ALGEBRA AND GEOMETRY: TO-AND-FRO, ONE MORE TIME

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Contents

Did people in the Middle-Ages think that connections were flat?

1. Introduction

In this work we present a characterization of certain algebraic object, namely (anti-)Yetter Drinfeld modules over a Hopf algebra H as modules involving a deep geometric property, such as having a flat connection. We provide an identification of the two later concepts.

Following [KK], we define a new algebra of noncommutative differential forms over any Hopf algebra H with an invertible antipode. The resulting differential calculus, denoted here by $\mathcal{K}^*(H)$, is intimately related to the class of anti-Yetter-Drinfeld modules over H. More precisely, we show that there is a one to one correspondence between anti-Yetter-Drinfeld modules over H and H–modules that admit a flat connection with respect to our differential calculus $\mathcal{K}^*(H)$.

Furthermore, in the last part of the work this characterization is considered once again and it is proven as a corollary of a more general theory, based on [B]. In this part, the relationship between comodules of a coring and flat connections is reviewed. In particular we specialize to corings which are built on a tensor product of algebra and a coalgebra. Such corings are in one-to-one correspondence with entwining structures, and their comodules are entwined modules. These include Yetter-Drinfeld and anti-Yetter-Drinfeld modules and their generalizations. In this way the interpretation of the latter as modules with flat connections given before is obtained as a corollary of a more general theory.

The work answers to the following organization.

In Sections 2 and 3 we introduce, respectively, the basic algebraic and geometric concepts that will be buildings block of our language during the exposition of the work.

In Section 4 we build the differential calculus $\mathcal{K}^*(H)$, for a Hopf algebra H and prove our main result identifying Anti Yetter Drinfeld modules over H and H modules with a flat connection with respect to this new calculus. An analogous result for Yetter Drinfeld modules is also developed.

In Section 5 we extend our algebraic concepts to a more general setting and we finally develop, in Section 6, a similar theory for these new objects to the one that raised in Section 4, including these previous results as corollaries.

2. Algebra I

To introduce the definition of Hopf algebras, we have to go gradually introducing a couple of previous concepts. For more details concerning these structures, we refer the reader to the books:

(1) *Hopf algebras.* Sweedler [S69].
(2) *Quantum groups.* Kassel [Kas95].
(3) *Hopf algebras and their action on rings.* Montgomery [Mon93]

Definition 2.1. *A counital, coassociative coalgebra over a field ⅂ is ⅂-vector space together with two maps, a multiplication $\Delta : H \to H \otimes H$ and a counit $\epsilon :\to$ ⅂ such that the following identities hold:*

$$(\Delta \otimes \mathrm{id})\Delta = (\mathrm{id} \otimes \Delta)\Delta, (\epsilon \otimes \mathrm{id})\Delta = (\mathrm{id} \otimes \epsilon)\Delta = \mathrm{id}_H,$$

where we identify $\Bbbk \otimes H \cong H \otimes \Bbbk \cong H$.

Definition 2.2. *A bialgebra over a field $\Bbbk$ is an algebra B which is also a coalgebra and such that the maps that provide it the structure of coalgebra are algebra maps.*

Definition 2.3. *A Hopf algebra H over a field $\Bbbk$ is a bialgebra together with a map $S : H \to H$, the "antipode" such that*

$$m(S \otimes \mathrm{id})\Delta = m(\mathrm{id} \otimes \Delta) = u\epsilon,$$

where $m : H \otimes H \to H$ and $u : \Bbbk \to H$ denote the multiplication and the unit in the algebra H.

Throughout this work, when referring to Hopf algebras, we will make use of the so called *Sweedler notation* [S69] for the coproduct. Therefore, if H is a Hopf algebra with comultiplication Δ and $x \in H$, we will, for example, write:

$$\Delta(x) = x_{(1)} \otimes x_{(2)},$$

and

$$(\mathrm{id} \otimes \Delta)\Delta(x) = (\Delta \otimes \mathrm{id})\Delta(x) = x_{(1)} \otimes x_{(2)} \otimes x_{(3)}.$$

We do not define here morphisms: coalgebra maps, Hopf algebra maps, which are linear maps that respect the structures involved.

Definition 2.4. *Let (C, Δ, ϵ) be a coalgebra. A (right) comodule M over C is a $\Bbbk$-vector space together with a map $\rho : N \to C \otimes N$ rendering the following diagrams commutative:*

$$\begin{array}{ccc} M & \xrightarrow{\varrho} & M \otimes_A C \\ \downarrow{\scriptstyle \varrho} & & \downarrow{\scriptstyle M \otimes_A \Delta} \\ M \otimes_A \mathrm{id} & \xrightarrow{\varrho \otimes_A \mathrm{id}} & M \otimes_A C \otimes_A C \end{array} \qquad \begin{array}{ccc} M & \xrightarrow{\varrho} & M \otimes_A C \\ & \searrow^{\simeq} & \downarrow{\scriptstyle M \otimes_A \epsilon} \\ & & M \otimes_A A \end{array}$$

The map ρ is said to be a coaction.

Analogously, one can define left comodules, bicomodules, etc. We leave to the imagination of the reader the obvious definition and properties of comodule morphisms.

