2 g_1^{-1} w$
G of type D_n	$d_1 = 4$	$d_2 = 2n$	$h_1 = -4g_1 y$	$h_2 = -2ng_2 y - ng_3 g_1^{-1} w$
G of type E_6	$d_1 = 6$	$d_2 = 8$	$h_1 = -6g_1 y$	$h_2 = -8g_2 y + \frac{4}{3} g_3 g_1^{-1} w$
G of type E_7	$d_1 = 8$	$d_2 = 12$	$h_1 = -8g_1 y$	$h_2 = -12g_2 y + 2g_3 g_1^{-1} w$
G of type E_8	$d_1 = 12$	$d_2 = 20$	$h_1 = -12g_1 y$	$h_2 = -20g_2 y - \frac{5}{3} g_3 g_1^{-1} w$

In conclusion, $B_2(\mathbb{C})^G$ is generated over K^G by h_1 and h_2. In other words $B_2(\mathbb{C})^G = \mathcal{U}_{\mathrm{Pois}}(K^G)$. The isomorphism $B_2(\mathbb{C})^G \simeq B_2(\mathbb{C})$ follows then from (82) by computational iterated changes of variables (see [45]). □

6 Completion: actions on noncommutative power series

6.1 Actions on skew Laurent series

We have already mentioned in 5.2.2 that a standard method in study of noncommutative fields of rational functions consists in embedding them into a field of noncommutative power series. We develop here this approach in connection with the open question of the structure of automorphism groups of Weyl skewfields and their quantum analogues. Partial results about $D_1(\mathbb{C})$ are cited at the end of 5.3.3; we concentrate now on the Weyl skewfield $D_1^q(\mathbb{C})$.

6.1.1 Automorphisms of skew Laurent series rings

We fix R a commutative domain. For any automorphism σ of R, the skew power series ring $A = R[[x\,;\,\sigma]]$ in one variable x over R twisted by σ is by definition the set of infinite sums $\sum_{i\geq 0} a_i x^i$ where the a_i's are in R, with usual addition and noncommutative multiplication defined from the law:

$$xa = \sigma(a)x \qquad \text{for all } a \in R. \tag{83}$$

Of course A contains the ring $T = R[x\,;\,\sigma]$ in the sense of 1.3.1, the elements of T bing the finite sums $\sum_i a_i t^i$, with usual addition and the same commutation law (83). It's clear that x generates a two-sided ideal in A; the localized ring of A with respect of the powers of x is denoted by $B = R(\!(x\,;\,\delta)\!)$. The elements of B are the Laurent series $\sum_{i>-\infty} a_i x^i$ where the a_i's are in R, with usual addition and noncommutative multiplication defined from (83) and

$$x^{-1}a = \sigma^{-1}(a)x^{-1} \qquad \text{for all } a \in R. \tag{84}$$

In particular $T \subset A \subset B$. For any nonzero element $f = \sum_{i>-\infty} a_i x^i \in B$, the integer $m \in \mathbb{Z}$ such that $a_m \neq 0$ and $a_j = 0$ for all $j < m$ is named the valuation of f, denoted by $v_x(f)$, and the element a_m is the coefficient of lowest valuation of f, denoted by $\varphi(f)$. By convention, we set $v_x(0) = +\infty$ and $\varphi(0) = 0$. It's easy to check that $v_x : B \to \mathbb{Z}$ is a discrete valuation and that $\varphi : B \to R$ satisfies $\varphi(fg) = \varphi(f)\sigma^{v_x(f)}(\varphi(g))$ for any $f, g \in B$. It follows that A and B are domains. We have $A = \{f \in B\,;\, v_x(f) \geq 0\}$ and any $f \in B$ can be written $f = hx^m$ for $m = v_x(f) \in \mathbb{Z}$ and $h \in A$. It is easy to prove (by the same argument than in the commutative setting) that an element of B is invertible in B if and only if its lowest valuation coefficient $\varphi(f)$ is invertible in R.
We concentrate in the following on the situation where R is a field K. Then its follows from all previous observation that in this case B is skewfield ; we will denote it by F :

$$\text{if } K \text{ is a field, } \sigma \in \operatorname{Aut} K, \text{ and } A = K[[x\,;\,\sigma]], \text{ then } F = K(\!(x\,;\,\sigma)\!) = \operatorname{Frac} A. \tag{85}$$

Then it follows from the trivial inclusion $T = K[x\,;\,\sigma] \subset A = K[[x\,;\,\sigma]]$ that the skewfield of rational functions $Q := \operatorname{Frac} T = K(x\,;\,\sigma)$ is a subfield of F:

$$T = K[x\,;\,\sigma] \subset \operatorname{Frac} T = K(x\,;\,\sigma) \subset K(\!(x\,;\,\sigma)\!). \tag{86}$$

The following theorem (appearing in [22]) asserts that any automorphism of the skewfield F is continuous for the x-adic topology, and then is an extension of an automorphism of the ring A. We need the preliminary technical lemma.

LEMMA. *Let p be a prime, $p \neq \operatorname{Char} K$. Then any element of A of the form $1+\sum_{i\geq 1} a_i x^i$ admits a p-th root in A.*

PROOF. Denote $f = \sum_{i\geq 0} b_i x^i \in A$ and $f^p = \sum_{i\geq 0} b_{p,i} x^i$ with $b_i, b_{p,i} \in K$. By straightforward calculations using (83), $b_{p,i} = [\sum_{0\leq j\leq p-1} b_0^{p-1-j} \sigma^i(b_0^j)]\, b_i + r_i$ where the rest r_i only depends on $b_{i-1}, \ldots, b_1, b_0$ (and their images by σ). Hence, for any sequence $(a_i)_{i\geq 1}$ of elements of K, we can find inductively a sequence $(b_i)_{i\geq 0}$ with $b_0 = 1$ such that $b_{p,i} = a_i$ for any $i \geq 1$, and then $(\sum_{i\geq 0} b_i x^i)^p = 1 + \sum_{i\geq 1} a_i x^i$. $\square$

THEOREM. *Let σ be an automorphism of a commutative field K. Let θ be an automorphism of $F = K(\!(x\,;\,\sigma)\!)$. Then $v_x(\theta(f)) = v_x(f)$ for all F. In particular θ restricts into an automorphism of $A = K[[x\,;\,\sigma]]$.*

PROOF. Let θ be an automorphism of F. Suppose that there exists $a \in K^\times$ such that $v_x[\theta(ax)] < 0$. Then $\theta(1+x^{-1}a^{-1}) = 1+\theta(x^{-1}a^{-1})$ with $v_x[\theta(x^{-1}a^{-1})] > 0$. We fix a prime $p \neq \operatorname{Char} K$ and apply the lemma: there exists $h \in A$ such that $\theta(1+x^{-1}a^{-1}) = h^p$. Hence $-1 = v_x(1+x^{-1}a^{-1}) = v_x[\theta^{-1}(h^p)] = v_x[\theta^{-1}(h)^p] \equiv 0$ modulo p. This is a contradiction. Thus we have proved:

