

# HOPF ALGEBRAS: FROM FINITE GROUPS TO ALGEBRAIC RENORMALIZATION

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ABSTRACT. These are informal notes from a course in UMA 2026, Mar del Plata.

## COURSE DESCRIPTION

This course provides an introduction to the structural theory of Hopf algebras and their role in bridging the gap between classical abstract algebra and modern combinatorial physics. We begin with a rigorous foundational framework, developing the properties of coalgebras, bialgebras, and antipodes through the familiar lens of finite group algebras.

The course reviews how these algebraic objects lie at the core of a precise mathematical foundation for renormalization—one of the most profound and groundbreaking insights of twentieth-century theoretical physics. By examining the structural architecture of rooted trees and combinatorial graphs, we translate the deep geometric intuition of the physics into the rigor of pure algebra.

We end the course with the Connes-Kreimer theorem, which shows that the celebrated algorithmic extraction of finite physical values from divergent Feynman diagrams is fundamentally a manifestation of pure algebra: the computation of a Birkhoff decomposition uniquely governed by the Hopf antipode.

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## 1. REGULARIZATION, AND RENORMALIZATION: A TOY MODEL FOR MATHEMATICIANS.

**1.1. A glimpse at Theoretical Physics Methodologies.** In real analysis, divergent improper integrals are typically treated as dead ends. If an integral diverges, the mathematical object does not exist in the classical sense. However, in Quantum Field Theory (QFT), divergent integrals arise naturally when modeling physical interactions at extreme scales.

For physicists, a divergence does not mean reality explodes; it signals that our **idealized mathematical model** breaks down at ultraviolet (short-distance) or infrared (long-distance) limits. To extract finite, measurable predictions, physics employs a rigorous two-step strategy:

- (1) **Regularization:** Deforming the divergent integral by introducing a control parameter ( $\epsilon$ ) to make the expression analytically well-behaved and finite.
- (2) **Renormalization:** Absorbing the isolated divergence ( $\frac{1}{\epsilon}$ ) into a redefinition of the physical parameters, anchoring the finite remainder to a real-world measurement scale ( $\mu$ ).

The goal of this lecture is to present a rigorous mathematical “toy model” of this workflow.

**1.2. The Physical Perspective: Dimensional Regularization and Subtraction.** Consider the simple 1-loop diagram (commonly referred to as the “bubble diagram”) in a scalar  $\phi^4$  theory. This graph consists of two 4-valent vertices connected by two internal propagators.

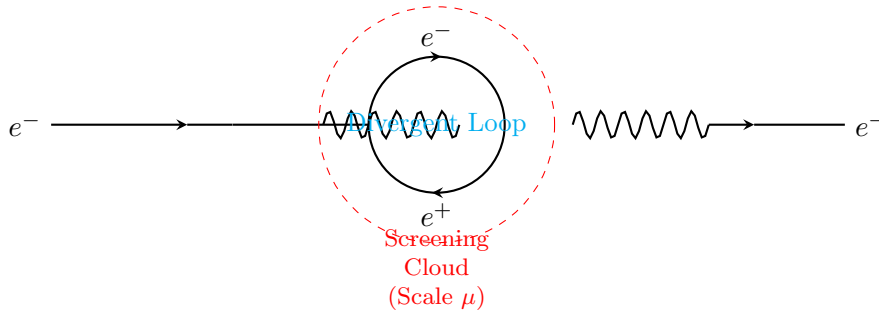


FIGURE 1. Feynman diagram representing vacuum polarization. The virtual loop acts as the physical counterpart to the mathematical pole singularity  $1/\epsilon$ , screening the bare charge at scale  $\mu$ .

**1.2.1. The Loop Integral in  $D = 4$ .** In exact four-dimensional spacetime, the Feynman rules associate to this loop the following integral in Euclidean momentum space:

$$(1) \quad I = \int \frac{d^4 k}{(2\pi)^4} \frac{1}{(k^2 + m^2)((k-p)^2 + m^2)}$$

where  $p$  is the external momentum entering the loop,  $k$  is the internal loop momentum, and  $m$  is the particle mass.

To evaluate the behavior at high momenta (Ultraviolet, UV), we perform a power-counting analysis for large values of  $k$  ( $k \gg p, m$ ):

- The volume element in 4D spherical coordinates scales as  $d^4 k \sim k^3 dk$ .
- The product of the two propagators in the denominator scales as  $k^2 \cdot k^2 = k^4$ .

Consequently, the integral behaves asymptotically as:

$$(2) \quad I \sim \int_0^\infty \frac{k^3}{k^4} dk = \int_0^\infty \frac{1}{k} dk = \ln(\infty)$$

The integral is *logarithmically divergent* in  $D = 4$  and lacks a well-defined numerical value.

To make the core mathematical mechanism transparent, we strip away the kinematical complexities and model this divergent loop using a simplified 1D improper integral  $\int_1^\infty \frac{1}{x} dx$ , which exhibits

the exact same logarithmic UV divergence as  $x \rightarrow \infty$ . Combinatorially, we represent this simple primitive loop as a graph  $L$  consisting of two vertices connected by two directed edges:

$$(3) \quad L = \begin{array}{c} \bullet \quad \bullet \\ \curvearrowright \quad \curvearrowleft \end{array}$$

We shall now go through the explicit process of regularization and renormalization, step by step.

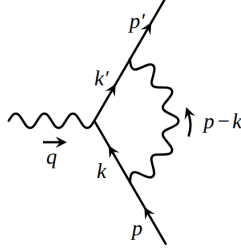


FIGURE 2. Another instance of a loop: as all possibilities must be considered, one integrates over all  $k$ .

1.2.2. *Deformation of the Exponent and Introduction of  $\mu$ .* Following the framework of Dimensional Regularization, we deform the integrand's exponent by introducing a small parameter  $\epsilon > 0$ :

$$(4) \quad \frac{1}{x} \longrightarrow \frac{1}{x^{1+\epsilon}}$$

Since modifying the exponent alters the analytical dimension of the integrand, we introduce an arbitrary mass scale parameter  $\mu > 0$  raised to the power  $\epsilon$  to maintain dimensional consistency:

$$(5) \quad I(\epsilon) = \mu^\epsilon \int_1^\infty \frac{1}{x^{1+\epsilon}} dx$$

1.2.3. *Evaluating the Regularized Integral.* Since  $\epsilon > 0$ , the integral now converges absolutely at the upper limit:

$$(6) \quad I(\epsilon) = \mu^\epsilon \left[ \frac{x^{-\epsilon}}{-\epsilon} \right]_1^\infty = \mu^\epsilon \left( 0 - \frac{1}{-\epsilon} \right) = \frac{\mu^\epsilon}{\epsilon} = \frac{1}{\epsilon} \mu^\epsilon$$

Notice how the unmanageable  $\ln(\infty)$  divergence has been mapped precisely into the isolated simple pole  $\frac{1}{\epsilon}$ .

1.2.4. *Laurent Series Expansion.* To isolate the pole and analyze the structure of the singularity near  $\epsilon = 0$ , we re-express  $\mu^\epsilon$  using its exponential Taylor expansion  $e^y = 1 + y + \frac{y^2}{2!} + \dots$ :

$$(7) \quad \mu^\epsilon = \exp(\epsilon \ln \mu) = 1 + \epsilon \ln \mu + \frac{\epsilon^2}{2} \ln^2 \mu + \mathcal{O}(\epsilon^3)$$

Substituting this back into  $I(\epsilon)$  yields:

$$(8) \quad I(\epsilon) = \frac{1}{\epsilon} \left[ 1 + \epsilon \ln \mu + \frac{\epsilon^2}{2} \ln^2 \mu + \mathcal{O}(\epsilon^3) \right]$$

Distributing the factor  $\frac{1}{\epsilon}$ , we arrive at the Laurent series expansion in  $\epsilon$ :

$$(9) \quad I(\epsilon) = \underbrace{\frac{1}{\epsilon}}_{\text{Divergent Part (Pole)}} + \underbrace{\ln \mu}_{\text{Finite Part}} + \underbrace{\frac{\epsilon}{2} \ln^2 \mu + \mathcal{O}(\epsilon^2)}_{\text{Vanishing Terms as } \epsilon \rightarrow 0}$$

To summarize our progress thus far:

- (1) **Regularization:** We transformed the divergent integral  $I$  into a well-defined analytical function  $I(\epsilon)$ .
- (2) **Laurent Decomposition:** We uniquely decomposed  $I(\epsilon)$  into:
  - A pure pole/divergent piece (belonging to  $\mathcal{A}_-$  in the language of Connes–Kreimer):  $\frac{1}{\epsilon}$ .
  - A holomorphic/finite piece (belonging to  $\mathcal{A}_+$ ):  $\ln \mu + \mathcal{O}(\epsilon)$ .

1.2.5. *The Renormalization Step.* Having regularized the integral, we proceed to renormalization. In the Minimal Subtraction (**MS**) scheme, the principle is to define a counter-term  $I_{\text{counter}}$  that is precisely the negative of the pure divergent pole, leaving the finite part untouched:

$$(10) \quad I_{\text{counter}}(\epsilon) = -\frac{1}{\epsilon}$$

The renormalized integral  $I_{\text{ren}}(\epsilon)$  is defined by adding this counter-term to the regularized expression:

$$(11) \quad I_{\text{ren}}(\epsilon) = I(\epsilon) + I_{\text{counter}}(\epsilon) = \left( \frac{1}{\epsilon} + \ln \mu + \mathcal{O}(\epsilon) \right) - \frac{1}{\epsilon}$$

The singular poles cancel identically in an algebraic fashion:

$$(12) \quad I_{\text{ren}}(\epsilon) = \ln \mu + \mathcal{O}(\epsilon)$$

With the divergence removed, we can safely remove the regularization regulator by taking the limit  $\epsilon \rightarrow 0$ :

$$(13) \quad I_{\text{ren}} = \lim_{\epsilon \rightarrow 0} I_{\text{ren}}(\epsilon) = \ln \mu$$

1.2.6. *The Physical Meaning of the Subtraction.* Let us reflect on the physical insight behind this calculation. The original divergent integral possessed no absolute numerical value; rather, its renormalized value depends explicitly on the mass scale  $\mu$  introduced during regularization. This is a direct manifestation of the **Renormalization Group**: physical observables depend inherently on the energy scale at which they are probed.

The parameter  $\mu$  represents the energy scale at which a physical experiment is conducted. In nature, no measurement takes place in an abstract vacuum; observations are tied to a specific energy scale  $\mu$  (such as collider center-of-mass energy or system temperature).

Re-examining our Laurent expansion:

$$(14) \quad I(\epsilon) = \underbrace{\frac{1}{\epsilon}}_{\text{Scale-independent Pole}} + \underbrace{\ln \mu}_{\text{Scale-dependent Physical Term}} + \mathcal{O}(\epsilon)$$

- **The term  $\ln \mu$  carries physical dynamics:** It governs how process amplitudes vary when changing the experimental energy scale  $\mu$ . Differentiating with respect to  $\ln \mu$  yields the Renormalization Group equations (such as the Callan–Symanzik equation), predicting how coupling constants run with energy.
- **The pole  $1/\epsilon$  contains no physical dynamics:** It is a static, rigid artifact that is insensitive to  $\mu$ . It carries no information about the scale of observation, acting purely as a kinematic pathology resulting from the unphysical assumption that our field theory holds up to infinitely short distances.

Therefore, subtracting  $1/\epsilon$  is not an arbitrary trick to hide infinities; it is the systematic removal of a non-dynamical mathematical artifact, preserving the true physical content: the functional dependence on the scale  $\mu$ .

In summary: while divergence is the analytical symptom, the lack of an energy scale ( $\mu$ ) is the ontological reason why the pole does not belong to observable physics and must be absorbed into bare parameters.

**1.3. The Mathematical Perspective: Hopf Algebra and Birkhoff Decomposition.** We now map this analytical subtraction process directly onto the algebraic structure of the Connes–Kreimer Hopf algebra.

The Connes–Kreimer Hopf algebra of Feynman graphs  $\mathcal{H}$ , our simple primitive loop from (3) is represented by a generator  $L$ . Since  $L$  contains no internal sub-divergences (i.e., it has no proper subgraphs that are themselves divergent),  $L$  is a **primitive element**:

$$\Delta(L) = L \otimes 1 + 1 \otimes L$$

By the Hopf algebra axioms for primitive elements, its antipode  $S : \mathcal{H} \rightarrow \mathcal{H}$  satisfies:

$$S(L) = -L$$

Feynman rules combined with dimensional regularization are represented mathematically as an algebra homomorphism (a character):

$$\phi : \mathcal{H} \longrightarrow \mathcal{A}$$

where  $\mathcal{A}$  is the algebra of Laurent series in  $\epsilon$ . As evaluated analytically for our model integral:

$$(15) \quad \phi(L) = I(\epsilon) = \frac{1}{\epsilon} + \ln \mu + \mathcal{O}(\epsilon) \in \mathcal{A}$$

The algebra  $\mathcal{A}$  admits a unique vector space direct sum decomposition:

$$\mathcal{A} = \mathcal{A}_- \oplus \mathcal{A}_+$$

where  $\mathcal{A}_- = \epsilon^{-1}\mathbb{C}[\epsilon^{-1}]$  consists of pure pole series (divergent parts) and  $\mathcal{A}_+ = \mathbb{C}[[\epsilon]]$  consists of regular power series (holomorphic at  $\epsilon = 0$ ). Let  $R : \mathcal{A} \rightarrow \mathcal{A}_-$  be the Rota–Baxter projection operator onto the pole part (corresponding to the **MS** scheme):

$$R\left(\frac{1}{\epsilon} + \ln \mu + \mathcal{O}(\epsilon)\right) = \frac{1}{\epsilon}$$

The algebraic Birkhoff decomposition ensures that the character  $\phi$  factorizes under the convolution product as:

$$(16) \quad \phi = \phi_-^{-1} * \phi_+ \implies \phi_+ = \phi_- * \phi$$

where  $\phi_- : \mathcal{H} \rightarrow \mathcal{A}_-$  is the counter-term character. For any element  $X \in \mathcal{H}$ , the Connes–Kreimer recursive formula for  $\phi_-(X)$  combines the antipode with the projector  $R$ :

$$\phi_-(X) = -R \left[ \phi(X) + \sum_{(X)} \phi_-(X') \phi(X'') \right]$$

Since  $\Delta(L) = L \otimes 1 + 1 \otimes L$ , the reduced sum over proper subgraphs  $\sum_{(L)} X' \otimes X''$  is empty, so:

$$\phi_-(L) = -R[\phi(L)]$$

Substituting our analytical evaluation  $\phi(L) = I(\epsilon)$ :

$$\phi_-(L) = -R \left[ \frac{1}{\epsilon} + \ln \mu + \mathcal{O}(\epsilon) \right] = -\frac{1}{\epsilon}$$

This connects the two frameworks:  $\phi_-(L)$  matches the physical counter-term  $I_{\text{counter}}(\epsilon) = -\frac{1}{\epsilon}$ .

In the Connes–Kreimer formalism, the counter-term character  $\phi_-$  can be written as a twisted composition using the antipode  $S$ :

$$\phi_-(L) = (R \circ \phi \circ S)(L)$$

Verifying this step by step:

- (1) Apply the antipode to the loop element:  $S(L) = -L$ .
- (2) Apply the Feynman character:  $\phi(S(L)) = \phi(-L) = -\phi(L) = -\frac{1}{\epsilon} - \ln \mu - \mathcal{O}(\epsilon)$ .

(3) Apply the projection  $R$  to extract the singular part in  $\mathcal{A}_-$ :

$$R(\phi(S(L))) = R\left(-\frac{1}{\epsilon} - \ln \mu - \mathcal{O}(\epsilon)\right) = -\frac{1}{\epsilon}$$

Therefore:

$$\phi_-(L) = (R \circ \phi \circ S)(L) = -\frac{1}{\epsilon} = I_{\text{counter}}(\epsilon)$$

The algebraic antipode  $S(L) = -L$  is the underlying algebraic operation responsible for generating the negative sign of the physical counterterm.

Finally, we compute the renormalized character  $\phi_+(L)$  using the convolution product  $*$ :

$$\phi_+(L) = (\phi_- * \phi)(L) = m_{\mathcal{A}}\left((\phi_- \otimes \phi)\Delta(L)\right)$$

Expanding over the coproduct  $\Delta(L) = L \otimes 1 + 1 \otimes L$ :

$$\phi_+(L) = m_{\mathcal{A}}\left(\phi_-(L) \otimes \phi(1) + \phi_-(1) \otimes \phi(L)\right) = \phi_-(L)\phi(1) + \phi_-(1)\phi(L)$$

Since  $\phi$  and  $\phi_-$  are algebra homomorphisms, they map the unit element  $1 \in \mathcal{H}$  to  $1 \in \mathcal{A}$ :

$$\phi_+(L) = \phi_-(L) \cdot 1 + 1 \cdot \phi(L) = \phi_-(L) + \phi(L)$$

Substituting our analytical expressions:

$$\phi_+(L) = \left(-\frac{1}{\epsilon}\right) + \left(\frac{1}{\epsilon} + \ln \mu + \mathcal{O}(\epsilon)\right) = \ln \mu + \mathcal{O}(\epsilon)$$

Evaluating at  $\epsilon \rightarrow 0$  (the evaluation in the holomorphic part  $\mathcal{A}_+$ ):

$$\lim_{\epsilon \rightarrow 0} \phi_+(L) = \ln \mu = I_{\text{ren}}$$

This illustrates how Birkhoff's algebraic factorization matches the physical subtraction procedure.