Definition 2.5. *Let H be a $\Bbbk-$Hopf algebra. A $\Bbbk-$vector space X is called a left-left* anti-Yetter-Drinfeld module *(AYD module, for short) if*

- *X is a left H-module,*
- *X is a left H-comodule,*
- *the following compatibility condition between the H–module and comodule structure on X holds:*

$$(hx)_{(-1)} \otimes (hx)_{(0)} = h_{(1)}x_{(-1)}S^{-1}(h_{(3)}) \otimes h_{(2)}x_{(0)} \tag{1}$$

for any $h \in H$ and $x \in X$.

A *Yetter Drinfeld module* Y over a Hopf algebra H is defined in an analogous manner, replacing the compatibility condition on the third item by

$$(hy)_{(-1)} \otimes (hy)_{(0)} = h_{(1)} y_{(-1)} S(h_{(3)}) \otimes h_{(2)} y_{(0)} \tag{2}$$

for any $h \in H$ and $y \in Y$.

Definition 2.6. *A morphism of AYD (resp, YD) modules $X \to Y$ is simply a k-linear map which is also a map of modules and comodules.*

Remark 2.7 ([Mon93]). The resulting category of AYD modules and YD-modules over H is an abelian category.

To understand these compatibility conditions better we proceed as follows. Let X be a left H module. We define a left H–action on $H \otimes X$ by letting

$$h(g \otimes x) := h_{(1)} g S^{-1}(h_{(3)}) \otimes h_{(2)} x \tag{3}$$

for any $h \in H$ and $g \otimes x \in H \otimes X$.

Lemma 2.8 ([Mon93])**.** *Let H be a Hopf algebra and X a left H-modules. Then $H \otimes X$ is a left H-module with the action*

$$h(g \otimes x) := h_{(1)} g S^{-1}(h_{(3)}) \otimes h_{(2)} x. \tag{4}$$

for any $h \in H$ and $g \otimes x \in H \otimes X$. Moreover, an H–module/comodule X is an anti-Yetter-Drinfeld module iff its comodule structure map $\rho_X : X \to H \otimes X$ is a morphism of H–modules.

Remark 2.9. There is a similar characterization of YD modules. The left action (4) should simply be replaced by the left action

$$h(g \otimes x) := h_{(1)} g S(h_{(3)}) \otimes h_{(2)} x \tag{5}$$

Let us give a characterization of (A)YD modules in a concrete example. Let G be a not necessarily finite, discrete group and let $H = k[G]$ be its groups algebra over k with its standard Hopf algebra structure, i.e. $\Delta(g) = g \otimes g$ and $S(g) = g^{-1}$ for all $g \in G$.

Definition 2.10. *Let G be a group, and consider G acting on itself by conjugation. Then, a G-graded G-module is a G module M with a decomposition as $M = \oplus_{g \in G} M_g$ such that $h \cdot M_g \subseteq M_{hgh^{-1}}$, for all $h, g \in G$.*

Proposition 2.11. *The category of (Anti-)Yetter-Drinfeld modules over $k[G]$ is isomorphic to the category of G-graded vector spaces. In particular, the categories of YD modules and AYD modules are equivalent in this case.*

Proof. Let M be a $k[G]$–module/comodule. Denote its structure morphisms by $\mu : k[G] \otimes M \to M$ and $\rho : M \to k[G] \otimes M$. Since we assumed k is a field, M has a basis of the form $\{e^i\}_{i \in I}$ for some index set I. Since M is a $k[G]$–comodule one has

$$e^i_{(-1)} \otimes e^i_{(0)} = \sum_{j \in I} \sum_{g \in G} c^i_{j,g} (g \otimes e^j)$$

where only finitely many $c_{j,g}$ is non-zero. One can chose a basis $\{m^\lambda\}_{\lambda \in \Lambda}$ for M such that

$$m^\lambda_{(-1)} \otimes m^\lambda_{(0)} = \sum_\alpha c_{\lambda,\alpha} (g_\lambda \otimes m^\alpha)$$

and since all comodules are counital and $k[G]$ has a counit $\varepsilon(g) = 1$ for any $g \in G$ we see that

$$m^\lambda = \sum_\alpha c_{\lambda,\alpha} m^\alpha$$

implying $c_{\lambda,\alpha}$ is uniformly zero except $c_{\lambda,\lambda}$ which is 1. In other words, one can split M as $\bigoplus_{g\in G} M_g$ such that $\rho(x) = g \otimes x$ for any $x \in M_g$. Now assume M is an (A)YD module. Then since

$$(hx)_{(-1)} \otimes (hx)_{(0)} = hgh^{-1} \otimes hx$$

for any $x \in M_g$ and $h \in G$ one can see that $L_h : M_g \to M_{hgh^{-1}}$ where L_h is the k–vector space endomorphism of M coming from the left action of h. This observation implies that the category of AYD modules over $k[G]$ and the category of $G-$graded $k[G]-$modules are isomorphic. $\square$

Definition 2.12. *Let (C, Δ, ϵ) be a coalgebra and L a C-bicomodule with coactions ${}^L\varrho : L \to C \otimes L$ and $\varrho^L : L \to L \otimes C$. A k-linear map $\lambda : L \to C$ is called a* coderivation*, provided*

$$\Delta \circ \lambda = (C \otimes \lambda) \circ {}^L\varrho + (\lambda \otimes C) \circ \varrho^L.$$

Note that in the previous definition $\epsilon \circ \lambda = 0$.