$$v_x[\theta(ax)] \geq 0 \quad \text{for any } a \in K^\times. \qquad (\star)$$

In particular $s := v_x[\theta(x)] \geq 0$. For any $a \in K^\times$, we have $0 \leq v_x[\theta(ax)] = v_x[\theta(a)] + v_x[\theta(x)]$ then $v_x[\theta(a)] \geq -s$. Suppose that there exists $a_0 \in K^\times$ such that $v_x[\theta(a_0)] = -m$ for some $0 < m \leq s$. For $a = a_0^{s+1}$, we deduce $-s \leq v_x[\theta(a_0^{s+1})] = -m(s+1)$. This is a contradiction because $s \geq 0, m \geq 1$. Thus we have proved that $v_x[\theta(a)] \geq 0$ for any $a \in K^\times$, and up to taking the inverse a^{-1}, we conclude:

$$v_x[\theta(a)] = 0 \quad \text{for any } a \in K^\times. \qquad (\star\star)$$

Any $t \in U(A)$ can be written $t = a(1+w)$ where $a \in K^\times$ and $v_x(w) \geq 1$. Applying the lemma for any prime $p \neq \operatorname{char} K$ there exists $g \in A$ such that $a^{-1}t = g^p$. Therefore $(\star\star)$ implies $v_x[\theta(t)] = v_x[\theta(a^{-1}t)] = v_x[\theta(g)^p] \equiv 0$ modulo p. It follows that $v_x[\theta(t)] = 0$; we have proved:

$$\text{for any } t \in U(A), \text{ we have } \theta(t) \in U(A). \qquad (\star\star\star)$$

Since $v_x[\theta(x)] = s \geq 0$, we have $\theta(x) = tx^s$ where $t \in U(A)$. Then $x = \theta^{-1}(t)\theta^{-1}(x)^s$. Using $(\star\star\star)$ for θ^{-1} it follows $1 = 0 + s\theta^{-1}(x)$. Thus $s = 1$ and the proof is complete. $\square$

6.1.2 Application to completion of the first quantum Weyl skewfield

We fix the following data and notations: $\Bbbk$ is a commutative base field, $K := \Bbbk(\!(y)\!)$, $q \in \Bbbk^\times$ is not a root of one, and σ is the $\Bbbk$-automorphism of K defined by $\sigma(y) = qy$. We denote $A = K[[x\,;\,\sigma]]$ and $F = \operatorname{Frac} A = K(\!(x\,;\,\sigma)\!)$. In particular F contains the Weyl skewfield $\Bbbk(y)(x\,;\,\sigma) \simeq D_1^q(\Bbbk)$ defined in (72). Since q is not a root of one, the center of F reduces to $\Bbbk$ (see the proof of the last lemma in 5.2.2). The next theorem describes the automorphism group of F. We need the following technical lemma.

Lemma. *For any $\theta \in \operatorname{Aut} F$, there exist $\beta \in \Bbbk^\times$ and two sequences $(a_i)_{i\geq 1}$ and $(b_i)_{i\geq 1}$ of elements of K, with $a_1 \neq 0$, such that:*

$$\theta(x) = \sum_{i\geq 1} a_i x^i \quad \text{and} \quad \theta(y) = \beta y + \sum_{i\geq 1} b_i x^i.$$

Moreover θ is an inner automorphism if and only if the two following conditions are satisfied:

(i) β *is a power of* q, (ii) *there exists* $u \in K^\times$ *such that* $a_1\sigma(u) = u$.

Proof. We know from theorem 6.1.1 that there exist $(a_i)_{i\geq 1}$ and $(b_i)_{i\geq 1}$ in K, with $a_1 \neq 0$ and $b_0 \neq 0$ such that: $\theta(x) = \sum_{i\geq 1} a_i x^i$ and $\theta(y) = \sum_{i\geq 0} b_i x^i$. The commutation relation $\theta(x)\theta(y) = q\theta(y)\theta(x)$ implies $\sigma(b_0) = qb_0$. We develop in K the series $b_0 = \sum_{i\geq n} \beta_i y^i$ with $n \in \mathbb{Z}$, $\beta_i \in \Bbbk$, $\beta_n \neq 0$. Since q is not a root of one, the support of this series reduces to $\{1\}$, then $b_0 = \beta y$ where $\beta_1 = \beta \in \Bbbk^\times$.

Now θ is inner if and only if there exist $f = \sum_{i\geq m} x^i \in F$, with $m \in \mathbb{Z}$, $u_i \in K$, $u_m \neq 0$, such that $fy = \sigma(y)f$ and $fx = \sigma(x)f$. By identification in $F = K(\!(x\,;\sigma)\!)$, the first relation is equivalent to $\beta = q^m$ and

$$u_{m+n}\, y\,(q^{m+n} - \beta) = \sum_{1\leq i\leq n} b_i \sigma^i(u_{m+n-i}) \quad \text{for any } n \geq 1.$$

The second equality implies in particular that $a_1\,\sigma(u_m) = u_m$. Hence conditions (i) and (ii) are necessary. Suppose conversely that θ satisfies assumptions (i) and (ii). Let $u \in K^\times$ solution of $a_1\,\sigma(u) = u$ and m the unique integer such that $\beta = q^m$. We define a sequence $(u_{m+n})_{n\geq 0}$ of elements of K by: $u_{m+n} = u$ and:

$$u_{m+n} = (q^{m+n} - \beta)^{-1}\, y^{-1} \sum_{1\leq i\leq n} b_i \sigma^i(u_{m+n-i}) \quad \text{for any } n \geq 1.$$

Then the so defined element $f := \sum_{i\geq m} u_i x^i$ satisfies $\theta(y) = fyf^{-1}$ and:

$$fxf^{-1} = [u_m x^{m+1} + \cdots][\sigma^{-m}(u_m^{-1})x^{-m} + \cdots] = u_m\sigma(u_m^{-1})x + \cdots = a_1 x + \cdots$$

Let us denote $\Delta := \theta(x) - fxf^{-1}$ and $s := v_x(\Delta)$. We compute:

$$\theta(y)^{-1}\Delta\theta(y) = \theta(y^{-1}xy) - fy^{-1}f^{-1}fxf^{-1}fyf^{-1} = q\Delta.$$

Suppose that $\Delta \neq 0$. We develop $\Delta = \sum_{i\geq s} w_i x^i$ where $w_i \in K$, $w_s \neq 0$. Hence: $(w_s x^s + \cdots)(\beta y + b_1 x + \cdots) = q(\beta + b_1 x + \cdots)(w_s x^s + \cdots)$. The identification of the terms of valuation s of each side gives $w_s \beta q^s\, y = q\beta\, y\, w_s$. Contradiction, therefore $\Delta = 0$, then $\theta(x) = fxf^{-1}$ and θ is inner.

Some automorphisms of F. For any $\alpha \in \Bbbk^\times$, we denote by w_α the $\Bbbk$-automorphism of $K = \Bbbk(\!(y)\!)$ defined by $w_\alpha(y) = \alpha y$. Let w be the injective morphism $\Bbbk^\times \to \operatorname{Aut} K$ defined by $\alpha \mapsto w_\alpha$, and $\Bbbk^\times \times_w K^\times$ the corresponding semidirect product.
For any $\alpha \in \Bbbk^\times$ and $f(y) \in K^\times$, we denote by $\theta_{\alpha,f}$ the $\Bbbk$-automorphism of F defined by:

$$\theta_{\alpha,f}(y) = \alpha y \text{ and } \theta_{\alpha,f}(x) = f(y)\,x.$$

We define in $\operatorname{Aut} F$ the subgroup

$$S := \{\,\theta_{\alpha,f}\,;\,\alpha \in \Bbbk^\times, f \in K^\times\,\} \simeq \Bbbk^\times \times_w K^\times.$$

Up to the subgroup $\operatorname{Inn} F$, this particular subgroup S contains all automorphisms:

Theorem. *We have:* $\operatorname{Aut} F/\operatorname{Inn} F \simeq S/(\operatorname{Inn} F \cap S)$.