## 2. ALGEBRAIC FOUNDATIONS AND CORE STRUCTURES

The goal of this class is to establish the foundational vocabulary of Hopf algebra theory. We build from the geometric intuition of the tensor product to the axioms of coalgebras, bialgebras, and antipodes, grounding each definition in classical ring theory and preparing the algebraic machinery required for subsequent classes.

**2.1. A Concise Review of the Tensor Product.** Let  $k$  be a field. All vector spaces, linear maps, and tensor products are assumed to be over  $k$  unless specified otherwise.

To move beyond classical linear combinations, we require a space capable of linearizing bilinear maps. Given two vector spaces  $V$  and  $W$ , their **tensor product**  $V \otimes W$  is defined as the vector space generated by the formal symbols  $v \otimes w$  (where  $v \in V, w \in W$ ) modulo the relations dictating bilinearity:

$$\begin{aligned} (v_1 + v_2) \otimes w &= v_1 \otimes w + v_2 \otimes w, \\ v \otimes (w_1 + w_2) &= v \otimes w_1 + v \otimes w_2, \\ \alpha(v \otimes w) &= (\alpha v) \otimes w = v \otimes (\alpha w), \quad \forall \alpha \in k. \end{aligned}$$

*Remark 2.1* (The Universal Property). If  $\beta : V \times W \rightarrow U$  is a bilinear map into a vector space  $U$ , there exists a unique linear map  $\tilde{\beta} : V \otimes W \rightarrow U$  such that  $\tilde{\beta}(v \otimes w) = \beta(v, w)$  for all  $v \in V, w \in W$ .

If  $\{v_i\}_{i \in I}$  is a basis for  $V$  and  $\{w_j\}_{j \in J}$  is a basis for  $W$ , then the set of pure tensors  $\{v_i \otimes w_j\}_{(i,j) \in I \times J}$  forms a basis for  $V \otimes W$ . We define the linear flip operator  $\tau : V \otimes W \rightarrow W \otimes V$  on pure tensors as  $\tau(v \otimes w) = w \otimes v$ .

**2.2. Dual Structures: Algebras and Coalgebras.** We define an associative algebra not by elements, but by linear maps over tensor products, allowing us to cleanly dualize the axioms into the definition of a coalgebra.

**Definition 2.2** (Algebra). An associative algebra over  $k$  is a triple  $(A, m, u)$ , where  $A$  is a  $k$ -vector space,  $m : A \otimes A \rightarrow A$  is a linear multiplication map, and  $u : k \rightarrow A$  is a linear unit map, satisfying the associativity and unit conditions:

$$(17) \quad m \circ (m \otimes id_A) = m \circ (id_A \otimes m), \quad m \circ (u \otimes id_A) = id_A = m \circ (id_A \otimes u).$$

Reversing the direction of all structural arrows yields the dual framework:

**Definition 2.3** (Coalgebra). A coalgebra over  $k$  is a triple  $(C, \Delta, \varepsilon)$ , where  $C$  is a  $k$ -vector space,  $\Delta : C \rightarrow C \otimes C$  is a linear *comultiplication* map, and  $\varepsilon : C \rightarrow k$  is a linear *counit* map, satisfying the coassociativity and counit conditions:

$$(18) \quad (\Delta \otimes id_C) \circ \Delta = (id_C \otimes \Delta) \circ \Delta, \quad (\varepsilon \otimes id_C) \circ \Delta = id_C = (id_C \otimes \varepsilon) \circ \Delta.$$

**Sweedler Notation:** Because writing explicit elements in  $C \otimes C$  is cumbersome, we use Sweedler's notation for the image of an element  $c \in C$  under comultiplication:

$$(19) \quad \Delta(c) = \sum_{(c)} c_{(1)} \otimes c_{(2)}$$

Coassociativity is beautifully written in this notation as

$$\sum (c_{(1)})_{(1)} \otimes (c_{(1)})_{(2)} \otimes c_{(2)} = \sum c_{(1)} \otimes (c_{(2)})_{(1)} \otimes (c_{(2)})_{(2)} \equiv \sum c_{(1)} \otimes c_{(2)} \otimes c_{(3)}.$$

The counit axiom reads  $\sum \varepsilon(c_{(1)})c_{(2)} = c = \sum c_{(1)}\varepsilon(c_{(2)})$ .

**Example 2.4** (The Running Example: The Group Algebra  $kG$ ). Let  $G$  be a finite group. The group algebra  $kG$  consists of formal linear combinations of elements of  $G$ .

- **Algebra structure:**  $m(g \otimes h) = gh$  and  $u(1_k) = 1_G$  for  $g, h \in G$ .

- **Coalgebra structure:** We make  $kG$  a coalgebra by setting  $\Delta(g) = g \otimes g$  and  $\varepsilon(g) = 1$  for all  $g \in G$ . Elements satisfying  $\Delta(x) = x \otimes x$  and  $\varepsilon(x) = 1$  are called *group-like elements*.

**2.3. Bialgebras and the Convolution Algebra.** A bialgebra merges these structural layers by demanding that the coalgebra maps behave compatibly with the algebra maps.

**Definition 2.5** (Bialgebra). A bialgebra  $(H, m, u, \Delta, \varepsilon)$  is a vector space  $H$  equipped with an algebra structure  $(H, m, u)$  and a coalgebra structure  $(H, \Delta, \varepsilon)$  such that  $\Delta$  and  $\varepsilon$  are algebra homomorphisms.

Using Sweedler notation, the assertion that  $\Delta$  preserves multiplication translates to the explicit compatibility identity for any  $a, b \in H$ :

$$(20) \quad \Delta(ab) = \Delta(a)\Delta(b) = \sum_{(a),(b)} a_{(1)}b_{(1)} \otimes a_{(2)}b_{(2)}$$

Additionally, we must have  $\Delta(1) = 1 \otimes 1$ . One can easily verify that the group algebra  $kG$  forms a valid bialgebra under these axioms.

To transition to Hopf algebras, we must first construct an algebraic framework on the space of linear maps themselves.

**Definition 2.6** (The Convolution Product). Let  $(C, \Delta, \varepsilon)$  be a coalgebra and  $(A, m, u)$  be an algebra. The vector space  $\text{Hom}_k(C, A)$  becomes an associative algebra under the *convolution product*  $*$  defined for any  $f, g \in \text{Hom}_k(C, A)$  by:

$$(21) \quad (f * g)(c) = m \circ (f \otimes g) \circ \Delta(c) = \sum_{(c)} f(c_{(1)})g(c_{(2)})$$

The identity element of this convolution algebra is the map  $u \circ \varepsilon : C \rightarrow A$ .

**2.4. The Antipode and Hopf Algebras.** A Hopf algebra is a bialgebra whose identity map possesses a unique generalized inverse within the convolution algebra.

**Definition 2.7** (Hopf Algebra). A Hopf algebra is a bialgebra  $H$  equipped with a linear map  $S : H \rightarrow H$ , called the *antipode*, which satisfies:

$$(22) \quad S * \text{id}_H = \text{id}_H * S = u \circ \varepsilon,$$

namely  $S = \text{id}^{-1}$  in  $\text{Hom}(H, H)$  with the product  $*$ .

Expressed explicitly in Sweedler notation for any  $x \in H$ :

$$(23) \quad \sum_{(x)} S(x_{(1)})x_{(2)} = \varepsilon(x)1 = \sum_{(x)} x_{(1)}S(x_{(2)})$$

For our running example  $kG$ , the antipode condition requires  $S(g)g = \varepsilon(g)1 = 1$ . Thus, the antipode is strictly determined by the group inversion:  $S(g) = g^{-1}$ .

**2.5. Connected bialgebras.** The following fact will be of importance later on. A grading  $H = \bigoplus_{n \geq 0} H_n$  on a bialgebra  $H$  is a decomposition in linear subspaces  $(H_n)_{n \geq 0}$  so that both the algebra and coalgebra structures are preserved, namely for  $n, m \geq 0$ :

$$H_n H_m \subseteq H_{n+m}, \quad \Delta(H_n) \subseteq \sum_{i+j=n} H_i \otimes H_j.$$

We say that a graded bialgebra  $H = \bigoplus_{n \geq 0} H_n$  is connected when  $H_0 \cong k$ .

**Theorem 2.8.** Let  $H = \bigoplus_{n \geq 0} H_n$  be a graded bialgebra over  $k$ . If  $H$  is connected, then  $H$  uniquely admits an antipode  $S : H \rightarrow H$ , making it a Hopf algebra.

*Proof.* The antipode must satisfy  $m(S \otimes id)\Delta = u\varepsilon$ . For  $x \in H_n$  ( $n > 0$ ),  $\varepsilon(x) = 0$ . The coproduct is  $\Delta(x) = x \otimes 1 + 1 \otimes x + \sum_{(x)} x' \otimes x''$ . Applying the axiom yields  $S(x)1 + S(1)x + \sum_{(x)} S(x')x'' = 0$ . Isolating  $S(x)$  provides the recursion:

$$(24) \quad S(x) = -x - \sum_{(x)} S(x')x''$$

Since  $x'$  has strictly lower degree than  $x$ ,  $S(x)$  is uniquely determined.

We have used only one of the axioms determining the antipode: which led us to define a *left* inverse  $S = S_L$  to  $id$  in the convolution algebra. If we chose to use the other axiom, we find a right inverse  $S_R$  which necessarily coincides with ours; hence the antipode is unique and does not depend on which axiom we choose to define it recursively.  $\square$

**2.6. Essential Structural Examples.** To demonstrate the versatility of the theory beyond group structures, we introduce some extra examples displaying different behaviors of the coproduct and antipode.

**2.6.1. The algebra of functions  $k^G$ .** Let  $G$  be a finite (non abelian) group. The vector space of functions  $k^G = \{f: G \rightarrow k\}$  admits an algebra structure with pointwise multiplication

$$(\phi\psi)(g) = \phi(g)\psi(g)$$

and unit  $\epsilon$ . It has a basis of idempotents  $(\delta_g)_{g \in G}$  so that  $\delta_g(h) = \delta_{g,h}$  and the following defines a coproduct on it:

$$\Delta(\delta_g) = \sum_{h \in G} \delta_{gh^{-1}} \otimes \delta_h$$

It is a Hopf algebra, with  $\epsilon(\delta_g) = \delta_{g,e}$  and antipode  $S(\delta_g) = \delta_{g^{-1}}$ .

**2.6.2. The Universal Enveloping Algebra  $U(\mathfrak{g})$ .** Let  $\mathfrak{g}$  be a Lie algebra over  $k$  with bracket  $[\cdot, \cdot]$ . The universal enveloping algebra  $U(\mathfrak{g})$  is the free associative algebra generated by the vector space  $\mathfrak{g}$  modulo the ideal generated by elements of the form  $xy - yx - [x, y]$ . We extend  $\mathfrak{g}$  to a Hopf algebra structure by defining the actions on elements  $x \in \mathfrak{g}$ :

$$(25) \quad \Delta(x) = x \otimes \mathbf{1} + \mathbf{1} \otimes x, \quad \varepsilon(x) = 0, \quad S(x) = -x.$$

Elements satisfying  $\Delta(x) = x \otimes \mathbf{1} + \mathbf{1} \otimes x$  are called *primitive elements*.  $U(\mathfrak{g})$  is an infinite-dimensional, cocommutative Hopf algebra.

**2.6.3. Cocommutative Hopf algebras over  $\mathbb{C}$ .** A Hopf algebra is called cocommutative if  $\tau\Delta = \Delta$ , for  $\tau(x \otimes y) = y \otimes x$ . The following theorem shows that the previous examples are quite ubiquitous.

**Theorem 2.9** (Cartier-Constant, Milnor Moore). *Let  $H$  be a cocommutative Hopf algebra over  $\mathbb{C}$ . Then there are a group  $G$  and a Lie algebra  $\mathfrak{g}$  so that  $H \simeq U(\mathfrak{g}) \rtimes \mathbb{C}G$ .*

**Corollary 2.10.** *Let  $H$  be a commutative finite-dimensional Hopf algebra over  $\mathbb{C}$ . Then there is a group  $G$  so that  $H \simeq k^G$ .*

*Proof.*  $H^*$  is cocommutative.  $\square$

**2.6.4. The Taft Algebra  $H_n(q)$ .** Our last example in this section is a non-commutative, and non-cocommutative Hopf algebra.

Let  $q \in k$  be a primitive  $n$ -th root of unity (requiring  $n \geq 2$ ). The Taft algebra  $H_n(q)$  is generated as an algebra by two elements,  $g$  and  $x$ , subject to the relations:

$$(26) \quad g^n = \mathbf{1}, \quad x^n = 0, \quad xg = qgx.$$

The Hopf algebra structure is given on generators by:

$$\Delta(g) = g \otimes g, \quad \varepsilon(g) = 1, \quad S(g) = g^{-1} = g^{n-1}$$

$$\Delta(x) = x \otimes \mathbf{1} + g \otimes x, \quad \varepsilon(x) = 0, \quad S(x) = -g^{-1}x = -q^{-1}g^{n-1}x.$$

Elements like  $x$  satisfying  $\Delta(x) = x \otimes \mathbf{1} + g \otimes x$  are known as *skew-primitive elements*.  $H_n(q)$  is a finite-dimensional ( $\dim = n^2$ ), non-commutative, and non-cocommutative Hopf algebra.

**2.7. Module Structures and Tensor Dualization.** We conclude this class by showing that the Hopf structure is categorically indispensable. In standard ring theory, the tensor product of two modules over a ring is not natively a module over that same ring. The coproduct and antipode resolve this limitation entirely.

**2.7.1. Tensor Products of Modules.** Let  $H$  be a bialgebra, and let  $M$  and  $N$  be left  $H$ -modules. We can equip the vector space  $M \otimes N$  with a left  $H$ -module action by pulling the operation back along the comultiplication map:

$$(27) \quad h \cdot (m \otimes n) = \Delta(h) \cdot (m \otimes n) = \sum_{(h)} (h_{(1)} \cdot m) \otimes (h_{(2)} \cdot n)$$

Coassociativity guarantees that for three modules,  $(M \otimes N) \otimes K \cong M \otimes (N \otimes K)$  as  $H$ -modules.

Additionally, the base field  $k$  becomes an  $H$ -module via the counit  $\epsilon$ :

$$h \cdot 1 = \epsilon(h)1, \quad h \in H.$$

We denote this module as  $\mathbb{1}$ , as it becomes a unit for the tensor product, in the sense that

$$\mathbb{1} \otimes M \simeq M, \quad M \otimes \mathbb{1} \simeq M,$$

as  $H$ -modules, for any  $H$ -module  $M$ .

**2.7.2. Dual Modules.** Let  $M$  be a left  $H$ -module. The dual vector space  $M^* = \text{Hom}_k(M, k)$  cannot easily be given a left  $H$ -module structure without an anti-homomorphism, because acting on the argument reverses the composition order. The antipode provides exactly this mechanism.

**Proposition 2.11.** *Let  $H$  be a Hopf algebra with antipode  $S$ . For any left  $H$ -module  $M$ , the dual space  $M^*$  becomes a left  $H$ -module under the action:*

$$(28) \quad (h \cdot f)(m) = f(S(h) \cdot m) \quad \forall h \in H, f \in M^*, m \in M.$$

*Proof.* We must verify the associative action requirement:  $(hk) \cdot f = h \cdot (k \cdot f)$  for all  $h, k \in H$ . Recall that the antipode  $S$  is always an algebra anti-homomorphism, meaning  $S(hk) = S(k)S(h)$ . Evaluating the left-hand action on an element  $m \in M$ :

$$(29) \quad ((hk) \cdot f)(m) = f(S(hk) \cdot m) = f((S(k)S(h)) \cdot m)$$

Now evaluating the right-hand formulation:

$$(30) \quad (h \cdot (k \cdot f))(m) = (k \cdot f)(S(h) \cdot m) = f(S(k) \cdot (S(h) \cdot m)) = f((S(k)S(h)) \cdot m)$$

Both expressions are identical, confirming that  $M^*$  is a well-defined left  $H$ -module.  $\square$

This structural closure—the ability to form tensor products of representations and construct their dual representatives—establishes the category of modules over a Hopf algebra as a tensor (monoidal) category with duals, mirroring the classical representations of finite groups.

### 3. THE HOPF ALGEBRA OF FEYNMAN GRAPHS

The Connes–Kreimer construction attaches to a class of graphs a commutative connected graded Hopf algebra whose generators are the one-particle irreducible graphs and whose coproduct records the subgraphs which may be contracted, together with the graphs obtained by contracting them. We work over a fixed field  $k$  of characteristic zero; the combinatorics is insensitive to this choice, and one may take  $k = \mathbb{Q}$ .

Everything below is purely combinatorial. In the physical theory one equips graphs with a superficial degree of divergence  $\omega$  and with a set of allowed external structures (*residues*), and one contracts a divergent subgraph to the elementary vertex or propagator carrying its external structure. Here we make the two simplest possible choices:

- a subgraph counts as *divergent* exactly when it is one-particle irreducible and contains at least one internal edge, equivalently when it contains a loop;
- the *contraction* of such a subgraph collapses each of its connected components to a single vertex.