In the following sections we will need the following general definition:

Definition 2.13. *Let $X_0, \dots, X_n$ be a finite set of H–bimodules. We define an H–bimodule structure on the k–module $X_0 \otimes \cdots \otimes X_n$ by*

$$\begin{aligned} h(x^0 \otimes \cdots \otimes x^n) =& h_{(1)}x^0 S^{-1}(h_{(2n+1)}) \otimes \cdots \otimes h_{(n)}x^{n-1}S^{-1}(h_{(n+2)}) \otimes h_{(n+1)}x^n, \\ (x^0 \otimes \cdots \otimes x^n)h =& x^0 \otimes \cdots \otimes x^{n-1} \otimes x^n h. \end{aligned}$$

for any $h \in H$ and $(x^0 \otimes \cdots \otimes x^n) \in X_0 \otimes \cdots \otimes X_n$. Checking the bimodule conditions is straightforward. We denote this bimodule by $X_0 \oslash \cdots \oslash X_n$.

3. Geometry

Definition 3.1. *A* differential ring *is a ring R equipped with a derivation*

$$d : R \to R,$$

i.e., an additive map satisfying the Leibniz rule:

$$d(ab) = ad(b) + bd(a), \ a, b \in R.$$

A differential ring which is also a domain, field, etc., will be called a differential domain, field, etc.

Definition 3.2. *A* differential module *over a differential ring (R, d) is a $R-$module M equipped with an additive map*

$$D : M \to M$$

satisfying

$$D(am) = aD(m) + d(a)m.$$

such a D will also be called a differential operator on M relative to d.

Example 3.3. *Let (R, d) be a differential ring. Then (R, d) is a differential module over itself. A differential module isomorphic to a direct sum of copies of (R, d) is said to be trivial.*

Definition 3.4. *A* differential ideal *of a differential ring R is a differential submodule of R itself, i.e., an ideal stable under d.*

Definition 3.5. *Let A be an associative unital algebra over a commutative ring $\Bbbk$. The* universal differential envelope *of A is a differential graded algebra $\Omega A = \oplus_{n=0}^{\infty} \Omega^n A$ over A (i.e. $A = \Omega^0 A$) defined as follows. The bimodule of one-forms is*

$$\Omega^1 A := \ker \mu = \{\sum_i a_i \otimes b_i \in A \otimes A \mid \sum_i a_i b_i = 0\}. \tag{6}$$

$\Omega^1 A$ has the obvious A-bimodule structure. The differential $d : A \to \Omega^1 A$ is defined as

$$d : a \mapsto 1 \otimes a - a \otimes 1 = (1 \otimes 1)a - a(1 \otimes 1). \tag{7}$$

One defines higher differential forms by iteration

$$\Omega^{n+1} A := \Omega^1 A \otimes_A \Omega^n A, \tag{8}$$

more precisely, ΩA is the tensor algebra of the A-bimodule $\Omega^1 A$, $\Omega A = T_A(\Omega^1 A)$. The differential d is extended to the whole of Ω by requiring the graded Leibniz rule (and that $d \circ d = 0$). This amounts to inserting the unit of the algebra A in all possible places in $\Omega^n A \subset A^{\otimes n+1}$ with alternating signs.

Definition 3.6. *Let A be a k–algebra. A differential calculus over A is a differential graded k–algebra (Ω^*, d) endowed with a morphism of algebras $\rho : A \to \Omega^0$. The differential d is assumed to have degree one.*

Since in our main examples we have $\Omega^0 = A$ and $\rho = id$, in the following we assume this is the case.

Definition 3.7. *Assume M is a left A–module. A morphism of k–modules $\nabla : M \to \Omega^1 \underset{A}{\otimes} M$ is called a connection with respect to the differential calculus (Ω^*, d) if one has a Leibniz rule of the form*

$$\nabla(am) = a\nabla(m) + d(a) \underset{A}{\otimes} m$$

for any $m \in M$ and $a \in A$.

Given any connection ∇ on M, there is a unique extension of ∇ to a map $\widehat{\nabla} : \Omega^* \underset{A}{\otimes} M \to \Omega^* \underset{A}{\otimes} M$ satisfying a graded Leibniz rule. It is given by

$$\widehat{\nabla}(\omega \otimes m) = d(\omega) \underset{A}{\otimes} m + (-1)^{|\omega|} \omega \nabla(m)$$

for any $m \in M$ and $\omega \in \Omega^*$.

We arrive to the following important definition:

Definition 3.8. *A connection $\nabla : M \to \Omega^1 \underset{A}{\otimes} M$ is called flat if its curvature $R := \widehat{\nabla}^2 = 0$.*

4. A=B and B=C

Our next goal is to find a noncommutative analogue of Proposition 2.11. To this end, we will replace the group algebra $\Bbbk[G]$ by a differential calculus $\mathcal{K}^*(H)$ naturally defined for any Hopf algebra H. The right analogue of G-graded vector spaces will be H–modules admitting flat connections with respect to the differential calculus $\mathcal{K}^*(H)$.