Proof. Let us consider $\theta \in \operatorname{Aut} F$; for $\beta \in \Bbbk^\times$ defined by the previous lemma, let us denote $\alpha = \beta^{-1}$. We introduce $\phi \in \operatorname{Aut} F$ defined by $\phi(y) = \alpha\, y$ and $\phi(x) = w_\alpha(a_1^{-1})x$. Thus $\phi\theta(y) = y + b'_1 x + b'_2 x^2 + \cdots$ and $\phi\theta(x) = x + a'_2 x^2 + \cdots$ where the a'_i, b'_i are in K. Hence conditions (i) and (ii) of the lemma are satisfied and $\phi\theta \in \operatorname{Inn} F$. We conclude that $\operatorname{Aut} F = (\operatorname{Inn} F)S$. □

The determination of $\operatorname{Aut} F$ is completed by the explicit description of the elements of $\operatorname{Inn} F \cap S$ (see proposition 2.8 in [22]).

Comment. Let us recall here that a similar theorem on the structure of automorphism groups is unknown for the rational skewfield $D_1^q(\mathbb{C})$. We have cited some partial results from [24] on the classical $D_1(\mathbb{C})$ in the final remarks of 5.3.3. Similar properties in the quantum case for $D_1^q(\mathbb{C})$ are proved in [24] using explicitly the embedding [in the sense of (86)] of $D_1^q(\mathbb{C})$ in the skew Laurent series field F considered here.

6.2 Actions on pseudo-differential operators and related invariants

6.2.1 Automorphisms of pseudo-differential operators rings

We fix R a commutative domain (related to forthcoming applications, we'll sometimes refer to R as the "ring of functions"). For any derivation d of R, the ring of formal operators in one variable t over R is by definition the Ore extension $T = R[t\,;\,d]$ in the sense of 1.3.1. Let us recall that the elements of T are the finite sums $\sum_i a_i t^i$ where the a_i's are in R, with usual addition and noncommutative product defined from relation:

$$ta = at + d(a) \qquad \text{for all } a \in R. \tag{87}$$

For any derivation δ of R, the ring $A = R[[x\,;\,\delta]]$ of formal power series in one variable x over R is by definition the set of infinite sums $\sum_{i\geq 0} a_i x^i$ where the a_i's are in R, with usual addition and noncommutative multiplication defined from the law:

$$xa = ax + \delta(a)x^2 + \delta^2(a)x^3 + \cdots \qquad \text{for all } a \in R. \tag{88}$$

It's clear that x generates a two-sided ideal in A; the localized ring of A with respect of the powers of x is named the ring of formal pseudo-differential operators in one variable x

with coefficients in R, and is denoted $B = R((x\,;\,\delta))$. The elements of B are the Laurent series $\sum_{i>-\infty} a_i x^i$ where the a_i's are in R, with usual addition and noncommutative multiplication defined from (88) and

$$x^{-1}a = ax^{-1} - \delta(a) \qquad \text{for all } a \in R. \tag{89}$$

It follows from (87) and (89), and we have already observed in 5.2.2, that $T = R[x^{-1}\,;\,-\delta]$ is a subring of $B = R((x\,;\,\delta))$.

For any nonzero series $f \in B$, there exist an integer $m \in \mathbb{Z}$ and a sequence $(a_i)_{i\geq m}$ of elements of R such that $f = \sum_{i\geq m} a_i x^i$ and $a_m \neq 0$. The integer m is the valuation of f, denoted by $v_x(f)$, and the element a_m is the coefficient of lowest valuation of f, denoted by $\varphi(f)$. By convention, we set $v_x(0) = +\infty$ and $\varphi(0) = 0$. It's easy to check that $v_x : B \to \mathbb{Z}$ is a discrete valuation and that $\varphi : B \to R$ is a multiplicative map. It follows that A and B are domains. We have $A = \{f \in B\,;\, v_x(f) \geq 0\}$ and

$$\text{for all } f \in B \text{ with } v_x(f) = m \in \mathbb{Z}, \text{ there exists } h \in A \text{ with } v_x(f) = 0 \text{ s.t. } f = hx^m. \tag{90}$$

For any integer $k \in \mathbb{Z}$, we denote $B_k = \{f \in B\,;\, v_x(f) \geq k\}$ and π_k the morphism $B_k \to R$ defined by $\pi_k(\sum_{i\geq k} a_i x^i) = a_k$. In particular $B_0 = A$.

Remarks

(i) Let $U(A)$ be the group of invertible elements of A. An element $f = \sum_{i\geq 0} a_i x^i$ of A lies in $U(A)$ if and only if $v_x(f) = 0$ and $\varphi(f) = a_0$ lies in the group $U(R)$ of invertible elements of R (although the calculations in A are twisted by δ, the proof is similar to the commutative case). In other words, an element of B lies in $U(B)$ if and only if its coefficient of lowest valuation lies in $U(R)$.

(ii) Let $f = \sum_{i\geq 0} a_i x^i$ be an element of A with $v_x(f) = 0$ and $\varphi(f) = a_0 = 1$. Then, for any positive integer p such that $p.1 \in U(R)$, there exist $h \in A$ satisfying $v_x(h) = 0$ and $\varphi(h) = 1$ such that $f = h^p$ (the proof is a simple calculation by identification and is left to the reader).

Proposition. *We assume here that R is a field. Then:*

(i) $B = R((x\,;\,\delta))$ *is a skewfield, and* $B = \operatorname{Frac} A$ *where* $A = R[[x\,;\,\delta]]$*;*

(ii) $R(x^{-1}\,;\,-\delta) = \operatorname{Frac} R[x^{-1}\,;\,-\delta]$ *is a subfield of B;*

(iii) *for any $f \in B$, we have $f \in A$, or $f \neq 0$ and $f^{-1} \in A$.*

Proof. Straightforward by remark (i) and (90). □

The following x-adic continuity lemma, which will be fundamental in the following, is an analogue of the previous theorem 6.1.1 under somewhat different assumptions: the domain R is not supposed here to be a field but the result only applies to automorphisms of B stabilizing R.

Lemma. *Let δ be a derivation of a commutative domain R. Let θ be an automorphism of $R(\!(x\,;\,\delta)\!)$ such that $\theta(R)=R$. Then $v_x(\theta(f))=v_x(f)$ for all $f\in R(\!(x\,;\,\delta)\!)$.*

Proof. It's clear that $\theta(x)\neq 0$. Denote $s=v_x(\theta(x))\in\mathbb{Z}$. First we prove that $s\geq 0$. Suppose that $s<0$. We set $u=1+x^{-1}\in B$. Since $v_x(\theta(x)^{-1})=-s>0$, we have $\theta(u)=1+\theta(x)^{-1}\in A$. We can apply to $\theta(u)$ the remark (ii) above. For an integer $p\geq 2$ such that $p.1$ is invertible in R, there exists $f\in A$ such that $\theta(u)=f^p$. Applying the automorphism θ^{-1}, we obtain $v_x(u)=pv_x(\theta^{-1}(f))$, so a contradiction since $v_x(u)=-1$ by definition. We have proved that $s\geq 0$. In particular the restriction of θ to A is an automorphism of A.

