# TIME ESTIMATES FOR THE PHASE-CHANGE PROCESS IN A MATERIAL WITH A PERFECT THERMAL CONTACT BOUNDARY CONDITION


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## ABSTRACT

In this paper, we consider a slab represented by the interval $0 < y < \ell$, at the initial temperature $v_0(y) = M > 0$ having a positive constant heat flux $q_0$ on the left face and a contact perfect condition, $kv_y(\ell,\tau) + M_f c_f v_\tau(\ell,\tau) = 0$, on the right face $y = \ell$. We consider the corresponding heat conduction problem and we assume that the phase-change temperature is $0^0 C$. We obtain time estimates for the occurrence of a phase-change by means of Laplace Transform and Method of Lines.


## Introduction an preliminaries

We consider a one-dimensional slab $[0,\ell]$ with its face $y = \ell$ in perfect thermal contact with mass $M_f$ per unit area of a well-stirred fluid (or a perfect conductor) of specific heat $c_f$. We consider the following heat conduction problem:

Problem P

$$kv_{yy} = \rho c v_\tau, \qquad D = \{(y,t) : 0 \le y \le \ell, \tau > 0\}, \tag{1}$$

$$v(y,0) = V_0 > 0, \qquad 0 \le y \le \ell, \tag{2}$$

$$kv_y(0,\tau) = q_0 > 0, \qquad \tau > 0, \tag{3}$$

$$kv_y(\ell,\tau) + M_f c_f v_\tau(\ell,\tau) = 0, \qquad \tau > 0, \tag{4}$$

where $k$ is the thermal conductivity, $\rho$ the density, $q_0$ the heat flux and $c$ the specific heat of the material, all of them positive constants.

We propose the following changes of variables:

$$x = \frac{y}{\ell}, \quad t = \frac{k\tau}{\rho c \ell^2}, \quad v(y,\tau) = cu(x,t).$$

The problem P is transformed in the following problem P1:

Problem P1

$$\begin{aligned} u_{xx} &= u_t, \qquad D = \{(x,t) : 0 \leq x \leq 1, t > 0\}, & (5)\\ u(x,0) &= M > 0, \qquad 0 \leq x \leq 1, & (6)\\ u_x(0,t) &= q > 0, \qquad t > 0, & (7)\\ u_x(1,t) &= \gamma u_t(1,t), \qquad t > 0, & (8) \end{aligned}$$

where:

$$M = cV_0, \quad q = \frac{c\ell q_0}{k}, \quad \gamma = -\frac{M_f c_f}{\rho c} < 0.$$

This paper was motived by [3,4,8,9,10,13]. In the case of models of heat conduction in material media it is natural to attempt to determine its temporary range of validity. Here an important limitation of this range is imposed by the change of phase phenomena. Hence the need of modifying the model to include this characteristic (free boundaries and mushy regions could now appear)[5,6,7,11].

A large bibliography on phase-change problems was given in [14].

Much of the contents of this paper will be concerned with the time estimates of the occurrence of the phase-change process in a material with a perfect contact boundary condition. In section 2 we show the exact solution of Problem P1 and we obtain an approximation for its solution at $x = 0$, with an estimate of the error. In section 3 we apply the Laplace Transform to the problem considered, we obtain the exact solution of the transformed problem and we use the asymptotic behavior to approximate the inverse of the solution of the transformed problem. We approximate the solution of the original problem in order to obtain time estimates for occurrence of the change-phase in the material. In section 4 we apply the method of lines to Problem P1 in which the partial differential equation is replaced by a sequence of ordinary differential equations at discrete time levels. We obtain a time estimate for phase-change process that depends on $M$, $q$ and $\gamma$.

Problem P1 satisfies the following minimum principle (section 1.1) that we use in following sections in order to consider only the behavior of $u(x,t)$ for $x = 0$. Then, we will call a phase change time, $t_{ch}$, a time such that $u(0, t_{ch}) = 0$.

