THREE LECTURES ON HYPERGEOMETRIC FUNCTIONS

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ABSTRACT. In this course we will study multivariate hypergeometric functions in the sense of Gel'fand, Kapranov, and Zelevinsky (GKZ systems). These functions generalize the classical hypergeometric functions of Gauss, Horn, Appell, and Lauricella. We will emphasize the algebraic methods of Saito, Sturmfels, and Takayama to construct hypergeometric series and the connection with deformation techniques in commutative algebra. We end with a brief discussion of the classification problem for rational hypergeometric functions.

RESUMEN. En este curso se estudiarán las funciones hipergeométricas multivaluadas en el sentido de Gel'fand, Kapranov, y Zelevinsky (sistemas GKZ). Estas funciones generalizan las funciones hipergeométricas de Gauss, Horn, Appell, y Lauricella. Se explorarán los métodos algebraicos de Saito, Sturmfels, y Takayama para construir series hipergeométricas y la aplicación de técnicas de algebra conmutativa. El curso concluye con una breve discusión del problema de caracterización de funciones hipergeométricas racionales.

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Introduction

The study of one-variable hypergeometric functions is more than 200 years old. They appear in the work of Euler, Gauss, Riemann, and Kummer. Their integral representations were studied by Barnes and Mellin, and special properties of them by Schwarz and Goursat. The famous Gauss hypergeometric equation is ubiquitous in mathematical physics as many well-known partial differential equations may be reduced to Gauss' equation via separation of variables.

There are three possible ways in which one can characterize hypergeometric functions: as functions represented by series whose coefficients satisfy certain recursion properties; as solutions to a system of differential equations which is, in an appropriate sense, holonomic and has mild singularities; as functions defined by integrals such as the Mellin-Barnes integral. For one-variable hypergeometric functions this interplay has been well understood for several decades. In the several variables case, on the other hand, it is possible to extend each one of these approaches but one may get slightly different results. Thus, there is no agreed upon definition of a multivariate hypergeometric function. For example, there is a notion due to Horn of multivariate hypergeometric series in terms of the coefficients of the series. The recursions they satisfy gives rise to a system of partial differential equations. It turns out that for more than two variables this system need not be holonomic, i.e. the space of local solutions may be infinite dimensional. On the other hand, there is a natural way to enlarge this system of PDE's into a holonomic system. The relation between these two systems is only well understood in the two variable case [13]. Even in the case of the classical Horn, Appell, Pochhammer, and Lauricella, multivariate hypergeometric functions it is only in 1970's and 80's that an attempt was made by W. Miller Jr. and his collaborators to study the Lie algebra of differential equations satisfied by these functions and their relationship with the differential equations arising in mathematical physics.

There has been a great revival of interest in the study of hypergeometric functions in the last two decades. Indeed, a search for the title word *hypergeometric* in the MathSciNet database yields 3181 articles of which 1530 have been published since 1990! This newfound interest comes from the connections between hypergeometric functions and many areas of mathematics such as representation theory, algebraic geometry and Hodge theory, combinatorics, D-modules, number theory, mirror symmetry, etc. A key new development is the work of Gel'fand, Graev, Kapranov, and Zelevinsky in the late 80's and early 90's which provided a unifying foundation for the theory of multivariate hypergeometric series. More recently, through the work of Oaku, Saito, Sturmfels, and Takayama, the algorithmic aspects of the theory of hypergeometric functions have been developed and the connections with the theory and techniques of computational algebra have been made apparent. It is this aspect of the theory which will be emphasized in this course. The book of Saito, Sturmfels, and Takayama [36] serves as the backbone for these lectures and we refer to it for many of the proofs.

It would be impossible to give even an introduction to this theory in just three lectures. That is the reason why these notes are called what they are, rather than "An Introduction to Hypergeometric Functions" or some other similar title. This emphasizes the fact that I have chosen to highlight a number of topics which I hope will make the reader interested in further study of this beautiful subject, but I have made no attempt to give a comprehensive view of the field.

The purpose of the first lecture will be to motivate the notion of GKZ system. This will be done through the work of Miller and his collaborators. In the second lecture we will discuss Frobenius' method for obtaining series solutions of an ODE around a regular singular point and the extension to systems of PDEs by Saito, Sturmfels, and Takayama. In the last lecture we will discuss the construction of series solutions without logarithmic terms and we will end with a brief discussion of rational hypergeometric functions and their connection with residue integrals.

There is no claim of originality in these notes; indeed, most of the non-classical material may be found in [36] and other sources. Moreover, these lectures are very much influenced by those given by Mutsumi Saito at last year's "Workshop on D-modules and Hypergeometric Functions" held in Lisbon. In fact, it was from Saito that I first learned about Miller's work on bivariate hypergeometric functions and how it motivates the definition of GKZ systems.

1. Hypergeometric Series and Differential Equations

1.1. **The Gamma Function and the Pochhammer Symbol.** We recall that the Gamma function $\Gamma(s)$ may be defined by the integral:

$$\Gamma(s) \;=\; \int_0^\infty e^{-t}\, t^{s-1} dt\,. \tag{1.1}$$

The integral (1.1) defines a holomorphic function in the half-plane $\mathrm{Re}(s) > 0$. Moreover it satisfies the functional equation

$$\Gamma(s+1) \;=\; s\,\Gamma(s)\;; \quad \mathrm{Re}(s) > 0. \tag{1.2}$$

Hence, since $\Gamma(1) = 1$, we have $\Gamma(n+1) = n!$ for all $n \in \mathbb{N}$.

We may use (1.2) to extend Γ to a meromorphic function in the whole complex plane with simple poles at the non-positive integers. For example, in the strip $\{-1 < \mathrm{Re}(s) \leq 0\}$ we define

$$\Gamma(s) \;:=\; \frac{\Gamma(s+1)}{s}.$$

Exercise 1.1. Compute the residue of Γ at $n \in \mathbb{Z}_{\leq 0}$.

The Γ-function has remarkable symmetries. For example, it satisfies the identity (see a good complex analysis textbook for a proof):

$$\Gamma(s)\cdot\Gamma(1-s) = \frac{\pi}{\sin \pi s}. \tag{1.3}$$

Exercise 1.2. Prove that $G(s) := 1/\Gamma(s)$ is an entire function whose only zeroes occur at the non-positive integers.

Definition 1.3. Given $\alpha \in \mathbb{C}\backslash\mathbb{Z}_{\leq 0}$ and $k \in \mathbb{N}$ we define the *Pochhammer symbol*:

$$(\alpha)_k \;:=\; \frac{\Gamma(\alpha+k)}{\Gamma(\alpha)} \tag{1.4}$$

Exercise 1.4. Prove that:

(1) $(\alpha)_k = \alpha(\alpha+1)\cdots(\alpha+k-1)$.

(2) $k\,(\alpha)_k = \alpha\,((\alpha+1)_k - (\alpha)_k)$.

(3) $\dfrac{k}{(\alpha)_k} = \dfrac{\alpha-1}{(\alpha-1)_k} - \dfrac{\alpha-1}{(\alpha)_k}$.

(4) $(\alpha)_{k+1} = (\alpha+k)(\alpha)_k = \alpha(\alpha+1)_k$.

(5) $(\alpha)_k = (-1)^k\,\dfrac{\Gamma(1-\alpha)}{\Gamma(1-k-\alpha)}$.

1.2. **Hypergeometric Series.** Let $n = (n_1, \ldots, n_r) \in \mathbb{N}^r$ be an r-tuple of non-negative integers. Given $x = (x_1, \ldots, x_r) \in \mathbb{C}^r$, we will denote by x^n the power product

$$x^n \;:=\; x_1^{n_1} \cdots x_r^{n_r},$$

and by e_j the j-th standard basis vector in $\mathbb{Q}^r$.

Definition 1.5. A (formal) multivariate power series

$$F(x_1, \ldots, x_r) \;=\; \sum_{n \in \mathbb{N}^r} A_n \, x^n$$

is said to be (Horn) *hypergeometric* if and only if, for all $j = 1, \ldots, r$, the quotient

$$R_j(n) \;:=\; \frac{A_{n+e_j}}{A_n}$$

is a rational function of $n = (n_1, \ldots, n_r)$.

Example 1.6. Let $r = 1$ and suppose we want $R(n) = R_1(n)$ to be a constant function. Then $A_n = A_0 c^n$ for some $c \in \mathbb{C}$ and therefore

$$F(x) \;=\; A_0 \sum_{n=0}^{\infty} c^n x^n \;=\; \frac{A_0}{1 - cx}.$$

Thus, in the simplest possible case, a hypergeometric series is just a geometric series.

Example 1.7. Let $r = 1$ and set $R(n) = 1/(n+1)$. Then $A_n = A_0/n!$ and therefore

$$F(x) \;=\; A_0 \sum_{n=0}^{\infty} \frac{x^n}{n!} \;=\; A_0 e^x.$$

Example 1.8. We consider now the most general one-variable case. Let

$$F(x) \;=\; \sum_{n=0}^{\infty} A_n x^n$$

be hypergeometric. We can then factor the rational function $R(n)$ as:

$$R(n) \;=\; c\, \frac{(n+\alpha_1)\cdots(n+\alpha_p)}{(n+\gamma_1)\cdots(n+\gamma_q)}\,.$$

Once A_0 is chosen, the coefficients A_n are recursively determined and by 4) in Exercise 1.4,

$$\frac{(\alpha)_{n+1}}{(\alpha)_n} \;=\; (n+\alpha).$$

Hence,

$$A_n \;=\; A_0\, c^n \, \frac{(\alpha_1)_n \cdots (\alpha_p)_n}{(\gamma_1)_n \cdots (\gamma_q)_n}. \tag{1.5}$$

Consequently, up to constant and with an appropriate change of variables, a univariate series is hypergeometric if and only if its coefficients may be written in terms of Pochhammer symbols.

The following series, usually called the *Gauss hypergeometric series* will be a running example throughout these lectures:

$${}_2F_1(\alpha, \beta, \gamma; x) \;:=\; \sum_{n=0}^{\infty} \frac{(\alpha)_n (\beta)_n}{(\gamma)_n n!}\, x^n \;; \quad \gamma \notin \mathbb{Z}_{\leq 0}. \tag{1.6}$$

Exercise 1.9. Consider the series:

$$\sum_{n=0}^{\infty} \frac{(p_1 n + k_1)! \cdots (p_r n + k_r)!}{(q_1 n + \ell_1)! \cdots (q_s n + \ell_s)!}\, x^n, \tag{1.7}$$

where the p_is and q_js are positive integers and the k_i and ℓ_j are non-negative integers.

(1) Prove that (1.7) has a finite, non-zero, radius of convergence if and only if:

$$p_1 + \cdots + p_r = q_1 + \cdots + q_s.$$

(2) Clearly (1.7) is hypergeometric. Write its coefficients as in (1.5).

Remark 1.10. A beautiful result by Fernando Rodriguez-Villegas [42] asserts that in the case $k_i = \ell_j = 0$, for all i, j, and $s + r > 0$, the series (1.7) defines an algebraic function if and only if $s - r = 1$ and, for all n, its coefficients are integers. On the other hand, it is shown in [9] that if (1.7) defines a rational function then $r = s$ and, after reordering if necessary, $p_1 = q_1, \ldots, p_r = q_r$.