We can write $\theta(x)=a(1+w)x^s$ with nonzero $a\in R$ and $w\in A$ such that $v_x(w)\geq 1$. Applying θ^{-1}, we obtain $x=\theta^{-1}(a)\theta^{-1}(1+w)\theta^{-1}(x)^s$, and then:

$$v_x(\theta^{-1}(a))+v_x(\theta^{-1}(1+w))+sv_x(\theta^{-1}(x))=1.$$

From the one hand, $\theta(R)=R$ implies $\theta^{-1}(R)=R$, thus $\theta^{-1}(a)$ is a nonzero element of R, and so $v_x(\theta^{-1}(a))=0$. From the other hand, it follows from remark (i) above that $1+w\in U(A)$; since $U(A)$ is stable by θ^{-1} (which is an automorphism of A by the first step of the proof), we deduce that $v_x(\theta^{-1}(1+w))=0$. We deduce that $sv_x(\theta^{-1}(x))=1$. As $s\geq 0$, we conclude that $s=1$ and the result follows. □

6.2.2 Extension of an action from functions to pseudo-differential operators.

We fix R a commutative domain and δ a nonzero derivation of R. We denote by $U(R)$ the multiplicative group of invertible elements in R. We consider a group Γ acting by automorphisms on R.

Definitions. We say that the action of Γ on R is δ-compatible if δ is an eigenvector for the action of Γ by conjugation on $\mathrm{Der}R$, i.e. equivalently when the following condition is satisfied:

$$\text{for all } \theta\in\Gamma,\ \text{there exists } p_\theta\in U(R),\ \text{such that } \theta\circ\delta=p_\theta\,\delta\circ\theta. \tag{91}$$

It's clear that $\theta\mapsto p_\theta$ defines then an application $p:\Gamma\to U(R)$ which is multiplicative 1-cocycle for the canonical action of Γ on $U(R)$, that means which satisfies:

$$p_{\theta\theta'}=p_\theta\theta(p_{\theta'})\qquad\text{for all } \theta,\theta'\in\Gamma. \tag{92}$$

It follows that, if we set

$$\langle\, f\,|_k\,\theta\,\rangle:=p_\theta^{-k}\theta(f)\qquad\text{for all } k\in\mathbb{Z},\ \theta\in\Gamma,\ f\in R, \tag{93}$$

then the map $(\theta,f)\mapsto\langle\, f\,|_k\,\theta\,\rangle$ defines a left action $\Gamma\times R\to R$. This action is named the left action of weight k of Γ on R. The weight 0 action is just the canonical action. For the weight one action, a 1-cocycle for the weight one action is a map $r:\Gamma\to R$ which satisfies:

$$r_{\theta\theta'}=r_\theta+p_\theta^{-1}\theta(r_{\theta'})=r_\theta+\langle\, r_{\theta'}\,|_1\,\theta\,\rangle\qquad\text{for all } \theta,\theta'\in\Gamma. \tag{94}$$

We denote by $Z^1(\Gamma,R)$ the left R^Γ-module of such 1-cocycles. For all $k\in\mathbb{Z}$, we define the additive subgroup of R of weight k invariants:

$$I_k := \{\ f\in R\,;\ \langle\, f\,|_k\,\theta\,\rangle = f\ \text{ for all } \theta\in\Gamma\}. \tag{95}$$

In particular, $I_0=R^\Gamma$ and $I_kI_\ell\subseteq I_{k+\ell}$.

Theorem ([43]). *With the previous data and notations, the action of Γ on R extends into an action by automorphisms on $B=R((x\,;\,\delta))$ if and only if this action is δ-compatible, and we have then:*

$$\theta(x^{-1}) = p_\theta x^{-1} + p_\theta r_\theta \qquad \text{for all } \theta\in\Gamma, \tag{96}$$

where $p:\Gamma\to U(R)$ is the multiplicative 1-cocycle uniquely determined by condition (91) of δ-compatibility and $r:\Gamma\to R$ is a 1-cocycle for the weight one action arbitrarily chosen in $Z^1(\Gamma,R)$.

Proof. Let θ be an automorphism of B such that the restriction of θ to R is an element of Γ. In particular, we have $\theta(R)=R$. We can apply the lemma of 6.2.1 to write $\theta(x^{-1}) = c_{-1}x^{-1}+c_0+c_1x+\cdots$, with $c_i\in R$ for any $i\geq -1$ and $c_{-1}\neq 0$. Moreover $x^{-1}\in U(B)$ implies $\theta(x^{-1})\in U(B)$ and then $c_{-1}\in U(R)$ by remark (i) of 6.2.1. Applying θ to (89), we obtain:

$$\theta(x^{-1})\theta(a)-\theta(a)\theta(x^{-1}) = -\theta(\delta(a)) \quad \text{ for any } a\in R.$$

Since $\theta(a)\in R$, we can develop this identity:

$$[c_{-1}x^{-1}\theta(a)-\theta(a)c_{-1}x^{-1}]+[c_0\theta(a)-\theta(a)c_0]+\textstyle\sum_{j\geq 1}[c_jx^j\theta(a)-\theta(a)c_jx^j] = -\theta(\delta(a)).$$

The first term is: $c_{-1}[x^{-1}\theta(a)-\theta(a)x^{-1}] = -c_{-1}\delta(\theta(a))\in R$. The second is zero by commutativity of R. The third is of valuation ≥ 1. So we deduce that:

$$-c_{-1}\delta(\theta(a)) = -\theta(\delta(a)) \qquad \text{and} \qquad \textstyle\sum_{j\geq 1}[c_jx^j\theta(a)-\theta(a)c_jx^j]=0.$$

Denote $p_\theta := c_{-1}$; we have $p_\theta\in U(R)$ and the first assertion above implies that $p_\theta\delta(\theta(a)) = \theta(\delta(a))$ for all $a\in R$. Now we claim that the second assertion implies that $c_j=0$ for all $j\geq 1$. To see that, suppose that there exists a minimal index $r\geq 1$ such that $c_r\neq 0$; then $\sum_{j\geq r}[c_jx^j\theta(a)-\theta(a)c_jx^j]=0$ implies by identification of the coefficients of lowest valuation that $c_r r\delta(\theta(a))x^{r+1}+\cdots=0$. Therefore $c_r r\delta(\theta(a))=0$. If we choose $a\in R$ such that $\delta(a)\neq 0$, then $\theta(\delta(a))\neq 0$; hence $\delta(\theta(a))\neq 0$ [by the condition $p_\theta\delta(\theta(a))=\theta(\delta(a))$ that we have proved previously], and we obtain a contradiction since R is a domain and $c_r\neq 0$. We conclude that $c_j=0$ for all $j\geq 1$.
We have finally checked that $\theta(x^{-1}) = c_{-1}x^{-1}+c_0$. We have already observed that $p_\theta=c_{-1}$ satisfies (91). Now we set $r_\theta=(c_{-1})^{-1}c_0$. We have $\theta(x^{-1})=p_\theta x^{-1}+p_\theta r_\theta$. Relations (92) and (94) follow then from a straightforward calculation of $\theta(\theta'(x^{-1}))$.