**Minimum Principle for the Problem P1**

**Lemma 1** *The solution of the Problem P1 holds:*

$$u_x(x,t) \geq 0 \quad and \quad u(0,t) \leq u(x,t), \qquad 0 \leq x \leq 1, \qquad t > 0.$$

**Proof**

We set $v = u_x$, the function $v(x,t)$ satisfies the following heat conduction problem:

$$v_{xx} = v_t, \qquad D = \{(x,t) : 0 \leq x \leq 1, t > 0\}, \tag{9}$$

$$v(x,0) = 0, \qquad 0 \leq x \leq 1, \tag{10}$$

$$v(0,t) = q > 0, \qquad t > 0, \tag{11}$$

$$v(1,t) = \gamma v_x(1,t), \qquad t > 0. \tag{12}$$

By using the maximum principle ([2],[12]) for $0 \leq x \leq 1$ and $t > 0$ we have:

$$\min v(x,t) = \min\{q, 0, v(1,t)\},$$

we suppose that $v(1,t) < 0$ (we remark that $q > 0$), then it follows that:

$$\min v(x,t) = v(1,t),$$

by using Hopf lemma [2] we deduce that:

$$v_x(1,t) < 0,$$

which contradicts the condition (12). Therefore $u_x(x,t) \geq 0$, since $\gamma < 0$, from which we obtain the thesis. □

## Exact solution of the Problem P1

In order to obtain an exact solution for the problem P1, we set $v = u_x$, the function $v(x,t)$ satisfies the equations (9),(10), (11) and (12). The solution of this problem is given by [15]:

$$\frac{v(x,t)}{q} = \frac{-\gamma + 1 - x}{-\gamma + 1} - \sum_{n=1}^{\infty} A_n sin(\beta_n x) e^{-\beta_n^2 t}, \tag{13}$$

where

$$A_n = \frac{2(\gamma^2 \beta_n^2 + 1)}{\beta_n(\gamma^2 \beta_n^2 - \gamma + 1)}, \tag{14}$$

and $\beta_n$ are different positive roots of the following transcendental equation:

$$\beta_n \cot \beta_n - \frac{1}{\gamma} = 0, \tag{15}$$

where the roots $\beta_n \in ((n-1)\pi, n\pi)$, for $n \geq 1$ independent of $\gamma$.

From (13) we have that:

$$v(1,t) = \frac{-q\gamma}{-\gamma+1}[1 + \sum_{n=1}^{\infty} B_n e^{-\beta_n^2 t}], \tag{16}$$

where

$$B_n = \frac{\frac{2}{\gamma}(\frac{1}{\gamma}-1)\sec \beta_n}{[(\frac{1}{\gamma}-1)\frac{1}{\gamma}+\beta_n^2]}. \tag{17}$$

We obtain the following expression for the solution of the Problem P1 using (13),(16),(8):

$$\begin{aligned} u(x,t) &= M + \frac{1}{\gamma}\int_0^t v(1,\tau)d\tau - \int_x^1 v(\xi,t)d\xi, && (18)\\ &= M + I_1 + I_2 && (19)\end{aligned}$$

where

$$I_1 = -\frac{q}{-\gamma+1}[t - \sum_{n=1}^{\infty}\frac{B_n}{\beta_n^2}(e^{-\beta_n^2 t}-1)], \tag{20}$$

$$I_2 = q\left[\frac{\gamma - \gamma x - \frac{(1-x)^2}{2}}{-\gamma+1} + \sum_{n=1}^{\infty}\frac{A_n}{\beta_n}e^{-\beta_n^2 t}(-\cos\beta_n + \cos\beta_n x)\right]. \tag{21}$$

By Lemma 1 we need to approximate $u(0,t)$ in order to estimate $u(x,t)$ for all $x \in [0,1]$. This function is given by:

$$u(0,t) = M - \frac{qt}{-\gamma+1} + \frac{q(\gamma-\frac{1}{2})}{-\gamma+1} + \frac{q}{-\gamma+1}\sum_{n=1}^{\infty}\frac{B_n}{\beta_n^2}(e^{-\beta_n^2 t}-1) + q\sum_{n=1}^{\infty}\frac{A_n}{\beta_n}e^{-\beta_n^2 t}(1-\cos\beta_n). \tag{22}$$