The point of Sections 3.4 and 3.5 is that with these choices the closure properties which are usually *imposed* on the class of graphs become *theorems*.

#### 3.1. Half-edge graphs.

**Definition 3.1.** A *finite half-edge graph* is a quadruple

$$\Gamma = (H, V, s, \iota),$$

where

- $H$  is a finite set of *half-edges*;
- $V$  is a finite set of *vertices*;
- $s : H \rightarrow V$  is the *incidence map*;
- $\iota : H \rightarrow H$  is an involution,  $\iota^2 = \text{id}$ .

The two-element orbits of  $\iota$  are the *internal edges* of  $\Gamma$ ; the fixed points of  $\iota$  are the *external half-edges*. Thus

$$E_{\text{int}}(\Gamma) = \{\{h, \iota(h)\} : h \in H, \iota(h) \neq h\}, \quad E_{\text{ext}}(\Gamma) = \{h \in H : \iota(h) = h\}.$$

We write

$$I(\Gamma) = |E_{\text{int}}(\Gamma)|, \quad V(\Gamma) = |V|, \quad N(\Gamma) = |E_{\text{ext}}(\Gamma)|.$$

The *underlying graph* of  $\Gamma$  has vertex set  $V$  and one edge joining  $s(h)$  and  $s(\iota(h))$  for every internal edge  $\{h, \iota(h)\}$ . Multiple edges and self-loops (internal edges  $\{h, \iota(h)\}$  with  $s(h) = s(\iota(h))$ ) are allowed. Connectedness always refers to the underlying graph; in particular a graph with no vertices is not connected, and a single vertex carrying any number of external half-edges is.

**Definition 3.2.** The *loop number* of a connected half-edge graph  $\Gamma$  is

$$L(\Gamma) = I(\Gamma) - V(\Gamma) + 1.$$

For a disjoint union,  $L$  is defined to be the sum of the loop numbers of the components; equivalently  $L(\Gamma) = I(\Gamma) - V(\Gamma) + c(\Gamma)$ , where  $c(\Gamma)$  is the number of connected components.

Note that  $L$  does not see the external half-edges.

#### 3.2. One-particle irreducibility.

**Definition 3.3.** Let  $e = \{h, \iota(h)\}$  be an internal edge of  $\Gamma$ . The graph  $\Gamma \setminus e$  has the same  $H$ ,  $V$  and  $s$ , and involution  $\iota'$  obtained from  $\iota$  by declaring  $h$  and  $\iota(h)$  to be fixed points. Thus  $\Gamma \setminus e$  is obtained by cutting  $e$  and keeping its two half-edges as external half-edges.

A connected half-edge graph  $\Gamma$  is *one-particle irreducible* (1PI) if  $\Gamma \setminus e$  is connected for every  $e \in E_{\text{int}}(\Gamma)$ . A disconnected graph is called 1PI if each of its connected components is.

So a connected graph is 1PI precisely when its underlying graph is bridgeless. A single vertex with external half-edges and no internal edge is vacuously 1PI; these graphs, which we call *corollas*, will be excluded from the generators.

**Lemma 3.4.** *Let  $\Gamma$  be a connected 1PI graph. Then*

$$L(\Gamma) \geq 1 \iff I(\Gamma) \geq 1,$$

*and  $L(\Gamma) = 0$  if and only if  $\Gamma$  is a corolla.*

*Proof.* If  $I(\Gamma) = 0$  then connectedness forces  $V(\Gamma) = 1$ , so  $\Gamma$  is a corolla and  $L(\Gamma) = 0$ . Conversely suppose  $I(\Gamma) \geq 1$  and  $L(\Gamma) = 0$ , i.e.  $I(\Gamma) = V(\Gamma) - 1$ . A connected graph with  $V - 1$  edges is a tree, and in a tree every edge is a bridge; cutting any one of them disconnects  $\Gamma$ , contradicting one-particle irreducibility. Finally  $L(\Gamma) \geq 0$  always holds for connected  $\Gamma$ , so  $I(\Gamma) \geq 1$  gives  $L(\Gamma) \geq 1$ .  $\square$

This is the first place where the physical input disappears: *for 1PI graphs, “contains a loop” and “is not a corolla” are the same condition*, and we may use either as the definition of divergence.

**Definition 3.5.** Let  $\mathcal{G}$  denote the set of isomorphism classes of connected 1PI half-edge graphs  $\Gamma$  with  $L(\Gamma) \geq 1$ .

### 3.3. Subgraphs.

**Definition 3.6.** Let  $\Gamma = (H, V, s, \iota)$  be a half-edge graph. A *subgraph*  $\gamma \subseteq \Gamma$  is determined by a subset

$$E_{\text{int}}(\gamma) \subseteq E_{\text{int}}(\Gamma).$$

Explicitly, let

$$V_\gamma = \{s(h) : \{h, \iota(h)\} \in E_{\text{int}}(\gamma)\}, \quad H_\gamma = s^{-1}(V_\gamma),$$

and let  $\iota_\gamma : H_\gamma \rightarrow H_\gamma$  be the involution which agrees with  $\iota$  on the half-edges of the selected internal edges and fixes every other element of  $H_\gamma$ . Then  $\gamma = (H_\gamma, V_\gamma, s|_{H_\gamma}, \iota_\gamma)$ .

Thus a subgraph retains *all* half-edges incident to its vertices; those not paired inside  $\gamma$  become external half-edges of  $\gamma$ . We write  $\gamma \subsetneq \Gamma$  when  $E_{\text{int}}(\gamma) \subsetneq E_{\text{int}}(\Gamma)$ , and  $\gamma = \emptyset$  for the subgraph with no internal edges (hence no vertices). A subgraph has no isolated vertices, and subgraphs are partially ordered by inclusion of their internal edge sets, the poset being the Boolean lattice  $2^{E_{\text{int}}(\Gamma)}$ .

*Remark 3.7.* If  $\alpha \subseteq \gamma \subseteq \Gamma$ , then the subgraph  $\alpha$  of  $\gamma$  and the subgraph  $\alpha$  of  $\Gamma$  coincide. Indeed  $V_\alpha \subseteq V_\gamma$  and  $\gamma$  contains every half-edge of  $\Gamma$  incident to a vertex of  $V_\gamma$ , so the two constructions produce the same half-edge set  $s^{-1}(V_\alpha)$ . Consequently all properties of  $\alpha$  used below — connectedness, one-particle irreducibility, loop number — are intrinsic and do not depend on the ambient graph.

We write  $\gamma = \gamma_1 \sqcup \dots \sqcup \gamma_r$  for the decomposition of a subgraph into connected components; these are again subgraphs of  $\Gamma$ .

### 3.4. Contraction.

**Definition 3.8.** Let  $\gamma \subseteq \Gamma$  be a subgraph with connected components  $\gamma_1, \dots, \gamma_r$  and vertex sets  $V_{\gamma_1}, \dots, V_{\gamma_r}$ . Let

$$H^\gamma = H \setminus \bigcup_{\{h, \iota(h)\} \in E_{\text{int}}(\gamma)} \{h, \iota(h)\}$$

be the set of half-edges of  $\Gamma$  which do not belong to a selected internal edge. The *contraction*  $\Gamma/\gamma$  is the half-edge graph with

$$\begin{aligned} \text{half-edges} & \quad H^\gamma, \\ \text{vertices} & \quad \left(V \setminus \bigsqcup_i V_{\gamma_i}\right) \sqcup \{v_1, \dots, v_r\}, \end{aligned}$$

$$\begin{aligned} \text{incidence} \quad h &\longmapsto \begin{cases} v_i, & s(h) \in V_{\gamma_i}, \\ s(h), & \text{otherwise,} \end{cases} \\ \text{involution} \quad \iota|_{H^\gamma}. \end{aligned}$$

The set  $H^\gamma$  is  $\iota$ -stable, so the last line makes sense. In words: each connected component of  $\gamma$  is collapsed to a single new vertex  $v_i$ , and every half-edge which was attached to a vertex of  $\gamma_i$  and not paired inside  $\gamma_i$  is reattached to  $v_i$ . By construction

$$E_{\text{int}}(\Gamma/\gamma) = E_{\text{int}}(\Gamma) \setminus E_{\text{int}}(\gamma), \quad E_{\text{ext}}(\Gamma/\gamma) = E_{\text{ext}}(\Gamma).$$

In particular contraction never creates or destroys external half-edges: they play no role whatsoever in what follows.

**Lemma 3.9** (Additivity of the loop number). *For every subgraph  $\gamma \subseteq \Gamma$  of a connected graph  $\Gamma$ ,*

$$L(\Gamma) = L(\gamma) + L(\Gamma/\gamma).$$

*Proof.* Write  $\gamma = \gamma_1 \sqcup \dots \sqcup \gamma_r$  and put  $i_j = I(\gamma_j)$ ,  $v_j = V(\gamma_j)$ . Then  $I(\Gamma/\gamma) = I(\Gamma) - \sum_j i_j$  and  $V(\Gamma/\gamma) = V(\Gamma) - \sum_j v_j + r$ , while  $\Gamma/\gamma$  is connected because  $\Gamma$  is. Hence

$$L(\Gamma/\gamma) = \left( I(\Gamma) - \sum_j i_j \right) - \left( V(\Gamma) - \sum_j v_j + r \right) + 1 = L(\Gamma) - \sum_{j=1}^r (i_j - v_j + 1),$$

and  $\sum_j (i_j - v_j + 1) = \sum_j L(\gamma_j) = L(\gamma)$ .  $\square$

**Lemma 3.10.** *Let  $\Gamma$  be connected and 1PI and let  $\gamma \subseteq \Gamma$  be any subgraph. Then  $\Gamma/\gamma$  is connected and 1PI.*

*Proof.* Connectedness is clear, since collapsing connected subgraphs of a connected graph leaves it connected. Let  $e \in E_{\text{int}}(\Gamma/\gamma)$ , so that  $e \in E_{\text{int}}(\Gamma)$  and  $e \notin E_{\text{int}}(\gamma)$ . Then  $\gamma$  is still a subgraph of  $\Gamma \setminus e$ , and directly from Definitions 3.6 and 3.8,

$$(\Gamma/\gamma) \setminus e = (\Gamma \setminus e)/\gamma.$$

Since  $\Gamma$  is 1PI,  $\Gamma \setminus e$  is connected, hence so is its contraction. Therefore  $(\Gamma/\gamma) \setminus e$  is connected for every  $e$ .  $\square$

**Lemma 3.11.** *Let  $\gamma \subseteq \Gamma$  be a subgraph whose connected components are connected. If  $\Gamma/\gamma$  is connected and each component of  $\gamma$  is connected, then  $\Gamma$  is connected. More precisely, if  $\gamma \subseteq \Gamma$  and  $\Gamma/\gamma$  is connected, then  $\Gamma$  is connected.*

*Proof.* Two vertices of  $\Gamma$  lying in the same component  $\gamma_i$  are joined inside  $\gamma_i$ , which is connected; two vertices whose images in  $\Gamma/\gamma$  are joined by a path lift to a path in  $\Gamma$  after inserting paths inside the collapsed components.  $\square$

### 3.5. Admissible subgraphs.

**Definition 3.12.** Let  $\Gamma$  be a half-edge graph. A subgraph  $\gamma \subseteq \Gamma$  is *admissible* if every connected component of  $\gamma$  is 1PI. We write

$$\mathcal{A}(\Gamma) = \{\gamma \subseteq \Gamma : \gamma \text{ admissible}\}, \quad \mathcal{D}(\Gamma) = \mathcal{A}(\Gamma) \setminus \{\emptyset, \Gamma\}.$$

Elements of  $\mathcal{D}(\Gamma)$  are the *proper admissible* subgraphs.

By Remark 3.7 admissibility of  $\gamma$  does not depend on the ambient graph, and the empty subgraph is admissible. Note also that  $\mathcal{A}(\Gamma)$  is finite, being a subset of  $2^{E_{\text{int}}(\Gamma)}$ .

*Remark 3.13.* Admissibility is a condition on the edge set of  $\gamma$  alone, and no disjointness hypothesis has to be imposed separately: the connected components of a graph are vertex-disjoint by definition, so two overlapping subgraphs can never both be components of a single admissible  $\gamma$ . This does not exclude overlapping divergences from the theory. Overlapping subgraphs occur in *different* terms of  $\Delta(\Gamma)$  — in Example 3.28 the three cycles pairwise share an edge and each contributes — and their union occurs in a term of its own whenever its components are 1PI. What never happens is that two overlapping subgraphs are contracted simultaneously. Nesting is handled by iterating  $\Delta$ , since the pairs indexing  $(\Delta \otimes \text{id})\Delta$  satisfy  $\alpha \subseteq \delta$ .

**Lemma 3.14.** *If  $\gamma \in \mathcal{A}(\Gamma)$  is nonempty with components  $\gamma_1, \dots, \gamma_r$ , then  $\gamma_i \in \mathcal{G}$  for every  $i$ .*

*Proof.* Each  $\gamma_i$  is connected and 1PI, and carries at least one internal edge because subgraphs have no isolated vertices. Apply Lemma 3.4.  $\square$

**Proposition 3.15** (Closure). *Let  $\Gamma \in \mathcal{G}$  and  $\gamma \in \mathcal{D}(\Gamma)$ . Then*

$$\Gamma/\gamma \in \mathcal{G}.$$

*Proof.* By Lemma 3.10 the graph  $\Gamma/\gamma$  is connected and 1PI. If it had loop number 0, then by Lemma 3.4 it would be a corolla, i.e.  $E_{\text{int}}(\Gamma/\gamma) = \emptyset$ , i.e.  $E_{\text{int}}(\gamma) = E_{\text{int}}(\Gamma)$ , contradicting  $\gamma \neq \Gamma$ . Hence  $L(\Gamma/\gamma) \geq 1$ .  $\square$

Proposition 3.15 and Lemma 3.14 are exactly the closure conditions which in the physical setting have to be imposed as renormalizability hypotheses on the class of graphs. Here they are automatic.

*Remark 3.16.* Both extreme cases are excluded from  $\mathcal{D}(\Gamma)$  on purpose:  $\gamma = \emptyset$  gives  $\Gamma/\emptyset = \Gamma$ , and  $\gamma = \Gamma$  gives a corolla, which is not a generator. We adopt once and for all the conventions

$$\Gamma/\emptyset = \Gamma, \quad \Gamma/\Gamma = 1,$$

the second one identifying the corolla with the unit of the algebra defined below.

### 3.6. The algebra.

**Definition 3.17.** The *graph algebra* is the free commutative  $k$ -algebra on  $\mathcal{G}$ :

$$\mathcal{H} = S(k[\mathcal{G}]) = k[\Gamma : \Gamma \in \mathcal{G}].$$

The product of two generators is their disjoint union,  $\Gamma_1\Gamma_2 = \Gamma_1 \sqcup \Gamma_2$ , and the unit 1 is the empty graph.

A monomial  $\Gamma_1\Gamma_2 \cdots \Gamma_r$  is thus a disjoint union of connected 1PI graphs, each with at least one loop. For a nonempty admissible subgraph  $\gamma$  with components  $\gamma_1, \dots, \gamma_r$  we write

$$[\gamma] = \gamma_1\gamma_2 \cdots \gamma_r \in \mathcal{H}, \quad [\emptyset] = 1,$$

which is legitimate by Lemma 3.14.

### 3.7. The coproduct.

**Definition 3.18.** For  $\Gamma \in \mathcal{G}$  set

$$\Delta(\Gamma) = \Gamma \otimes 1 + 1 \otimes \Gamma + \sum_{\gamma \in \mathcal{D}(\Gamma)} [\gamma] \otimes \Gamma/\gamma,$$

and extend  $\Delta$  to  $\mathcal{H}$  as an algebra homomorphism,  $\Delta(XY) = \Delta(X)\Delta(Y)$ . The counit  $\epsilon : \mathcal{H} \rightarrow k$  is the algebra homomorphism with

$$\epsilon(1) = 1, \quad \epsilon(\Gamma) = 0 \quad (\Gamma \in \mathcal{G}).$$

The sum is finite, and all its terms lie in  $\mathcal{H} \otimes \mathcal{H}$  by Lemma 3.14 and Proposition 3.15. With the conventions above the formula compresses to

$$(31) \quad \Delta(\Gamma) = \sum_{\gamma \in \mathcal{A}(\Gamma)} [\gamma] \otimes [\Gamma/\gamma].$$

*Remark 3.19.* The sum is over *subgraphs*, not over isomorphism classes of subgraphs. This is well defined on isomorphism classes of graphs: an isomorphism  $\Gamma \rightarrow \Gamma'$  induces a bijection  $\mathcal{A}(\Gamma) \rightarrow \mathcal{A}(\Gamma')$  compatible with  $\gamma \mapsto [\gamma]$  and with contraction, so the right-hand side of (31) depends only on the class of  $\Gamma$ . In particular  $\text{Aut}(\Gamma)$  acts on  $\mathcal{A}(\Gamma)$  and isomorphic subgraphs in one orbit contribute equal terms.