4.1. AYD.

Definition 4.1. *For each $n \geq 0$, let $\mathcal{K}^n(H) = H^{\oslash n+1}$. We define a differential $d : \mathcal{K}^n(H) \to \mathcal{K}^{n+1}(H)$ by*

$$\begin{aligned} d(h^0 \otimes \cdots \otimes h^n) = &- (1 \otimes h^0 \otimes \cdots \otimes h^n) + \sum_{j=0}^{n-1} (-1)^j (h^0 \cdots \otimes h^j_{(1)} \otimes h^j_{(2)} \otimes \cdots \otimes h^n) \\ &+ (-1)^n (h^0 \otimes \cdots \otimes h^{n-1} \otimes h^n_{(1)} S^{-1}(h^n_{(3)}) \otimes h^n_{(2)}). \end{aligned}$$

We also define an associative graded product structure by

$$\begin{aligned} &(x^0 \otimes \cdots \otimes x^n)(y^0 \otimes \cdots \otimes y^m) \\ &\quad = x^0 \otimes \cdots \otimes x^{n-1} \otimes x^n_{(1)} y^0 S^{-1}(h_{(2m+1)}) \otimes \cdots \otimes x^n_{(m)} y^{m-1} S^{-1}(x^n_{(m+2)}) \otimes x^n_{(m+1)} y^m \end{aligned}$$

for any $(x^0 \otimes \cdots \otimes x^n)$ in $\mathcal{K}^n(H)$ and $(y^0 \otimes \cdots \otimes y^m)$ in $\mathcal{K}^m(H)$.

Proposition 4.2 ([KK]). *$\mathcal{K}^*(H)$ is a differential graded k–algebra.*

Proof. For any $x \in \mathcal{K}^0(H)$ one has

$$d(x) = -(1 \otimes x) + (x_{(1)} S^{-1}(x_{(3)}) \otimes x_{(2)}) = [x, (1 \otimes 1)],$$

and for $(y \otimes 1)$ in $\mathcal{K}^1(H)$ and $x \in \mathcal{K}^0(H)$ we see

$$\begin{aligned} d(x(y \otimes 1)) = &d(x_{(1)} y S^{-1}(x_{(3)}) \otimes x_{(2)}) \\ = &- (1 \otimes x_{(1)} y S^{-1}(x_{(3)}) \otimes x_{(2)}) + (x_{(1)} y_{(1)} S^{-1}(x_{(5)}) \otimes x_{(2)} y_{(2)} S^{-1}(x_{(4)}) \otimes x_{(3)}) \\ &- (x_{(1)} y S^{-1}(x_{(5)}) \otimes x_{(2)} S^{-1}(x_{(4)}) \otimes x_{(3)}) \\ = &d(x)(y \otimes 1) + x d(y \otimes 1) \end{aligned}$$

We also see for $(x \otimes y)$ in $\mathcal{K}^1(H)$ the we have

$$\begin{aligned} d((x \otimes 1)y) = d(x \otimes y) = &-(1 \otimes x \otimes y) + (x_{(1)} \otimes x_{(2)} \otimes y) - (x \otimes y_{(1)} S^{-1}(y_{(3)}) \otimes y_{(2)}) \\ = &- (1 \otimes x \otimes 1)y + (x_{(1)} \otimes x_{(2)} \otimes 1)y - (x \otimes 1 \otimes 1)y + (x \otimes 1)(1 \otimes y) \\ &- (x \otimes 1)(y_{(1)} S^{-1}(y_{(3)}) \otimes y_{(2)}) \\ = &d(x \otimes 1)y - (x \otimes 1)d(y). \end{aligned}$$

Note that with the product structure on $\mathcal{K}^*(H)$ one has

$$(x^0 \otimes \cdots \otimes x^n) = (x^0 \otimes 1) \cdots (x^{n-2} \otimes 1)(x^{n-1} \otimes 1)x^n$$

for any $x^0 \otimes \cdots \otimes x^n$ in $\mathcal{K}^n(H)$. Now, one can inductively show that

$$d(\Psi\Phi) = d(\Psi)\Phi + (-1)^{|\Psi|} \Psi d(\Phi)$$

for any Ψ and Φ in $\mathcal{K}^*(H)$. Since the algebra is generated by degree zero and degree one terms, all that remains is to show that for all $x \in H$ we have $d^2(x) = 0$ and $d^2(x \otimes 1) = 0$. For the first assertion we see that

$$\begin{aligned} d^2(x) &= -d(1 \otimes x) + d(x_{(1)}S^{-1}(x_{(3)}) \otimes x_{(2)}) \\ &=(1 \otimes 1 \otimes x) - (1 \otimes 1 \otimes x) + (1 \otimes x_{(1)}S^{-1}(x_{(3)}) \otimes x_{(2)}) - (1 \otimes x_{(1)}S^{-1}(x_{(3)}) \otimes x_{(2)}) \\ &\quad + (x_{(1)(1)}S^{-1}(x_{(3)(2)}) \otimes x_{(1)(2)}S^{-1}(x_{(3)(1)}) \otimes x_{(2)}) - (x_{(1)}S^{-1}(x_{(3)}) \otimes x_{(2)(1)}S^{-1}(x_{(2)(3)}) \otimes x_{(2)(2)}) \\ &=0 \end{aligned}$$