We remark that $\beta_n \to +\infty$ when $n \to +\infty$, moreover $A_n = \mathcal{O}(1/\beta_n)$ and $B_n = \mathcal{O}(1/\beta_n)$. Since we are interested in obtaining estimates for the time where $u(0,t) = 0$, we set the following approximation for the temperature $u(0,t)$ given by:

$$U_{J,P}(t) = M - \frac{qt}{-\gamma+1} + \frac{q(\gamma-\frac{1}{2})}{-\gamma+1} + \frac{q}{-\gamma+1}\sum_{n=1}^{J}\frac{B_n}{\beta_n^2}(e^{-\beta_n^2 t}-1) + q\sum_{n=1}^{P}\frac{A_n}{\beta_n}e^{-\beta_n^2 t}(-\cos\beta_n + 1). \tag{23}$$

We can express the temperature $u(0,t)$ as:

$$u(0,t) = U_{J,P}(t) + E_{J,P}(t). \tag{24}$$

In order to estimate the error $E_{M,N}(t)$ we note that the coefficients $A_n$, $B_n$ can be bounded in the following way:

$$\begin{aligned} \left|\frac{A_n}{\beta_n}\right| &= \left|\frac{2(\gamma^2\beta_n^2+1)}{\beta_n^2(\gamma^2\beta_n^2-\gamma+1)}\right| & (25)\\ &\leq \left|\frac{2(\gamma^2\beta_n^2+1)}{\beta_n^2(\gamma^2\beta_n^2+1)}\right| = \frac{2}{\beta_n^2}. & (26)\end{aligned}$$

From the equation (15) we have that

$$\frac{|\sec\beta_n|}{|\beta_n|} = \frac{|\gamma|}{|\sin\beta_n|} \tag{27}$$

and since $|\sin\beta_n| \to 1$ when $n\to\infty$ , then there exist $i_0 \in N$ such that $|\sin\beta_n| \leq \frac{1}{2}$ for all $i \geq i_0$. Therefore we obtain that:

$$\frac{|\sec\beta_n|}{|\beta_n|} \leq \frac{|\gamma|}{d}, \tag{28}$$

where

$$d = \max_{i\leq i_0}\{\frac{1}{2}, \sin\beta_i\} \leq 1.$$

Therefore, we have the following inequality:

$$\begin{aligned} \left|\frac{B_n}{\beta_n}\right| &= \left|\frac{\frac{2}{\gamma}(\frac{1}{\gamma}-1)\sec\beta_n}{\beta_n[(\frac{1}{\gamma}-1)\frac{1}{\gamma}+\beta_n^2]}\right| & (29)\\ &\leq \left|\frac{\frac{2}{\gamma}(\frac{1}{\gamma}-1)|\gamma|}{d[(\frac{1}{\gamma}-1)\frac{1}{\gamma}+\beta_n^2]}\right| & (30)\\ &\leq \frac{2(1-\gamma)}{|\gamma| d\beta_n^2}. & (31)\end{aligned}$$

These bounds for $A_n$ and $B_n$ allow us to prove the following lemma:

**Lemma 2** *We have the following estimation for the error $E_m(t)$:*

$$|E_{J,P}(t)| \leq \frac{q}{\pi^3 d|\gamma|}\frac{1}{J^2} + \frac{4q}{\pi^2}\frac{1}{P}.$$

**Proof**

First, we remark that $(n-1)\pi \le \beta_n \le n\pi$. Then we use the following inequalities:

$$\sum_{J+1}^{\infty} \frac{1}{\beta_n^3} \le \frac{1}{\pi^3} \int_J^{\infty} \frac{1}{x^3} dx = \frac{1}{2\pi^3 J^2},$$

$$\sum_{P+1}^{\infty} \frac{1}{\beta_n^2} \le \frac{1}{\pi^2} \int_P^{\infty} \frac{1}{x^2} dx = \frac{1}{\pi^2 P}$$

and

$$|e^{-\beta_n^2 t} - 1| \le 1, \quad |e^{-\beta_n^2 t}| \le 1, \qquad |-\cos\beta_n + 1| \le 2.$$

Now from the equation (24) we have that $|E_{M,N}(t)|$ satisfies

$$\begin{aligned} |E_{J,P}(t)| &\le \frac{2q}{d|\gamma|} \sum_{J+1}^{\infty} \frac{1}{\beta_n^3} + 4q \sum_{P+1}^{\infty} \frac{1}{\beta_n^2} \\ &\le \frac{q}{\pi^3 d|\gamma|} \frac{1}{J^2} + \frac{4q}{\pi^2} \frac{1}{P} \square \end{aligned}$$

**Numerical Example**

Below we present a table with the time $t_{ch}$ such that $U_{M,N}(t_{ch}) = 0$ and $E_{J,P}$ differents values of $J, P \in N$.