**Lemma 3.20.** *Formula (31) holds for every monomial: if  $X = \Gamma_1 \cdots \Gamma_n$  with  $\Gamma_i \in \mathcal{G}$ , then  $\mathcal{A}(X) = \prod_i \mathcal{A}(\Gamma_i)$  and*

$$\Delta(X) = \sum_{\alpha \in \mathcal{A}(X)} [\alpha] \otimes [X/\alpha].$$

*Proof.* A subgraph of  $X = \Gamma_1 \sqcup \cdots \sqcup \Gamma_n$  is a subset of  $E_{\text{int}}(X) = \bigsqcup_i E_{\text{int}}(\Gamma_i)$ , i.e. a tuple  $(\alpha_1, \dots, \alpha_n)$  with  $\alpha_i \subseteq \Gamma_i$ ; its components are the components of the  $\alpha_i$ , so it is admissible iff each  $\alpha_i$  is. Moreover  $[\alpha] = \prod_i [\alpha_i]$  and  $X/\alpha = \bigsqcup_i \Gamma_i/\alpha_i$ . Expanding the product  $\prod_i \Delta(\Gamma_i)$  gives the claim.  $\square$

**3.8. Coassociativity.** The whole content of coassociativity is the following pair of statements about iterated contraction.

**Lemma 3.21** (Transitivity of contraction). *Let  $\alpha \subseteq \delta \subseteq \Gamma$  be subgraphs. Then  $\delta/\alpha$  is the subgraph of  $\Gamma/\alpha$  with internal edge set  $E_{\text{int}}(\delta) \setminus E_{\text{int}}(\alpha)$ , and*

$$(\Gamma/\alpha)/(\delta/\alpha) = \Gamma/\delta.$$

*Moreover  $\delta \mapsto \delta/\alpha$  is an isomorphism of posets from  $\{\delta : \alpha \subseteq \delta \subseteq \Gamma\}$  onto the subgraphs of  $\Gamma/\alpha$ .*

*Proof.* The poset statement is immediate from  $E_{\text{int}}(\Gamma/\alpha) = E_{\text{int}}(\Gamma) \setminus E_{\text{int}}(\alpha)$ . For the displayed identity, both sides have the same half-edge set, namely the half-edges of  $\Gamma$  not belonging to an internal edge of  $\delta$ , with the involution restricted from  $\iota$ . Both sides have vertex set the quotient of  $V$  by the partition whose nonsingleton blocks are the vertex sets of the connected components of  $\delta$ : on the left one first collapses the components of  $\alpha$  and then collapses the components of  $\delta/\alpha$ , and a component of  $\delta$  is exactly the union of the components of  $\alpha$  it contains together with a component of  $\delta/\alpha$  (or is itself a component of  $\alpha$ , if it is contained in  $\alpha$ ). The incidence maps agree.  $\square$

**Lemma 3.22** (Admissibility is preserved and reflected). *Let  $\alpha \subseteq \delta \subseteq \Gamma$  with  $\alpha \in \mathcal{A}(\Gamma)$ . Then*

$$\delta \in \mathcal{A}(\Gamma) \quad \Longleftrightarrow \quad \delta/\alpha \in \mathcal{A}(\Gamma/\alpha).$$

*Proof.* First note that, since  $\alpha \subseteq \delta$ , every connected component of  $\alpha$  is contained in a single connected component of  $\delta$ . Consequently the connected components of  $\delta/\alpha$  are precisely the graphs  $D/(\alpha \cap D)$ , where  $D$  ranges over the components of  $\delta$  not contained in  $\alpha$ , and  $\alpha \cap D$  denotes the union of the components of  $\alpha$  lying in  $D$ .

( $\Rightarrow$ ) Suppose every component  $D$  of  $\delta$  is 1PI. Fix  $D \not\subseteq \alpha$ . Then  $D$  is connected and 1PI, and  $\alpha \cap D$  is a subgraph of  $D$ , so Lemma 3.10 gives that  $D/(\alpha \cap D)$  is connected and 1PI. Hence  $\delta/\alpha$  is admissible.

( $\Leftarrow$ ) Suppose every component of  $\delta/\alpha$  is 1PI, and let  $D$  be a component of  $\delta$ . If  $D \subseteq \alpha$  then  $D$  is a component of  $\alpha$ , hence 1PI by hypothesis. Otherwise put  $A = \alpha \cap D$  and  $B = D/A$ , a component of  $\delta/\alpha$ , so  $B$  is connected and 1PI. Note first that  $D$  is connected by Lemma 3.11. Let  $e \in E_{\text{int}}(D)$ ; we show  $D \setminus e$  is connected.

*Case 1:*  $e \notin E_{\text{int}}(A)$ . Then, as in Lemma 3.10,  $(D \setminus e)/A = (D/A) \setminus e = B \setminus e$ , which is connected because  $B$  is 1PI. By Lemma 3.11,  $D \setminus e$  is connected.

*Case 2:*  $e \in E_{\text{int}}(A)$ . Let  $A_0$  be the connected component of  $A$  containing  $e$ ; by hypothesis  $A_0$  is 1PI, so  $A_0 \setminus e$  is connected and has the same vertex set as  $A_0$ . Therefore the subgraph  $A \setminus e \subseteq D \setminus e$  has connected components  $A_0 \setminus e$  and the other components of  $A$ , and collapsing them identifies exactly the same vertices of  $D$  as collapsing  $A$  does. Hence

$$(D \setminus e)/(A \setminus e)$$

is obtained from  $B$  by adjoining a self-loop at the vertex coming from  $A_0$ ; in particular it is connected. By Lemma 3.11,  $D \setminus e$  is connected.

Thus  $D$  is 1PI in either case, and  $\delta$  is admissible.  $\square$

**Theorem 3.23.**  $(\Delta \otimes \text{id})\Delta = (\text{id} \otimes \Delta)\Delta$  and  $(\epsilon \otimes \text{id})\Delta = \text{id} = (\text{id} \otimes \epsilon)\Delta$ . Hence  $(\mathcal{H}, \cdot, 1, \Delta, \epsilon)$  is a commutative bialgebra.

*Proof.* Both  $\Delta$  and the two composites are algebra maps, so it suffices to check the identities on a generator  $\Gamma \in \mathcal{G}$ .

For the counit, the only term of (31) whose left factor has  $\epsilon \neq 0$  is  $\gamma = \emptyset$ , giving  $(\epsilon \otimes \text{id})\Delta(\Gamma) = \Gamma$ ; similarly only  $\gamma = \Gamma$  survives on the other side.

For coassociativity, using (31) and Lemma 3.20,

$$\begin{aligned} (\Delta \otimes \text{id})\Delta(\Gamma) &= \sum_{\delta \in \mathcal{A}(\Gamma)} \sum_{\alpha \in \mathcal{A}(\delta)} [\alpha] \otimes [\delta/\alpha] \otimes [\Gamma/\delta], \\ (\text{id} \otimes \Delta)\Delta(\Gamma) &= \sum_{\alpha \in \mathcal{A}(\Gamma)} \sum_{\beta \in \mathcal{A}(\Gamma/\alpha)} [\alpha] \otimes [\beta] \otimes [(\Gamma/\alpha)/\beta]. \end{aligned}$$

By Remark 3.7, the index set of the first double sum is

$$P = \{(\alpha, \delta) : \alpha \subseteq \delta \subseteq \Gamma, \alpha \in \mathcal{A}(\Gamma), \delta \in \mathcal{A}(\Gamma)\},$$

and that of the second is

$$Q = \{(\alpha, \beta) : \alpha \in \mathcal{A}(\Gamma), \beta \in \mathcal{A}(\Gamma/\alpha)\}.$$

Lemmas 3.21 and 3.22 show that  $(\alpha, \delta) \mapsto (\alpha, \delta/\alpha)$  is a bijection  $P \rightarrow Q$ , and Lemma 3.21 further gives  $(\Gamma/\alpha)/(\delta/\alpha) = \Gamma/\delta$ . The two sums therefore agree term by term.  $\square$

### 3.9. The loop grading.

**Definition 3.24.** Assign to a generator  $\Gamma \in \mathcal{G}$  the degree  $|\Gamma| = L(\Gamma) \geq 1$ , and to a monomial the sum of the degrees of its factors. Let  $\mathcal{H}_n$  be the span of the monomials of degree  $n$ .

**Proposition 3.25.**  $\mathcal{H} = \bigoplus_{n \geq 0} \mathcal{H}_n$  is a connected graded bialgebra:

$$\mathcal{H}_p \mathcal{H}_q \subseteq \mathcal{H}_{p+q}, \quad \Delta(\mathcal{H}_n) \subseteq \bigoplus_{p+q=n} \mathcal{H}_p \otimes \mathcal{H}_q, \quad \mathcal{H}_0 = k \cdot 1.$$

*Proof.* Multiplicativity of the degree is the definition. Compatibility with  $\Delta$  is Lemma 3.9:  $L(\Gamma) = L(\gamma) + L(\Gamma/\gamma)$  for every  $\gamma \in \mathcal{A}(\Gamma)$ , and  $||[\gamma]|| = L(\gamma)$  since the degree of a monomial is the sum over its components. Finally every generator has degree  $\geq 1$  by definition of  $\mathcal{G}$ , so the only monomial of degree 0 is the empty one.  $\square$

It is here that corollas had to be excluded: a one-vertex graph with external half-edges is connected and vacuously 1PI, but has loop number 0, and admitting it as a generator would destroy the connectedness of the grading.

*Remark 3.26.* Each  $\mathcal{H}_n$  is infinite dimensional, because the number of external half-edges and the valences of the vertices are unconstrained: already for  $n = 1$  one has all cycles of all lengths with arbitrarily many external half-edges. This does not affect anything below. If finite dimensionality is wanted, fix in advance a finite set  $\mathcal{V} \subseteq \mathbb{Z}_{\geq 1}$  of allowed vertex valences and an upper bound on

$N(\Gamma)$ ; one must then check that the resulting subclass is stable under contraction, which is where the physical notion of renormalizability reappears.

### 3.10. The antipode.

**Theorem 3.27.**  *$\mathcal{H}$  admits a unique antipode  $S$ , making it a commutative Hopf algebra. On a generator  $\Gamma \in \mathcal{G}$  it is given recursively by*

$$S(\Gamma) = -\Gamma - \sum_{\gamma \in \mathcal{D}(\Gamma)} S([\gamma]) \Gamma / \gamma,$$

and extended as an algebra antihomomorphism (equivalently, since  $\mathcal{H}$  is commutative, as an algebra homomorphism).

*Proof.* A connected graded bialgebra is a Hopf algebra, the antipode being the inverse of  $\text{id}$  in the convolution algebra  $\text{End}(\mathcal{H})$ , which exists and is unique because  $\text{id} - u\epsilon$  is locally nilpotent for the convolution product. Concretely,  $m(S \otimes \text{id})\Delta(\Gamma) = \epsilon(\Gamma) = 0$  reads

$$S(\Gamma) + \Gamma + \sum_{\gamma \in \mathcal{D}(\Gamma)} S([\gamma]) \Gamma / \gamma = 0,$$

which is the stated recursion. It terminates because every  $\gamma \in \mathcal{D}(\Gamma)$  satisfies  $1 \leq L(\gamma) < L(\Gamma)$ , the strict inequality by Lemma 3.9 together with Proposition 3.15.  $\square$

Expanding the recursion produces a signed sum over chains of nested admissible subgraphs; this is the combinatorial shadow of the forest formula.

**3.11. Comparison with the physical construction.** For orientation, we record what has been simplified. In the setting of a fixed renormalizable quantum field theory one has in addition:

- a set  $\mathcal{R}$  of *residues*, the elementary propagator and vertex types of the theory, and a map  $\text{res}$  assigning to a graph the residue with the same external structure;
- a superficial degree of divergence  $\omega : \mathcal{G} \rightarrow \mathbb{Z}$ , required to factor as  $\omega = \Omega \circ \text{res}$ ; a subgraph counts as divergent when  $\omega \geq 0$ ;
- contraction defined by replacing a divergent component by a copy of its residue rather than by a single vertex.

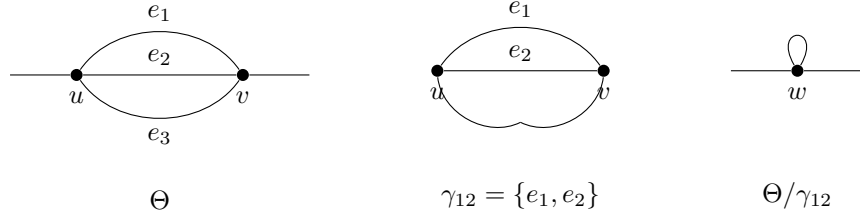
The two constructions differ exactly for subgraphs with two external half-edges (self-energies), which physically must contract to a propagator, i.e. to an *edge*, not to a vertex. Keeping track of that forces one to carry  $\mathcal{R}$  along, and then the two conditions “ $\omega$  factors through  $\text{res}$ ” and “the class of graphs is stable under contraction of divergent subgraphs” must be imposed as axioms, since they encode renormalizability of the theory. In the version developed above all subgraphs contract to a point, and Lemma 3.4, Lemma 3.10 and Proposition 3.15 replace those axioms by proofs.

**3.12. Reading a subgraph off an edge set.** The single point which causes the most confusion is that a subgraph is determined by a set of *internal edges*, not by a set of vertices. In particular a proper subgraph may well contain *all* the vertices of  $\Gamma$ . The following example should be worked through in complete detail before anything else.

**Example 3.28** (The theta graph). Let  $\Theta$  have two vertices  $u, v$ , three internal edges  $e_1, e_2, e_3$  joining them, and one external half-edge at each vertex. Thus

$$I(\Theta) = 3, \quad V(\Theta) = 2, \quad N(\Theta) = 2, \quad L(\Theta) = 3 - 2 + 1 = 2.$$

$\Theta$  is 1PI: deleting the pairing of any  $e_i$  leaves the other two joining  $u$  and  $v$ .



Take  $\gamma = \gamma_{12}$  with  $E_{\text{int}}(\gamma) = \{e_1, e_2\}$ . Then  $V_\gamma = \{u, v\} = V(\Theta)$  and  $H_\gamma = s^{-1}(V_\gamma)$  is *all eight* half-edges of  $\Theta$ . But  $\iota_\gamma$  pairs only the four half-edges of  $e_1, e_2$ ; the two half-edges of  $e_3$  are fixed points of  $\iota_\gamma$ . Hence

$$I(\gamma) = 2, \quad V(\gamma) = 2, \quad N(\gamma) = 4, \quad L(\gamma) = 1,$$

the four external half-edges being the two external legs of  $\Theta$  together with the two severed ends of  $e_3$ . In particular  $\gamma \neq \Theta$ , even though  $V_\gamma = V(\Theta)$ .

Contracting collapses  $u, v$  to a single vertex  $w$ ; the surviving internal edge  $e_3$  becomes a self-loop at  $w$  and the two external legs remain, so  $\Theta/\gamma$  is the tadpole, with  $L = 1$ . Note  $N(\Theta/\gamma) = 2 = N(\Theta)$ , as it must be, while  $N(\gamma) = 4$ : the external legs of a *subgraph* have nothing to do with those of the ambient graph.

*Remark 3.29.* Had we used *induced* subgraphs — a subgraph determined by a set of vertices, carrying all edges between them — the subgraph on  $\{u, v\}$  would be  $\Theta$  itself and the coproduct of Example 3.28 would be trivial. The edge-set convention is what makes the theory nontrivial, and severing an edge is exactly what the half-edge formalism is designed to record.

### 3.13. Two structural shortcuts.

**Lemma 3.30.** *Every  $\Gamma \in \mathcal{G}$  with  $L(\Gamma) = 1$  is primitive.*

*Proof.* If  $\gamma \in \mathcal{D}(\Gamma)$  then  $L(\gamma) \geq 1$  by Lemma 3.14 and  $L(\Gamma/\gamma) \geq 1$  by Proposition 3.15, while  $L(\gamma) + L(\Gamma/\gamma) = 1$  by Lemma 3.9.  $\square$

**Lemma 3.31.** *Let  $\Gamma \in \mathcal{G}$  with  $L(\Gamma) = 2$ . Then  $\mathcal{D}(\Gamma)$  is precisely the set of cycles of  $\Gamma$  (self-loops and 2-cycles included), and every such cycle  $C$  satisfies  $L(C) = L(\Gamma/C) = 1$ .*

*Proof.* A  $\gamma \in \mathcal{D}(\Gamma)$  has  $L(\gamma) = L(\Gamma/\gamma) = 1$  by Lemmas 3.9 and 3.4, so  $\gamma$  is connected with  $L(\gamma) = 1$ : it has  $I = V$ , hence contains exactly one cycle, and one-particle irreducibility forbids any edge outside that cycle. So  $\gamma$  is a cycle. Conversely a cycle is connected and 1PI, and it is proper because  $L(\Gamma) = 2$ . Finally  $\Gamma$  contains no two vertex-disjoint cycles, since an edge on a path joining them would be a bridge; hence no disconnected admissible subgraph exists.  $\square$

So in loop order 2 the coproduct simply enumerates cycles. Since the cycle space of  $\Gamma$  has dimension  $L(\Gamma) = 2$  over  $\mathbb{F}_2$ , there are at most  $2^2 - 1 = 3$  candidates.

**3.14. Examples in loop order two.** Throughout,  $T_j$  denotes the graph with one vertex and  $j$  self-loops (so  $L(T_j) = j$ ), and  $B_k$  the “banana” with two vertices joined by  $k$  internal edges (so  $L(B_k) = k - 1$ ). External legs may be attached anywhere; they are inert and we suppress them.