for any $x \in H$. For the second assertion we see

$$\begin{aligned} &d^2(x \otimes 1) = -d(1 \otimes x \otimes 1) + d(x_{(1)} \otimes x_{(2)} \otimes 1) - d(x \otimes 1 \otimes 1) \\ &= -(1 \otimes 1 \otimes x \otimes 1) + (1 \otimes 1 \otimes x \otimes 1) - (1 \otimes x_{(1)} \otimes x_{(2)} \otimes 1) + (1 \otimes x \otimes 1 \otimes 1) \\ &\quad - (1 \otimes x_{(1)} \otimes x_{(2)} \otimes 1) + (x_{(1)} \otimes x_{(2)} \otimes x_{(3)} \otimes 1) - (x_{(1)} \otimes x_{(2)} \otimes x_{(3)} \otimes 1) + (x_{(1)} \otimes x_{(2)} \otimes 1 \otimes 1) \\ &\quad + (1 \otimes x \otimes 1 \otimes 1) - (x_{(1)} \otimes x_{(2)} \otimes 1 \otimes 1) + (x \otimes 1 \otimes 1 \otimes 1) - (x \otimes 1 \otimes 1 \otimes 1) \\ &=0 \end{aligned}$$

for any $(x \otimes 1)$ in $\mathcal{K}^*(H)$. The result follows. □

Note that the calculus $\mathcal{K}^*(H)$ is determined by

(1) the H–bimodule $\mathcal{K}^1(H) = H \otimes H$
(2) the differential $d_0 : H \to H \otimes H$ and $d_1 : H \otimes H \to H \otimes H \otimes H$ and
(3) the Leibniz rule $d(\Psi\Phi) = d(\Psi)\Phi + (-1)^{|\Psi|}\Psi d(\Phi)$.

Theorem 4.3 ([KK])**.** *The category of AYD modules over H is isomorphic to the category of H–modules admitting a flat connection with respect to the differential calculus $\mathcal{K}^*(H)$.*

Proof. Assume M is a H–module which admits a morphism of k–modules of the form $\nabla : M \to \mathcal{K}^1(H) \underset{H}{\otimes} M \cong H \otimes M$. Define $\rho_M(m) = \nabla(m) + (1 \otimes m)$ and denote $\rho_M(m)$ by $(m_{(-1)} \otimes m_{(0)})$ for any $m \in M$. First we see that

$$\nabla(hm) =(hm)_{(-1)} \otimes (hm)_{(0)} - (1 \otimes hm)$$

and also

$$\begin{aligned} d(h) \underset{H}{\otimes} m + h\nabla(m) =& -(1 \otimes hm) + (h_{(1)}S^{-1}(h_{(3)}) \otimes h_{(2)}m) \\ &+ (h_{(1)}m_{(-1)}S^{-1}(h_{(3)}) \otimes h_{(2)}m_{(0)}) - (h_{(1)}S^{-1}(h_{(3)}) \otimes h_{(2)}m) \\ =&(h_{(1)}m_{(-1)}S^{-1}(h_{(3)}) \otimes h_{(2)}m_{(0)}) - (1 \otimes hm) \end{aligned}$$

for any $h \in H$ and $m \in M$. This means ∇ is a connection iff the H–module M together with $\rho_X : M \to H \otimes M$ satisfy the AYD condition. The flatness condition will hold iff for any $m \in M$ one has

$$\begin{aligned} \widehat{\nabla}^2(m) =&d(m_{(-1)} \otimes 1) \underset{H}{\otimes} m_{(0)} - (m_{(-1)} \otimes 1)\nabla(m_{(0)}) - d(1 \otimes 1)m + (1 \otimes 1)\nabla(m) \\ =&(m_{(-1)(1)} \otimes m_{(-1)(2)} \otimes m_{(0)}) - (m_{(-1)} \otimes m_{(0)(-1)} \otimes m_{(0)(0)}) = 0, \end{aligned}$$

meaning ∇ is flat iff $\rho_M : M \to H \otimes M$ defines a coassociative coaction of H on M. □

4.2. **YD.** Instead of the AYD condition, one can consider the YD condition and form a differential calculus $\widehat{\mathcal{K}}^*(H)$ using the YD condition.

Definition 4.4. *As before, assume H is a Hopf algebra, but this time we do not require the antipode to be invertible. We define a new differential calculus $\widehat{\mathcal{K}}^*(H)$ over H as follows: let $\widehat{\mathcal{K}}^n(H) = H^{\otimes n+1}$ and define the differentials as*

$$d(x^0 \otimes \cdots \otimes x^n) = -(1 \otimes x^0 \otimes \cdots \otimes x^n) + \sum_{j=0}^{n-1} (-1)^j (x^0 \otimes \cdots \otimes x^j_{(1)} \otimes x^j_{(2)} \otimes \cdots x^n) \\ + (-1)^n (x^0 \otimes \cdots \otimes x^{n-1} \otimes x^n_{(1)} S(x^n_{(3)}) \otimes x^n_{(2)})$$

for any $x^0 \otimes \cdots \otimes x^n$ in $\widehat{\mathcal{K}}^n(H)$. The multiplication is defined as

$$(x^0 \otimes \cdots \otimes x^n)(y^0 \otimes \cdots \otimes y^m) \\ = x^0 \otimes \cdots \otimes x^{n-1} \otimes x^n_{(1)} y^0 S(x_{(2m+1)}) \otimes \cdots \otimes x^n_{(m)} y^{m-1} S(x^n_{(m+1)}) \otimes x^n_{(m+1)} y^m$$

for any $x^0 \otimes \cdots \otimes x^n$ and $y^0 \otimes \cdots \otimes y^m$ in $\widehat{\mathcal{K}}^*(H)$.