Example 1.11. The following two-variable hypergeometric series are particular cases of the so called Horn series [15, page 224]. This is a list of 34 bivariate hypergeometric series for which the ratios $R_i(m, n) = P_i(m, n)/Q_i(m, n)$ satisfy that:

- $(m + 1)$ divides Q_1 and $(n + 1)$ divides Q_2.
- The maximum of the degrees of P_i, Q_i is 2.

$$F_1(\alpha, \beta, \beta', \gamma; x_1, x_2) = \sum_{m,n=0}^{\infty} \frac{(\alpha)_{m+n}(\beta)_m(\beta')_n}{(\gamma)_{m+n} m! n!}\, x_1^m x_2^n. \qquad \text{(Appell's } F_1) \tag{1.8}$$

$$F_2(\alpha, \beta, \beta', \gamma, \gamma'; x_1, x_2) = \sum_{m,n=0}^{\infty} \frac{(\alpha)_{m+n}(\beta)_m(\beta')_n}{(\gamma)_m(\gamma')_n m! n!}\, x_1^m x_2^n. \qquad \text{(Appell's } F_2) \tag{1.9}$$

$$F_4(\alpha, \beta, \gamma, \gamma'; x_1, x_2) = \sum_{m,n=0}^{\infty} \frac{(\alpha)_{m+n}(\beta)_{m+n}}{(\gamma)_m(\gamma')_n m! n!}\, x_1^m x_2^n. \qquad \text{(Appell's } F_4) \tag{1.10}$$

1.3. **Differential Equations.** The recursion properties of the coefficients of a hypergeometric series imply that they are formal solutions of ordinary or partial differential equations. We begin by deriving the second order ordinary differential equation satisfied by Gauss' hypergeometric function. In what follows we will use the following notation: For functions of a single variable x, we will write ∂_x for the differentiation operator d/dx; for functions of several variables $x_1, \ldots, x_n$, we will write ∂_j for the partial differentiation operator $\partial/\partial x_j$. We shall also consider the *Euler operators*:

$$\theta_x := x\partial_x\ ; \quad \theta_j := x_j \partial_j\ . \tag{1.11}$$

Consider now Gauss' hypergeometric series (1.6), where for simplicity of notation we write F for ${}_2F_1$. We have:

$$\theta_x F(\alpha, \beta, \gamma; x) = \sum_{n=0}^{\infty} \frac{(\alpha)_n(\beta)_n}{(\gamma)_n n!}\, n\, x^n.$$

But, according to 2) in Exercise 1.4: $n(\alpha)_n = \alpha((\alpha + 1)_n - (\alpha)_n)$ and therefore

$$\begin{aligned}\theta_x F(\alpha,\beta,\gamma;x) &= \alpha \sum_{n=0}^{\infty}\left(\frac{(\alpha+1)_n(\beta)_n}{(\gamma)_n n!} - \frac{(\alpha)_n(\beta)_n}{(\gamma)_n n!}\right) x^n \\ &= \alpha\left(F(\alpha+1,\beta,\gamma;x) - F(\alpha,\beta,\gamma;x)\right).\end{aligned}$$

Hence:

$$(\theta_x+\alpha)\cdot F(\alpha,\beta,\gamma;x) \;=\; \alpha\cdot F(\alpha+1,\beta,\gamma;x). \tag{1.12}$$

Similarly we have:

$$(\theta_x+\beta)\cdot F(\alpha,\beta,\gamma;x) \;=\; \beta\cdot F(\alpha,\beta+1,\gamma;x), \tag{1.13}$$

while an analogus argument shows that 3) in Exercise 1.4 implies that

$$(\theta_x+(\gamma-1))\cdot F(\alpha,\beta,\gamma;x) \;=\; (\gamma-1)\cdot F(\alpha,\beta,\gamma-1;x). \tag{1.14}$$

Finally, we note that 4) in Exercise 1.4 gives that:

$$\partial_x F(\alpha,\beta,\gamma;x) \;=\; \frac{\alpha\beta}{\gamma} F(\alpha+1,\beta+1,\gamma+1;x). \tag{1.15}$$

Combining these four equations we have that the Gauss hypergeometric series satisfies the following ODE:

$$(\theta_x+\alpha)(\theta_x+\beta)F = (\theta_x+\gamma)\partial_x F. \tag{1.16}$$

Exercise 1.12. Show that (1.16) is equivalent to the well-known equation:

$$x(x-1)\partial_x^2 F + \left((\alpha+\beta+1)x-\gamma\right)\partial_x F + \alpha\beta F = 0. \tag{1.17}$$

Exercise 1.13. Show that Appell's F_1-series (1.8) satisfies the system of partial differential equations:

$$\begin{aligned} x_1\,(\theta_1+\theta_2+\alpha)(\theta_1+\beta)F_1 &= \theta_1(\theta_1+\theta_2+\gamma-1)F_1 \\ x_2\,(\theta_1+\theta_2+\alpha)(\theta_2+\beta')F_1 &= \theta_2(\theta_1+\theta_2+\gamma-1)F_1\end{aligned}$$

Exercise 1.14. Let

$$F(x_1,x_2) \;=\; \sum_{m,n=0}^{\infty} A(m,n)\, x_1^m x_2^n$$

be a hypergeometric series. Write:

$$R_1(m,n) \;=\; \frac{P_1(m,n)}{Q_1(m+1,n)}\;;\quad R_2(m,n) \;=\; \frac{P_2(m,n)}{Q_2(m,n+1)}.$$

Show that F satisfies the system of PDE:

$$(Q_i(\theta_1,\theta_2) - x_i P_i(\theta_1,\theta_2))F = 0\;;\quad i=1,2.$$

The equation (1.17) is a second order ODE with singularities at $0, 1$, and ∞ [25, 11]. Gauss' function ${}_2F_1(\alpha,\beta,\gamma;x)$ is a solution defined for $|x|<1$. This equation and its solutions are ubiquitous in mathematics and physics. In particular, via separation of variables, the solution of many physically meaningful partial differential equations may be written in terms of Gauss' hypergeometric function.

Consider, for example, Laplace's equation:

$$\left(\frac{\partial^2}{\partial u_1^2}+\frac{\partial^2}{\partial u_2^2}+\frac{\partial^2}{\partial u_3^2}+\frac{\partial^2}{\partial u_4^2}\right)\Phi \;=\; 0.$$

The change of variables:

$$u_1 = x_1+x_2;\quad u_2 = ix_1 - ix_2;\quad u_3 = x_3 - x_4;\quad u_4 = ix_3+ix_4,$$

transforms Laplace's equation into the form:

$$(\partial_1\partial_2 - \partial_3\partial_4)\,\Phi \;=\; 0, \tag{1.18}$$

where, as before, $\partial_j = \partial/\partial x_j$.

Proposition 1.15. *For any α, β, γ, where $\gamma \notin \mathbb{Z}_{\leq 0}$, the function*

$$\Phi_{\alpha,\beta,\gamma}(x_1,\dots,x_4) \;:=\; x_1^{-\alpha}x_2^{-\beta}x_3^{\gamma-1}\,{}_2F_1\left(\alpha,\beta,\gamma;\frac{x_3x_4}{x_2x_1}\right)$$

is a solution of (1.18) and, hence, gives rise to a solution of Laplace's equation.

This statement may be verified by a straightforward, though tedious, computation. In what remains of this lecture, we will explain this result following the work of Willard Miller and his school [32, 33, 26, 27]. This approach, which was motivated by the desire to find systems of PDEs whose solutions could be expressed in terms of generalized hypergeometric series, leads naturally to the notion of GKZ systems.

According to (1.12), the operator $\theta_x+\alpha$ may be viewed as an index-raising operator. Similarly for $\theta_x+\beta$, while $\theta_x+\gamma$ lowers the third index in Gauss' function. One can generate other raising and lower operators using the recursion properties of the Pochhammer symbols. However, the dependence of these operators on the parameters makes it hard to study, for example, their composition properties and thus the algebra they generate. Miller's idea is to replace the operators: multiplication by α, β and γ by Euler operators θ_u, θ_v, θ_w, corresponding to new variables u, v, w. Now, the Euler operator, θ_u acts as multiplication by α on functions which are homogeneous of degree α on u. Hence, we define:

$$\Phi_{\alpha,\beta,\gamma} \;:=\; u^\alpha v^\beta w^{\gamma-1}\,{}_1F_2(\alpha,\beta,\gamma;x). \tag{1.19}$$

Note that automatically

$$\theta_u(\Phi_{\alpha,\beta,\gamma}) = \alpha\,\Phi_{\alpha,\beta,\gamma};\quad \theta_v(\Phi_{\alpha,\beta,\gamma}) = \beta\,\Phi_{\alpha,\beta,\gamma};\quad \theta_w(\Phi_{\alpha,\beta,\gamma}) = (\gamma-1)\,\Phi_{\alpha,\beta,\gamma}$$

Hence:

$$(\theta_x+\theta_u)\Phi_{\alpha,\beta,\gamma} = u^\alpha v^\beta w^{\gamma-1}(\theta_x+\alpha)_1F_2(\alpha,\beta,\gamma;x) = \alpha u^\alpha v^\beta w^{\gamma-1}\,{}_1F_2(\alpha+1,\beta,\gamma;x)$$

and therefore

$$u\,(\theta_x+\theta_u)\Phi_{\alpha,\beta,\gamma} = \alpha\Phi_{\alpha+1,\beta,\gamma} \tag{1.20}$$

Similarly, the identities (1.13), (1.14), and (1.15) may be written in parameter-free form as:

$$v\,(\theta_x+\theta_v)\Phi_{\alpha,\beta,\gamma} = \beta\Phi_{\alpha,\beta+1,\gamma}, \tag{1.21}$$

$$w^{-1}\,(\theta_x+\theta_w)\Phi_{\alpha,\beta,\gamma} = (\gamma-1)\Phi_{\alpha,\beta,\gamma-1}, \tag{1.22}$$

$$uvw\,\frac{\partial}{\partial x}(\Phi_{\alpha,\beta,\gamma}) = \frac{\alpha\beta}{\gamma}\Phi_{\alpha+1,\beta+1,\gamma+1} \tag{1.23}$$

We say that these are parameter-free forms of the equations (1.12)-(1.15) because the operators on the left-hand side now make sense for any (nice) function on the variables u, v, w, x, while the previous form depended on the non-intrinsic parameters α, β, γ. Moreover, the following proposition shows that we can go even further:

Proposition 1.16. *The operators $E_1 := u\,(\theta_x+\theta_u)$, $E_2 := v\,(\theta_x+\theta_v)$, $E_3 := w^{-1}\,(\theta_x+\theta_w)$, and $E_4 := uvw\,\partial/\partial x$ commute. Consequently, there exist coordinates $z_1,\dots,z_4$ in $\mathbb{C}^4$ such that $E_j = \partial/\partial z_j$.*

Proof. It is clear that E_1, E_2, and E_3 are commuting operators. Thus we need to verify that they each commute with E_4. We have:

$$\begin{aligned}[E_1, E_4] &= [u(\theta_x + \theta_u), uvw\,\partial/\partial x] = u^2vw\,[\theta_x, \partial/\partial x] + vw\,[u\theta_u, u\,\partial/\partial x] \\ &= -u^2vw\,\partial/\partial x + u^2vw\,\partial/\partial x = 0.\end{aligned}$$

The verification that $[E_2, E_4] = 0$ is identical and:

$$\begin{aligned}[E_3, E_4] &= [w^{-1}(\theta_x + \theta_w), uvw\,\partial/\partial x] = uv\,[\theta_x, \partial/\partial x] + uv\,[w^{-1}\theta_w, w\,\partial/\partial x] \\ &= -uv\,\partial/\partial x + uv\,\partial/\partial x = 0.\end{aligned}$$

The existence of the coordinates z_i follows from Frobenius' Theorem. However, in this case we can write them explicitly as:

$$z_1 = -u^{-1}, \quad z_2 = -v^{-1}, \quad z_3 = w, \quad z_4 = u^{-1}v^{-1}w^{-1}x$$

Indeed, we have:

$$\frac{\partial}{\partial u} = u^{-2}\,\frac{\partial}{\partial z_1} - u^{-1}v^{-1}w^{-1}x\,\frac{\partial}{\partial z_4}$$

and therefore

$$\theta_u = -\theta_1 - \theta_4,$$

where $\theta_i = z_i\,\frac{\partial}{\partial z_i}$. Similarly: $\theta_v = -\theta_2 - \theta_4$, $\theta_w = \theta_3 - \theta_4$, and $\theta_x = \theta_4$. It is now an easy verification that $E_i = \frac{\partial}{\partial z_i}$. □

Theorem 1.17. *Given complex numbers α, β, γ, with $\gamma \notin \mathbb{Z}_{\leq 0}$, the function $\Phi_{\alpha,\beta,\gamma}$ defined by (1.19) satisfies the system of partial differential equations:*

$$\begin{gathered}(\theta_1 + \theta_4 + \alpha)\Phi_{\alpha,\beta,\gamma} = 0 \\ (\theta_2 + \theta_4 + \beta)\Phi_{\alpha,\beta,\gamma} = 0 \\ (-\theta_3 + \theta_4 + \gamma - 1)\Phi_{\alpha,\beta,\gamma} = 0 \\ \left(\frac{\partial}{\partial z_1}\frac{\partial}{\partial z_2} - \frac{\partial}{\partial z_3}\frac{\partial}{\partial z_4}\right)\Phi_{\alpha,\beta,\gamma} = 0\end{gathered}$$

Proof. As we saw in the proof of Proposition 1.16, the operator $\theta_1 + \theta_4$ equals $-\theta_u$. Since $\Phi_{\alpha,\beta,\gamma}$ is homogeneous of degree α on u, it follows that $(\theta_1 + \theta_4 + \alpha)\Phi_{\alpha,\beta,\gamma} = 0$. The verification of the second and third equations are similar. Finally, we note that the last equation is simply (1.16) written in parameter-free form and replacing E_i by $\frac{\partial}{\partial z_i}$. □

Remark 1.18. The operators $\theta_1+\theta_4$, $\theta_2+\theta_4$, and $-\theta_3+\theta_4$ appearing in the first three equations in Theorem 1.17 may be viewed as weighted Euler operators. Recall that given $\omega = (\omega_1, \dots, \omega_n) \in \mathbb{R}^n$, a function $f(z_1, \dots, z_n)$ is said to be ω-homogeneous of degree λ if and only if

$$f(t^{\omega_1}z_1, \dots, t^{\omega_n}z_n) \;=\; t^{\lambda}f(z_1, \dots, z_n), \quad \text{for all } t \in \mathbb{R}.$$

Moreover, if f is ω-homogeneous of degree λ then f satisfies the weighted Euler equation:

$$(\omega_1\theta_1 + \cdots + \omega_n\theta_n)f = \lambda f$$

Thus, the first equation in Theorem 1.17 means that the function $\Phi_{\alpha,\beta,\gamma}$ is homogeneous of degree $-\alpha$ relative to the weight $(1, 0, 0, 1)$ and similarly for the second and third equations.