**Example 3.32** (Theta graph). By Lemma 3.31 the admissible subgraphs of  $\Theta = B_3$  are the three 2-cycles  $\gamma_{12}, \gamma_{13}, \gamma_{23}$ , all isomorphic to  $B_2$ , each contracting to  $T_1$ . Hence

$$\Delta(B_3) = B_3 \otimes 1 + 1 \otimes B_3 + 3 B_2 \otimes T_1.$$

The coefficient 3 is the number of *subgraphs*, which here is the size of a single  $\text{Aut}(B_3)$ -orbit.

**Example 3.33** (Double tadpole). Let  $T_2$  have one vertex and two self-loops  $e_1, e_2$ . Its cycles are  $\{e_1\}$  and  $\{e_2\}$ , each a copy of  $T_1$ ; the set  $\{e_1, e_2\}$  is all of  $T_2$  and hence excluded. Contracting a self-loop collapses its single vertex and leaves the other loop, so

$$\Delta(T_2) = T_2 \otimes 1 + 1 \otimes T_2 + 2 T_1 \otimes T_1.$$

**Example 3.34** (Bubble inside a bubble). Let  $\Gamma$  have vertices  $A, B, C, D$  and internal edges

$$f = AB, \quad g_1 = AC, \quad g_2 = DB, \quad h_1 = h_2 = CD,$$

so  $I = 5$ ,  $V = 4$ ,  $L = 2$ . Its cycles are

$$\gamma = \{h_1, h_2\}, \quad \delta_1 = \{f, g_1, h_1, g_2\}, \quad \delta_2 = \{f, g_1, h_2, g_2\},$$

again three, but now *not* all isomorphic:  $\gamma \cong B_2$  while  $\delta_1 \cong \delta_2$  is a 4-cycle. Contracting:  $\Gamma/\gamma$  merges  $C, D$  and leaves the triangle on  $A, B, w$ ; each  $\delta_i$  uses all four vertices, so  $\Gamma/\delta_i$  is a single vertex carrying the surviving loop, i.e.  $T_1$ . Hence

$$\Delta(\Gamma) = \Gamma \otimes 1 + 1 \otimes \Gamma + \gamma \otimes \Delta + 2 \delta \otimes T_1.$$

Note again that  $V_{\delta_i} = V(\Gamma)$  while  $\delta_i \subsetneq \Gamma$ , exactly as in Example 3.28. Comparing with the theta graph shows that the multiplicities in the coproduct are orbit counts and need not be equal.

### 3.15. A loop order three example, and coassociativity.

**Example 3.35** (The 4-banana). Let  $\Gamma = B_4$ : two vertices  $u, v$  joined by  $e_1, e_2, e_3, e_4$ . Then  $L(\Gamma) = 4 - 2 + 1 = 3$ . A nonempty subset  $S \subseteq \{e_1, \dots, e_4\}$  spans a connected 1PI subgraph if and only if  $|S| \geq 2$  (a single edge is a bridge in the subgraph it spans), and every such subgraph is  $B_{|S|}$  with  $L = |S| - 1$ . No disconnected admissible subgraph exists, since every subgraph uses both vertices. Contracting  $B_{|S|}$  merges  $u$  and  $v$ , turning the  $4 - |S|$  remaining edges into self-loops, so  $\Gamma/S = T_{4-|S|}$ . Therefore

$$\Delta(B_4) = B_4 \otimes 1 + 1 \otimes B_4 + 6 B_2 \otimes T_2 + 4 B_3 \otimes T_1,$$

the coefficients being  $\binom{4}{2}$  and  $\binom{4}{3}$ . Degrees check:  $1 + 2 = 3$  and  $2 + 1 = 3$ .

**Example 3.36** (Coassociativity, explicitly). We verify  $(\Delta \otimes \text{id})\Delta(B_4) = (\text{id} \otimes \Delta)\Delta(B_4)$ . The ingredients are Example 3.35 together with

$$\Delta(B_2) = B_2 \otimes 1 + 1 \otimes B_2, \quad \Delta(T_1) = T_1 \otimes 1 + 1 \otimes T_1,$$

$$\Delta(B_3) = B_3 \otimes 1 + 1 \otimes B_3 + 3 B_2 \otimes T_1, \quad \Delta(T_2) = T_2 \otimes 1 + 1 \otimes T_2 + 2 T_1 \otimes T_1.$$

All terms containing a 1 in some tensor slot agree on both sides by the counit identities, so it suffices to compare the terms with no slot equal to 1. On the left, such a term can only come from the reduced part of  $\Delta(B_3)$ :

$$4 \cdot (3 B_2 \otimes T_1) \otimes T_1 = 12 B_2 \otimes T_1 \otimes T_1.$$

On the right, only from the reduced part of  $\Delta(T_2)$ :

$$6 B_2 \otimes (2 T_1 \otimes T_1) = 12 B_2 \otimes T_1 \otimes T_1.$$

The two agree. Structurally, the left count is “choose a 3-element subset, then a 2-element subset inside it” and the right count is “choose a 2-element subset, then one of the two remaining edges”, i.e.  $4 \cdot 3 = 6 \cdot 2$ ; this is exactly the bijection of Theorem 3.23 for  $\Gamma = B_4$ .

**Example 3.37** (Antipode of  $B_3$ ). Using  $S(X) = -X - \sum_{\gamma \in \mathcal{D}(X)} S([\gamma]) X/\gamma$  and  $S(B_2) = -B_2$ ,  $S(T_1) = -T_1$ :

$$S(B_3) = -B_3 - 3 S(B_2) T_1 = -B_3 + 3 B_2 T_1.$$

One checks  $m(S \otimes \text{id})\Delta(B_3) = S(B_3) + B_3 + 3 S(B_2) T_1 = 0$ .

### 3.16. What to take away.

- A subgraph is a set of internal edges. Its vertex set is whatever those edges touch, and may be all of  $V(\Gamma)$  without the subgraph being  $\Gamma$ .
- The external half-edges of  $\gamma$  are the half-edges at  $V_\gamma$  that  $\gamma$  does not pair: the severed ends of the edges of  $\Gamma$  left out of  $\gamma$ , plus any external legs of  $\Gamma$  sitting at  $V_\gamma$ .
- Contraction never changes  $N$ , but passing to a subgraph usually does.
- Multiplicities in  $\Delta(\Gamma)$  count subgraphs, i.e. they are sums of  $\text{Aut}(\Gamma)$ -orbit sizes, and are equal only when the subgraphs in question are all isomorphic.

**3.17. Example: four-dimensional  $\varphi^4$  theory.** Consider the four-dimensional scalar theory  $\varphi_4^4$ . In this case, the allowed interaction vertex has four incident half-edges, while the propagator is the allowed two-point elementary structure.

Accordingly, not every half-edge graph is an admissible Feynman graph of the theory. We restrict to graphs whose vertices are four-valent.

**Definition 3.38.** A half-edge graph  $\Gamma$  is  $\varphi^4$ -admissible if

$$|s^{-1}(v)| = 4 \quad \text{for every } v \in V(\Gamma).$$

Thus the class of  $\varphi^4$ -admissible graphs is a proper subclass of the class of all half-edge graphs. In particular, the combinatorial notion of a graph is more general than the notion of a Feynman graph allowed by a given theory.

For a connected  $\varphi^4$ -admissible graph, the external structure is determined, at the level relevant here, by the number  $N(\Gamma)$  of external half-edges. The residue map therefore takes the form

$$\text{res}(\Gamma) = \begin{cases} \text{a propagator,} & N(\Gamma) = 2, \\ \text{a four-valent vertex,} & N(\Gamma) = 4, \end{cases}$$

for superficially divergent graphs.

For four-dimensional  $\varphi^4$  theory, the superficial degree of divergence is given by

$$\omega(\Gamma) = 4 - N(\Gamma).$$

Hence a connected 1PI graph is superficially divergent precisely when

$$N(\Gamma) \leq 4.$$

In the sector of graphs relevant for the renormalization of the theory, the divergent cases are therefore the two-point and four-point graphs:

| $N(\Gamma)$ | $\omega(\Gamma)$ | $\text{res}(\Gamma)$ |
|-------------|------------------|----------------------|
| 2           | 2                | propagator           |
| 4           | 0                | four-valent vertex.  |

The role of the residue can be seen directly in the contraction operation. If  $\gamma \subseteq \Gamma$  is a superficially divergent subgraph, then  $\gamma$  is replaced by its residue. Thus a two-point component is replaced by a propagator joining its two boundary slots, whereas a four-point component is replaced by a four-valent vertex carrying its four boundary slots.

In either case, the resulting graph is again  $\varphi^4$ -admissible. Thus, in this example, the abstract closure condition under contraction takes the concrete form

$$\gamma \in \mathcal{D}(\Gamma) \implies \Gamma/\gamma \text{ is } \varphi^4\text{-admissible.}$$

This illustrates the graph-theoretic content of renormalizability: the external structures of superficially divergent subgraphs are among the elementary structures allowed by the theory, so their contraction remains within the same class of graphs.

**Example 3.39.** A two-loop graph with a proper subdivergence

Consider the two-loop four-point  $\varphi^4$  graph  $\Gamma_{\text{nest}}$ , with an admissible subgraph  $\gamma \subset \Gamma_{\text{nest}}$ .

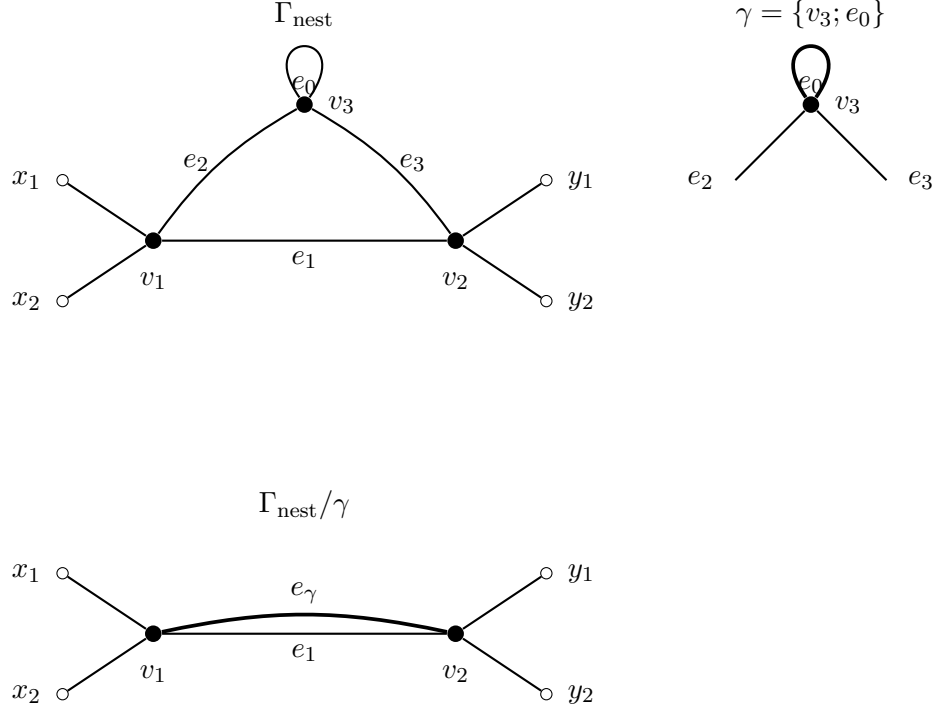


FIGURE 3. The superficially divergent subgraph  $\gamma = \{v_3; e_0\}$  and its contraction in  $\Gamma_{\text{nest}}$ .

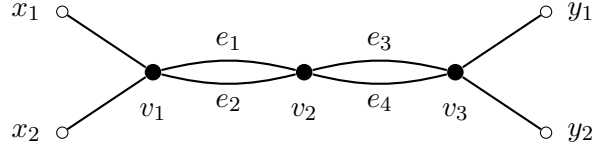
Since  $\gamma$  is a two-point graph, its residue is a propagator. Contracting  $\gamma$  replaces  $v_3$  and the loop  $e_0$  by a single internal edge joining  $v_1$  to  $v_2$ . We denote this new edge by  $e_\gamma$ . Thus  $\Gamma_{\text{nest}}/\gamma$  is the one-loop four-point graph.

The complete coproduct is

$$\Delta(\Gamma_{\text{nest}}) = \Gamma_{\text{nest}} \otimes 1 + 1 \otimes \Gamma_{\text{nest}} + \gamma \otimes (\Gamma_{\text{nest}}/\gamma).$$

**Example 3.40.** (A two-loop graph with overlapping subdivergences)

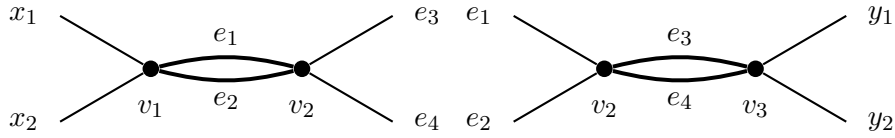
Consider now the graph  $\Gamma_{\text{ov}}$ :



It has two divergent admissible subgraphs  $\gamma_L$  and  $\gamma_R$ :

$$\gamma_L = \{v_1, v_2; e_1, e_2\}$$

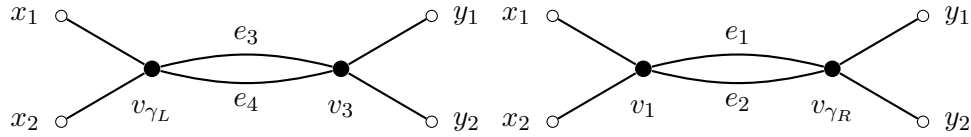
$$\gamma_R = \{v_2, v_3; e_3, e_4\}$$



These give rise to the following contractions

$$\Gamma_{\text{ov}}/\gamma_L$$

$$\Gamma_{\text{ov}}/\gamma_R$$



The complete coproduct is

$$\Delta(\Gamma_{\text{ov}}) = \Gamma_{\text{ov}} \otimes 1 + 1 \otimes \Gamma_{\text{ov}} + \gamma_1 \otimes (\Gamma_{\text{ov}}/\gamma_1) + \gamma_2 \otimes (\Gamma_{\text{ov}}/\gamma_2).$$

Observe that there is no term

$$\gamma_1 \gamma_2 \otimes \Gamma_{\text{ov}}/(\gamma_1 \cup \gamma_2),$$

because  $\gamma_1$  and  $\gamma_2$  overlap and hence cannot occur simultaneously in a forest.

Also, we remark that as abstract graphs the reductions are isomorphic, but they arise from different subgraphs of  $\Gamma_{\text{ov}}$ .

*Remark 3.41.* The two examples illustrate the two basic configurations.

For a nested inclusion

$$\gamma_1 \subseteq \gamma_2 \subseteq \Gamma,$$

successive contractions are compatible:

$$(\Gamma/\gamma_1)/(\gamma_2/\gamma_1) \cong \Gamma/\gamma_2.$$

This compatibility is what appears algebraically as coassociativity of the coproduct.

For overlapping subgraphs

$$\gamma_1 \not\subseteq \gamma_2, \quad \gamma_2 \not\subseteq \gamma_1, \quad \gamma_1 \cap \gamma_2 \neq \emptyset,$$

the two subgraphs occur as separate terms

$$\gamma_1 \otimes \Gamma/\gamma_1, \quad \gamma_2 \otimes \Gamma/\gamma_2,$$

but not as a product

$$\gamma_1 \gamma_2.$$

Thus the coproduct itself already distinguishes the combinatorial relations that are later organized by forests.

## 4. ALGEBRAIC RENORMALIZATION AND BIRKHOFF DECOMPOSITION

Let  $\mathcal{H}$  be a connected Hopf algebra. Let us fix  $\mathcal{A}$  to be the field of formal Laurent series in a complex parameter  $z$  (representing the dimensional regularization parameter  $\varepsilon$ ):

$$\mathcal{A} = \mathbb{C}((z)) = \left\{ \sum_{n=-k}^{\infty} a_n z^n \mid a_n \in \mathbb{C}, k \in \mathbb{Z} \right\}$$

Let  $G(\mathcal{A}) = \text{Hom}_{\text{alg}}(\mathcal{H}, \mathcal{A})$  be the set of  $k$ -algebra homomorphisms (characters) from our  $\mathcal{H}$  to the target algebra  $\mathcal{A}$ . Recall that  $G_*(\mathcal{A})$  inherits a group structure under the convolution product:

$$(\phi_1 * \phi_2)(x) = m_{\mathcal{A}} \circ (\phi_1 \otimes \phi_2) \circ \Delta(x) = \sum_{(x)} \phi_1(x_{(1)}) \phi_2(x_{(2)})$$

**Definition 4.1.** Define the minimal subtraction projection  $R : \mathcal{A} \rightarrow \mathcal{A}$  which extracts the polar part of a Laurent series:

$$R \left( \sum_{n=-k}^{\infty} a_n z^n \right) = \sum_{n=-k}^{-1} a_n z^n$$

Our target algebra  $\mathcal{A}$  decomposes into a direct sum of subalgebras

$$\mathcal{A} = \mathcal{A}_- \oplus \mathcal{A}_+$$

where  $\mathcal{A}_- = \text{Im}(R) = z^{-1}\mathbb{C}[z^{-1}]$  (the pure pole parts) and  $\mathcal{A}_+ = \text{Ker}(R) = \mathbb{C}[[z]]$  (the finite Taylor series).