Conversely, let us assume that the action of Γ on R is δ-compatible. Denote by p the map $\Gamma\to U(R)$ uniquely determined by (91), which satisfies necessarily (92). Let us choose a 1-cocycle $r:\Gamma\to R$ arbitrarily in $Z^1(\Gamma,R)$. We consider any $\theta\in\Gamma$; denoting $q_\theta=p_\theta r_\theta$, we calculate for all $a\in R$:

$$(p_\theta x^{-1}+q_\theta)\theta(a)-\theta(a)(p_\theta x^{-1}+q_\theta)=p_\theta(x^{-1}\theta(a)-\theta(a)x^{-1})=-p_\theta\delta(\theta(a))=-\theta(\delta(a)).$$

Hence we can define an automorphism θ_r of $T=R[t\,;\,-\delta]=R[x^{-1}\,;\,-\delta]$ such that the restriction of θ_r to R is θ and $\theta_r(t)=p_\theta t+p_\theta r_\theta$; (observe that $p_\theta\in U(R)$ implies the bijectivity of θ_r). Since $p_\theta\in U(R)$, the element $\theta_r(x^{-1})=p_\theta x^{-1}+q_\theta$ is invertible in B by remark (i) of 6.2.1. Then we define: $\theta_r(x)=\theta_r(x^{-1})^{-1}=x(p_\theta+q_\theta x)^{-1}$ with $p_\theta+q_\theta x$ which is invertible in $A=R[[x\,;\,\delta]]$. So we have built for any $\theta\in\Gamma$ an automorphism θ_r of B which extends θ. It follows immediately from the assumptions (92) on p and (94) on r that $(\theta\theta')_r=\theta_r\theta'_r$ for all $\theta,\theta'\in\Gamma$. □

Remark. Computing $(p_\theta+q_\theta x)^{-1}=\big(\sum_{j\geq 0}(-1)^j(p_\theta^{-1}q_\theta x)^j\big)p_\theta^{-1}\in A$, we deduce that, under the hypothesis of the theorem, we have:

$$\theta(x)=x\Big(\sum_{j\geq 0}(-1)^j(r_\theta x)^j\Big)p_\theta^{-1}=p_\theta^{-1}x+\cdots\quad\text{for all }\theta\in\Gamma\tag{97}$$

In particular, the restriction to B_k of the action of Γ on B defines an action on B_k for any $k\in\mathbb{Z}$.

Corollary. *Under the assumptions of the theorem, the action of Γ on R extends into an action by automorphisms on $B=R((x\,;\,\delta))$ if and only if it extends into an action by automorphisms on $T=R[x^{-1}\,;\,-\delta]$.*

Examples. We suppose that the action of Γ on R is δ-compatible; thus the map $p:\Gamma\to U(R)$ defined by (91) is uniquely determined and satisfies (92), and we consider here various examples for the choice of $r\in Z^1(\Gamma,R)$.

1. If we take $r=0$, the action of Γ on B is defined by $\theta(x^{-1})=p_\theta x^{-1}$, and then $\theta(x)=xp_\theta^{-1}=\sum_{j\geq 0}\delta^j(p_\theta^{-1})x^{j+1}$ for any $\theta\in\Gamma$.

2. If r is a coboundary (i.e. there exists $f\in R$ such that: $r_\theta=\langle\, f\,|_1\,\theta\,\rangle-f=p_\theta^{-1}\theta(f)-f$ for any $\theta\in\Gamma$), then the element $y=(x^{-1}-f)^{-1}$ satisfies $B=R((x\,;\,\delta))=R((y\,;\,\delta))$ and $\theta(y^{-1})=p_\theta y^{-1}$ for any $\theta\in\Gamma$. Thus we find the situation of example 1.

3. We can take for r the map $\Gamma\to R$ defined by: $r_\theta=-p_\theta^{-1}\delta(p_\theta)$ for any $\theta\in\Gamma$, which is an element of $Z^1(\Gamma,R)$ by (91) and (92). The corresponding action of Γ on B is given by: $\theta(x^{-1})=p_\theta x^{-1}-\delta(p_\theta)=x^{-1}p_\theta$ for any $\theta\in\Gamma$.

4. For any $r\in Z^1(\Gamma,G)$ and any $f\in R$, the map $\theta\mapsto r_\theta+p_\theta^{-1}\theta(f)-f$ is an element of $Z^1(\Gamma,R)$. The corresponding action of Γ on B is defined by $\theta(x^{-1})=p_\theta x^{-1}+p_\theta r_\theta+\theta(f)-p_\theta f$. As in example 2, $y=(x^{-1}-f)^{-1}$ satisfies $B=R((x\,;\,\delta))=R((y\,;\,\delta))$ and allows to express the action by $\theta(y^{-1})=p_\theta y^{-1}+p_\theta r_\theta$ for any $\theta\in\Gamma$.

5. Since $Z^1(\Gamma,R)$ is a left R^Γ-module, the map $\kappa\,r$ is an element of $Z^1(\Gamma,R)$ for any $r\in Z^1(\Gamma,G)$ and any $\kappa\in R^\Gamma$. The corresponding action of Γ on B is given by: $\theta(x^{-1})=p_\theta x^{-1}+\kappa\,p_\theta r_\theta$ for any $\theta\in\Gamma$. If we suppose moreover that $\kappa\in U(R)$, then $y=(\kappa^{-1}x^{-1})^{-1}$ satisfies $B=R((x\,;\,\delta))=R((y\,;\,\kappa^{-1}\delta))$, and we find $\theta(y^{-1})=p_\theta y^{-1}+p_\theta r_\theta$ for any $\theta\in\Gamma$.

6.2.3 Invariant pseudo-differential operators

We fix R a commutative domain, δ a nonzero derivation of R, and Γ a group acting by automorphisms on R. We suppose that the action of Γ is δ-compatible and so extends to $B = R(\!(x\,;\,\delta)\!)$ by (96) where r is an arbitrarily chosen element of $Z^1(\Gamma, R)$. We denote by $B^{\Gamma,r}$ (respectively $A^{\Gamma,r}$) the subring of invariant elements of B (respectively A) under this action.

Remarks. For any $k \in \mathbb{Z}$, we denote $B_k^{\Gamma,r} = B_k \cap B^{\Gamma,r}$. The following observations precise some relations between invariant pseudo-differential operators of valuation k (i.e. elements of $B_k^{\Gamma,r}$) and weight k invariant functions (i.e. elements of I_k, see (95)).

(i) If $B^{\Gamma,r} \neq R^\Gamma$, then there exists some nonzero integer k such that $I_k \neq \{0\}$.

> Proof. Suppose that there exists $y \in B^{\Gamma,r}$ such that $y \notin R^\Gamma$. Set $k = v_x(y)$, thus $y \in B_k^{\Gamma,r}$. If $k \neq 0$, then $\pi_k(y)$ is a non zero element of I_k by remark (i). If $k = 0$, then $\pi_0(y) \in I_0 = R^\Gamma$, thus $y' = y - \pi_0(y)$ is a nonzero element of $B_{k'}^{\Gamma,r}$ for some integer $k' > 0$ and we apply the first case.

(ii) For any $k \in \mathbb{Z}$ and $y \in B$, we have: ($y \in B_k^{\Gamma,r} \Rightarrow \pi_k(y) \in I_k$); this is a straightforward consequence of (93), (95), (96) and (97). If we assume that

$$0 \longrightarrow B_{k+1}^{\Gamma,r} \xrightarrow[\text{inj}]{\text{can}} B_k^{\Gamma,r} \xrightarrow{\pi_k} I_k \longrightarrow 0$$

is a split exact sequence, then $B^{\Gamma,r} \neq R^\Gamma$ if and only if there exists some nonzero integer k such that $I_k \neq \{0\}$.