Exercise 1.19. Let $D := \frac{\partial}{\partial z_1}\frac{\partial}{\partial z_2} - \frac{\partial}{\partial z_3}\frac{\partial}{\partial z_4}$ and $\theta_\omega := \omega_1\theta_1 + \cdots + \omega_4\theta_4$ be a weighted Euler operator in $\mathbb{C}^4$. Show that

$$\theta_\omega\,D \;=\; D\,\theta_\omega - (\omega_1 + \omega_2)\partial_1\partial_2 + (\omega_3 + \omega_4)\partial_3\partial_4.$$

Deduce that if $\omega_1 + \omega_2 = \omega_3 + \omega_4$ then D maps ω-weighted homogeneous functions to functions which are also ω-weighted homogeneous and that the shift in weight is given by $-(\omega_1 + \omega_2)$.

Exercise 1.20. Generalize Exercise 1.19 as follows. Given $u \in \mathbb{N}^n$, let ∂^u denote the operator:

$$\partial^u \ := \ \partial_1^{u_1} \cdots \partial_n^{u_n}. \tag{1.24}$$

Let $D = \partial^u - \partial^v$, $u, v \in \mathbb{N}^n$ and let $\theta_\omega := \omega_1\theta_1 + \cdots + \omega_n\theta_n$ be a weighted Euler operator in $\mathbb{C}^n$. Show that if

$$u_1\omega_1 + \cdots + u_n\omega_n = v_1\omega_1 + \cdots + v_n\omega_n$$

then D maps ω-weighted homogeneous functions to functions which are also ω-weighted homogeneous and that the shift in weight is given by $-(u_1\omega_1 + \cdots + u_n\omega_n)$.

As noted earlier, the operator $D := \frac{\partial}{\partial z_1}\frac{\partial}{\partial z_2} - \frac{\partial}{\partial z_3}\frac{\partial}{\partial z_4}$ is linearly equivalent to Laplace's operator. On the other hand, it follows from Exercise 1.19 that D maps common eigenfunctions for the commuting Euler operators $\theta_1 + \theta_4$, $\theta_2 + \theta_4$, and $-\theta_3 + \theta_4$ to common eigenfunctions. Hence, if F is a solution of the partial differential equation $DF = 0$ and we decompose F into a sum of common eigenfunctions for the commuting Euler operators, then each component is annihilated by D. Our discussion shows that if F is a solucion of Laplace's equation and is weighted homogeneous with respect to the Euler operators then it is associated to a Gauss hypergeometric function with parameters corresponding to the eigenvalues of the Euler operators. Thus we may view the Gauss hypergeometric functions as furnishing solutions of the Laplace's equation which are weighted homogeneous relative to the three Euler operators. Notice that, automatically, a function which is weighted homogeneous of degree α, β, γ relative to $\theta_1 + \theta_4$, $\theta_2 + \theta_4$, and $-\theta_3 + \theta_4$, may be written as

$$z_1^\alpha\, z_2^\beta\, z_3^{-\gamma}\varphi(\frac{z_1\, z_2}{z_3\, z_4})$$

and this allow us to transform Laplace's equation into a second order ODE on the variable $x = \frac{z_1\, z_2}{z_3\, z_4}$. This procedure may be seen as a prototypical example of the method of separation of variables.

Exercise 1.21. Consider Appell's function $F_1(x_1, x_2)$ defined in (1.8). Let $\partial_i = \partial/\partial x_i$, $\theta_i = x_i\partial_i$. Generalize (1.12)- (1.15) to show:

(1) $(\theta_1+\theta_2+\alpha)F_1 = \alpha\, F_1(\alpha+1)$, where we are only indicating the parameter that changes.
(2) $(\theta_1 + \beta)F_1 = \beta\, F_1(\beta + 1)$;
(3) $(\theta_2 + \beta')F_1 = \beta'\, F_1(\beta' + 1)$;
(4) $(\theta_1 + \theta_2 + \gamma - 1)F_1 = (\gamma - 1)\, F_1(\gamma - 1)$;
(5) $\partial_1 F_1 = (\alpha\beta/\gamma)\, F_1(\alpha + 1, \beta + 1, \gamma + 1)$;
(6) $\partial_2 F_1 = (\alpha\beta'/\gamma)\, F_1(\alpha + 1, \beta' + 1, \gamma + 1)$;

Show that these equations imply that F_1 satisfies the equations in Exercise 1.13.

Exercise 1.22. Mimicking our discussion in the Gauss case, introduce new variables u, v, v', w whose associated Euler operators $\theta_u, \theta_v, \theta_{v'}, \theta_w$, will correspond to multiplication by $\alpha, \beta, \beta', \gamma$ respectively. Show that the operators

$$E_1 = u(\theta_1 + \theta_2 + \theta_u); \quad E_2 = v(\theta_1 + \theta_v); \quad E_3 = v'(\theta_2 + \theta_{v'})$$
$$E_4 = w^{-1}(\theta_1 + \theta_2 + \theta_w - 1); \quad E_5 = uvw\partial_1; \quad E_6 = uv'w\partial_2$$

commute.

Exercise 1.23. Show that the operators $E_1, \ldots, E_6$, together with $\theta_u, \theta_v, \theta_{v'}, \theta_w$ define a ten-dimensional Lie algebra.

Exercise 1.24. Let $\Phi(x_1, x_2, u, v, v', w) = u^\alpha v^\beta v'^{\beta'} w^\gamma F_1(\alpha, \beta, \beta', \gamma, x_1, x_2)$. Show that Φ satisfies the PDEs:

$$(E_1E_2 - E_3E_4)\Phi = 0; \quad (E_1E_5 - E_3E_6)\Phi = 0$$

Exercise 1.25. Show that there exist coordinates $z_1, \dots, z_6$ in an open subset of $\mathbb{C}^6$ such that $E_j = \partial/\partial z_j$ and, if abusing notation we denote $\theta_j = z_j \partial/\partial z_j$ then:

$$\theta_u = -\theta_1 - \theta_4 - \theta_6; \quad \theta_v = -\theta_2 - \theta_4$$
$$\theta_{v'} = -\theta_5 - \theta_6; \quad \theta_w = \theta_3 - \theta_4 - \theta_6 + 1$$

1.4. **GKZ Systems.** In [26], Kalnins, Manocha, and Miller listed what they called the canonical equations for the 34 Horn series described in Example 1.11. They also noted that there were only 21 sets of different canonical series. They did this on a case-by-case basis. We will now present a uniform approach due to Gel'fand, Graev, Kapranov, and Zelevinsky [16, 19, 20, 17]. Our main reference will be [36].

We will denote by $\mathcal{D}_n$ the Weyl algebra of partial differential operators in n variables with polynomial coefficients. That is, $\mathcal{D}_n$ is generated by variables $x_1, \dots, x_n, \partial_1, \dots \partial_n$. All of these variables commute with the exception of x_i, ∂_i which satisfy the relation $[\partial_i, x_i] = 1$. Let $A = (a_{ij})$ be a $d \times n$ matrix of rank d with coefficients in $\mathbb{Z}$. Throughout these lectures we will assume that

Assumption 1.26. *The column vectos of A span $\mathbb{Z}^d$ over $\mathbb{Z}$.*

Assumption 1.27. *The vector $(1, \dots, 1) \in \mathbb{Q}^n$ lies in the row span of A.*

It should be noted that while we could easily dispense with the first Assumption, the second one is essential for what follows.

We define the *toric ideal* associated with A as the ideal in the commutative polynomial ring $\mathbb{C}[\partial_1, \dots, \partial_n]$

$$I_A \;:=\; \{\partial^u - \partial^v : Au = Av\,;\ u, v \in \mathbb{N}^n\}. \tag{1.25}$$

We note that the toric ideal is weighted homogeneous relative to any weight ω in the row-span of A. In particular, Assumption 1.27 implies that I_A is homogeneous in the usual sense. It follows from Exercise 1.20 that if $D = \partial^u - \partial^v \in I_A$ and ω is in the row-span of A then D preserves ω-homogeneity with a shift of $-(\omega_1 u_1 + \cdots + \omega_n u_n)$. Given $\delta \in \mathbb{C}^d$, we denote by $\langle A \cdot \theta - \delta \rangle$ the ideal in $\mathcal{D}_n$ generated by the Euler operators associated to each of the rows of A:

$$a_{j1}\theta_1 + \cdots + a_{jn}\theta_n \; - \delta_j; \quad j = 1, \dots, d,$$

where, as before, $\theta_j = x_j \partial_j$.

Definition 1.28. Given A and δ, the GKZ hypergeometric system is the left ideal $H_A(\delta)$ in the Weyl algebra generated by the union

$$I_A \cup \langle A \cdot \theta - \delta \rangle$$

Example 1.29. It follows from Theorem 1.17 that the function $\Phi_{\alpha,\beta,\gamma}$ is annihilated by the GKZ hypergeometric system associated with

$$A = \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & -1 & 1 \end{pmatrix} \;;\quad \delta = \begin{pmatrix} -\alpha \\ -\beta \\ 1-\gamma \end{pmatrix}. \tag{1.26}$$

In this case $I_A = \langle \partial_1\partial_2 - \partial_3\partial_4 \rangle$.

Example 1.30. Consider the matrix

$$A = \begin{pmatrix} 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & -1 & 0 & -1 \end{pmatrix}. \tag{1.27}$$

There are several algorithms that may be used to compute the toric ideal I_A. They have been implemented in computer algebra systems such as Singular [23] or Macaulay 2 [22]. In this case we obtain:

$$I_A = \langle \partial_1\partial_2 - \partial_3\partial_4, \partial_1\partial_5 - \partial_3\partial_6, \partial_2\partial_6 - \partial_4\partial_5 \rangle.$$

According to Exercise 1.24, the function Φ constructed from the Appell function F_1 is annihilated by the first two generators of the ideal I_A. One can also verify that it is also annihilated by the third generator (if written back in the x, y coordinates, the third generator corresponds to an Euler-Darboux equation [26]). Also, it follows from Exercise 1.25 that Φ is A-homogeneous of degree $\delta = (-\alpha, -\beta, -\beta', \gamma - 1)$.

Every element $p \in \mathcal{D}_n$ has a unique expression of the form

$$p = \sum_{u,v \in \mathbb{N}^n} c_{uv} x^u \partial^v,$$

where $c_{uv} = 0$ for all but finitely many pairs (u, v). Associated with $\mathcal{D}_n$ we consider the commutative polynomial ring $R = \mathbb{C}[x_1, \ldots, x_n, \xi_1, \ldots, \xi_n]$. Given any $p \in \mathcal{D}_n$, let $\nu(p) := \max\{|v| : c_{uv} \neq 0 \text{ for some } u \in \mathbb{N}^n\}$ be the *order* of p and set

$$\sigma(p) := \sum_{u \in \mathbb{N}^n, |v| = \nu} c_{uv} x^u \xi^v \in R.$$

The polynomial $\sigma(p)$ is called the (principal) symbol of the differential operator p.

Definition 1.31. Given a left ideal $I \subset \mathcal{D}_n$, its *characteristic variety* is the affine variety in $\mathbb{C}^{2n}$ defined by the *characteristic ideal* $\mathrm{ch}(I) := \langle \sigma(p) : p \in I \rangle \subset R$.

Remark 1.32. For readers familiar with the theory of D-modules, it should be pointed out that this notion of characteristic variety agrees with the standard one (cf. [36, Theorem 1.4.1]).

An algorithm for the computation of characteristic varieties is implemented in Macaulay 2 [22]. Using it we find that in the Gaussian case, Example 1.29, the characteristic ideal is generated by the symbols of the generator of the toric ideal and the Euler operators. In particular, it is independent of the vector $\delta \in \mathbb{C}^d$. This is also true in the Appell case discussed in Example 1.30. Indeed, this is a consequence of a general result (cf. [36, Theorem 4.3.8]) which asserts that this property will hold whenever the toric ideal I_A is Cohen-Macaulay.

Definition 1.33. A left ideal $I \subset \mathcal{D}_n$ is called *holonomic* if and only if its characteristic ideal has (Krull) dimension n. In such case, the vector space over the field of rational functions $\mathbb{C}(x)$

$$\mathbb{C}(x)[\xi]/(\mathbb{C}(x)[\xi] \cdot \mathrm{ch}(I))$$

is finite-dimensional and its dimension is called the *holonomic rank* of I.