**Proposition 4.2.** *The map  $R$  is a Rota-Baxter operator of weight  $-1$ . That is, for all  $x, y \in \mathcal{A}$ :*

$$(32) \quad R(x)R(y) = R(R(x)y + xR(y) - xy)$$

*Proof.* Any  $x, y \in \mathcal{A}$  can be uniquely decomposed as  $x = x_- + x_+$  and  $y = y_- + y_+$ , where  $x_-, y_- \in \mathcal{A}_-$  and  $x_+, y_+ \in \mathcal{A}_+$ . By definition,  $R(x) = x_-$  and  $R(y) = y_-$ .

The left-hand side of (32) is exactly  $x_-y_-$ .

For the right-hand side, we expand the argument of  $R$ :

$$\begin{aligned} R(x)y + xR(y) - xy &= x_-(y_- + y_+) + (x_- + x_+)y_- - (x_- + x_+)(y_- + y_+) \\ &= x_-y_- + x_-y_+ + x_-y_- + x_+y_- - (x_-y_- + x_-y_+ + x_+y_- + x_+y_+) \\ &= x_-y_- - x_+y_+ \end{aligned}$$

Observe that  $\mathcal{A}_-$  is a non-unital algebra (the product of polynomials in  $z^{-1}$  without constant terms also has no constant term), so  $x_-y_- \in \mathcal{A}_-$ . Similarly,  $\mathcal{A}_+$  is an algebra, so  $x_+y_+ \in \mathcal{A}_+$ . Applying the projection  $R$  isolates the  $\mathcal{A}_-$  component:

$$R(x_-y_- - x_+y_+) = x_-y_-$$

Thus, both sides equal  $x_-y_-$ , proving the relation.  $\square$

**4.1. Connes-Kreimer Algebraic Renormalization.** We now state the crowning result: renormalization is strictly equivalent to the Birkhoff factorization of a loop in the character group.

**Theorem 4.3** (Connes-Kreimer (2000)). *The renormalization  $\phi_+ \in G(\mathcal{A}_+)$  of a character  $\phi \in G(\mathcal{A})$  is obtained via*

$$\phi_+ = \phi_- * \phi$$

where the counterterm character  $\phi_-$  is defined recursively by deforming the antipode with the Rota-Baxter operator  $R$ . More precisely, for any  $x \in \text{Ker}(\varepsilon)$ :

$$\phi_-(x) = -R(\bar{\phi}(x)) \quad \text{where} \quad \bar{\phi}(x) = \phi(x) + \sum_{(x)'} \phi_-(x_{(1)})\phi(x_{(2)})$$

Here,  $\bar{\phi}(x)$  is the Bogoliubov preparation map.

*Proof.* We must prove (1) that  $\phi_+$  maps entirely into the finite space  $\mathcal{A}_+$ , and (2) that  $\phi_-$  is a true algebra homomorphism (the character property), which guarantees that counterterms of disjoint graphs multiply correctly.

**Step 1: Finiteness of  $\phi_+$**

By the definition of convolution on  $x \in \text{Ker}(\varepsilon)$ :

$$\phi_+(x) = (\phi_- * \phi)(x) = \phi_-(x) + \phi(x) + \sum_{(x)'} \phi_-(x_{(1)})\phi(x_{(2)})$$

The second and third terms constitute exactly the Bogoliubov map  $\bar{\phi}(x)$ . Thus:

$$\phi_+(x) = \phi_-(x) + \bar{\phi}(x) = -R(\bar{\phi}(x)) + \bar{\phi}(x) = (id - R)\bar{\phi}(x)$$

Since  $R$  projects onto  $\mathcal{A}_-$ , the operator  $(id - R)$  is the projection onto  $\mathcal{A}_+$ . Therefore,  $\phi_+(x) \in \mathcal{A}_+$  for all  $x$ , guaranteeing the extraction of finite values.

**Step 2:  $\phi_-$  is an algebra homomorphism**

We must show  $\phi_-(xy) = \phi_-(x)\phi_-(y)$ . We proceed by induction on the grading (loop number) of  $\mathcal{H}$ . Assume the property holds for elements of degree strictly less than  $n$ . Let  $x, y$  be such that  $\deg(xy) = n$ .

We make the following claim, and postpone the proof for the last paragraph.

**Claim.** For any  $x, y \in \text{Ker}(\varepsilon)$  of positive degree, the Bogoliubov preparation map satisfies:

$$\bar{\phi}(xy) = \bar{\phi}(x)\phi_+(y) + \phi_-(x)\bar{\phi}(y)$$

Next, we apply the counterterm definition  $\phi_-(xy) = -R(\bar{\phi}(xy))$ :

$$\phi_-(xy) = -R(\bar{\phi}(x)\phi_+(y) + \phi_-(x)\bar{\phi}(y))$$

Using  $\phi_+(y) = \bar{\phi}(y) + \phi_-(y)$  and the Rota-Baxter relation (32), this expression evaluates precisely to:

$$\phi_-(xy) = \phi_-(x)\phi_-(y)$$

This proves that the Birkhoff factorization resides within the character group  $G(\mathcal{A})$ .

*Proof of the Claim* We proceed by induction on the total degree  $\deg(xy) = n$ , assuming the character property  $\phi_-(ab) = \phi_-(a)\phi_-(b)$  holds for all elements of degree strictly less than  $n$ .

By definition, the Bogoliubov preparation map is:

$$\bar{\phi}(xy) = \phi(xy) + \sum_{(xy)'} \phi_-(x_{(1)})\phi(x_{(2)})$$

Since  $\phi$  is an algebra character ( $\phi(xy) = \phi(x)\phi(y)$ ) and  $\Delta$  is an algebra homomorphism ( $\Delta(xy) = \Delta(x)\Delta(y)$ ), we expand  $\Delta(x)\Delta(y)$  using Sweedler notation ( $\tilde{\Delta}(x) = \sum x_{(1)} \otimes x_{(2)}$ ):

$$\begin{aligned} \Delta(xy) &= \left( x \otimes \mathbf{1} + \mathbf{1} \otimes x + \sum x_{(1)} \otimes x_{(2)} \right) \left( y \otimes \mathbf{1} + \mathbf{1} \otimes y + \sum y_{(1)} \otimes y_{(2)} \right) \\ (33) \quad &= xy \otimes \mathbf{1} + \mathbf{1} \otimes xy + x \otimes y + y \otimes x + \text{(cross terms)} \dots \end{aligned}$$

Multiplying all nine tensor components via  $(a \otimes b)(c \otimes d) = ac \otimes bd$  and applying the functional  $(\phi_- \otimes \phi)$  to the reduced coproduct components yields the exact expanded sum:

$$\begin{aligned} \bar{\phi}(xy) &= \phi(x)\phi(y) + \phi_-(x)\phi(y) + \phi_-(y)\phi(x) \\ &\quad + \sum \phi_-(xy_{(1)})\phi(y_{(2)}) + \sum \phi_-(y_{(1)})\phi(xy_{(2)}) \\ &\quad + \sum \phi_-(x_{(1)}y)\phi(x_{(2)}) + \sum \phi_-(x_{(1)})\phi(x_{(2)}y) \\ (34) \quad &\quad + \sum \phi_-(x_{(1)}y_{(1)})\phi(x_{(2)}y_{(2)}) \end{aligned}$$

Using the induction hypothesis for lower-degree products (e.g.,  $\phi_-(xy_{(1)}) = \phi_-(x)\phi_-(y_{(1)})$  and  $\phi(x_{(2)}y_{(2)}) = \phi(x_{(2)})\phi(y_{(2)})$ ), we substitute and distribute:

$$\begin{aligned}
 \bar{\phi}(xy) &= \phi(x)\phi(y) + \phi_-(x)\phi(y) + \phi_-(y)\phi(x) \\
 &\quad + \phi_-(x) \sum \phi_-(y_{(1)})\phi(y_{(2)}) + \phi(x) \sum \phi_-(y_{(1)})\phi(y_{(2)}) \\
 &\quad + \phi_-(y) \sum \phi_-(x_{(1)})\phi(x_{(2)}) + \phi(y) \sum \phi_-(x_{(1)})\phi(x_{(2)}) \\
 (35) \quad &\quad + \left( \sum \phi_-(x_{(1)})\phi(x_{(2)}) \right) \left( \sum \phi_-(y_{(1)})\phi(y_{(2)}) \right)
 \end{aligned}$$

By grouping terms containing  $\phi_-(x)$  versus those containing  $\phi(x)$  and factoring, this eight-term polynomial rearranges precisely into:

$$\begin{aligned}
 \bar{\phi}(xy) &= \left( \phi(x) + \sum \phi_-(x_{(1)})\phi(x_{(2)}) \right) \left( \phi(y) + \phi_-(y) + \sum \phi_-(y_{(1)})\phi(y_{(2)}) \right) \\
 (36) \quad &\quad + \phi_-(x) \left( \phi(y) + \sum \phi_-(y_{(1)})\phi(y_{(2)}) \right)
 \end{aligned}$$

Recognizing the definitions  $\bar{\phi}(x) = \phi(x) + \sum \phi_-(x_{(1)})\phi(x_{(2)})$ ,  $\phi_+(y) = \phi_-(y) + \bar{\phi}(y)$ , we obtain:

$$\bar{\phi}(xy) = \bar{\phi}(x)\phi_+(y) + \phi_-(x)\bar{\phi}(y)$$

This completes the proof of the claim. □

**4.2. The Free Commutative Algebra of Forests.** We begin by defining the underlying vector space and its algebra structure over a field  $k$  (typically  $\text{char}(k) = 0$ ). We recall that a graph  $\Gamma$  consists of a finite set of vertices  $V(\Gamma)$  and a finite set of edges.

**Definition 4.4** (Rooted Tree). A non-planar rooted tree  $T$  is a finite, connected, simply-connected graph with a distinguished vertex  $r \in V(T)$  called the root. Edges are implicitly oriented away from the root. A leaf is any vertex connected to a single other vertex.

**Definition 4.5** (Forest). A forest  $F$  is a finite disjoint union of rooted trees. The empty forest is denoted by  $\mathbf{1}$ .

**Definition 4.6** (The Algebra  $\mathcal{H}_{CK}$ ). Let  $\mathcal{T}$  be the set of all isomorphism classes of non-planar rooted trees. We define  $\mathcal{H}_{CK}$  as the free commutative  $k$ -algebra generated by  $\mathcal{T}$ .

- **Basis:** The underlying vector space has a basis given by all forests.
- **Multiplication:** The product  $m : \mathcal{H}_{CK} \otimes \mathcal{H}_{CK} \rightarrow \mathcal{H}_{CK}$  of two forests  $F_1, F_2$  is their disjoint union  $F_1 F_2$ .
- **Unit:** The unit element is the empty forest  $\mathbf{1}$ .

*Remark 4.7* (The Grafting Operator). To systematically generate all trees, we define a linear operator  $B_+ : \mathcal{H}_{CK} \rightarrow \mathcal{H}_{CK}$ . For any forest  $F = T_1 T_2 \dots T_n$ , the operator  $B_+(F)$  creates a new tree by adding a single new vertex  $r_{\text{new}}$  (the new root) and drawing exactly one edge from  $r_{\text{new}}$  to the root of each tree  $T_i$  in  $F$ .

**Grading:** We grade  $\mathcal{H}_{CK}$  by the number of vertices. Let  $|F|$  denote the number of vertices in a forest  $F$ , with  $|\mathbf{1}| = 0$ . Since  $|F_1 F_2| = |F_1| + |F_2|$ ,  $\mathcal{H}_{CK}$  is a graded algebra:

$$(37) \quad \mathcal{H}_{CK} = \bigoplus_{n \geq 0} (\mathcal{H}_{CK})_n$$

Notice that  $(\mathcal{H}_{CK})_0 = k\mathbf{1}$ . Therefore, the algebra is connected.

**4.2.1. Admissible Cuts.** The coproduct on  $\mathcal{H}_{CK}$  relies entirely on the combinatorial process of slicing trees. Let  $T$  be a non-trivial rooted tree.

**Definition 4.8** (Cut). A cut  $c$  is a subset of the edges of  $T$ ,  $c \subset E(T)$ .

**Definition 4.9** (Admissible Cut). A cut  $c$  is *admissible* if, for every directed path starting from the root of  $T$  and ending at any leaf, the path contains **at most one** edge belonging to  $c$ .

We denote the set of all admissible cuts of  $T$  as  $\text{Adm}(T)$ . This set strictly includes the empty cut  $c = \emptyset$ .

**Definition 4.10** (Pruned Forest and Root Tree). Let  $c \in \text{Adm}(T)$  be a non-empty admissible cut. The removal of the edges in  $c$  disconnects the graph into several connected components.

- **The Root Tree  $R_c(T)$ :** The unique connected component that contains the original root of  $T$ .
- **The Pruned Forest  $P_c(T)$ :** The disjoint union of all other connected components.

*Convention:* For the empty cut  $c = \emptyset$ , we define  $P_\emptyset(T) = \mathbf{1}$  and  $R_\emptyset(T) = T$ . For the formal full cut  $c = V(T)$ , we define  $P_{\text{full}}(T) = T$  and  $R_{\text{full}}(T) = \mathbf{1}$ .

**4.2.2. The Coproduct and Bialgebra Structure.**

**Definition 4.11** (The Coproduct). We define a  $k$ -linear map  $\Delta : \mathcal{H}_{CK} \rightarrow \mathcal{H}_{CK} \otimes \mathcal{H}_{CK}$ . For a single tree  $T \in \mathcal{T}$ :

$$(38) \quad \Delta(T) = \mathbf{1} \otimes T + T \otimes \mathbf{1} + \sum_{c \in \text{Adm}(T), c \neq \emptyset} P_c(T) \otimes R_c(T)$$

We extend  $\Delta$  to all forests as an algebra morphism:  $\Delta(F_1 F_2) = \Delta(F_1) \Delta(F_2)$ .

**Definition 4.12** (The Counit). The map  $\varepsilon : \mathcal{H}_{CK} \rightarrow k$  is defined by  $\varepsilon(\mathbf{1}) = 1$  and  $\varepsilon(T) = 0$  for any non-empty tree  $T$ .

**Theorem 4.13.**  $(\mathcal{H}_{CK}, m, \mathbf{1}, \Delta, \varepsilon)$  is a graded connected bialgebra.

*Proof.* We demonstrate coassociativity:  $(\Delta \otimes id)\Delta(T) = (id \otimes \Delta)\Delta(T)$ . We reduce the general coassociativity of forests to the fundamental structural identity on a single tree  $T$ . By definition, an admissible cut on a tree  $T$  is a set of edges  $c$  such that every path from the root to a leaf contains at most one edge in  $c$ .

It is easy to see that it is enough to check that the reduced coproduct  $\tilde{\Delta} : \mathcal{H}_{CK} \rightarrow \mathcal{H}_{CK} \otimes \mathcal{H}_{CK}$ , defined on a rooted tree  $T$  by

$$(39) \quad \tilde{\Delta}(T) = \sum_{c \in C(T)} P_c(T) \otimes R_c(T)$$

where  $C(T)$  is the set of proper admissible cuts, is coassociative:  $(\tilde{\Delta} \otimes id) \circ \tilde{\Delta} = (id \otimes \tilde{\Delta}) \circ \tilde{\Delta}$ .

Evaluating the left-hand side on  $T$ ,  $(\tilde{\Delta} \otimes id)\tilde{\Delta}(T)$ , corresponds to choosing an initial admissible cut  $c_{lower}$ , which yields the tensor  $P_{c_{lower}}(T) \otimes R_{c_{lower}}(T)$ , and then applying  $\tilde{\Delta}$  to the pruned forest  $P_{c_{lower}}(T)$ . Choosing an admissible cut  $c_{upper}$  inside  $P_{c_{lower}}(T)$  is equivalent to specifying an **\*\*admissible double cut\*\***  $(c_{lower}, c_{upper})$  where  $c_{lower}$  and  $c_{upper}$  are disjoint, and any path from the root that intersects both encounters  $c_{lower}$  first. This yields the triple tensor:

$$(40) \quad (\tilde{\Delta} \otimes id)\tilde{\Delta}(T) = \sum_{(c_{lower}, c_{upper})} P_{c_{upper}}(P_{c_{lower}}(T)) \otimes R_{c_{upper}}(P_{c_{lower}}(T)) \otimes R_{c_{lower}}(T)$$

Conversely, evaluating the right-hand side,  $(id \otimes \tilde{\Delta})\tilde{\Delta}(T)$ , corresponds to choosing an initial admissible cut  $c_{upper}$  on  $T$ , and then applying  $\tilde{\Delta}$  to the root component  $R_{c_{upper}}(T)$ . Choosing a subsequent cut  $c_{lower}$  strictly inside  $R_{c_{upper}}(T)$  generates the exact same set of admissible double cuts  $(c_{lower}, c_{upper})$ , yielding:

$$(41) \quad (id \otimes \tilde{\Delta})\tilde{\Delta}(T) = \sum_{(c_{lower}, c_{upper})} P_{c_{upper}}(T) \otimes P_{c_{lower}}(R_{c_{upper}}(T)) \otimes R_{c_{lower}}(R_{c_{upper}}(T))$$

To establish equality term-by-term for any fixed double cut  $(c_{lower}, c_{upper})$ , we analyze the three tensor components:

- (1) **The Top Component (Leftmost tensor):** The operation  $P_{c_{upper}}(P_{c_{lower}}(T))$  removes the edges of  $c_{upper}$  from the forest  $P_{c_{lower}}(T)$ . Because  $c_{upper}$  is entirely contained within the branches of  $P_{c_{lower}}(T)$ , this operation is identical to pruning  $c_{upper}$  directly from the full tree  $T$ . Thus,  $P_{c_{upper}}(P_{c_{lower}}(T)) = P_{c_{upper}}(T)$ .
- (2) **The Root Component (Rightmost tensor):** The root component  $R_{c_{lower}}(T)$  is formed by the vertices connected to the root up to the lower cut  $c_{lower}$ . Because  $c_{lower}$  lies closer to the root than  $c_{upper}$ , this region is entirely unaffected by whether  $c_{upper}$  is subsequently cut. Thus,  $R_{c_{lower}}(T) = R_{c_{lower}}(R_{c_{upper}}(T))$ .
- (3) **The Intermediate Component (Middle tensor):** Both expressions,  $R_{c_{upper}}(P_{c_{lower}}(T))$  and  $P_{c_{lower}}(R_{c_{upper}}(T))$ , describe the exact same intermediate subgraph: the collection of vertices and edges isolated above  $c_{lower}$  but lying below  $c_{upper}$ .