The proofs of the following facts are similar to the corresponding statements for the differential calculus $\mathcal{K}^*(H)$ and AYD modules.

Proposition 4.5 ([KK])**.** *$\widehat{\mathcal{K}}^*(H)$ is a differential graded k–algebra.*

Theorem 4.6 ([KK])**.** *The category of YD modules over H is isomorphic to the category of H–modules admitting a flat connection with respect to the differential calculus $\widehat{\mathcal{K}}^*(H)$.*

5. Algebra II

Let A be an associative unital algebra over a commutative ring $\Bbbk$.

Definition 5.1. *An A-bimodule $\mathcal{C}$ is called an A-*coring *iff there are A-bimodule maps $\Delta_{\mathcal{C}} : \mathcal{C} \to \mathcal{C} \otimes_A \mathcal{C}$, $\epsilon_{\mathcal{C}} : \mathcal{C} \to A$ rendering the following diagrams commutative:*

(9)
$$\begin{array}{ccc} \mathcal{C} & \xrightarrow{\Delta_{\mathcal{C}}} & \mathcal{C} \otimes_A \mathcal{C} \\ {\scriptstyle \Delta_{\mathcal{C}}}\downarrow & & \downarrow{\scriptstyle \mathcal{C} \otimes_A \Delta_{\mathcal{C}}} \\ \mathcal{C} \otimes_A \mathcal{C} & \xrightarrow{\Delta_{\mathcal{C}} \otimes_A \mathcal{C}} & \mathcal{C} \otimes_A \mathcal{C} \otimes_A \mathcal{C}\,, \end{array}$$

(10)
$$\begin{array}{ccc} \mathcal{C} & \xrightarrow{\Delta_{\mathcal{C}}} & \mathcal{C} \otimes_A \mathcal{C} \\ & {\scriptstyle \simeq}\searrow & \downarrow{\scriptstyle \epsilon_{\mathcal{C}} \otimes_A \mathcal{C}} \\ & & A \otimes_A \mathcal{C}\,, \end{array} \qquad \begin{array}{ccc} \mathcal{C} & & \\ {\scriptstyle \Delta_{\mathcal{C}}}\downarrow & \searrow{\scriptstyle \simeq} & \\ \mathcal{C} \otimes_A \mathcal{C} & \xrightarrow{\mathcal{C} \otimes_A \epsilon_{\mathcal{C}}} & \mathcal{C} \otimes_A A\,. \end{array}$$

As for coalgebras, $\Delta_{\mathcal{C}}$ is called a *coproduct* and $\epsilon_{\mathcal{C}}$ is called a *counit*. The coring $\mathcal{C} = A \otimes A$ is known as the *Sweedler* or *canonical* coring associated to the ring extension $k \to A$. Note in passing that A itself is an A-coring. Thus the notion of a coring includes that of a ring. In the case of a general A-coring $\mathcal{C}$ we can distinguish elements which have above properties and thus arrive at the following

Definition 5.2. *An element g of an A-coring $\mathcal{C}$ is called a* group-like element *provided that*

$$\Delta_{\mathcal{C}}(g) = g \otimes_A g, \qquad \epsilon_{\mathcal{C}}(g) = 1.$$

We will study corepresentation of corings. Specifically,

Definition 5.3. *A right A-module M together with a right A-linear map $\varrho^M : M \to M \otimes_A \mathcal{C}$ rendering the following diagrams*

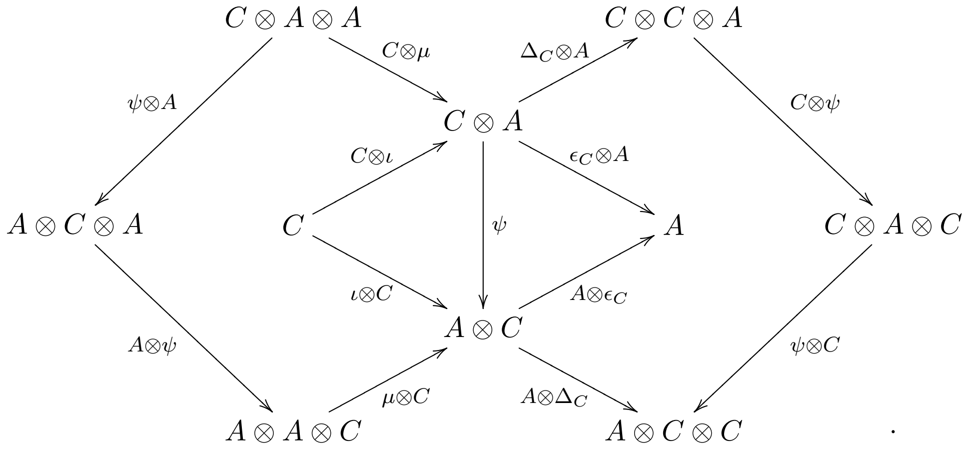
$$\begin{array}{ccc} M & \xrightarrow{\varrho^M} & M \otimes_A \mathcal{C} \\ {\scriptstyle \varrho^M}\downarrow & & \downarrow{\scriptstyle M \otimes_A \Delta_{\mathcal{C}}} \\ M \otimes_A \mathcal{C} & \xrightarrow{\varrho^M \otimes_A \mathcal{C}} & M \otimes_A \mathcal{C} \otimes_A \mathcal{C} \end{array} \qquad \begin{array}{ccc} M & \xrightarrow{\varrho^M} & M \otimes_A \mathcal{C} \\ & {\scriptstyle \simeq}\searrow & \downarrow{\scriptstyle M \otimes_A \epsilon_{\mathcal{C}}} \\ & & M \otimes_A A \end{array}$$

commutative is called a right $\mathcal{C}$-comodule*.*

As for coalgebras, the map ϱ^M is called a *coaction*. When needed one refers to map ϱ^M which obeys the square but not the triangle condition in Definition 5.3 as to a *non-counital coaction*.