For the TABLE 1 we consider $M = 1$, $q = 1$, $\gamma = -2$ and for tha TABLE 2 we set the following values $M = 0.1$, $q = 1$, $\gamma = -2$.

| J | P | tch | $E_{J,P}$ |
|---|---|---|---|
| 20 | 5 | 0.7905 | 0.0209 |
| 20 | 10 | 0.7894 | 0.0204 |
| 20 | 20 | 0.7895 | 0.0203 |
| 25 | 5 | 0.7905 | 0.0168 |
| 30 | 5 | 0.7905 | 0.0141 |
| 30 | 10 | 0.7894 | 0.0136 |
| 30 | 20 | 0.7895 | 0.0135 |
| 100 | 20 | 0.7895 | 0.004 |

| J | P | tch | $E_{J,P}$ |
|---|---|---|---|
| 20 | 5 | 0.007635 | 0.0209 |
| 20 | 10 | 0.007562 | 0.0204 |
| 20 | 20 | 0.007575 | 0.0203 |
| 25 | 5 | 0.007635 | 0.0168 |
| 30 | 5 | 0.007635 | 0.0141 |
| 30 | 10 | 0.007562 | 0.0136 |
| 30 | 20 | 0.007575 | 0.0135 |
| 100 | 20 | 0.007575 | 0.004 |

In the following sections we will show some alternatives in order to obtain the phase change time estimates.

## Estimates through the Laplace Transform

In this section we apply the Laplace Transform to the problem considered, we obtain the exact solution of the transformed problem and we use the asymptotic behavior to approximate the inverse of the solution of the transformed problem. We approximate the solution of the original problem in order to obtain time estimates of the occurrence of the change-phase in the material. The Laplace Transform is defined by:

$$U(s,t) = L(u(x,t)) = \int_0^\infty u(x,t)e^{-st}dt,$$

where $s$ is a positive parameter.

We apply the Laplace Transform to Problem I, and taking into account the following properties

$$L(u_x(x,t)) = \frac{dU(s,x)}{dx}, \quad L(u_{xx}(x,t)) = \frac{d^2U(s,x)}{dx^2}.$$

$$L(u_t(x,t)) = -M + sU(s,x), \quad L(q) = \frac{q}{s}.$$

the following problem is obtained:

Problem P2

$$U_{xx}(s,x) - sU(s,x) = -M, \tag{32}$$

$$U_x(s,0) = \frac{q}{s}, \tag{33}$$

$$\gamma s U(s,1) - U_x(s,1) = \gamma M. \tag{34}$$

The solution of Problem P2 is given by:

$$U(s,x) = A(s)\exp(-\sqrt{s}x) + B(s)\exp(\sqrt{s}x) + \frac{M}{s}, \tag{35}$$

where

$$A(s) = \frac{-(\gamma s - \sqrt{s})\exp(\sqrt{s})q}{2s(\gamma s^{\frac{3}{2}}\cosh(\sqrt{s}) - s\sinh(\sqrt{s}))}, \tag{36}$$

$$B(s) = \frac{(\gamma s + \sqrt{s})\exp(-\sqrt{s})q}{2s(\gamma s^{\frac{3}{2}}\cosh(\sqrt{s}) - s\sinh(\sqrt{s}))}. \tag{37}$$

After straightforward computations we obtain the following expression for $U(s,x)$:

$$U(s,x)=\left[\frac{-\gamma s\sinh\left(\sqrt{s}(1-x)\right)+\sqrt{s}\cosh\left(\sqrt{s}(1-x)\right)}{\gamma s^{\frac{5}{2}}\cosh\left(\sqrt{s}\right)-s^2\sinh\left(\sqrt{s}\right)}\right]q+\frac{M}{s}. \tag{38}$$

**Remark 1** *If we consider $q=0$, in (38) then $U(s,x)=\dfrac{M}{s}$. In this case $u(x,t)=M$ for $0\leq x\leq 1$ and $t>0$, which can be observed directly from Problem P1.*

We will need a useful property from Davies, [1], in order to estimate the time of change of phase.