□

4.2.3. *The Hopf Structure and the Antipode.* A direct consequence of Theorem 2.8 is the following.

**Corollary 4.14.**  $\mathcal{H}_{CK}$  is a Hopf algebra with antipode:

$$(42) \quad S(T) = -T - \sum_{c \in \text{Adm}(T), c \neq \emptyset} S(P_c(T))R_c(T)$$

4.2.4. *Explicit Computations.* Let  $\bullet$  denote the trivial tree.  $\Delta(\bullet) = \mathbf{1} \otimes \bullet + \bullet \otimes \mathbf{1}$  and  $S(\bullet) = -\bullet$ . Let  $T_2$  be the tree with two vertices ( $\bullet \rightarrow \bullet$ ).  $\text{Adm}(T_2)$  contains one proper cut.

$$\begin{aligned}\Delta(T_2) &= \mathbf{1} \otimes T_2 + T_2 \otimes \mathbf{1} + \bullet \otimes \bullet \\ S(T_2) &= -T_2 - S(\bullet)\bullet = -T_2 + \bullet\bullet\end{aligned}$$

Let  $T_V$  be the "V-shaped" tree with a root and two independent leaves.  $\text{Adm}(T_V)$  contains three proper cuts (cutting either branch, or both).

$$\begin{aligned}\Delta(T_V) &= \mathbf{1} \otimes T_V + T_V \otimes \mathbf{1} + 2(\bullet \otimes T_2) + \bullet\bullet \otimes \bullet \\ S(T_V) &= -T_V - 2S(\bullet)T_2 - S(\bullet\bullet)\bullet = -T_V + 2\bullet T_2 - \bullet\bullet\bullet\end{aligned}$$

Let  $T_3$  be the linear tree with three vertices (root  $\rightarrow$  node  $\rightarrow$  leaf).

$$\begin{aligned}\Delta(T_3) &= \mathbf{1} \otimes T_3 + T_3 \otimes \mathbf{1} + \bullet \otimes T_2 + T_2 \otimes \bullet \\ S(T_3) &= -T_3 - S(\bullet)T_2 - S(T_2)\bullet = -T_3 + 2\bullet T_2 - \bullet\bullet\bullet\end{aligned}$$

**A Four-Vertex Example:** Let  $T_Y$  be the "Y-shaped" tree: a root connected to a middle node, which then branches to two independent leaves. The edges are  $e_r$  (bottom) and  $e_1, e_2$  (top branches). The proper admissible cuts are:

- $c = \{e_r\}$ : yields pruned forest  $T_V$  and root tree  $\bullet$ .
- $c = \{e_1\}$ : yields pruned forest  $\bullet$  and root tree  $T_3$ .
- $c = \{e_2\}$ : yields pruned forest  $\bullet$  and root tree  $T_3$ .
- $c = \{e_1, e_2\}$ : yields pruned forest  $\bullet\bullet$  and root tree  $T_2$ .

Note that  $\{e_r, e_1\}$  is **not** admissible, as a path from root to the first leaf would cross two cuts.

$$(43) \quad \Delta(T_Y) = \mathbf{1} \otimes T_Y + T_Y \otimes \mathbf{1} + T_V \otimes \bullet + 2(\bullet \otimes T_3) + \bullet\bullet \otimes T_2$$

To compute the antipode, we apply the recursion:

$$(44) \quad S(T_Y) = -T_Y - S(T_V)\bullet - 2S(\bullet)T_3 - S(\bullet\bullet)T_2$$

Substituting the previously computed antipodes (and using  $S(\bullet\bullet) = \bullet\bullet$ ):

$$\begin{aligned}S(T_Y) &= -T_Y - (-T_V + 2\bullet T_2 - \bullet\bullet\bullet)\bullet - 2(-\bullet)T_3 - (\bullet\bullet)T_2 \\ &= -T_Y + T_V\bullet - 2\bullet\bullet T_2 + \bullet\bullet\bullet\bullet + 2\bullet T_3 - \bullet\bullet T_2 \\ &= -T_Y + T_V\bullet - 3\bullet\bullet T_2 + \bullet\bullet\bullet\bullet + 2\bullet T_3\end{aligned}$$

4.3. **Connection to trees.** We now make precise the relation between the Hopf algebra of Feynman graphs and the Hopf algebra of rooted trees.

Let  $\mathcal{H}_{FG}$  denote the Hopf algebra of Feynman graphs and  $\mathcal{H}_{CK}$  the Hopf algebra of rooted trees whose vertices are decorated by *primitive* 1PI graphs, where  $\pi$  is primitive when  $\mathcal{D}(\pi) = \emptyset$ , i.e. when  $\Delta(\pi) = \pi \otimes 1 + 1 \otimes \pi$ .

The purpose of the rooted-tree representation is to record the nesting and disjointness relations among the divergent subgraphs of a graph. Overlapping subgraphs cannot occur simultaneously in a forest, and a graph with overlapping subdivergences therefore gives rise to several different rooted trees. The main result of this subsection is that, after summing over all of them, one obtains a morphism of Hopf algebras

$$\mathcal{T} : \mathcal{H}_{FG} \longrightarrow \mathcal{H}_{CK},$$

which is however neither injective nor surjective: the two Hopf algebras are related, not equivalent.

**Hypothesis.** *Throughout we use only the following property of the notion of divergence, which we call  $(\star)$ : if  $\alpha \subseteq \delta$  are subgraphs of  $\Gamma$  with  $\alpha$  admissible, then*

$$\delta \text{ is admissible in } \Gamma \quad \Longleftrightarrow \quad \delta/\alpha \text{ is admissible in } \Gamma/\alpha.$$

In the undecorated (core) Hopf algebra this is Lemma 3.22. In the decorated setting it is that lemma together with the invariance of the boundary under contraction,  $\text{res}(\delta/\alpha) = \text{res}(\delta)$ , which is what makes divergence a property of the residue alone.

Everything below is therefore valid verbatim for both versions of the graph Hopf algebra.

#### 4.3.1. Forests and maximal forests.

**Definition 4.15.** A *forest* of a 1PI graph  $\Gamma$  is a collection

$$F = \{\gamma_1, \dots, \gamma_r\}$$

of *connected* proper divergent 1PI subgraphs of  $\Gamma$  such that any two elements are either disjoint or nested. The empty collection is a forest. A forest is *maximal* if it is not properly contained in any other forest of  $\Gamma$ .

Requiring the elements to be connected is what makes the Hasse diagram below a tree; a disconnected admissible subgraph enters as the set of its components. Since  $\Gamma$  is finite, every forest is contained in a maximal one.

For a maximal forest  $F$ , consider the poset  $F \cup \{\Gamma\}$  ordered by inclusion. Its Hasse diagram is a rooted tree: the root is  $\Gamma$ , and the children of  $\gamma$  are the maximal elements of  $F$  properly contained in  $\gamma$ . The vertices are decorated not by the elements of  $F \cup \{\Gamma\}$  themselves but by the corresponding *primitive quotients*: if  $\gamma \in F \cup \{\Gamma\}$  has children  $\gamma_1, \dots, \gamma_s$ , the decoration of the corresponding vertex is

$$\pi_\gamma := \gamma / (\gamma_1 \cdots \gamma_s).$$

We denote the resulting decorated rooted tree by  $t_F$ . Thus the leaves are decorated by primitive graphs, and every internal vertex is decorated by the graph obtained after contracting all the divergent subgraphs represented by its children. If  $\Gamma$  is primitive, the empty forest gives the one-vertex tree  $\bullet_\Gamma$ .

That the decorations are primitive is exactly where maximality is used, and it deserves a proof.

**Lemma 4.16.** *Let  $F$  be a maximal forest of  $\Gamma$ , let  $\gamma \in F \cup \{\Gamma\}$ , and let  $\gamma_1, \dots, \gamma_s$  be the maximal elements of  $F$  properly contained in  $\gamma$ . Then  $\gamma_1, \dots, \gamma_s$  are pairwise disjoint and  $\pi_\gamma$  is primitive.*

*Proof.* The  $\gamma_i$  are pairwise incomparable elements of a forest, hence pairwise disjoint, so  $\pi_\gamma$  is defined and is a connected 1PI graph.

Suppose  $\pi_\gamma$  had a proper divergent 1PI subgraph  $\delta'$ . Let  $v_1, \dots, v_s$  be the vertices of  $\pi_\gamma$  obtained by contracting  $\gamma_1, \dots, \gamma_s$ , and let  $\delta \subseteq \gamma$  be the lift of  $\delta'$ : the subgraph of  $\gamma$  whose internal edges are those of  $\delta'$  together with those of every  $\gamma_i$  with  $v_i \in V(\delta')$ . Then:

- $\delta$  is admissible in  $\gamma$ , by the reflection half of  $(\star)$  applied to  $\delta' = \delta / (\delta \cap \gamma_1 \cdots \gamma_s)$ ;
- $\delta \subsetneq \gamma$ , since  $\delta'$  is proper;
- $\delta$  is nested or disjoint with each  $\gamma_i$ : if  $v_i \in V(\delta')$  then  $\gamma_i \subseteq \delta$ , and otherwise  $\delta \cap \gamma_i = \emptyset$ ;
- $\delta$  is nested or disjoint with every other  $\eta \in F$ : if  $\eta \subsetneq \gamma$  then  $\eta$  lies inside some  $\gamma_i$ , hence inside  $\delta$  or disjoint from it; if  $\eta \supseteq \gamma$  then  $\eta \supseteq \delta$ ; and if  $\eta$  is disjoint from  $\gamma$  it is disjoint from  $\delta$ .

Hence  $F \cup \{\delta\}$  is a forest strictly containing  $F$ , contradicting maximality.  $\square$

#### 4.3.2. The tree map.

**Definition 4.17.** Let  $\mathfrak{M}(\Gamma)$  be the set of maximal forests of  $\Gamma$ . For a connected 1PI graph  $\Gamma$  put

$$\mathcal{T}(\Gamma) = \sum_{F \in \mathfrak{M}(\Gamma)} t_F,$$

extend multiplicatively,  $\mathcal{T}(\Gamma_1 \cdots \Gamma_r) = \mathcal{T}(\Gamma_1) \cdots \mathcal{T}(\Gamma_r)$ , and set  $\mathcal{T}(1) = 1$ .

So  $\mathcal{T}$  is by construction an algebra morphism  $\mathcal{H}_{FG} \rightarrow \mathcal{H}_{CK}$ . The sum over maximal forests is what handles overlapping subdivergences: if all divergent subgraphs are mutually nested or disjoint there is a single maximal forest and a single tree, whereas overlapping subgraphs force incompatible choices and  $\mathcal{T}(\Gamma)$  becomes a sum.

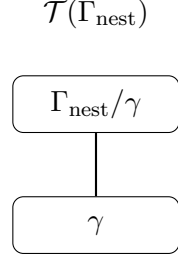


FIGURE 4. The tree associated with the unique maximal forest  $F = \{\gamma\}$  of  $\Gamma_{\text{nest}}$ .

**Example 4.18** (Nested case, schematic).

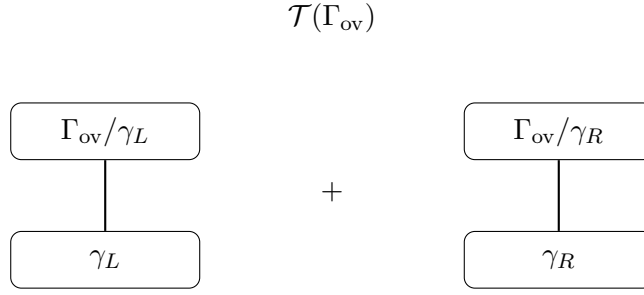


FIGURE 5. The tree representation of  $\Gamma_{\text{ov}}$ . The two maximal forests are  $F_L = \{\gamma_L\}$  and  $F_R = \{\gamma_R\}$ .

**Example 4.19** (Overlapping case, schematic).

**Example 4.20** (The theta graph, concretely). Let  $\Theta = B_3$  be the graph with two vertices joined by three edges  $e_1, e_2, e_3$ , and write  $B_2$  for the bubble and  $T_1$  for the tadpole. Its proper divergent subgraphs are the three cycles  $\gamma_{12}, \gamma_{13}, \gamma_{23}$ , which pairwise overlap, so

$$\mathfrak{M}(\Theta) = \{\{\gamma_{12}\}, \{\gamma_{13}\}, \{\gamma_{23}\}\},$$

the empty forest not being maximal. Each  $t_F$  is the two-vertex chain with root decorated by  $\Theta/\gamma_{ij} \cong T_1$  and leaf decorated by  $\gamma_{ij} \cong B_2$ ; both decorations have loop number 1, hence are primitive. The three trees are isomorphic, so

$$\mathcal{T}(\Theta) = 3 B_+^{T_1}(\bullet_{B_2}),$$

matching  $\Delta(\Theta) = \Theta \otimes 1 + 1 \otimes \Theta + 3 B_2 \otimes T_1$ .

**4.3.3. Cuts and the splitting of maximal forests.** Recall that an *admissible cut* of a rooted tree  $t$  is a set  $c$  of edges meeting every root-to-leaf path at most once; equivalently, an antichain of non-root vertices, namely the lower endpoints of the cut edges. One writes  $P_c(t)$  for the product of the branches removed and  $R_c(t)$  for the trunk containing the root, so that

$$\Delta_{CK}(t) = t \otimes 1 + 1 \otimes t + \sum_{c \text{ proper admissible}} P_c(t) \otimes R_c(t).$$

The proof of compatibility rests on two statements: an admissible cut of  $t_F$  is the same thing as an admissible subgraph of  $\Gamma$  all of whose components lie in  $F$ , and the maximal forests containing a given admissible subgraph split as a product.

**Proposition 4.21.** *Let  $\Gamma$  be connected 1PI and let  $\gamma = \gamma_1 \sqcup \dots \sqcup \gamma_k \in \mathcal{D}(\Gamma)$ . For a forest  $F$  write  $F|_{\gamma_i} = \{\eta \in F : \eta \subsetneq \gamma_i\}$  and  $F/\gamma = \{\eta/\gamma : \eta \in F, \eta \not\subseteq \gamma\}$ . Then*

$$F \longmapsto (F|_{\gamma_1}, \dots, F|_{\gamma_k}, F/\gamma)$$

*is a bijection*

$$\{F \in \mathfrak{M}(\Gamma) : \gamma_1, \dots, \gamma_k \in F\} \xrightarrow{\sim} \mathfrak{M}(\gamma_1) \times \dots \times \mathfrak{M}(\gamma_k) \times \mathfrak{M}(\Gamma/\gamma).$$

*Proof.* Let  $F$  be a forest containing every  $\gamma_i$ . Each  $\eta \in F$  is nested or disjoint with each  $\gamma_i$ , so exactly one of the following holds:  $\eta \subsetneq \gamma_i$  for a unique  $i$ ;  $\eta = \gamma_i$ ; or  $\eta \not\subseteq \gamma$ . In the last case  $\eta$  contains some of the  $\gamma_i$  and is disjoint from the others, so  $\eta/\gamma$  is a connected divergent 1PI subgraph of  $\Gamma/\gamma$  by  $(\star)$ . Nestedness and disjointness are preserved in both directions, so the three kinds of pieces are forests and the map is well defined; the inverse sends a tuple to the union of the  $F|_{\gamma_i}$ , the  $\gamma_i$  themselves, and the lifts of the elements of  $F/\gamma$ .