Definition 5.4. *Let (A, μ, ι) be an algebra and $(C, \Delta_C, \epsilon_C)$ a coalgebra. A map $\psi : C \otimes A \to A \otimes C$ is said to be an* entwining map *provided the commutativity of the following* bow-tie diagram

(11)



In this case, C and A are said to be entwined *by ψ, and the triple (A, C, ψ) is called an* entwining structure.

Proposition 5.5 ([B])**.** *Let A and C as in the Definition above. Let $\psi : C \otimes A \to A \otimes C$ be entwining map. Then, there is a A-coring structure on $\mathcal{C} = A \otimes C$.*

Proof. $\mathcal{C}$ has an obvious left A-multiplication and counit:

$$a(a' \otimes c) := aa' \otimes c,\ \epsilon_{\mathcal{C}} := A \otimes \epsilon. \tag{12}$$

In view of the identification $\mathcal{C} \otimes_A \mathcal{C} = (A \otimes C) \otimes_A (A \otimes C) \simeq A \otimes C \otimes C$, the map $\Delta_{\mathcal{C}} := A \otimes \Delta$, is an obvious candidate for a coproduct for $\mathcal{C}$. To make $A \otimes C$ into an A-coring with this already specified structures we need to introduce a suitable right A-multiplication. Since $A \otimes C$ must be an A-bimodule, any such a right A-multiplication is determined by a map $\theta : C \otimes A \to A \otimes C$,

$$\psi(c \otimes a) := (1 \otimes c)a. \tag{13}$$

The map θ must satisfy (four) conditions corresponding to unitality and associativity of the right A-multiplication and to the facts that both $\Delta_{\mathcal{C}}$ and $\epsilon_{\mathcal{C}}$ are right A-linear maps. It is straightforward to check that this conditions are equivalent to the commutativity of a bow-tie diagram, and therefore, taking $\theta = \psi$ we get the coring structure desired. □

Right comodules of the A-coring $\mathcal{C} = A \otimes C$ associated to an entwining structure are known as *entwined modules* (or (A, C, ψ)-entwined modules). It is easy to check that these modules are simply k-modules M which are both right A-modules with multiplication $\varrho_M : M \otimes A \to M$ and right C-comodules with comultiplication $\varrho^M : M \to M \otimes C$ rendering commutative the following diagram

$$\begin{array}{ccccc} M \otimes A & \xrightarrow{\varrho^M \otimes A} & M \otimes C \otimes A & \xrightarrow{M \otimes \psi} & M \otimes A \otimes C \\ \downarrow{\scriptstyle \varrho_M} & & & & \downarrow{\scriptstyle \varrho_M \otimes C} \\ M & & \xrightarrow{\quad\quad\varrho^M\quad\quad} & & M \otimes C\,. \end{array} \tag{14}$$

6. A=C

The following remarkable result of Roiter [Roi] states that any differential graded algebra of certain kind comes from a coring with a group-like element.

Theorem 6.1 ([Roi]).
- *Any A-coring $\mathcal{C}$ with a group-like element g gives rise to a differential graded algebra ΩA defined as follows: $\Omega^1 A = \ker \epsilon_{\mathcal{C}}$, $\Omega^{n+1} A = \Omega^1 A \otimes_A \Omega^n A$ and the multiplication is given by the tensor product (i.e., ΩA is the tensor algebra $\Omega A = T_A(\ker \epsilon_{\mathcal{C}})$). The differential is defined by $d(a) = ga - ag$, for all $a \in A$, and, for all $c^1 \otimes_A \cdots \otimes_A c^n \in (\ker \epsilon_{\mathcal{C}})^{\otimes_A n}$,*

$$\begin{aligned} d(c^1 \otimes_A \cdots \otimes_A c^n) &= g \otimes_A c^1 \otimes_A \cdots \otimes_A c^n + (-1)^{n+1} c^1 \otimes_A \cdots \otimes_A c^n \otimes_A g \\ &+ \sum_{i=1}^{n} (-1)^i c^1 \otimes_A \cdots \otimes_A c^{i-1} \otimes_A \Delta_{\mathcal{C}}(c^i) \otimes_A c^{i+1} \otimes_A \cdots \otimes_A c^n. \end{aligned}$$

- *A differential graded algebra ΩA over A such that $\Omega A = T_A(\Omega^1 A)$ (that is $\Omega^{n+1} A = \Omega^1 A \otimes_A \Omega^n A$; a differential graded algebra with this property is said to be* semi-free*), defines a coring with a grouplike element.*
- *The operations described in items (1) and (2) are mutual inverses.*