> Proof. Suppose that there exists a nonzero integer k and a nonzero element f in I_k. By assumption, we can consider $\psi_k : I_k \to B_k^{\Gamma,r}$ such that $\pi_k \circ \psi_k = \mathrm{id}_{I_k}$. Then $\psi_k(f) = fx^k + \cdots$ lies in $B_k^{\Gamma,r}$ with valuation $k \neq 0$. Thus $\psi_k(f) \notin R^\Gamma$.

The following theorem gives an explicit description of the ring $B_k^{\Gamma,r}$ when the functions ring R is a field of characteristic zero. It can be viewed as an analogue for noncommutative power series of the theorem previously proved in 5.3.1 for noncommutative rational functions.

Theorem ([43]). *Let R be a commutative field of characteristic zero. Let δ be a nonzero derivation of R, $A = R[[x\,;\,\delta]]$ and $B = R(\!(x\,;\,\delta)\!) = \operatorname{Frac} A$. For any δ-compatible action of a group Γ on R and for any $r \in Z^1(\Gamma, R)$, we have:*

(i) if $A^{\Gamma,r} \subseteq R$, then $A^{\Gamma,r} = B^{\Gamma,r} = R^\Gamma$;

(ii) if $A^{\Gamma,r} \not\subseteq R$ and $R^\Gamma \subset \ker\delta$, then there exist elements of positive valuation in $A^{\Gamma,r}$ and, for any $u \in A^{\Gamma,r}$ of valuation $e = \min\{v_x(y)\,;\, y \in A^{\Gamma,r}, v_x(y) \geq 1\}$, we have $A^{\Gamma,r} = R^\Gamma[[u]]$ and $B^{\Gamma,r} = \operatorname{Frac}(A^{\Gamma,r}) = R^\Gamma(\!(u)\!)$;

(iii) if $A^{\Gamma,r} \not\subseteq R$ and $R^\Gamma \not\subset \ker\delta$, then there exists an element u of valuation 1 in $A^{\Gamma,r}$ and a nonzero derivation δ' of R^Γ such that $A^{\Gamma,r} = R^\Gamma[[u\,;\,\delta']]$ and $B^{\Gamma,r} = \mathrm{Frac}\,(A^{\Gamma,r}) = R^\Gamma(\!(u\,;\,\delta')\!)$.

The proof of this theorem is somewhat long and technical and cannot take place here (see [43]). It uses in an essential way the notion of higher derivation (see [42] for a survey).

Some comments.

1. In point (iii) of the theorem, $\delta' = c_1^{-1}\delta$ where $u = c_1x + c_2x^2 + \cdots$ with $c_i \in R$, $c_1 \neq 0$.

2. The equality $\mathrm{Frac}\,(A^{\Gamma,r}) = (\mathrm{Frac}\,A)^{\Gamma,r}$, which can be nontrivial in some cases (see the proof of 5.3.1 and remark 1 in 5.2.1) follows here immediately from point (iii) of the proposition in 6.2.1.

3. Under the assumptions of the theorem, if r and r' are two 1-cocycles in $Z^1(\Gamma, R)$ such that $B^{\Gamma,r} \not\subseteq R$ and $B^{\Gamma,r'} \not\subseteq R$, then $B^{\Gamma,r} \simeq B^{\Gamma,r'}$.

4. Under the assumptions of the theorem, if the exact sequence of remark (ii) is split for r and r' two 1-cocycles in $Z^1(\Gamma, R)$, then $B^{\Gamma,r} \simeq B^{\Gamma,r'}$.

5. If we don't assume that R is a field, we don't have a general theorem, but some particular results can be useful for further arithmetical applications. In particular it is proved in [43] that: *if there exists in $B^{\Gamma,r}$ an element $w = bx^{-1} + c$ with $b \in U(R)$ and $c \in R$, then the derivation $D = b\delta$ restricts into a derivation of R^Γ, and we have then $A^{\Gamma,r} = R^\Gamma[[u\,;\,D]]$ and $B^{\Gamma,r} = R^\Gamma(\!(u\,;\,D)\!)$ for $u = w^{-1}$.*

6.2.4 Application to completion of the first Weyl skewfield

We take here $R = \mathbb{C}(z)$ and $\delta = \partial_z$. We consider the ring $A = R[[x\,;\,\delta]]$ and its skewfield of fractions $F = R(\!(x\,;\,\delta)\!)$. Then $Q = R(t\,;\,d)$ where $t = x^{-1}$ and $d = -\delta$ is a subfield of F [see point (ii) of proposition 6.2.1] which is clearly isomorphic to the Weyl skewfield $D_1(\mathbb{C})$ (see 5.2.3). We have:

$$xz - zx = x^2, \quad \text{or equivalently} \quad zt - tz = 1.$$

We name F the first local skewfield. It's well known that any $\mathbb{C}$-automorphism θ of R is of the form $z \mapsto \frac{az+b}{cz+d}$ with $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{GL}(2, \mathbb{C})$. For any $f(z) \in R$, we compute:

$$\partial_z(\theta(f)) = \partial_z(f(\tfrac{az+b}{cz+d})) = \tfrac{ad-bc}{(cz+d)^2} f'(\tfrac{az+b}{cz+d}) = \tfrac{ad-bc}{(cz+d)^2}\,\theta(\partial_z(f)).$$

By (91), it follows that the action of any $\theta \in \mathrm{Aut}\,R$ is δ-compatible, with $p_\theta = \frac{(cz+d)^2}{ad-bc}$. We conclude with the theorem of 6.2.2 that any automorphism θ of F which restricts into an automorphism of R is of the form:

$$\theta \;:\; z \mapsto \tfrac{az+b}{cz+d}, \quad x^{-1} \mapsto \tfrac{(cz+d)^2}{ad-bc} x^{-1} + q_\theta(z).$$

where $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{GL}(2, \mathbb{C})$ and $q_\theta(z) \in \mathbb{C}(z)$. Then, using remark 2 of 5.2.1, we can prove that point (iii) of the theorem of 6.2.3 applies and it's easy to deduce with Lüroth's theorem that:

Proposition. *For any finite subgroup Γ of $\mathbb{C}$-automorphisms of $F = \mathbb{C}(z)(\!(x\,;\,\partial_z)\!)$ stabilizing $\mathbb{C}(z)$, we have $F^\Gamma \simeq F$.*

6.3 Applications to modular actions

We give here an overview about some applications of the previous results in number theory (see [43] for a more complete lecture).

6.3.1 Modular forms

In the following, Γ is a subgroup of $\mathrm{SL}(2,\mathbb{C})$, and R is a commutative $\mathbb{C}$-algebra R of functions in one variable z such that:

(i) Γ acts (on the right) by homographic automorphisms on R

$$(f\,|_0\,\gamma) = f(\tfrac{az+b}{cz+d}) \quad \text{for all } f \in R \text{ and } \gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma,$$

(ii) the function $z \mapsto cz+d$ is invertible in R for any $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma$,

(iii) R is stable by the derivation ∂_z.