Definition 1.34. Let $I \subset \mathcal{D}_n$ be a left ideal and $\mathbb{V}(\mathrm{ch}(I)) \subset \mathbb{C}^{2n}$ its characteristic variety. The singular locus $\mathrm{Sing}(I)$ is defined as the Zariski closure of the projection on $\mathbb{C}^n_x$ of

$$\mathbb{V}(\mathrm{ch}(I)) \backslash \{\xi_1 = \cdots = \xi_n = 0\}.$$

We note that given the characteristic ideal, the ideal of the singular variety may be computed by commutative algebra algorithms. For example, by a direct computation or using Macaulay 2's command *singLocus* one obtains that in Gauss' Example 1.29

$$\mathrm{Sing}(H_A(\delta)) = \mathbb{V}\left(\langle x_1 x_2 x_3 x_4 (x_1 x_2 - x_3 x_4) \rangle\right).$$

Example 1.35. Let A be the configuration given by the matrix

$$A = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \end{pmatrix}. \tag{1.28}$$

The toric ideal I_A is generated by the operator $\partial_1\partial_3 - \partial_2^2$. By direct computation or applying [36, Theorem 4.3.8] we have that

$$ch(H_A(\delta)) = \langle \xi_1\xi_3 - \xi_2^2, x_1\xi_1 + x_2\xi_2 + x_3\xi_3, x_2\xi_2 + 2x_3\xi_3 \rangle.$$

Moreover,

$$\mathrm{Sing}(H_A(\delta)) = \mathbb{V}(\langle x_1x_3(x_2^2 - 4x_1x_3)\rangle).$$

We recall that given an integral $d \times n$ matrix A satisfying Assumption 1.27, its *sparse discriminant* D_A is the polynomial, well-defined up to sign, defining the dual variety of the projective variety $\mathbb{V}(I_A)$. Of course, this only makes sense if that dual variety is a hypersurface, otherwise we declare $D_A = 1$. A subset $A' \subset A$ is said to be *facial* if it is the intersection of A with a face of its convex hull. The following theorem collects the main properties of GKZ systems.

Theorem 1.36. *Let $H_A(\delta)$ be a GKZ hypergeometric system.*

(1) *$H_A(\delta)$ is always holonomic.*
(2) *The singular locus of $H_A(\delta)$ is independent of $\delta \in \mathbb{C}^d$ and agrees with the zero locus of the* principal A-determinant $E_A(x)$.
(3) *For arbitrary A and generic δ, the holonomic rank of $H_A(\delta)$ equals the normalized volume of the convex hull of A,* $\mathrm{vol}(\mathrm{conv}(A))$. *Here, normalized means that the volume of the standard simplex is* 1.
(4) *For arbitrary A and δ,* $\mathrm{rank}(H_A(\delta)) \geq \mathrm{vol}(\mathrm{conv}(A))$.
(5) *Given A,* $\mathrm{rank}(H_A(\delta)) = \mathrm{vol}(\mathrm{conv}(A))$ *for all $\delta \in \mathbb{C}^d$ if and only if the ideal I_A is Cohen-Macaulay.*

Proof. The first three statements are due to Gel'fand, Kapranov, and Zelevinsky [19, 20, 17] under Assumptions 1.26 and 1.27 and to Adolphson [1] in the general case. We refer to [18] for the definition and properties of the principal A-determinant. In particular, Theorem 10.1.2 in [18] gives a factorization of $E_A(x)$ whose whose irreducible factors are the sparse discriminant of facial subsets A' of A. The fourth statement is due to Saito, Sturmfels and Takayama [36, Theorem 3.5.1]. The if part of the last statement is due to Gel'fand, Kapranov, and Zelevinsky[20, 21] and to Adolphson[1]. The only if part was conjectured by Sturmfels based on the results of [41, 5] and recently proved by Matusevich, Miller, and Walther [31]. Special cases of this conjecture had been proved in [5, 28, 29]. □

Remark 1.37. We note that in Example 1.28, the configuration A consists of the three integral points in the segment from $(1, 0)$ to $(1, 2)$. In the expression for the singular locus, the monomials x_1 and x_3 correspond to the two vertices (faces) of A and the term $x_2^2 - 4x_1x_3$ to the discriminant D_A which coincides with the usual discriminant of the generic quadratic polynomial:

$$f(t) \;=\; x_1 + x_2t + x_3t^2.$$

2. Solutions of Hypergeometric Differential Equations

2.1. Regular Singularities.

We consider a linear n-th order ODE in the complex plane:

$$a_0(z)\, w^{(n)} + a_1(z)\, w^{(n-1)} + \cdots + a_{n-1}(z)\, w' + a_n(z)\, w \;=\; 0, \tag{2.1}$$

where the $a_j(z)$ are holomorphic in an open set $U \subset \mathbb{C}$. We have the basic result [11, 25]:

Theorem 2.1. [11, 25] *Let $z_0 \in U$ be such that $a_0(z_0) \neq 0$. Then, there exists $\delta > 0$ such that in the disk $\{|z - z_0| < \delta\}$ the vector space of holomorphic solutions of (2.1) has dimension n.*

Definition 2.2. A point $z_0 \in U$ is said to be a *singular point* of (2.1) if $a_0(z) = 0$.

In general, the behavior of the solutions of (2.1) around a singular point can be quite wild. For example, the equation $z^2 w' + w = 0$ has a solution near zero, $w = exp(1/z)$, with an essential singularity at the origin. There is however a special kind of well-behaved singular points. They are described by the following theorem whose proof may be found in [11, Chapter 4] or [25].

Theorem 2.3. *Let z_0 be a singular point of (2.1). Then the following are equivalent:*

(1) *The functions $b_k(z) := \dfrac{a_k(z)}{a_0(z)}$ have at worst a pole of order k at z_0.*
(2) *The vector space of multivalued holomorphic functions in a sufficiently small punctured disk $\{0 < |z - z_0| < \delta\}$, which are solutions of (2.1), has dimension n and is generated by functions of the form*

$$(z - z_0)^\lambda \, (\ln(z - z_0))^j \, f(z),$$

where $\lambda \in \mathbb{C}$, $j \in \mathbb{Z}$, $0 \leq j \leq n - 1$, and $f(z)$ is holomorphic in the disk $\{|z - z_0| < \delta\}$ and $f(z_0) \neq 0$.

If these conditions hold, we say that z_0 is a *regular singular point* of (2.1).

Example 2.4. The Euler equation:

$$z^n \, w^{(n)} + b_1 \, z^{n-1} \, w^{(n-1)} + \cdots + b_{n-1} \, z \, w' + b_n \, w \;=\; 0, \tag{2.2}$$

where $b_1, \ldots, b_n$ are constant, has a regular singular point at the origin. Writing $\theta = z \, \partial_z$, we have:

$$z^j \partial_z^j \;=\; \theta \cdot (\theta - 1) \cdots (\theta - j + 1)$$

and consequently we may rewrite (2.2) as $L(w) = 0$, where L is the differential operator

$$L \;:=\; \sum_{j=0}^{n} b_{n-j} \; \theta \cdot (\theta - 1) \cdots (\theta - j + 1). \tag{2.3}$$

Now, for any $\lambda \in \mathbb{C}$, $\theta(z^\lambda) \;=\; \lambda \, z^\lambda$. Hence

$$L(z^\lambda) \;=\; p(\lambda) \, z^\lambda, \tag{2.4}$$

where $p(\lambda)$ is the polynomial

$$p(\lambda) \;=\; \sum_{k=0}^{n} \; b_{n-k} \; \prod_{j=0}^{k-1} (\lambda - j). \tag{2.5}$$

Clearly, if $p(\lambda_0) = 0$ then $w = z^{\lambda_0}$ is a (multivalued) holomorphic solution of (2.2) near zero. Moreover, if λ_0 is a root of p of multiplicity ℓ, then the functions $(\ln z)^j \, z^{\lambda_0}$ are solutions of (2.2) for $j = 0, \ldots, \ell - 1$. This may be easily verified: consider (2.4) as an identity of functions of z and λ. If λ_0 is a root of p of multiplicity ℓ, then

$$\frac{\partial^j}{\partial \lambda^j} \left(p(\lambda) \, z^\lambda \right)$$

vanishes at λ_0 for $j = 0, \ldots, \ell - 1$. On the other hand, $(\partial^j / \partial \lambda^j)(z^\lambda) = (\ln z)^j \, z^\lambda$ which implies the result.

Remark 2.5. We may extend the notion of regular singularity to the point at infinity in the Riemann sphere in the usual way, i.e. by making the change of variable $z' = 1/z$. In this manner, we say that ∞ is a regular singular point of (2.1) if and only if $b_j(z)$ is holomorphic at ∞ and has a zero of order at least j there. The following result is Theorem 6.4 in [11, Chapter 4]:

Theorem 2.6. *The equation (2.1) has regular singularities at the distinct points $z_1, \ldots, z_k, \infty$ and no other singularities if and only if:*

$$b_j(z) \;=\; \prod_{i=1}^{k} (z - z_i)^j \, c_j(z),$$

where $c_j(z)$ is a polynomial of degree at most $j(k-1)$.

An equation satisfying the conditions of Theorem 2.6 is called a *Fuchsian equation*. Theorem 2.3 describes the structure of the solutions of a Fuchsian equation around a singular point. Euler's equation discussed in Example 2.4 is a Fuchsian equation whose only singularities are at 0 and ∞. The method used in Example 2.4 to construct the local solutions of the Euler equation around the origin may be generalized to construct local solutions around a regular singular point. This is discussed in the next section.

2.2. **The Frobenius Method.** Suppose (2.1) has a regular singular point at z_0. Let us assume without loss of generality that z_0 is the origin. Multiplying through by z^n we may rewrite (2.1) as $L(w) = 0$, where L denotes the differential operator:

$$L \;:=\; \sum_{k=0}^{n} b_{n-k}(z)\; \theta \cdot (\theta - 1) \cdots (\theta - k + 1)\,,$$

where $b_0(z) = 1$ and $b_k(z)$ is holomorphic in a neighborhood of the origin.

We will look for a solution of (2.1) of the form

$$w(z) \;=\; z^{\lambda} \sum_{r=0}^{\nu} \sum_{j=0}^{\infty} u_{rj} (\ln z)^r \; z^j \;; \quad u_{00} = 1,\; \lambda \in \mathbb{C}, \tag{2.6}$$

where the u_{rj} are complex coefficients to be determined. It turns out that if a formal series (2.6) is a solution of (2.1) around a regular singular point, then it converges in a punctured neighborhood of that point (cf. Theorem 3.1 in [11, Chapter 4]).

Let us define, in analogy with (2.5), the polynomials

$$p^{(j)}(\lambda) \;=\; \sum_{k=0}^{n} \frac{b^{(j)}_{n-k}(0)}{j!} \prod_{j=0}^{k-1} (\lambda - j). \tag{2.7}$$

We will call the polynomial $p(\lambda) := p^{(0)}(\lambda)$ the *indicial polynomial* of the operator L. The following is a simple form of Frobenius' Theorem:

Theorem 2.7. *Let $\lambda_0 \in \mathbb{C}$ be a root of $p(\lambda)$ of multiplicity μ. Suppose, moreover, that $p(\lambda + j) \neq 0$ for all $j \in \mathbb{N}$. Then, there exist μ linearly independent series (2.6), with $\lambda = \lambda_0$ and $\nu < \mu$ such that $L(w) = 0$.*

Proof. Let $w(z)$ be as in (2.6) with $r = 0$. In this case we drop the second index in the coefficients u_j. We have:

$$\begin{aligned} L(w) &= \sum_{j=0}^{\infty} u_j \, L(z^{\lambda+j}) \\ &= \sum_{j=0}^{\infty} \sum_{k=0}^{j} u_{j-k} \, p^{(k)}(\lambda + k - j) \, z^{\lambda+j} \end{aligned}$$

Thus, the coefficient of $z^{\lambda+j}$ is given by:

$$u_j \, p(\lambda + j) + \sum_{k=1}^{j} u_{j-k} \, p^{(k)}(\lambda + k - j) \tag{2.8}$$

and, therefore we can recursively find coefficients u_j so that (2.8) will vanish for all $j \geq 1$. For that choice of coefficients,

$$L(w) \;=\; p(\lambda) \, z^{\lambda}.$$

Moreover, if follows from (2.8) that the coefficients u_j will depend rationally on λ with poles at the roots of $p(\lambda + k)$, $k = 1, \ldots, j$. Hence, if λ_0 is a root of p and $p(\lambda_0 + j) \neq 0$ for all $j \in \mathbb{N}$, then we can construct a unique w as in (2.6) with $r = 0$ and $\lambda = \lambda_0$ which is a solution of $L(w) = 0$.

If λ is a root of multiplicity μ, then as in Example 2.4, we have that the specialization of $\partial_\lambda^\nu w(\lambda, z)$ is annihilated by L for all $\nu < \mu$. This gives a series (2.6) which is a solution of $Lw = 0$ and has $\nu < \mu$. □

Remark 2.8. The construction of series solutions of the form (2.6) may be extended to the case where the indicial polynomial has roots separated by integers. We sketch the idea and leave the details to the interested reader. Suppose for example that $p(\lambda)$ has roots at λ_3, $\lambda_2 = \lambda_3 + a$ and $\lambda_1 = \lambda_3 + b$, where $b > a$ are positive integers, and no other roots of this form. Suppose moreover that these roots have multiplicity μ_3, μ_2, and μ_1 respectively. Clearly, the construction in the proof of Theorem 2.7 fails at the value λ_3 and since $p(\lambda_3 + a) = p(\lambda_3 + b) = 0$. On the other hand, if we modify our construction by choosing

$$u_0 = (\lambda - \lambda_3)^{\mu_1 + \mu_2}$$

Then, it is easy to check by induction that for all $j = 0, \ldots, a-1$, u_j has a factor of $(\lambda - \lambda_3)^{\mu_1 + \mu_2}$ and therefore, given (2.8), u_a is well defined at λ_3 since $p(\lambda + a)$ has a zero of order μ_2 at λ_3 but that's taken care by the factor $(\lambda - \lambda_3)^{\mu_1 + \mu_2}$ in all u_j, $j < a$. Similarly, for all $a \leq j < b$, u_j will have a factor of $(\lambda - \lambda_3)^{\mu_1}$ and we can argue in the same way to show that the coefficient u_b is well-defined.