**Lemma 3** *Suppose that the Laplace transform $F(s)=L(f(t))$ has an asymptotic expansion:*

$$F(s)\approx\sum_{\nu=1}^{\infty}a_\nu s^{-\lambda_\nu}\qquad as\quad s\to+\infty,\quad with\quad \lambda_1\leq\lambda_2\leq\lambda_3\leq...$$

*then we have*

$$f(t)\approx\sum_{\nu=1}^{\infty}\frac{a_\nu t^{\lambda_\nu-1}}{(\lambda_\nu-1)!}\qquad as\quad t\to 0.$$

**Theorem 1** *An estimate of the phase change time, for the Problem P1, is given by the following approximation:*

$$t_{ch}\approx\left(\frac{\sqrt{\pi}M}{2q}\right)^2.$$

**Proof**

We only need to consider the behavior of (38) for large $s$ and $x=0$, since (Lemma 1 ), so that

$$U(s,0)\approx\frac{M}{s}-\frac{q}{s^{\frac{3}{2}}},\qquad s\to+\infty. \tag{39}$$

Therefore by Lemma 3, we can obtain the asymptotic behavior for $u(0,t)$ for small $t$:

$$u(0,t)\approx M-\frac{2}{\sqrt{\pi}}qt^{\frac{1}{2}},\qquad t\to 0. \tag{40}$$

We need $u(0,t_{ch})=0$ for some $t_{ch}>0$ in order to have a change phase in the material for $t>t_{ch}$, that is:

$$t_{ch}\approx\left(\frac{\sqrt{\pi}M}{2q}\right)^2. \tag{41}$$

□

**Remark 2** *The estimate given by theorem 1 is valid for* $t \approx 0$ *which is equivalent to* $q \gg M$*. We can see that the approximation does not depend on the constant* $\gamma$*. The time of phase-change for* $M = 1$, $q = 1$, $\gamma = -2$ *given by Theorem 1 is* $t_{ch} = 0.7853$ *(see TABLE 1) and for* $M = 0.1$, $q = 1$, $\gamma = -2$ *is* $t_{ch} = 0.007853$ *(see TABLE 2).*

**Remark 3** *Roughly speaking, we expect that the solution of P1 converges when* $\gamma \to -\infty$ *to the solution of following problem:*

$$u_{xx} = u_t, \qquad D = \{(x,t) : 0 \leq x \leq 1, t > 0\}, \tag{42}$$

$$u(x,0) = M > 0, \qquad 0 \leq x \leq 1, \tag{43}$$

$$u_x(0,t) = q > 0, \qquad t > 0, \tag{44}$$

$$u(1,t) = C, \qquad t > 0, \tag{45}$$

*where* $C$ *is a constant to be determined. This problem was studied in [3] where the same expression (41) for the time of change of phase was obtained.*

**Remark 4** *If we consider the case where the domain is seminfinite (i.e.* $0 < x < +\infty$*), then the exact solution for the Problem P1 is given by:*

$$u(x,t) = M - 2q\sqrt{t} \quad ierfc(\frac{x}{2\sqrt{t}}) \tag{46}$$

*where* $erf(x) = \frac{2}{\sqrt{\pi}} \int_0^x \exp(-t^2) dt$*, and*

$$erfc(x) = 1 - erf(x), \qquad ierfc(x) = \frac{\exp(-x^2)}{\sqrt{\pi}} - xerfc(x).$$

*The phase change time (i.e.* $u(0,t) = 0$*) is given by [3]:*

$$t = (\frac{\sqrt{\pi}M}{2q})^2. \tag{47}$$

*In this case, we consider the following problem* $P2_\infty$ *(the Laplace Transform):*

$$U_{xx}(s,x) - sU(s,x) = -M, \tag{48}$$

$$U_x(s,0) = \frac{q}{s}. \tag{49}$$

*Now, the exact solution for problem* $P2_\infty$ *in* $x = 0$ *is given by:*

$$U(0,s) = -\frac{q}{s^{3/2}} + \frac{M}{s}.$$

*Therefore, we obtain that:*

$$u(0,t) = -\frac{2q}{\sqrt{\pi}}\sqrt{t} + M,$$

*and the phase change time is equal to (47).*

In order to obtain an expression for the phase change time which depends on $\gamma$, we will use the method of lines in the next section.