Proof. (1) and (3) are proven by straightforward calculations, so we only indicate how to construct a coring from a differential graded algebra (i.e. sketch the proof of (2)). Starting with ΩA, define

$$\mathcal{C} = Ag \oplus \Omega^1 A,$$

where g is an indeterminate. In other words we define $\mathcal{C}$ to be a direct sum of A and $\Omega^1 A$ as a left A-module. We now need to specify a compatible right A-module structure. This is defined by

$$(ag + \omega)a' := aa'g + ada' + \omega a'.$$

The coproduct is specified by

$$\Delta_{\mathcal{C}}(ag) = ag \otimes_A g, \qquad \Delta_{\mathcal{C}}(\omega) = g \otimes_A \omega + \omega \otimes_A g - d(\omega),$$

and the counit

$$\epsilon_{\mathcal{C}}(ag + \omega) := a,$$

for all $a \in A$ and $\omega \in \Omega^1 A$. Note that this structure is chosen in such a way that g becomes the required group-like element. □

The following theorem is proven via straightforward calculation and it will provide us with an identification which will allow us to generalize the results in the previous sections.

Theorem 6.2 ([B]). *Assume that $\mathcal{C}$ is an A-coring with a group-like element g, and write ΩA for the associated differential graded algebra.*

- *If (M, ϱ^M) is a right $\mathcal{C}$-comodule, then the map*

$$\nabla : M \to M \otimes_A \Omega^1 A, \qquad m \mapsto \varrho^M(m) - m \otimes_A g,$$

  is a flat connection.
- *If M is a right A-module with a flat connection $\nabla : M \to M \otimes_A \Omega^1 A$, then M is a right $\mathcal{C}$-comodule with the coaction*

$$\varrho^M : M \to M \otimes_A \mathcal{C}, \qquad m \mapsto \nabla(m) + m \otimes_A g.$$

- *The operations described in items (1) and (2) are mutual inverses.*

Yetter-Drinfeld modules and anti-Yetter-Drinfeld modules are objects known as *Hopf-type modules*. They are characterized by having action and a coaction of a Hopf algebra or, more generally, with an action of an algebra and a coaction of a coalgebra which are compatible one with the other through an action/coaction of a Hopf algebra.

Essentially, compatibility conditions for all known Hopf-type modules can be recast in the form of an entwining structure and are of the form of equation (14).

If we restrict ourselves to the example of anti-Yetter-Drinfeld modules and apply the general theory of this section to this particular case, we recover our fundamental Theorem 4.3 on Section 4. We provide the details below.

Take $A = C = H$, where H is a Hopf algebra with a bijective antipode S. Then one can define an entwining map $\psi : H \otimes H \to H \otimes H$ by

$$\psi(c \otimes a) = a_{(2)} \otimes S^{-1}(a_{(1)}) c a_{(3)}, \tag{15}$$

for all $a, c \in H$. The fact that ψ is an entwining map follows explicitly because the antipode is an anti-algebra and anti-coalgebra map.

Consequently, there is an H-coring $\mathcal{C} = H \otimes H$ with the right H-multiplication

$$(b \otimes c)a = ba_{(2)} \otimes S^{-1}(a_{(1)}) c a_{(3)}. \tag{16}$$

The compatibility (14) for right H-module and H-comodule M comes out as, for all $a \in H$,

$$\varrho^M(ma) = m_{(0)} a_{(2)} \otimes S^{-1}(a_{(1)}) m_{(1)} a_{(3)}, \tag{17}$$

where $\varrho^M(m) = m_{(0)} \otimes m_{(1)}$ is the C-coaction on M, i.e. we have that

Lemma 6.3. *Entwined modules for* (15) *coincide with (right-right) anti-Yetter-Drinfeld modules.*

Since $C = H$ is a Hopf algebra, 1_H is a group-like element in H, and hence $1_H \otimes 1_H$ is a group-like element in the H-coring $\mathcal{C}$. By the Roiter theorem there is the associated differential graded algebra and by Theorem 6.2 anti-Yetter-Drinfeld modules are modules with a flat connection with respect to this differential graded structure. Explicitly,

$$\Omega^1 H = \{\sum_i a_i \otimes c_i \in H \otimes H \mid \sum_i a_i \epsilon(c_i) = 0\}.$$

Thus, in particular $\Omega^1 H = H \otimes H^+$, where $H^+ := \ker \epsilon$, provided H is a flat k-module. The right H-action on $\Omega^1 H$ is given by the formula (16). The differential comes out as

$$d(a) = (1 \otimes 1)a - a(1 \otimes 1) = a_{(2)} \otimes S^{-1}(a_{(1)}) a_{(3)} - a \otimes 1.$$

Remark 6.4. Anti-Yetter-Drinfeld modules are an example of *(α, β)-equivariant C-comodules*. In this case A is a bialgebra, C is an A-bimodule coalgebra, $\alpha : A \to A$ is a bialgebra map and $\beta : A \to A$ is an anti-bialgebra map (i.e. β is both an anti-algebra and anti-coalgebra map). All these data give rise to an entwining map $\psi : C \otimes A \to A \otimes C$ defined by

$$\psi(c \otimes a) = a_{(2)} \otimes \beta(a_{(1)}) c \alpha(a_{(3)}).$$

We do not work out explicitly the form of the corresponding coring $\mathcal{C} = A \otimes C$ and of the compatibility condition (14), which are rather straightforward. If, in addition, C has a group-like element e, then $1 \otimes e$ is a group-like element in $\mathcal{C}$ and we can derive the associated differential graded algebra.

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