The case where $R = \mathbb{C}(z)$ corresponds to the formal situation studied at the end of 6.2.3. In many arithmetical situations, R is some particular subalgebra of $\mathcal{F}_{\mathrm{der}}(\Delta,\mathbb{C})$ with $\Delta \subseteq \mathbb{C}$ stable by the homographic action of a subgroup Γ of $\mathrm{SL}(2,\mathbb{Z})$. We denote:

$$(f\,|_k\,\gamma)(z) = (cz+d)^{-k} f(\tfrac{az+b}{cz+d}) \qquad \text{for all } f \in R,\ \gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma,\ k \in \mathbb{Z}. \tag{98}$$

Let us observe that $(\,(f\,|_k\,\gamma')\,|_k\,\gamma\,) = (f\,|_k\,\gamma'\gamma)$ for all $\gamma,\gamma' \in \Gamma$ and $f \in R$. For any $k \in \mathbb{Z}$, we define the $\mathbb{C}$-vector space of weight k modular forms:

$$M_k(\Gamma, R) = \{\ f \in R\,;\ (f\,|_k\,\gamma) = f \text{ for all } \gamma \in \Gamma\ \} \tag{99}$$

Remarks.

1. $M_0(\Gamma,R) = R^\Gamma$.
2. If $\Gamma \ni \left(\begin{smallmatrix} -1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$, then $M_k(\Gamma,R) = (0)$ for any odd k.
3. If Γ contains at least one element $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ such that $(c,d) \notin \{0\} \times \mathbb{U}_\infty$, we have $M_k(\Gamma,R) \cap M_\ell(\Gamma,R) = (0)$ pour $k \neq \ell$.
4. For all $f \in M_k(\Gamma,R)$ and $g \in M_\ell(\Gamma,R)$, we have $fg \in M_{k+\ell}(\Gamma,R)$.
5. For any $f \in M_k(\Gamma,R)$, the function $f' = \partial_z(f)$ satisfies $(f'\,|_{k+2}\,\gamma)(z) = f'(z) + k\frac{c}{cz+d}f(z)$. Thus f' is not necessarily a modular form (unless for $k=0$).

Comment: Rankin-Cohen brackets (see [39]). It follows from remark 5 above that, for $f \in M_k(\Gamma,R)$ and $g \in M_\ell(\Gamma,R)$, and r,s nonnegative integers, the product $f^{(r)}g^{(s)}$ is not necessarily an element of $M_{k+\ell+2r+2s}(\Gamma,R)$. For any integer $n \geq 0$, we denote by $[\ ,\ ]_n$ the n-th Rankin-Cohen bracket, defined as the linear combination:

$$[f,g]_0 = fg,$$

$$[f,g]_1 = kfg' - \ell f'g,$$

$[f,g]_2 = k(k+1)fg'' - (k+1)(\ell+1)f'g' + \ell(\ell+1)f''g,$

...

$[f,g]_n = \sum_{r=0}^{n} (-1)^r \binom{k+n-1}{n-r} \binom{\ell+n-1}{r} f^{(r)} g^{(n-r)},$

and satisfies the characteristic property:

for $f \in M_k(\Gamma, R)$ and $g \in M_\ell(\Gamma, R)$, we have $[f,g]_n \in M_{k+\ell+2n}(\Gamma, R)$.

(More precisely it is possible to prove that any linear combination of $f^{(r)}g^{(s)}$ satisfying this property is a scalar multiple of the n-th Rankin-Cohen bracket). It follows from the definition that $[g,f]_n = (-1)^n [f,g]_n$, and that $[\ ,\ ]_1$ satisfies Jacobi identity.

6.3.2 Associated invariant pseudo-differential operators

• Extension of the modular action.

For $\delta = -\partial_z$, we compute: $\delta(f\,|_0\,\gamma)(z) = -\partial_z(f(\frac{az+b}{cz+d})) = -f'(\frac{az+b}{cz+d}) \times \frac{1}{(cz+d)^2}$, and thus: $(\,\delta(f)\,|_0\,\gamma\,)(z) = (cz+d)^2 \delta(\,f\,|_0\,\gamma\,)(z)$. Then the homographic action of Γ on R is δ-compatible. The associated multiplicative 1-cocycle $p : \Gamma \to U(R)$ defined by (91) is:

$$p_\gamma = (cz+d)^2 \text{ for any } \gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma \tag{100}$$

For any $k \in \mathbb{Z}$, the weight k action in the sense of (94) corresponds to the weight $2k$ action in the sense (98) of modular forms:

$$\langle f\,|_k\,\gamma \rangle(z) = (cz+d)^{-2k} f(\frac{az+b}{cz+d}) = (f\,|_{2k}\,\gamma)(z) \quad \text{for all } \ \gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma,\ f \in R. \tag{101}$$

and then $I_k = M_{2k}(\Gamma, R)$.

We know by example 3 of 6.2.2 that $r'_\gamma = -p_\gamma^{-1}\delta(p_\gamma) = (cz+d)^{-2}\partial_z((cz+d)^2) = 2c(cz+d)^{-1}$ defines an additive 1-cocycle $r' : \Gamma \to R$. Then by example 5 of 6.2.2, we can consider for any $\kappa \in \mathbb{C}$ the additive 1-cocycle $r = \frac{\kappa}{2} r'$:

$$r_\gamma = \kappa\, c(cz+d)^{-1} \quad \text{for all } \ \gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma. \tag{102}$$

Applying the theorem of 6.2.2, the action of Γ on R extends for any $\kappa \in \mathbb{C}$ into an action by automorphisms on $B = R(\!(x\,;\,-\partial_z)\!)$ by

$$\gamma(x^{-1}) = (cz+d)^2 x^{-1} + \kappa\, c(cz+d) \quad \text{for all } \ \gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma. \tag{103}$$

We denote by $B^{\Gamma,\kappa}$ the subalgebra of invariant elements of B under this action.

• Invariant pseudo-differential operators.

We fix $\kappa \in \mathbb{C}$. For any $f \in R$ and any integer $k \geq 1$, we define:

$$\psi_k(f) = fx^k + \sum_{n\geq 1} (-1)^n \tfrac{(n+k-1)!}{n!\,(n+2k-1)!} \times k!\,(-\kappa+k+1)(-\kappa+k+2)\cdots(-\kappa+k+n)\, f^{(n)} x^{k+n} \in B,$$

$$\psi_0(f) = f \ \in \ R,$$

$$\psi_{-k}(f) = fx^{-k} + \sum_{n=1}^{k} \tfrac{(2k-n)!}{n!\,(k-n)!} \times \tfrac{(\kappa+k-n)(\kappa+k-n+1)\cdots(\kappa+k-1)}{(k-1)!}\, f^{(n)} x^{-k+n} \in B,$$

with the notation $f^{(n)} = \partial_z^n(f)$. The following two results by P. Cohen, Y. Manin and Don Zagier allow to define a vector space isomorphism between the invariant pseudo-differential operators and the product of even weight modular forms.

Lemma ([39]). *For all $f \in R$, $k \in \mathbb{Z}$, $\gamma \in \Gamma$, we have: $\psi_k(\,(f\,|_{2k}\,\gamma)\,) = \gamma(\,\psi_k(f)\,)$, thus:*

$$(\ f \in M_{2k}(R;\Gamma)\ ) \ \Leftrightarrow \ (\ \psi_k(f) \in B_k^{\Gamma,\kappa}\ ),$$

and then:

$$0 \longrightarrow B_{k+1}^{\Gamma,r} \xrightarrow[\text{inj}]{\text{can}} B_k^{\Gamma,r} \underset{\psi_k}{\overset{\pi_k}{\rightleftarrows}} M_{2k}(\Gamma,R) \longrightarrow 0$$

is a split exact sequence.

• Theorem ([39]).

(i) *For any $j \in \mathbb{Z}$, the map*

$$\Psi_{2j} : \mathcal{M}_{2j} := \prod_{k\geq j} M_{2k}(\Gamma,R) \longrightarrow B_j^{\Gamma,\kappa}\ ;\ (f_{2k})_{k\geq j} \longmapsto \sum_{k\geq j} \psi_k(f_{2k})$$

is a vector space isomorphism.