Note however, that setting $\lambda = \lambda_3$ would have the effect of annihilating all coefficients u_j, $j < b$, and hence we would get a series whose leading term is z^{λ_1}. On the other hand, if we consider the series

$$\partial_\lambda^b (w(\lambda, z))$$

and evaluate it at λ_3, we get a series of the form (2.6) which is a solution of $Lw = 0$ and whose leading term is z^{λ_3}. Taking further derivatives we may construct $\mu_3 - 1$ series solutions.

We leave it up to the reader to generalize these constructions to the case of the point at infinity. In this manner, given a Fuchsian equation, we may construct a basis of series solutions around each singular point.

Example 2.9. Consider Gauss' hypergeometric equation

$$z(z-1)\partial_z^2 F + ((\alpha+\beta+1)z-\gamma)\,\partial_z F + \alpha\beta F = 0.$$

It has regular singularities at $0, 1$ and ∞. The indicial polynomial at 0 is

$$p(\lambda) \;=\; \lambda(\lambda-1)+\gamma\lambda \;=\; \lambda(\lambda-(1-\gamma))$$

Thus, if $\gamma \notin \mathbb{Z}_{\leq 0}$, then 0 is a root of the indicial polynomial and no positive integer is a root. Hence, we can proceed as in the proof of Theorem 2.7 to construct a power series solution of Gauss' equation. Moreover, if $\gamma \neq 1$, 0 is a simple root of p.

In order to construct the solution around the origin corresponding to the root 0 it is better to work with the expression coming from (1.16) for Gauss' differential operator. Since we are interested in the behavior around a punctured neighborhood of zero, we may multiply (1.16) by x and obtain the differential equation

$$((\theta_z - 1 + \gamma)\theta_z - (\theta_z+\alpha)(\theta_z+\beta))\, F = 0.$$

Applying this differential operator to the series

$$\sum_{j=0}^{\infty} u_j\, z^j$$

we get

$$\sum_{j=1}^{\infty} \left((j-1+\gamma)\, j\, u_j - (j-1+\alpha)\,(j-1+\beta)\, u_{j-1}\right)\, z^j.$$

The term of degree zero vanishes because 0 is a root of the indicial polynomial. Thus, the given series will be a solution if and only if

$$\frac{u_j}{u_{j-1}} \;=\; \frac{(j-1+\alpha)\,(j-1+\beta)}{(j-1+\gamma)\, j}\;,$$

that is, assuming that $u_0 = 1$, if and only if

$$u_j \;=\; \frac{(\alpha)_j\,(\beta)_j}{(\gamma)_j\, j!}.$$

In other words, we recover Gauss' hypergeometric series ${}_1F_2(\alpha,\beta,\gamma;z)$.

Exercise 2.10. Suppose $\gamma = 1$, then 0 is a double root of the indicial polynomial. Theorem 2.7 implies that there is a second (multivalued) solution in a punctured neighborhood of zero of the form

$$\sum_{j=0}^{\infty} (u_j\, \ln z + v_j)\, z^j$$

(1) Set up the linear equations to compute u_j, v_j.
(2) Show that $u_j = (\alpha)_j(\beta)_j/(\gamma)_j j!$.
(3) Compute v_1, v_2, and v_3.

Exercise 2.11. Show that for $\gamma \neq 1$, the solution of Gauss' hypergeometric equation corresponding to the root $1-\gamma$ is given by

$$w(z) \;=\; z^{1-\gamma}\; {}_1F_2(\alpha-\gamma+1, \beta-\gamma+1, 2-\gamma; z).$$

Exercise 2.12. Show that 0 is a root of the indicial polynomial of Gauss' hypergeometric equation at $z=1$ and show that the corresponding solution is

$$w(z) \;=\; {}_1F_2(\alpha,\beta,\alpha+\beta-\gamma+1; 1-z).$$

2.3. **Multivariate Logarithmic Series.** The GKZ hypergeometric systems $H_A(\delta)$ share many of the special properties of Fuchsian equations. Indeed, as we will see below, their holonomicity implies that, away from the singular locus, the vector space of local holomorphic solutions is finite-dimensional of dimension equal to the holonomic rank. Moreover, Assumption 1.27 implies that $H_A(\delta)$ is regular, in a sense which will not be defined here, but that guarantees that the local solutions have at worst logarithmic singularities near the singular locus. This generalizes the statement in Theorem 2.3 about the behavior of solutions around a regular singular point.

There are many ways to construct logarithmic series solutions of GKZ systems. We refer to [20, 37] for the construction of Γ-series and to [3] for a very recent new approach. We will discuss the Gröbner deformation technique of Saito, Sturmfels, and Takayama which generalizes Frobenius' method.

We begin by collecting some general results about the solutions of GKZ hypergeometric systems. The following is a special case of the Cauchy-Kovalewskii-Kashiwara Theorem (cf. [36, Theorem 1.4.19]) and generalizes Theorem 2.1:

Theorem 2.13. *Let $U \subset \mathbb{C}^n \backslash \mathrm{Sing}(H_A(\delta))$ be a simply connected open set. Then the vector space of functions, holomorphic in U, which are annihilated by the ideal $H_A(\delta)$ has dimension equal to the holonomic rank of $H_A(\delta)$. Such a function is called an A-hypergeometric function of degree δ.*

As in the one variable case we will look for logarithmic series solutions of $H_A(\delta)$. More precisely:

Definition 2.14. A logarithmic series is a series of the form

$$\sum_{\alpha \in A} \sum_{\beta \in B} c_{\alpha\beta}\, x^{\alpha}\, (\ln x)^{\beta},$$

where $A \subset \mathbb{C}^n$ is a discrete set and $B \subset \{0, \ldots, r\}^n$, for some $r \in \mathbb{N}$.

Given a Fuchsian equation, the logarithmic series solutions constructed via Frobenius' method converged in a neighborhood of one of the singular points. In the GKZ case, the singular locus is described by the zero locus of the principal A determinant. Thus, it is reasonable to expect that the geometry of the singular locus should be closely related to the regions of convergence of logarithmic series solutions of the GKZ system.

We recall that given a Laurent polynomial f, its Newton polytope $\mathcal{N}(f)$ is defined as the convex hull of the exponents corresponding to terms in f with non-zero coefficient. Let ν be a vertex of the Newton polytope of E_A (this is the so-called secondary polytope of A), and let $v_1, \ldots, v_n$ be a lattice basis of $\mathbb{Z}^n$ such that

$$\mathcal{N}(E_A) \subset \nu + \mathcal{C}(v),$$

where $\mathcal{C}(v)$ denotes the positive cone spanned by $v_1, \ldots, v_n$. Let $\mathcal{U}(v, \epsilon)$ be the open set

$$\mathcal{U}(v, \epsilon) \;:=\; \{x \in \mathbb{C}^n : 0 < |x^{v_i}| < \epsilon, i = 1, \ldots, n\}. \tag{2.9}$$

The following theorem –Corollary 2.4.16 in [36]– says that the open sets (2.9) play the role of the punctured neighborhoods around a regular singular point in the one-variable case.

Theorem 2.15. *If F is an A-hypergeometric function in $\mathcal{U}(v, \epsilon)$ then F may be written as logarithmic series converging to F in that region.*

2.4. **The Indicial Ideal.** Our goal is to generalize Frobenius' method to construct logarithmic series solutions of GKZ hypergeometric systems. The construction in Theorem 2.7 proceeds in two stages:

(1) Given an equation (2.1) with a regular singularity at the origin, consider the Euler equation whose indicial polynomial is the same as the indicial polynomial of the original equation. Its solutions are of the form $x^\lambda(\ln x)^j$, where λ is a root of the indicial polynomial and j is strictly smaller than the multiplicity of λ.
(2) Given a solution $x^\lambda(\ln x)^j$ of the Euler equation construct a logarithmic series solution for which $x^\lambda(\ln x)^j$ is the initial term.

To carry out the generalization of Frobenius' method we need to identify the "leading" term of a logarithmic series. In analogy with Gröbner basis theory in commutative algebra, we consider initial terms with respect to a weight.

Definition 2.16. Given a weight vector $w \in \mathbb{R}^n$ we define a partial order on the terms of a logarithmic series by

$$x^\alpha\,(\log x)^\beta \leq x^{\alpha'}\,(\log x)^{\beta'} \text{ if and only if } \mathrm{Re}(\langle w, \alpha\rangle) \leq \mathrm{Re}(\langle w, \alpha'\rangle).$$

The w-initial form of a logarithmic series is the set of terms where $\mathrm{Re}(\langle w, \alpha\rangle)$ reaches its minimum.

We can refine the partial order defined by w to a total order $\prec_w$ by using the lexicographic order to break ties.

Our next task it to define a system of equations which will play the role of the Euler equation. Note that the data of an Euler equation and of its indicial polynomial are equivalent, i.e., the study of the Euler equation is really an algebraic problem. This motivates the definition of the *indicial ideal*.

Let $w \in \mathbb{R}^n$, then given $p \in \mathcal{D}_n$ written in canonical form:

$$p \;=\; \sum_{u,v\in\mathbb{N}^n} c_{uv}\, x^u\, \partial^v,$$

we define $\mathrm{in}_{(-w,w)}(p)$ as the sum of the terms in p where the expression $-\langle w, u\rangle + \langle w, v\rangle$ attains its maximum. Given a left ideal $I \subset \mathcal{D}_n$ we set

$$\mathrm{in}_{(-w,w)}(I) \;:=\; \left\langle \mathrm{in}_{(-w,w)}(p) : p \in I \right\rangle.$$

The ideal $\mathrm{in}_{(-w,w)}(I) \subset \mathcal{D}_n$ is called a Gröbner deformation of I.

Exercise 2.17. Let $I \subset \mathcal{D}_n$ be an ideal and let f be a logarithmic series which is a solution of I in the sense that $p(f) = 0$ for all $p \in I$. Show that for any weight w, $\mathrm{in}_w(f)$ is a solution of $\mathrm{in}_{(-w,w)}(I)$.

We will denote by $\hat{\mathcal{D}}_n := \mathbb{C}(x) \cdot \mathcal{D}_n$, i.e., the ring of differential operators whose coefficients are rational functions of $(x_1, \ldots, x_n)$. Given an ideal $I \in \mathcal{D}_n$, we denote by $\hat{I}$ the ideal $\mathbb{C}(x) \cdot I$ in $\hat{\mathcal{D}}_n$.

Definition 2.18. The *indicial ideal* $\mathrm{ind}_w(I)$ of I, relative to the weight $w \in \mathbb{R}^n$, is the ideal in the commutative ring $\mathbb{C}[\theta] = \mathbb{C}[\theta_1, \ldots, \theta_n]$:

$$\mathrm{ind}_w(I) \;:=\; \widehat{\mathrm{in}}_{(-w,w)}(I) \cap \mathbb{C}[\theta]. \tag{2.10}$$

We verify, first of all, that the indicial ideal generalizes the notion of indicial polynomial.

Example 2.19. Consider a differential operator D in the complex plane with a regular singularity at the origin and rational coefficients. We can multiply D by a rational function so that it takes the form:

$$z^n\partial_z^n + b_1(z)z^{n-1}\partial_z^{n-1} + \cdots + b_{n-1}(z)z\partial_z + b_n(z),$$

where the $b_j(z)$ are holomorphic in a neighborhood of the origin, i.e.,

$$\mathbb{C}(z)\cdot\langle D\rangle = \mathbb{C}(z)\cdot\langle z^n\partial_z^n + b_1(z)z^{n-1}\partial_z^{n-1} + \cdots b_n(z)\rangle$$

Let $w = 1$, then

$$\mathbb{C}(z)\cdot \mathrm{in}_{(-1,1)}(\langle D\rangle) \;=\; \mathbb{C}(z)\cdot\langle z^n\partial_z^n + b_1(0)z^{n-1}\partial_z^{n-1} + \cdots + b_{n-1}(0)z\partial_z + b_n(0)\rangle.$$

But since this $z^j\partial_z^j \in \mathbb{C}[\theta]$, we have that the indicial ideal of $\langle D\rangle$ is generated by $p(\theta)$ where p is the indicial polynomial of D.

Given an ideal $I \subset \mathcal{D}_n$, and a weight $w \in \mathbb{R}^n$, we can view the indicial ideal in two different ways: as the $\mathcal{D}_n$-module $\mathcal{D}_n \cdot \mathrm{ind}_w(I)$ or as an ideal in the commutative polynomial ring $\mathbb{C}[\theta_1, \dots, \theta_n]$.