## Estimates through the Method of Lines

In this section we apply the method of lines to Problem P1 in which the partial differential equation is replaced by a sequence of ordinary differential equations at discrete time levels. For this purpose, we shall define a partition $\{0 = t_0 < t_1 < ... < t_N = T\}$ of the interval $[0, T]$, with equal subintervals $\Delta t = t_i - t_{i-1}$ and $i = 1, .., N$. The simplest, and most commonly used, method of lines approximation for Problem P1 requires the substitution

$$u_t(x, t_n) \approx \frac{u(x, t_n) - u(x, t_{n-1})}{\Delta t},$$

which reduces the partial differential equation (1) to a second order differential equation

$$\Delta t\, u_n^{''}(x) - u_n(x) = -u_{n-1}(x), \tag{50}$$

for $n = 1, .., N$ and $\Delta t = \frac{T}{N}$, where $u_n = u(x, t_n)$ and $u_n^{''} = \frac{d^2 u_n(x)}{d^2 x}$. The boundary conditions are transformed in the following equations:

$$\begin{aligned} u_0(x) &= M, && (51)\\ u_n^{'}(0) &= q && (52)\\ -\Delta t\, u_n^{'}(1) + \gamma u_n(1) &= \gamma u_{n-1}(1). && (53) \end{aligned}$$

The method of lines approximation for the heat conduction problem P1 is given now by (50),(51), (52) and (53) which is called Problem P3(n).

The solution of Problem P3(n) has the representation:

$$u_n(x) = A_{n,k} \exp\left(-\frac{1}{k}x\right) + B_{n,k} \exp\left(\frac{1}{k}x\right) + g_{n,k}(x), \tag{54}$$

where

$$A_{n,k} = \frac{q(-k+\gamma)\exp(\frac{1}{k}) - \frac{1}{k}(\gamma u_{n-1}(1) + k^2 g_n'(1) - g_n(1))}{2(\sinh(\frac{1}{k}) - \frac{\gamma}{k}\cosh(\frac{1}{k}))}, \tag{55}$$

$$B_{n,k} = \frac{-\frac{1}{k}(\gamma u_{n-1}(1) + k^2 g_n'(1) - g_n(1)) - q(k+\gamma)\exp(-\frac{1}{k})}{2(\sinh(\frac{1}{k}) - \frac{\gamma}{k}\cosh(\frac{1}{k}))}, \tag{56}$$

$$k = \sqrt{\Delta t}, \tag{57}$$

and $g_{n,k}(x)$ is a particular solution of Problem P3(n). We remark that $A$ and $B$ depend on $n$ and $k$. The particular solution $g_{n,k}(x)$ is given by:

$$g_{n,k}(x) = \frac{1}{k}\int_0^x \sinh(\frac{1}{k}(s-x))u_{n-1}(s)ds. \tag{58}$$

Henceforth, in order to simplify the notation we omit the indices $k$. We consider one iteration (i.e. $n = 1$), in this case the solution of problem P3(1) is given by:

$$u_1(x) = A_1\exp(-\frac{1}{k}x) + B_1\exp(\frac{1}{k}x) + \frac{M}{k}\int_0^x \sinh(\frac{1}{k}(s-x))ds, \tag{59}$$

where

$$\begin{aligned} g_1(x) &= \frac{M}{k}\int_0^x \sinh(\frac{1}{k}(s-x))ds \\ &= M(1-\cosh(\frac{x}{k})). \end{aligned} \tag{60}$$

and its derivative $g_1'(x)$ is given by :

$$g_1'(x) = -\frac{M}{k}\sinh(\frac{x}{k}). \tag{61}$$

We use this last expressions at $x = 1$ and that $g_n(0) = 0$, (58), in order to obtain an expression for $u_1(0)$ (By using Lemma 1 we are only interested in the behavior at $x = 0$) . After some algebraic manipulation, we obtain:

$$u_1(0) = G_\gamma(k)M + F_\gamma(k)q, \tag{62}$$

where

$$F_\gamma(k) = \frac{\gamma \sinh(\frac{1}{k}) - k\cosh(\frac{1}{k})}{\sinh(\frac{1}{k}) - \frac{\gamma}{k}\cosh(\frac{1}{k})}, \tag{63}$$

$$G_\gamma(k) = \frac{\sinh(\frac{1}{k}) - k\cosh(\frac{1}{k}) + \frac{1-\gamma}{k}}{\sinh(\frac{1}{k}) - \frac{\gamma}{k}\cosh(\frac{1}{k})}. \tag{64}$$