(ii) *The map $\Psi_{2*} : \mathcal{M}_{2*} := \bigcup_{j\in\mathbb{Z}} \mathcal{M}_{2j} \longrightarrow \bigcup_{j\in\mathbb{Z}} B_j^{\Gamma,\kappa} = B^{\Gamma,\kappa} = R(\!(x\,;\,-\partial_z)\!)^{\Gamma,\kappa}$ canonically induced by the Ψ_{2j}'s is vector space isomorphism.*

It's not possible to give here the proofs of these results and we can only refer the reader to the original article [39]. In order to illustrate the construction, let us give some explicit calculations for Ψ_0 in the particular case where $\kappa = 0$.

Example.

$\Psi_0 : \mathcal{M}_0 = \prod_{k\geq 0} M_{2k}(\Gamma,R) \longrightarrow A^{\Gamma,0} = R[[x\,;\,-\partial_z]]^{\Gamma,0} = B_0^{\Gamma,0}\ ;\ (f_{2k})_{k\geq 0} \longmapsto \sum_{k\geq 0} \psi_k(f_{2k})$

For any $(f_0, f_2, f_4, \ldots) \in \mathcal{M}_0$, we have:

$$\psi_0(f_0) = f_0$$

$$\psi_1(f_2) = f_2 x - f_2' x^2 + f_2'' x^3 - f_2''' x^4 + \cdots = x f_2$$

$$\psi_2(f_4) = \tfrac{1}{3} f_4 x^2 - \tfrac{1}{2} f_4' x^3 + \tfrac{3}{5} f_4'' x^4 + \cdots$$

$$\psi_3(f_6) = \tfrac{1}{10} f_6 x^3 - \tfrac{1}{5} f_6' x^4 + \cdots$$

$$\psi_4(f_8) = \tfrac{1}{35} f_8 x^4 + \cdots$$

....

thus:

$$\Psi_0 : \mathcal{M}_0 \longrightarrow A^{\Gamma,0} \quad ; \quad (f_{2k})_{k\geq 0} \longmapsto \sum_{n\geq 0} h_n x^n$$
$$\Psi_0^{-1} : A^{\Gamma,0} \longrightarrow \mathcal{M}_0 \quad ; \quad \sum_{n\geq 0} h_n x^n \longmapsto (f_{2k})_{k\geq 0},$$

with:

$$\begin{array}{ll}
h_0 = f_0 & f_0 = h_0 \\
h_1 = f_2 & f_2 = h_1 \\
h_2 = \frac{1}{3} f_4 - f_2' & f_4 = 3h_2 + 3h_1' \\
h_3 = \frac{1}{10} f_6 - \frac{1}{2} f_4' + f_2'' & f_6 = 10h_3 + 15h_2' + 5h_1'' \\
h_4 = \frac{1}{35} f_8 - \frac{1}{5} f_6' + \frac{3}{5} f_4'' - f_2''' & f_8 = 35h_4 + 70h_3' + 42h_2'' + h_1''' \\
\ldots. & \ldots \\
h_n = \sum\limits_{r=0}^{n-1} (-1)^r \frac{n!\,(n-1)!}{r!\,(2n-r-1)!}\, f_{2(n-r)}^{(r)} & f_{2k} = \sum\limits_{r=0}^{k-1} \frac{(2k-1)\,(2k-2-r)!}{r!\,(k-r)!\,(k-r-1)!}\, h_{k-r}^{(r)}
\end{array}$$

6.3.3 Non commutative structure on even weight modular forms

By transfer of structures, the vector space isomorphisms

$$\Psi_{2*} : \mathcal{M}_{2*} \to B^{\Gamma,\kappa} \quad \text{et} \quad \Psi_{2*}^{-1} : B^{\Gamma,\kappa} \to \mathcal{M}_{2*}$$

resulting of point (ii) of the theorem of 6.3.2 allow to equip M_{2*} with a structure of non commutative $\mathbb{C}$-algebra. We denote by $\mathcal{M}_{2*}^{\kappa}$ which depends in principle on the parameter κ fixed in the definition of the extension of the action form R to B.

$$\mathcal{M}_{2*}^{\kappa} \simeq B^{\Gamma,\kappa} \text{ for any } \kappa \in \mathbb{C}.$$

The description given in 6.2.3 of the rings $B^{\Gamma,\kappa}$ allows to deduce some algebraic properties (center, centralizers,...) of the algebras $\mathcal{M}_{2*}^{\kappa}$. In particular, supposing that R is a field of characteristic zero, the corollary of the theorem on 6.2.3 given in the comment 4 applies by the lemma of 6.3.2, and we prove so that:

Theorem. *If R is a commutative field of characteristic zero, then $\mathcal{M}_{2*}^{\kappa} \simeq \mathcal{M}_{2*}^{\kappa'}$ for all $\kappa, \kappa' \in \mathbb{C}$.*

Application to the noncommutative product of two modular forms. Let us fix $f \in M_{2k}(\Gamma, R)$ and $g \in M_{2\ell}(\Gamma, R)$. With the identifications:

$$f \equiv (f, 0, 0, \ldots) \in \mathcal{M}_{2k} \quad \text{and} \quad g \equiv (g, 0, 0, \ldots) \in \mathcal{M}_{2\ell},$$

the noncommutative product of f by g in $\mathcal{M}_{2*}^{\kappa}$, for an arbitrary choice of $\kappa \in \mathbb{C}$, is given by:

$$\mu^{\kappa}(f,g) = \Psi_{2*}^{-1}\big(\Psi_{2*}(f).\Psi_{2*}(g)\big) = \Psi_{2(k+\ell)}^{-1}\big(\psi_k(f).\psi_\ell(g)\big) \in \mathcal{M}_{2(k+\ell)}.$$

The authors of [39] prove then that:

$$\mu^\kappa(f,g) = \sum_{n\geq 0} t_n^\kappa(k,\ell)\,[f,g]_n,$$

where $[\ ,\ ]_n : M_{2k}(\Gamma,R)\times M_{2\ell}(\Gamma,R) \to M_{2(k+\ell+n)}(\Gamma,R)$ is the n-th Rankin-Cohen bracket (see comment in 6.3.1), and $t_n^\kappa(k,\ell)\in\mathbb{Q}$ is defined by:

$$t_n^\kappa(k,\ell) = \frac{1}{\binom{-2\ell}{n}} \sum_{r+s=n} \frac{\binom{-k}{r}\binom{-k-1+\kappa}{r}}{\binom{-2k}{r}} \frac{\binom{n+k+\ell-\kappa}{s}\binom{n+k+\ell-1}{s}}{\binom{2n+2k+2\ell-2}{s}}$$

These coefficients satisfy $t_n^\kappa(k,\ell) = t_n^{2-\kappa}(k,\ell)$. In particular for $\kappa=\frac{1}{2}$ or $\kappa=\frac{3}{2}$, the product $\mu^{\frac{1}{2}}(f,g)$ is the well known associative Eholzer product $f\star g = \mu^{\frac{1}{2}}(f,g) = \sum_{n\geq 0}[f,g]_n$.

Final observation. The results of 6.3.2 and 6.3.3 can be extended to general (with even or odd weight) modular forms by a more sophisticated construction where the pseudo-differential operator rings are replaced by more general kind of power series (see [43]).

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