Theorem 2.20. *Let $I \subset \mathcal{D}_n$ be a left ideal in the Weyl algebra and $w \in \mathbb{R}^n$ a generic weight.*

(1) *If I is holonomic then $\mathcal{D}_n \cdot \mathrm{ind}_w(I)$ is holonomic and*

$$\mathrm{rank}(\mathcal{D}_n \cdot \mathrm{ind}_w(I)) \;=\; \mathrm{rank}(\mathrm{in}_{(-w,w)}(I))$$

(2) *If I is holonomic then $\mathrm{ind}_w(I)$ is a zero-dimensional ideal in $\mathbb{C}[\theta_1, \dots, \theta_n]$ and*

$$\mathrm{rank}(\mathcal{D}_n \cdot \mathrm{ind}_w(I)) \;=\; \dim_{\mathbb{C}}\left(\mathbb{C}[\theta]/\mathrm{ind}_w(I)\right).$$

(3) *If I is regular holonomic then*

$$\mathrm{rank}(I) \;=\; \mathrm{rank}(\mathrm{in}_{(-w,w)}(I))$$

Proof. This is a collection of results from [36]. For a precise characterization of generic weights we refer to [36, Page 68]. The first statement is Theorem 2.3.9, the second is Proposition 2.3.6, while the third is Theorem 2.5.1. We should point out that this latter statement holds for every weight $w \in \mathbb{R}^n$. □

Theorem 2.20 together with Theorem 2.13 imply that for a regular holonomic system, the dimension of the space of local solutions away from the singular locus agrees with the number of roots of the indicial ideal counted with multiplicity. Saito, Sturmfels, and Takayama show in [36] that, given a generic weight w it is possible to construct a basis of logarithmic series solutions for a regular holonomic ideal I which converge in a common open set analogous to $\mathcal{U}(v,\epsilon)$ and whose w-leading term is of the form $x^\lambda(\ln x)^j$, where λ is a root of the indicial ideal. Moreover, x^λ appears with as many different powers of $\ln x$ as the multiplicity of λ as a root of the indicial ideal. A full discussion of these results is far beyond the scope of these lectures. In the next section we will discuss the construction of logarithm-free series for GKZ hypergeometric systems $H_A(\delta)$.

3. Logarithm-free Series Solutions for Hypergeometric Systems

3.1. Fake Indicial Ideal.

Let $H_A(\delta)$ be a GKZ hypergeometric system. Then, $H_A(\delta)$ is the ideal in $\mathcal{D}_n$ generated by the toric ideal I_A and the Euler operators $A\cdot\theta - \delta$. We can consider the indicial ideal of I_A relative to a weight $w \in \mathbb{R}^n$:

$$\mathrm{ind_w(I_A)} = \widehat{\mathrm{in}}_{(-w,w)}(I_A) \cap \mathbb{C}[\theta].$$

Note that I_A is, in fact, an ideal in the commutative polynomial ring $\mathbb{C}[\partial_1, \dots, \partial_n]$ so that $\mathrm{in}_{(-w,w)}(I_A) = \mathrm{in}_w(I_A)$ and for generic weights this initial ideal is a monomial ideal. We shall assume from now on that we have chosen the weight so that this condition is satisfied. Note also that every Euler operator is homogeneous of weight zero for every weight $(-w, w)$.

Definition 3.1. The *fake indicial ideal* of $H_A(\delta)$ with respect to w is the subideal of the indicial ideal:

$$\mathrm{find}_w(H_A(\delta)) \;:=\; \langle \mathrm{ind}_w(I_A), A\theta - \delta\rangle \subset \mathrm{ind}_w(H_A(\delta)) \tag{3.1}$$

Theorem 3.2. *For a generic weight $w \in \mathbb{R}^n$ and generic degree $\delta \in \mathbb{C}^s$:*

(1) $\mathrm{in}_{(-w,w)}(H_A(\delta)) \;=\; \langle \mathrm{in}_w(I_A), A\theta - \delta\rangle$.
(2) $\mathrm{find}_w(H_A(\delta)) \;=\; \mathrm{ind}_w(H_A(\delta))$.

Proof. The first statement is Theorem 3.1.3 in [36] while the second Corollary 3.1.6. □

Lemma 3.3. *Let $M \subset \mathbb{C}[\partial_1, \ldots, \partial_n]$ be a monomial ideal in $\mathcal{D}_n$. Then,*

$$\hat{M} \cap \mathbb{C}[\theta] \;=\; \langle [\theta]_u \rangle\,, \tag{3.2}$$

where u runs over the exponents of a set of generators of M and

$$[\theta]_u \;:=\; \prod_{i=1}^{n} \prod_{j=0}^{u-1} (\theta_i - j). \tag{3.3}$$

Proof. This is a special case of Theorem 2.3.4 in [36]. We note, first of all that if $\partial^u \in M$ then

$$x^u \partial^u = [\theta]_u \in \hat{M} \cap \mathbb{C}[\theta].$$

Conversely, suppose $p(\theta) \in \mathbb{C}[\theta]$ can be written as

$$p(\theta) = \sum_u \; R_u(x, \partial)\, \partial^u,$$

where the R_u are polynomial in ∂ and rational in x. Clearly, we can also write

$$p(\theta) = \sum_u \; \tilde{R}_u(x, \partial)\, [\theta]_u. \tag{3.4}$$

Pick a generic positive weight $\rho \in \mathbb{Z}^n_{>0}$ and replace in the above expression $x_i \mapsto t^{\rho_i} x_i$ and $\partial_i \mapsto t^{-\rho_i} \partial_i$. We can then expand the right hand side of (3.4) as a Laurent series in t. But since the left hand side has degree 0 in t we must have

$$p(\theta) = \sum_u \; P_u(x, \partial)\, [\theta]_u$$

where P_u is a polynomial in x and ∂ of ρ-degree zero. Since ρ is generic this implies that $P(x, \partial) \in \mathbb{C}[\theta]$. □

Although Lemma 3.3 gives a simple characterization of the generators of the fake indicial ideal, we can do a lot better using the notion of standard pairs from commutative algebra.

3.2. Standard Pairs.

Given a monomial ideal in a commutative ring $R = \mathbb{C}[x_1, \ldots, x_n]$ we can associate a combinatorial object, the set of standard pairs, in terms of which we may express many of the properties of the ideal.

Definition 3.4. Let $M \subset R$ be a monomial ideal. A *standard pair* for M is a pair (x^a, K), where $a \in \mathbb{N}^n$ and K is a subset of the index set $\{1, \ldots, n\}$ such that

(1) $a_k = 0$ if $k \in K$.
(2) $\{x^v : v \in a + \mathbb{N}^K\} \cap M = \emptyset$, where $\mathbb{N}^K := \{w \in \mathbb{N}^n : w_\ell = 0, \ell \notin K\}$.
(3) For each $\ell \notin K$, $\{x^v : v \in a + \mathbb{N}^{K \cup \{\ell\}}\} \cap M \neq \emptyset$.

Let $\mathcal{S}(M)$ denote the set of standard pairs of M. If we also denote by M the set of points in $\mathbb{N}^n$ which are exponents of monomials en M, the standard pairs allow us to decompose the complement of M as

$$\mathbb{N}^n \backslash M \;=\; \bigcup_{(x^a,K)\in\mathcal{S}(M)} (a+\mathbb{N}^K).$$

Algorithm 3.2.5 in [36] for the computation of standard pairs has been implemented by G. Smith in Macaulay 2. We refer to the chapter by S. Hosten and G. Smith in [14] for details.

Exercise 3.5. Let $M \subset R$ be a monomial ideal and $(x^a, K) \in Ss(M)$. Prove that if $x^m \in M$ then there exists $i \notin K$ such that $0 \leq a_i < m_i$.

Example 3.6. Consider the toric ideal of Example 1.30. With respect to $w = (1,1,0,0,0,0)$, the initial ideal $\mathrm{in_w}(I_A)$ is the monomial ideal $\langle \partial_2\partial_6, \partial_1\partial_5, \partial_1\partial_2\rangle$. By direct inspection or using the *standardPairs* command in Macaulay 2 we find that

$$\mathcal{S}(\mathrm{in_w}(I_A)) \;=\; \{(1,\{3,4,5,6\}),(1,\{1,3,4,6\}),(1,\{2,3,4,5\})\}.$$

Exercise 3.7. Let M be a monomial ideal in the ring $\mathbb{C}[\partial_1,\ldots,\partial_n]$ and consider the system of partial differential equations

$$\partial^u F = 0\;;\quad \partial^u \in M. \tag{3.5}$$

Show that the solutions of (3.5) are the functions of the form:

$$F(x) = \sum_{(\partial^a,K)\in\mathcal{S}(M)} x^a \cdot F_K(x),$$

where the function F_K depends only on the variables $x_k, k \in K$.

Remark 3.8. Exercise 3.7 exhibits a very simple connection between monomial systems of partial differential equations and commutative algebra. There is also a beautiful connection between binomial systems of PDEs and the corresponding binomial ideals. We refer to [40, Chapter 10] for details.

Let $I_A \subset \mathbb{C}[\partial]$ be the toric ideal associated with $A \in \mathbb{Z}^{d\times n}$ and let $w \in \mathbb{R}^n$ be a generic weight, i.e., so that $\mathrm{in_w}(I_A)$ is a monomial ideal. Then

$$\max\{|K| : (\partial^a, K) \in \mathcal{S}(\mathrm{in_w}(I_A)) \text{ for some } x^a\}$$

is the dimension of the affine subvariety of $\mathbb{C}^n$ defined by $\mathrm{in_w}(I_A)$. But that agrees with the dimension of the affine variety defined by I_A, that is d. On the other hand, the number of index sets K of cardinality d appearing in the standard pairs gives the degree of the affine variety defined by $\mathrm{in_w}(I_A)$ and, consequently the degree of the toric variety $\mathbb{V}(I_A)$ which is known to coincide with the normalized volume of the convex hull of A.

Example 3.9. In the Gaussian case (Example 1.29), $I_A = \langle \partial_1\partial_2 - \partial_2\partial_4\rangle$. For $w = (0,0,1,0)$, $\mathrm{in_w}(I_A) = \langle \partial_3\partial_4\rangle$ and

$$\mathcal{S}(\mathrm{in_w}(I_A)) \;=\; \{(1,\{1,2,3\}),(1,\{1,2,4\})\}.$$

Clearly I_A defines a hypersurface of degree 2 in $\mathbb{C}^4$. The convex hull of A is a unit square which has normalized volume 2.

In the case of Example 1.30, choosing w as in Example 3.6 we have that the affine variety $\mathbb{V}(I_A)$ is of codimension 2 and degree 3 in $\mathbb{C}^6$.

We can now give an irredundant decomposition of the ideal $\mathrm{ind}_w(I_A)$ into prime ideals. The following result is Corollary 3.2.3 in [36].

Theorem 3.10. *Let $I_A(\delta)$ be a GKZ hypergeometric system and $w \in \mathbb{R}^n$ a generic weight, then*

$$(1)\quad \mathrm{ind}_w(I_A) \;=\; \bigcap_{(\partial^a,K)\in\mathcal{S}(\mathrm{in}_w(I_A))} \langle(\theta_j - a_j), j \notin K\rangle\,.$$

$$(2)\quad \mathrm{find}_w(H_A(\delta)) \;=\; \bigcap_{(\partial^a,K)\in\mathcal{S}(\mathrm{in}_w(I_A))} \langle(\theta_j - a_j), j \notin K\rangle + \langle A\cdot\theta - \delta\rangle.$$

Example 3.11. For A as in Example 1.29 and $w = (0,0,1,0)$

$$\mathrm{ind}_w(I_A) \;=\; \langle\theta_3\rangle \cap \langle\theta_4\rangle$$

and if $\delta = (\delta_1,\delta_2,\delta_3)$ then the roots of the fake indicial ideal are $(\delta_1,\delta_2,-\delta_3,0)$ and $(\delta_1-\delta_3,\delta_2-\delta_3,0,\delta_3)$.

For A as in Example 1.30 and $w = (1,1,0,0,0,0)$ we have

$$\mathrm{ind}_w(I_A) \;=\; \langle\theta_1,\theta_2\rangle \cap \langle\theta_2,\theta_5\rangle \cap \langle\theta_1,\theta_6\rangle.$$

Exercise 3.12. Let A be as in Example 1.30 and $w = (1,1,0,0,0,0)$. Compute the roots of the fake indicial ideal $\mathrm{find}_w(H_A(\delta))$.

Definition 3.13. The roots of the fake indicial ideal $\mathrm{find}_w(H_A(\delta))$ are called *fake exponents* of the hypergeometric system $H_A(\delta)$ relative to the weight w.

Exercise 3.14. Let $v \in \mathbb{C}^n$ be a fake exponent of $H_A(\delta)$ relative to w. Prove that there exists $(\partial^a, K) \in \mathcal{S}(\mathrm{in}_w(I_A))$ such that $v_i = a_i$ for all $i \notin K$.