We look for $k$ satisfying the equation $u_1(0) = 0$, which is equivalent to:

$$H_\gamma(k) = -\frac{M}{q}, \tag{65}$$

where

$$H_\gamma(k) = \frac{\gamma \sinh(\frac{1}{k}) - k\cosh(\frac{1}{k})}{\sinh(\frac{1}{k}) - k\cosh(\frac{1}{k}) + \frac{1-\gamma}{k}}. \tag{66}$$

It is easy to verify that $H_\gamma(k)$ satisfies the following properties:

**Lemma 4** *The function $H_\gamma(k)$ holds the following properties:*

1. $\lim H_\gamma(k) = 0$ *when* $k \to 0^+$ *for all* $\gamma < 0$.

2. $H_\gamma(k) \leq 0$ *for* $k \approx 0$ *for all* $\gamma < 0$.

**Proof**

We have the equivalent expression

$$\begin{aligned} H_\gamma(k) &= \frac{\gamma k \sinh(\frac{1}{k}) - k^2\cosh(\frac{1}{k})}{k\sinh(\frac{1}{k}) - \cosh(\frac{1}{k}) + 1 - \gamma} \\ &= \frac{(-k^2+\gamma k)e^{1/k} - (k^2+\gamma k)e^{-1/k}}{(k-1)e^{1/k} - (k-1)e^{-1/k} + 2(1-\gamma)}. \end{aligned}$$

Hence, the behavior of $H_\gamma(k)$ for $k \approx 0$ is given by

$$H_\gamma(k) \approx \frac{\gamma k}{1-\gamma} \qquad as \quad k \approx 0, \tag{67}$$

From (67), then (1) and (2) hold.□

We will use the expression (67) which holds for $k \approx 0$ in order to solve the equation (65).

**Theorem 2** *An estimate of the time of change of phase for the Problem P1, is given by the following approximation :*

$$t_{ch} \approx \frac{M^2(1-\gamma)^2}{q^2\gamma^2}.$$

**Proof**

The time given by the method of lines is given by (65) where

$$H_\gamma \approx \frac{\gamma k}{1-\gamma} \qquad k \approx 0,$$

therefore

$$t_{ch} = \frac{M^2(1-\gamma)^2}{\gamma^2 q^2},$$

we remark that $t = k^2$.□

## Concluding Remarks

In the last two sections we have obtained two different estimates for the time of phase-change process. The first one was computed by means of Laplace transform. This approximation depends on $M$, $q$ but it does not depend on $\gamma$. In fact this is the difference with the approximation derived by the Methods of lines. We remark that the estimates are the same order. From theorem 1 we have that $t_{ch}$ holds:

$$-\frac{M}{q} = -\sqrt{\frac{4t_{ch}}{\pi}}, \tag{68}$$

where we note that $t = k^2$. Now we define the following function:

$$R(k) = -\frac{2k}{\sqrt{\pi}}. \tag{69}$$

We may immediately verify the following lemma which implies that the times given by theorem 1 and 2 are comparable.

**Lemma 5** *We have that*

$$\lim_{k\to 0^+} \frac{H_\gamma(k)}{R(k)} = -\frac{-\gamma\sqrt{\pi}}{2(1-\gamma)}.$$

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**Nomenclature**

$k$: thermal conductivity, $[Wm/m^2K]$.

$\rho$: density, $[Kg/m^3]$.

$q_0$: heat flux on the fixed face $x = 0$, $[Ws/m^2]$.

$c$: specific heat, $[J/KgK]$.

$c_f$: specific heat of tthe fluid, $[J/KgK]$.

$M_f$: mass of the fluid, $[Kg]$.

$\tau$: time variable, $[s]$.

$v$: temperature of material, $[K]$.

$y$: spatial variable, $[m]$.

$t$: dimensionless time.

$x$: dimensionless spatial variable.

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