It remains to match maximality. Suppose  $F$  is maximal. If some  $F|_{\gamma_i}$  were not maximal in  $\gamma_i$ , a compatible  $\eta \subsetneq \gamma_i$  could be added, and  $\eta$  is automatically compatible with the rest of  $F$ , since every other element of  $F$  either contains  $\gamma_i$ , is contained in  $\gamma_i$ , or is disjoint from it; this contradicts maximality of  $F$ . The same argument applied to lifts shows  $F/\gamma$  is maximal. Conversely, if all  $k+1$  pieces are maximal and  $\eta$  could be added to  $F$ , then  $\eta$  is nested or disjoint with each  $\gamma_i$ , so either  $\eta \subsetneq \gamma_i$  for some  $i$ , contradicting maximality of  $F|_{\gamma_i}$ , or  $\eta \not\subseteq \gamma$  and  $\eta/\gamma$  contradicts maximality of  $F/\gamma$ .  $\square$

**Lemma 4.22.** *Let  $F \in \mathfrak{M}(\Gamma)$ . The proper admissible cuts of  $t_F$  are in bijection with the sets  $\{\gamma_1, \dots, \gamma_k\}$ ,  $k \geq 1$ , of pairwise disjoint elements of  $F$ . For the cut  $c$  corresponding to such a set, with  $\gamma = \gamma_1 \sqcup \dots \sqcup \gamma_k \in \mathcal{D}(\Gamma)$ ,*

$$P_c(t_F) = t_{F|_{\gamma_1}} \cdots t_{F|_{\gamma_k}}, \quad R_c(t_F) = t_{F/\gamma}.$$

*Proof.* The non-root vertices of  $t_F$  are the elements of  $F$ , and two of them are incomparable in the tree order precisely when they are incomparable in  $F$ , hence disjoint. This gives the bijection between antichains and pairwise disjoint families; the union of such a family is admissible because its members are connected divergent 1PI subgraphs and are pairwise disjoint.

The branch cut off above  $\gamma_i$  has vertex set  $\{\gamma_i\} \cup F|_{\gamma_i}$  with the induced order, which is the Hasse diagram of the maximal forest  $F|_{\gamma_i}$  of  $\gamma_i$ , and its decorations are computed from elements contained in  $\gamma_i$  only, so they agree with those of  $t_{F|_{\gamma_i}}$ .

For the trunk, its vertices are  $\Gamma$  together with the  $\eta \in F$  with  $\eta \not\subseteq \gamma$ , which correspond to the vertices of  $t_{F/\gamma}$ . The decorations agree: if  $\eta$  has children  $\gamma_{i_1}, \dots, \gamma_{i_m}$  among the cut vertices and  $\eta_1, \dots, \eta_n$  among the others, its decoration in  $t_F$  is  $\eta/(\gamma_{i_1} \cdots \gamma_{i_m} \eta_1 \cdots \eta_n)$ , whereas the vertex  $\eta/\gamma$  of  $t_{F/\gamma}$  is decorated by  $(\eta/\gamma)/(\eta_1/\gamma \cdots \eta_n/\gamma)$ . These coincide by transitivity of contraction,  $(\Gamma/\delta)/(\gamma/\delta) \cong \Gamma/\gamma$  for  $\delta \subseteq \gamma \subseteq \Gamma$ .  $\square$

In words: an admissible cut separates a compatible forest of divergent subgraphs from the contracted remainder,

$$P_c(t_F) \longleftrightarrow \gamma, \quad R_c(t_F) \longleftrightarrow \Gamma/\gamma.$$

#### 4.3.4. Compatibility with the coproducts.

**Theorem 4.23.**

$$\Delta_{CK} \mathcal{T} = (\mathcal{T} \otimes \mathcal{T}) \Delta_{FG}$$

and  $\varepsilon_{CK} \mathcal{T} = \varepsilon_{FG}$ . Hence  $\mathcal{T} : \mathcal{H}_{FG} \rightarrow \mathcal{H}_{CK}$  is a morphism of Hopf algebras; in particular

$$\boxed{\mathcal{T} \circ S_{FG} = S_{CK} \circ \mathcal{T}.$$

*Proof.*  $\mathcal{T}$  is an algebra morphism by definition, and  $\varepsilon_{CK} \mathcal{T} = \varepsilon_{FG}$  because  $\mathcal{T}$  sends a nonempty graph to a sum of nonempty trees. Since both coproducts are multiplicative it suffices to verify the identity on a connected 1PI graph  $\Gamma$ . The terms  $\mathcal{T}(\Gamma) \otimes 1$  and  $1 \otimes \mathcal{T}(\Gamma)$  match, so it remains to compare

$$\sum_{F \in \mathfrak{M}(\Gamma)} \sum_c P_c(t_F) \otimes R_c(t_F) \quad \text{and} \quad \sum_{\gamma \in \mathcal{D}(\Gamma)} \mathcal{T}(\gamma) \otimes \mathcal{T}(\Gamma/\gamma).$$

By Lemma 4.22 the left-hand sum is indexed by pairs  $(F, \gamma)$  with  $\gamma = \gamma_1 \sqcup \cdots \sqcup \gamma_k \in \mathcal{D}(\Gamma)$  and  $F$  a maximal forest containing every  $\gamma_i$ , the corresponding term being  $\prod_i t_{F|_{\gamma_i}} \otimes t_{F/\gamma}$ . Fixing  $\gamma$  and applying Proposition 4.21, the inner sum becomes

$$\sum_{F_1 \in \mathfrak{M}(\gamma_1)} \cdots \sum_{F_k \in \mathfrak{M}(\gamma_k)} \sum_{F_0 \in \mathfrak{M}(\Gamma/\gamma)} t_{F_1} \cdots t_{F_k} \otimes t_{F_0} = \mathcal{T}(\gamma) \otimes \mathcal{T}(\Gamma/\gamma),$$

using  $\mathcal{T}(\gamma) = \prod_i \mathcal{T}(\gamma_i)$ . Summing over  $\gamma \in \mathcal{D}(\Gamma)$  gives the right-hand side. Finally, a morphism of bialgebras between Hopf algebras automatically commutes with the antipodes, by uniqueness of the convolution inverse of the identity.  $\square$

*Remark 4.24* (The right grading on trees).  $\mathcal{T}$  does *not* respect the grading of  $\mathcal{H}_{CK}$  by number of vertices: distinct maximal forests of the same  $\Gamma$  may have different cardinalities, so  $\mathcal{T}(\Gamma)$  is in general inhomogeneous for that grading. It does respect the *weight* grading

$$w(t) = \sum_{v \in V(t)} L(\text{dec}(v)),$$

since additivity of the loop number under contraction gives  $w(t_F) = L(\Gamma)$  for every  $F \in \mathfrak{M}(\Gamma)$ . With this grading  $\mathcal{H}_{CK}$  is again connected graded, every primitive decoration having  $L \geq 1$ , and  $\mathcal{T}$  is a morphism of graded connected Hopf algebras. The antipode identity above does not need the grading.

**Example 4.25** (Verification on the 4-banana). Let  $\Gamma = B_4$ , with  $T_j$  the graph with one vertex and  $j$  self-loops and  $B_k$  the two-vertex graph with  $k$  edges. Recall

$$\Delta(B_4) = B_4 \otimes 1 + 1 \otimes B_4 + 6 B_2 \otimes T_2 + 4 B_3 \otimes T_1.$$

Every proper divergent subgraph of  $B_4$  is a  $B_k$  with  $k \in \{2, 3\}$  and any two of the same size overlap, so the maximal forests are the chains  $B_2 \subsetneq B_3$ : there are  $4 \cdot 3 = 12$  of them, and each  $t_F$  is the three-vertex chain decorated, from the leaf up, by  $B_2, T_1, T_1$ . Hence  $\mathcal{T}(B_4) = 12 B_+^{T_1} B_+^{T_1}(\bullet_{B_2})$ .

Cutting that chain at its two edges gives

$$\tilde{\Delta}_{CK} \mathcal{T}(B_4) = 12 \bullet_{B_2} \otimes B_+^{T_1}(\bullet_{T_1}) + 12 B_+^{T_1}(\bullet_{B_2}) \otimes \bullet_{T_1}.$$

On the other side,  $\mathcal{T}(B_2) = \bullet_{B_2}$ ,  $\mathcal{T}(T_1) = \bullet_{T_1}$ ,  $\mathcal{T}(T_2) = 2 B_+^{T_1}(\bullet_{T_1})$  (the two self-loops share a vertex, so the two maximal forests are the two singletons), and  $\mathcal{T}(B_3) = 3 B_+^{T_1}(\bullet_{B_2})$  by Example 4.20. Therefore

$$(\mathcal{T} \otimes \mathcal{T}) \tilde{\Delta}_{FG}(B_4) = 6 \cdot 2 \bullet_{B_2} \otimes B_+^{T_1}(\bullet_{T_1}) + 4 \cdot 3 B_+^{T_1}(\bullet_{B_2}) \otimes \bullet_{T_1},$$

which agrees. The two ways of counting,  $6 \cdot 2$  and  $4 \cdot 3$ , are the two sides of Proposition 4.21 for this graph.

4.3.5.  $\mathcal{T}$  is neither injective nor surjective.

**Proposition 4.26.**  $\mathcal{T}$  is in general not injective.

*Proof.* Let  $p$  be a primitive graph possessing two internal edges  $e_1, e_2$  of the same type lying in different orbits of  $\text{Aut}(p)$ , and let  $q$  be a primitive graph whose residue is the propagator residue of that type. Let  $\Gamma_i$  be obtained from  $p$  by inserting  $q$  into  $e_i$ , and assume  $q$  is the only divergent proper subgraph of each  $\Gamma_i$ . Then  $\Gamma_1 \not\cong \Gamma_2$ , while each  $\Gamma_i$  has the single maximal forest  $\{q\}$  and  $\Gamma_i/q \cong p$ , so

$$\mathcal{T}(\Gamma_1) = \mathcal{T}(\Gamma_2) = B_+^p(\bullet_q).$$

The tree records *what* was inserted, never *where*. □

**Proposition 4.27.**  $\mathcal{T}$  is in general not surjective.

*Proof.* Two independent obstructions. First, if  $p, q$  are primitive graphs such that  $\text{res}(q)$  is the residue of no internal edge or vertex of  $p$ , then no graph  $\Gamma$  satisfies  $\Gamma/\gamma \cong p$  with  $\gamma \cong q$ , so  $B_+^p(\bullet_q) \in \mathcal{H}_{CK}$  is not in the image. Second, when  $\Gamma$  has overlapping divergences  $\mathcal{T}(\Gamma)$  is a sum of trees with multiplicities — three copies of one tree for  $\Theta$ , twelve for  $B_4$  — and in general no element of  $\mathcal{H}_{FG}$  maps to a single summand. □

4.3.6. Two maps, in opposite directions.

*Remark 4.28.* The tree Hopf algebra also maps to  $\mathcal{H}_{FG}$ . Writing  $B_+^\pi$  for the insertion operator attaching the components of its argument at the insertion places of a primitive graph  $\pi$ , the family  $\{B_+^\pi\}$  consists of Hochschild 1-cocycles, so the universal property of  $\mathcal{H}_{CK}$  produces a unique morphism

$$\rho : \mathcal{H}_{CK} \longrightarrow \mathcal{H}_{FG}, \quad \rho \circ B_+^\pi = B_+^\pi \circ \rho.$$

Neither composite is the identity:  $\rho \mathcal{T}(\Gamma)$  contains  $\Gamma$  with multiplicity  $|\mathfrak{M}(\Gamma)|$  together with every other graph sharing its nesting data. The two Hopf algebras are therefore genuinely different.  $\mathcal{H}_{CK}$  is free on the primitives;  $\mathcal{H}_{FG}$  additionally remembers where each insertion took place and allows overlapping subdivergences, which no single tree can express. On the subalgebra generated by graphs with a unique insertion place and no overlaps — ladders and rainbows —  $\mathcal{T}$  does restrict to an isomorphism onto the corresponding ladder trees.

*Remark 4.29.* Note finally what the forest condition does and does not exclude. Within a single term of  $\Delta_{FG}$  the components are disjoint, and within a single forest the elements are nested or disjoint; but overlapping subgraphs are not excluded from the theory. They appear in *different* terms of  $\Delta_{FG}(\Gamma)$  and, on the tree side, in different maximal forests. This is precisely the discrepancy that forces  $\mathcal{T}(\Gamma)$  to be a sum: the graph coproduct needs no global compatibility condition, since each of its terms carries only one antichain's worth of information, whereas a forest imposes pairwise compatibility on a whole collection at once.

## 5. APPENDIX I: HISTORY

**5.1. Act I: Scribbles in Pocono and the Birth of Diagrams.** It is the spring of 1948. At the secluded Pocono Manor Inn in Pennsylvania, the elite of theoretical physics gathers to confront a crisis: quantum electrodynamics (QED) is plagued by unmanageable mathematical infinities. Julian Schwinger, a mathematical prodigy, takes the floor and fills the blackboards with hours of dense, masterful analytical derivations to calculate the magnetic moment of the electron. The audience is exhausted.

Shortly after, it is Richard Feynman's turn. Instead of endless equations, Feynman begins drawing straight lines for electrons and wavy lines for photons on the board. He is presenting a visual, intuitive tool that directly maps the terms of the perturbative expansion of the  $S$ -matrix.

The initial reaction is almost universal rejection. Niels Bohr reprimands him, mistakenly believing that Feynman is ignoring the Pauli exclusion principle. J. Robert Oppenheimer interrupts constantly. However, Feynman's drawings conceal a profound mathematical truth. In 1949, Freeman Dyson rigorously proves that the analytical methods of Schwinger and Tomonaga are mathematically equivalent to Feynman diagrams.

- Feynman, R. P. (1949). *Space-Time Approach to Quantum Electrodynamics*. Physical Review, 76(6), 769–789.
- Dyson, F. J. (1949). *The Radiation Theories of Tomonaga, Schwinger, and Feynman*. Physical Review, 75(3), 486–502.

**5.2. Act II: The Subtraction of Infinities and the Bitter “Argentine Component”.**

Feynman diagrams simplified the formulation, but they did not cure the disease: closed loops continued to yield divergent integrals in the ultraviolet limit ( $k \rightarrow \infty$ ). To make sense of the theory, it had to be “renormalized,” which first required parametrically “regularizing” the integrals.

In the early 1970s, the problem of regularizing non-Abelian gauge field theories was the Holy Grail. In 1972, in the corridors of the Physics Department at the National University of La Plata (Argentina), Carlos Bollini and Juan José Giambiagi propose a radical, abstract, and mathematically elegant idea: abandoning the fixed nature of four spacetime dimensions. They analytically extend the integral to a complex dimension  $D$ , setting  $D = 4 - \epsilon$ . The divergences then manifest as simple poles of the form  $1/\epsilon$  in the Laurent series, flawlessly separating the finite part from the infinite.

They submit their manuscript to \*Physics Letters\* in late 1971. The referee rejects it, arguing that a non-integer dimension makes no physical sense. After appealing and restructuring the work, they manage to publish it first as a brief note in *Physics Letters B* and later in full in *Il Nuovo Cimento*.

Almost simultaneously, at Utrecht University, a young student named Gerard 't Hooft and his supervisor Martinus Veltman, entirely unaware of the work in La Plata, develop the exact same technique (which 't Hooft arrives at while seeking to preserve gauge symmetry). They publish it in *Nuclear Physics B* in 1972.

The subsequent history is well known in academic circles. The scientific machinery of the Global North adopted the names of 't Hooft and Veltman for “dimensional regularization”. In 1999, the Royal Swedish Academy of Sciences awarded both of them the Nobel Prize in Physics. Bollini and Giambiagi, pioneers from the Global South, were marginalized from the ultimate accolade—a testament to the asymmetries between peripheral and central science, leaving a bitter taste but an undeniable sense of pride in the history of Argentine science.

- Bollini, C. G., & Giambiagi, J. J. (1972). *Dimensional Renormalization: The Number of Dimensions as a Regularizing Parameter*. Il Nuovo Cimento B, 12(1), 20-36.
- 't Hooft, G., & Veltman, M. (1972). *Regularization and Renormalization of Gauge Fields*. Nuclear Physics B, 44(1), 189-213.

**5.3. Act III: Abstract Algebra Reclaims its Territory.** For decades, renormalization worked beautifully, but for pure mathematicians, it was an obscure algorithm—a recipe of nested subtractions (like the Zimmermann forest) lacking a neat underlying mathematical structure.

In 1997, physicist Dirk Kreimer attends a conference where he realizes that the combinatorics of nested subdivergences in Feynman diagrams obey strict factorization rules. Kreimer discovers that Feynman diagrams, organized as rooted trees, form a graded and connected Hopf algebra  $\mathcal{H}$ .

Alain Connes, a Fields Medalist, grasps the magnitude of Kreimer’s discovery. Together, between 1998 and 2000, they publish a series of foundational papers proving that the Bogoliubov-Parasiuk-Hepp-Zimmermann (BPHZ) renormalization algorithm is not an ad hoc trick, but is equivalent to the Birkhoff decomposition of a character over the Hopf algebra of Feynman diagrams, valued in the algebra of Laurent series.

Mathematically, dimensional regularization and minimal subtraction operate by defining a character  $\phi : \mathcal{H} \rightarrow \mathcal{A}$ . The coproduct  $\Delta$  of the Hopf algebra, which encodes the extraction of subdivergences, allows for the definition of the convolution product. Feynman’s counterterm (the subtraction term) is simply the divergent part obtained by applying the antipode  $S$  of the algebra, and the renormalized (finite and observable) value is the holomorphic part of the Birkhoff factorization:  $\phi_+ = \phi_- * \phi$ .

The circle is complete. What began as heuristic drawings on a blackboard in Pennsylvania and transitioned through analytical extensions conceived in La Plata, found its permanent home in the profound structural beauty of Hopf algebras.

- Kreimer, D. (1998). *On the Hopf algebra structure of perturbative quantum field theories*. Advances in Theoretical and Mathematical Physics, 2(2), 303-334.
- Connes, A., & Kreimer, D. (2000). *Renormalization in quantum field theory and the Riemann-Hilbert problem I: The Hopf algebra structure of graphs and the main theorem*. Communications in Mathematical Physics, 210(1), 249-273.

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