3.3. **Logarithm-free Hypergeometric Series.** Let $H_A(\delta)$ be a GKZ system. Let us denote by $\mathcal{L} \subset \mathbb{Z}^n$ the lattice of rank $m = n - d$:

$$\mathcal{L} \;:=\; \{u \in \mathbb{Z}^n : A\cdot u = 0\}.$$

If $u \in \mathcal{L}$ we can write $u = u_+ - u_-$, where $u_+, u_- \in \mathbb{N}^n$ have disjoint support. For any $w \in \mathbb{N}^n$ and $v \in \mathbb{C}^n$, we set

$$[v]_w \;:=\; \prod_{i:w_i\neq 0}\prod_{k=1}^{w_i}(v-k+1) \;=\; \prod_{i:w_i\neq 0}(-1)^{w_i}\,(v)_{w_i}. \tag{3.6}$$

Exercise 3.15. Show that for any $w \in \mathbb{N}^n$, $v \in \mathbb{C}^n$

$$\partial^w(x^v) \;=\; [v]_w\, x^{v-w}.$$

Let $u \in \mathbb{Z}^n$ and write $u = u_+ - u_-$. Prove that

$$\partial^{u_+}x^{v+u} \;=\; [u+v]_{u_+}\, x^{v-u_-}.$$

The following is Proposition 3.4.1 in [36]:

Theorem 3.16. *With notation as above, let $v \in \mathbb{C}^n$ and suppose that no coordinate of v is a negative integer. Then the formal series*

$$\phi_v \;:=\; \sum_{u\in\mathcal{L}} \frac{[v]_{u_-}}{[v+u]_{u_+}}\; x^{v+u} \tag{3.7}$$

is a solution of the GKZ hypergeometric system $H_A(\delta)$, where $\delta = A\cdot v$.

Proof. It is clear that all terms in (3.7) are A-homogeneous of degree $\delta = A\cdot v$. Thus, it suffices to show that ϕ_v is annihilated by the operators in the toric ideal I_A, i.e., if $w = w_+ - w_- \in \mathcal{L}$ then $\partial^{w_+}\phi_v = \partial^{w_-}\phi_v$. Comparing the coefficient for the term x^{v+u-w_-} and using Exercise 3.15, this is equivalent to verifying the identity

$$\frac{[v]_{u_-+w_-}[v+u+w]_{w_+}}{[v+u+w]_{u_++w_+}} \;=\; \frac{[v]_{u_-}[v+u]_{w_-}}{[v+u]_{u_+}}.$$

This can be checked case-by-case depending on the support of each of the vectors involved. For example, suppose $w_i > 0$ and $u_i > 0$, then the contribution of the i-th coordinate to the left hand side of the above identity is:

$$\frac{\prod_{k=1}^{w_i}(v+u+w-k+1)}{\prod_{\ell=1}^{u_i+w_i}(v+u+w-\ell+1)} = \frac{1}{\prod_{\ell=w_i+1}^{u_i+w_i}(v+u+w-\ell+1)},$$

while the contribution to the left hand side is

$$\frac{1}{\prod_{k=1}^{u_i}(v+u-k+1)}$$

and, clearly, the two expressions agree. Other cases are completely analogous. □

Of course, for most choices of $v \in \mathbb{C}^n$, the series ϕ_v will not define a holomorphic function in any open set in $\mathbb{C}^n$ so this formal solution will not define a hypergeometric function. On the other hand, the following result shows that if v is suitably chosen we will get a convergent series.

Lemma 3.17. *Let v be a fake exponent for $H_A(\delta)$ relative to a generic weight $w \in \mathbb{R}^n$. If $u \in \mathcal{L}$ is such that $\langle w, u\rangle < 0$ then*

$$[v]_{u_-} \;=\; 0.$$

Proof. By Exercise 3.14 there exists a standard pair $(\partial^a, K) \in \mathcal{S}(\mathrm{in}_w(I_A))$ such that $v_i = a_i$ for all $i \notin K$. On the other hand if $u \in \mathcal{L}$ then $\partial^{u_+} - \partial^{u_-} \in I_A$ and since $\langle w, u\rangle < 0$, $\mathrm{in}_w(\partial^{u_+} - \partial^{u_-}) = \partial^{u_-}$. Hence $\partial^{u_-} \in \mathrm{in}_w(I_A)$ and, by Exercise 3.5 there exists $i \notin K$ such that

$$0 \;\leq\; v_i \;=\; a_i \;<\; (u_-)_i,$$

but this implies $[v]_{u_-} = 0$. □

Given the toric ideal I_A and a generic weight $w \in \mathbb{R}^n$ we can consider the *Gröbner cone*

$$\mathcal{C}_w \;:=\; \{w' \in \mathbb{R}^n : \mathrm{in}_{w'}(I_A) = \mathrm{in}_w(I_A)\}. \tag{3.8}$$

and its dual

$$\mathcal{C}_w^* \;:=\; \{u \in \mathbb{R}^n : \langle u, w'\rangle \geq 0 \text{ for all } w' \in \mathcal{C}_w\}. \tag{3.9}$$

Corollary 3.18. *Let v be a fake exponent of $H_A(\delta)$ relative to w and suppose that no component of v is a negative integer. Then the series ϕ_v is supported in the cone $\mathcal{C}_w^*$. Moreover, for every $w' \in \mathcal{C}_w$, $\mathrm{in}_{w'}(\phi_v) = x^v$.*

Let $\mathcal{C}_\nu = \mathcal{C}(\nu_1, \ldots, \nu_n) \subset \mathcal{C}_w$ be a unimodular cone, that is the positive cone spanned by a $\mathbb{Z}$-basis of $\mathbb{Z}^n$, then $\mathcal{C}_w^* \subset \mathcal{C}_\nu^*$ and under the assumptions of Corollary 3.18, we have the following result which is a restatement of [36, Theorem 2.5.16].

Theorem 3.19. *The series ϕ_v converges in the open set*

$$\mathcal{U}_\nu(\epsilon) \;=\; \{0 < |x^{\nu_i}| < \epsilon\,;\; i = 1, \ldots, n\} \tag{3.10}$$

for ϵ sufficiently small. Moreover, for generic δ these series give a basis of holomorphic solutions of $H_A(\delta)$ in $\mathcal{U}_\nu(\epsilon)$.

Example 3.20. We saw in Example 3.11 that for the Gauss system $H_A(\delta)$ and $w = (0,0,1,0)$, the fake exponents were $v = (\delta_1, \delta_2, -\delta_3, 0)$ and $v' = (\delta_1 - \delta_3, \delta_2 - \delta_3, 0, \delta_4)$. Suppose that neither $\delta_1, \delta_2, -\delta_3$ is a negative integer. Since $\mathcal{L} = \mathbb{Z} \cdot (1,1,-1,-1)$, it follows that for $u = n \cdot (1,1,-1,-1)$ with $n \geq 1$, the expression $[v]_{u_-}$ vanishes since $v_4 = 0$. Hence

$$\begin{aligned}
\phi_v &= x^v \cdot \sum_{n=0}^{\infty} \frac{[v]_{(n,n,0,0)}}{[v + (-n,-n,n,n)]_{(0,0,n,n)}} \left(\frac{x_3 x_4}{x_1 x_2}\right)^n \\
&= x_1^{\delta_1} x_2^{\delta_2} x_3^{-\delta_3} \cdot \sum_{n=0}^{\infty} \frac{(-\delta_1)_n (-\delta_2)_n}{(-\delta_3+1)_n n!} \left(\frac{x_3 x_4}{x_1 x_2}\right)^n \\
&= x_1^{\delta_1} x_2^{\delta_2} x_3^{-\delta_3} \cdot {}_1F_2\left(\delta_1, \delta_2, -\delta_3+1; \frac{x_3 x_4}{x_1 x_2}\right),
\end{aligned}$$

i.e., we recover the series $\Phi_{\alpha,\beta,\gamma}$ of Proposition 1.15 with $\alpha = \delta_1$, $\beta = \delta_2$ and $\gamma = -\delta_3 + 1$.

Example 3.21. Consider A as in (1.28) in Example 1.35. The toric ideal I_A is generated by $\partial_1\partial_3 - \partial_2^2$. Choosing, for example, $w = (1,0,0)$ we have that $\mathrm{in}_w(I_A) = \langle \partial_1\partial_3 \rangle$ and $\mathrm{ind}_w(I_A) = \langle \theta_1\theta_3 \rangle$. Therefore, the standard pairs are

$$\mathcal{S}(H_A(\delta)) = \{(1,\{1,2\}), (1,\{2,3\}).$$

This means that the fake exponents of $H_A(\delta)$ relative to w are

$$v = (\delta_1 - \delta_2, \delta_2, 0) \quad \text{and} \quad v' = (0, 2\delta_1 - \delta_2, \delta_2 - \delta_1).$$

If we take $\delta = (0,-1)$ in the above example then the fake exponents are $v = (1,-1,0)$ and $v' = (0,1,-1)$ so, in both cases, they contain entries in $\mathbb{Z}_{<0}$ and we cannot use the construction in Theorem 3.16 to get a series solution. On the other hand, it is easy to verify that the series

$$\frac{x_1}{x_2} \sum_{n=0}^{\infty} \frac{(2n)!}{(n+1)!n!} \left(\frac{x_1 x_3}{x_2^2}\right)^n \tag{3.11}$$

and the (Laurent) monomial x_2/x_3 are solutions of $H_A((0,-1))$.

Exercise 3.22. Let $f(t) = x_1 + x_2 t + x_3 t^2$. Prove that the roots of f as functions of the coefficients x_1, x_2, x_3:

$$\rho_\pm = \frac{-x_2 \pm \sqrt{x_2^2 - 4x_1x_3}}{2x_3}$$

are A-hypergeometric function for A as in (1.28) and $\delta = (0,-1)$. What is the relationship between these solutions of $H_A((0,-1))$ and those discussed above?

Remark 3.23. One can generalize the result of Exercise 3.22 to univariate polynomials of arbitrary degree. Indeed, the roots of the generic polynomial

$$f(t) = x_0 + x_1 t + \cdots + x_d t^d$$

viewed as functions of the coefficients x_i are A-hypergeometric functions of degree $(0,-1)$ for:

$$A = \begin{pmatrix} 1 & 1 & \cdots & 1 & 1 \\ 0 & 1 & \cdots & d-1 & d \end{pmatrix}. \tag{3.12}$$

Indeed, the generators of the toric ideal I_A are the quadratic binomials

$$\partial_i\partial_j - \partial_k\partial_\ell\ ; \quad 0 \leq i,j,k,l \leq d \text{ such that } i+j = k+l,$$

and it was shown by K. Mayr in 1937 that the roots of a generic polynomial satisfies those partial differential equations. We refer to [39] for a modern treatment and the construction of Γ-series solutions in the sense of Gelfand, Kapranov, and Zelevinsky. In fact, in this case, it is possible to write all A-hypergeometric functions with integral degrees in terms of the roots of $f(t)$. We refer to [5] for details.

We will now generalize the above discussion and prove that under certain conditions we can construct series solutions even if the fake exponent contains an entry in $\mathbb{Z}_{<0}$.

Definition 3.24. Given $v \in \mathbb{C}^n$, we define the *negative support* of v as

$$\mathrm{nsupp}(v) \; := \; \{i \in \{1,\ldots,n\} \; : \; v_i \in \mathbb{Z}_{<0}\}. \tag{3.13}$$

If, as before, $\mathcal{L}$ denotes the lattice $\ker_{\mathbb{Z}}(A) \subset \mathbb{Z}^n$ we set:

$$\mathcal{N}_v \; : \; \{u \in \mathcal{L} \; : \; \mathrm{nsupp}(v+u) = \mathrm{nsupp}(v)\}.$$

We now redefine the series ϕ_v in (3.7) as:

$$\phi_v \; := \; \sum_{u \in \mathcal{N}_v} \frac{[v]_{u_-}}{[v+u]_{u_+}} \, x^{v+u} \tag{3.14}$$

The following result which completely characterizes logarithm-free hypergeometric series combines Lemma 3.4.12, Proposition 3.4.13, Theorem 3.4.14, and Corollary 3.15 of [36].

Theorem 3.25. *Let $H_A(\delta)$ be a hypergeometric system and $v \in \mathbb{C}^n$.*

1. *If* $\mathrm{nsupp}(v) = \emptyset$*, then the series (3.14) and (3.7) agree.*
2. *If v is a fake exponent of $H_A(\delta)$ relative to a generic weight w, then the series (3.14) is a formal solution of $H_A(\delta)$ if and only if there is no element $u \in \mathcal{L}$ such that* $\mathrm{nsupp}(u+v)$ *is a proper subset of* $\mathrm{nsupp}(v)$*. In this case we will say that v has* minimal negative support.
3. *If v is a fake exponent with minimal negative support then the series ϕ_v defined by (3.14) is supported in $\mathcal{C}_w$ and for any $w' \in \mathcal{C}_w$, $\mathrm{in}_{w'}(\phi_v) = x^v$. In particular ϕ_v defines a holomorphic function in the open set $\mathcal{U}_\nu(\epsilon)$ defined in (3.10).*
4. *All logarithm-free series solutions of $H_A(\delta)$ are of the form ϕ_v where v is a fake exponent with minimal negative support relative to some generic weight $w \in \mathbb{R}^n$.*

Example 3.26. Let A be as in (1.28), $\delta = (0,-1)$, and $w = (0,0,1,0)$. Then the fake exponents relative to w are $v = (1,-1,0)$ and $v' = (0,1,-1)$, and they have minimal negative support. Indeed, $\mathcal{L} = \{(n,-2n,n) : n \in \mathbb{Z}\}$ and therefore no point in $v + \mathcal{L}$ or $v' + \mathcal{L}$ has empty negative support. We also have:

$$\mathcal{N}_v = \{(n,-2n,n) : n \in \mathbb{Z}_{\geq 0}\}; \quad \mathcal{N}_{v'} = \{(0,0,0)\}.$$

Clearly, $\phi_{v'} = x_2/x_3$ and it is easy to check that ϕ_v agrees with the series (3.11).

Example 3.27. We return to Example 3.11 and choose $\delta = (-1,-1,0)$. For $w = (0,0,1,0)$ there is a unique root of the fake indicial ideal $v = (-1,-1,0,0)$ with multiplicity two. Since $\mathcal{L} = \{(-n,-n,n,n) : n \in \mathbb{Z}\}$ it is easy to see that v has minimal negative support and

$$\mathcal{N}_v = \{(-n,-n,n,n) : n \in \mathbb{N}\}.$$

Therefore

$$\phi_v \; = \; \frac{1}{x_1 x_2} \sum_{n=0}^{\infty} \frac{(-1)^{2n} n! n!}{n! n!} \left(\frac{x_3 x_4}{x_1 x_2}\right)^n \; = \; \frac{1}{x_1 x_2 - x_3 x_4}.$$

The reader may verify directly that ϕ_v is annihilated by the toric operator $\partial_1\partial_2 - \partial_3\partial_4$. Clearly, it has degree δ.

Remark 3.28. In both of these examples, the toric ideal I_A is generated by a single binomial so that there are only two possible choices of initial ideals and hence two possible regions of convergence for the logarithm-free series ϕ_v. For the general case, the study of the possible initial ideals and of the various regions of convergence, is a very interesting problem which is beyond the scope of these notes. We refer the reader to [36, §3.3] for more details.

3.4. **Rational Hypergeometric Functions.** We conclude these lectures with a brief discussion of joint work with Alicia Dickenstein and Bernd Sturmfels [9, 10, 6] on rational hypergeometric functions.

Consider a GKZ hypergeometric system $H_A(\delta)$ and suppose $F(x) = P(x)/Q(x)$, $P, Q \in \mathbb{C}[x_1, \ldots, x_n]$, is a rational A-hypergeometric function. Since by (2) in Theorem 1.36 the singular locus of $H_A(\delta)$ is the zero locus of the principal A-determinant, it follows that

$$Q(x) \;=\; \prod_{A'} D_{A'}(x),$$

where A' runs over the facial subsets of A and $D_{A'}$ denotes the sparse discriminant of A'.

All configurations admit polynomial solutions; indeed, these are related to integer programming (cf. [35]). Also, all configurations admit Laurent polynomial solutions. These are also interesting in the study of the holonomic rank of a GKZ hypergeometric system [5]. We would like to consider the "extreme" case where the denominator of the rational solution is as "large" as possible. More precisely, we have the following definition from [9]:

Definition 3.29. A configuration A is said to be *gkz-rational* if and only if D_A is not a monomial and there exists a rational A-hypergeometric function $F(x) = P(x)/Q(x)$ such that $D_A(x)$ divides $Q(x)$.

Example 3.30. The Gaussian configuration A defined in (1.26) is gkz-rational since $D_A(x) = x_1x_2 - x_3x_4$ and we saw in Example 3.27 that $1/D_A(x)$ is A-hypergeometric. On the other hand, it is shown in [5] that if A is as in (3.12) with $d \geq 2$, then A is not gkz-rational, that is, the only rational A-hypergeometric functions are Laurent polynomials.

Definition 3.31. Let $A_0, \ldots, A_d \subset \mathbb{Z}^d$ be configurations such that for every proper subset $I \subset 0, \ldots, n$,

$$\dim\left(\mathrm{conv}\left(\sum_{i\in I} A_i\right)\right) \geq |I|, \tag{3.15}$$

where sum means the usual Minkowski sum. Then the configuration

$$A \;:=\; \{e_0\} \times A_0 \cup \cdots \cup \{e_d\} \times A_d \subset \mathbb{Z}^{2d+1} \tag{3.16}$$

is called an *essential Cayley configuration.*

Exercise 3.32. Prove that the Gauss configuration (1.26) is affinely equivalent to the Cayley configuration defined by $A_0 = A_1 = \{0, 1\} \subset \mathbb{Z}$.

Exercise 3.33. Let $A = \{a_1, \ldots, a_n\}$ be a codimension-one essential Cayley configuration, i.e. $n = 2d + 2$.

(1) Show that each $A_i = \{\alpha_i, \beta_i\}$, $\alpha_i, \beta_i \in \mathbb{Z}^d$, and that for each proper subset $I \subset \{0, \ldots, d\}$, the vectors $\{\beta_i - \alpha_i : i \in I\}$ are linearly independent.
(2) Show that after reordering the elements of A, the lattice $\mathcal{L} = \ker_{\mathbb{Z}} A$ is of the form:

$$\mathcal{L} = \mathbb{Z} \cdot (b_0, -b_0, b_1, -b_1, \ldots, b_d, -b_d). \tag{3.17}$$

Theorem 3.34. *An essential Cayley configuration is gkz-rational.*

Proof. This is contained in Theorem 1.5 in [9]. One may exhibit an explicit rational A-hypergeometric function via the notion of toric residues. Very briefly, note that the points in the configuration A may be indexed by a pair (i,j) with $i = 0,\dots,d$ and $j = 1,\dots,|A_i|$. Let $n = |A|$. We can then consider the generic Laurent polynomials

$$f_i(t_1,\dots,t_d) \;:=\; \sum_{a_i^j \in A_i} u_i^j\, t^{a_i^j} \ ,$$

where $i = 0,\dots,d$, $j = 1,\dots,|A_i|$. Generically, on the coefficients, the sets

$$V_i \;:=\; \{t \in (\mathbb{C}^*)^d : f_k(t) = 0 \text{ for all } k \neq i\}$$

are finite and $V_0 \cap \cdots \cap V_d = \emptyset$.

Let $a \in \mathbb{Z}^d$ be a point in the interior of the Minkowski sum

$$A_0 + \cdots + A_d.$$

Then, one can show [4] that the expression

$$R_f(a) \;:=\; (-1)^i \sum_{\xi \in V_i} \mathrm{Res}_\xi \left(\frac{t^a/f_i(t)}{f_0(t)\cdots \widehat{f_i(t)} \cdots f_d(t)}\, \frac{dt_1}{t_1} \wedge \cdots \wedge \frac{dt_d}{t_d} \right),$$

where Res_ξ denotes the Grothendieck residue at the point ξ relative to $f_0(t),\cdots,\widehat{f_i(t)},\cdots,f_d(t)$ (cf. [24]), is independent of i and defines a rational function of the variables u_i^j which is A-hypergeometric of degree $\delta = (-1,\dots,-1,-a_1,\dots,-a_d) \in \mathbb{Z}^{2d+1}$. The function $R_f(a)$ is called a *toric residue*. We refer to [12, 4, 8, 7] for definitions and details.

As shown in [8], the denominator of the rational function $R_f(a)$ is a multiple of the sparse resultant of $f_0,\dots,f_d$ which, in turn, agrees with the sparse discriminant D_A. The fact that D_A is not a monomial is proved in [18, Proposition 9.1.17] in the case when all A_i have dimension d and extended to the essential case in [9, Proposition 5.1]. ☐

Example 3.35. Let A be the Cayley configuration

$$A = \{e_0\} \times \{0,1\} \cup \{e_1\} \times \{0,1\}$$

and let $a = 1$. Then setting

$$f_0(t) \;:=\; u_0^0 + u_0^1 t\ ; \quad f_1(t) \;:=\; u_1^0 + u_1^1 t\,,$$

we have

$$R_f(a) \;=\; \mathrm{Res}_{-u_1^0/u_1^1} \left(\frac{1/(u_0^0 + u_0^1 t)}{u_1^0 + u_1^1 t}\, dt \right) \;=\; \frac{1}{u_1^1}\frac{1}{(u_0^0 - u_0^1 u_1^0/u_1^1)} \;=\; \frac{1}{u_1^1 u_0^0 - u_0^1 u_1^0}.$$

Note that in this example we are computing the usual one-variable residue. We leave it up to the reader to check that under the affine isomorphism of Exercise 3.32, the function $R_f(a)$ agrees with the function ϕ_v in Example 3.27.

The following is Conjecture 1.3 in [9]:

Conjecture 3.36. *If A is gkz-rational then A is a Cayley essential configuration and every rational A-hypergeometric function whose denominator is a multiple of D_A is obtained from a toric residue.*

We will not specify the meaning of "obtained from" and, instead, refer the reader to [9] for details.

Theorem 3.37. *Let $A = \{a_1,\dots,a_n\} \subset Z^d$. Then Conjecture 3.36 holds in the following cases*
(1) $d \leq 3$.
(2) $n - d = 1$.
(3) $n - d = 2$ *and* $n = 7$.

Proof. The case $d = 1$ is proved in [5]. The cases $d = 2, 3$ and $n - d = 1$ are proved in [9]. The last case is studied in [6].

In order to illustrate the techniques of this section we sketch an argument to prove the second assertion. Suppose $A \subset \mathbb{Z}^d$ is a codimension one configuration, that is the lattice $\mathcal{L} = \ker_{\mathbb{Z}} A$ has rank one. If A is gkz rational then D_A is not a monomial and there exists a rational A-hypergeometric function $F = P/Q$, where $P, Q \in \mathbb{C}[x_1, \ldots, x_n]$ are polynomials and D_A divides Q. Given a generic weight w, some series expansion of F must converge in $\mathcal{U}_w(\epsilon)$ for ϵ sufficiently small and, therefore, it must correspond to a series ϕ_v as in (3.14).

Now, after relabeling the variables we may assume that

$$\mathcal{L} \;=\; \mathbb{Z} \cdot (-p_1, \ldots, -p_r, q_1, \ldots, q_s),$$

where $r + s = n$ and $p_1, \ldots, q_s$ are positive integers. (It is not hard to check that if one of the entries of a generator of $\mathcal{L}$ vanishes then $D_A = 1$ which is not possible.)

Since ϕ_v defines a rational function, its degree $\delta \in \mathbb{Z}^d$ and therefore we may assume that $v \in \mathbb{Z}^n$. Moreover, $\operatorname{nsupp}(v) \subset \{1, \ldots, r\}$ since otherwise v would not have minimal negative support. In fact, we must have $\operatorname{nsupp}(v) = \{1, \ldots, r\}$ since, otherwise, the series ϕ_v would have only finitely many terms which is not possible since D_A is not a monomial and divides Q. In particular,

$$\mathcal{N}_v \;=\; \{u(n) := (-p_1 n, \ldots, -p_r n, q_1 n, \ldots, q_s n) : n \in \mathbb{N}\}$$

and, given $u \in \mathcal{N}_v$ we have:

$$u(n)_- \;=\; (p_1 n, \ldots, p_r n, 0, \ldots, 0)\,; \quad u(n)_+ \;=\; (0, \ldots, 0, q_1 n, \ldots, q_s n).$$

Therefore, writing

$$v \;=\; (-k_1, \ldots, -k_r, \ell_1, \ldots, \ell_s),$$

where $k_1, \ldots, k_r$ are positive integers and $\ell_1, \ldots, \ell_s$ are non-negative integers, we have

$$\phi_v(x) \;=\; x^v \sum_{n=0}^{\infty} (-1)^{\rho n} \, \frac{\prod_{i=1}^{r}(p_i n + k_i - 1)!}{\prod_{j=1}^{s}(q_j n + \ell_j)!} \, x^{u(n)}, \tag{3.18}$$

where $\rho = p_1 + \cdots + p_r$.

Note that the series ϕ_v in (3.18) defines a rational function if and only if the series:

$$\sum_{n=0}^{\infty} \frac{\prod_{i=1}^{r}(p_i n + k_i - 1)!}{\prod_{j=1}^{s}(q_j n + \ell_j)!} \, t^n$$

defines a rational function in one-variable t. Indeed this follows, since $v \in \mathbb{Z}^n$, by taking

$$t \;= (-1)^{\rho} \, \frac{x_{r+1}^{q_1} \cdots x_n^{q_s}}{x_1^{p_1} \cdots x_r^{p_r}}.$$

As noted in Remark 1.10, it is shown in [9, Theorem 2.3] that the above series defines a rational function if and only if $r = s$ and after reordering, if necessary, $p_i = q_i$.But, by Exercise 3.33, this characterizes codimension one essential Cayley configurations. □

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